ALL HEREDITARY TORSION THEORIES ARE DIFFERENTIAL

CHRISTIAN LOMP AND JOHN VAN DEN BERG

Abstract. Let α and β be automorphisms on a ring R and $\delta : R \to R$ an (α, β)-derivation. It is shown that if $\mathfrak{F}$ is a right Gabriel filter on R then $\mathfrak{F}$ is δ-invariant if it is both α and β-invariant. A consequence of this result is that every hereditary torsion theory on the category of right R-modules is differential in the sense of Bland(2006). This answers in the affirmative a question posed by Vaš(2007) and strengthens a result due to Golan(1981) on the extendability of a derivation map from a module to its module of quotients at a hereditary torsion theory.

Introduction

In [4, Corollary 1] Golan proves that if R is a ring endowed with a derivation map δ, M a right R-module with δ-derivation map $d : M \to M$, and τ a hereditary torsion theory on the category of right R-modules such that $d[t_\tau(M)] \subseteq t_\tau(M)$, then d extends to a δ-derivation map on the module of quotients $Q_\tau(M)$ of M at τ. This result is sharpened by Bland [3] who calls a hereditary torsion theory *differential* if the aforementioned containment $d[t_\tau(M)] \subseteq t_\tau(M)$ holds for all M and δ-derivations $d : M \to M$, and then proves that the differential hereditary torsion theories are *precisely* those hereditary torsion theories τ for which all δ-derivation maps are extendable in the above sense [3, Proposition 2.3].

In a recent paper Vaš [9] identifies several classes of hereditary torsion theories that are differential and poses the question [9, page 852]: is every hereditary torsion theory differential? In this paper we shall answer this question in the affirmative by proving a slightly more general result on skew-derivations.

1. Preliminaries

Throughout this paper R will denote an associative ring with identity and Mod-R the category of unital right R-modules. If $N, M \in$ Mod-R we write $N \leqslant M$ if N is a submodule of M. If X, Y are nonempty subsets of M we define $(X : Y) = \{r \in R \mid Yr \subseteq X\}$. If $X, Y \subseteq R$, then $(X : Y)$ will be taken as above with R interpreted as a right module over itself.

If $d : R \to R$ is an additive map, we say that a nonempty family $\mathfrak{F}$ of right ideals of R is *d-invariant* if, for any $I \in \mathfrak{F}$, there exists $J \in \mathfrak{F}$ such that $d[J] \subseteq I$.

A *hereditary torsion theory* on Mod-R is a pair $\tau = (\mathcal{T}, \mathcal{F})$ where $\mathcal{T}$ is a class of right R-modules that is closed under submodules, homomorphic images, direct sums and module extensions, and $\mathcal{F}$ comprises all $N \in$ Mod-R such that $\mathrm{Hom}_R(M, E(N)) = 0$ for all $M \in \mathcal{T}$. The modules in $\mathcal{T}$ are called *τ-torsion* and those in $\mathcal{F}$ *τ-torsion-free*. For each $M \in$ Mod-R there is a largest τ-torsion submodule of M that we shall denote by $t_\tau(M)$.

A nonempty family $\mathfrak{F}$ of right ideals of a ring R is called a *right Gabriel filter* on R if it satisfies the following two conditions:

(G1) if $I \in \mathfrak{F}$ then $(I : r) \in \mathfrak{F}$ for all $r \in R$;
(G2) if $I \in \mathfrak{F}$ and $J \leqslant R_R$ is such that $(J : a) \in \mathfrak{F}$ for all $a \in I$, then $J \in \mathfrak{F}$.

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If $\tau = (\mathcal{T}, \mathcal{F})$ is an arbitrary hereditary torsion theory on Mod-R, then

$$\mathfrak{F}_\tau := \{I \leqslant R_R \mid R/I \in \mathcal{T}\}$$

is a right Gabriel filter on Mod-R. If $\mathfrak{F}$ is an arbitrary right Gabriel filter on R, then there is a (unique) hereditary torsion theory, denoted by $\tau_{\mathfrak{F}}$, whose torsion class $\mathcal{T}$ is given by

$$\mathcal{T} = \{M \in \text{Mod-}R \mid (0:x) \in \mathfrak{F} \text{ for all } x \in M\}.$$

For every ring R the maps $\tau \mapsto \mathfrak{F}_\tau$ and $\mathfrak{F} \mapsto \tau_{\mathfrak{F}}$ constitute a pair of mutually inverse maps between the sets of hereditary torsion theories on Mod-R and right Gabriel filters on R (see [8, Theorem VI.5.1, page 146]).

We refer the reader to [1], [5] and [8] for further background information on torsion theories and Gabriel filters.

2. Differential torsion theories

Let α and β be automorphisms on a ring R. An additive map $\delta : R \to R$ is called an (α, β)-*derivation on* R if

$$\delta(ab) = \delta(a)\alpha(b) + \beta(a)\delta(b) \text{ for all } a, b \in R.$$

If α and β coincide with the identity map on R, it is customary to omit the prefix (α, β) and speak simply of a *derivation on* R.

If δ is a derivation on R and $M \in$ Mod-R, then an additive map $d : M \to M$ is called a δ-*derivation on* M if

$$d(xr) = d(x)r + x\delta(r) \text{ for all } x \in M \text{ and } r \in R.$$

The following result is due to Bland [3, Lemma 1.5].

Theorem 1. *Let δ be a derivation on a ring R. The following conditions are equivalent for a hereditary torsion theory τ on* Mod-R*:*
(i) *for every $M \in$ Mod-R and δ-derivation d on M, $d[t_\tau(M)] \subseteq t_\tau(M)$;*
(ii) *$\mathfrak{F}_\tau$ is δ-invariant.*

A hereditary torsion theory τ satisfying the equivalent conditions of Theorem 1 is called *differential*. Differential torsion theories have the important property that every δ-derivation on a module M extends uniquely to a derivation on the module of quotients of M at the given torsion theory, as shown in [4, Corollary 1] and [3, Proposition 2.1].

We refer the reader to [4], [3], [2] and [9] as sources of further information on torsion theories in the context of rings endowed with a derivation map.

We now prove our main theorem from which it shall follow that *all* hereditary torsion theories are differential thus answering in the affirmative a question posed by Vaš [9, page 852].

Theorem 2. *Let α and β be automorphisms on a ring R and $\delta : R \to R$ an (α, β)-derivation on R. If $\mathfrak{F}$ is a right Gabriel filter on R that is both α and β-invariant, then $\mathfrak{F}$ is δ-invariant.*

Proof. Let $I \in \mathfrak{F}$. We have to show that there exists $J \in \mathfrak{F}$ with $\delta[J] \subseteq I$. Since $\mathfrak{F}$ is α and β-invariant, $L = \alpha^{-1}[I] \cap \beta^{-1}[I] \in \mathfrak{F}$. Let

$$J = \{x \in L \mid \delta(x) \in I\}.$$

Since δ is additive, J is an additive subgroup of R. Take any $x \in J$ and $r \in R$. Then $\delta(xr) = \delta(x)\alpha(r) + \beta(x)\delta(r) \in I$, because $\beta(x) \in \beta[L] \subseteq I$ and $\delta(x) \in I$ by definition, whence $xr \in J$. We conclude that J is a right ideal.

For each $x \in L$ we claim that

$$(L : \alpha^{-1}(\delta(x))) \subseteq (J : x). \tag{1}$$

To prove (1) note that

$$y \in (L : \alpha^{-1}(\delta(x))) \Leftrightarrow \alpha^{-1}(\delta(x))y \in L \Rightarrow \delta(x)\alpha(y) \in \alpha[L] \subseteq I.$$

Since $x \in L$ we also have $\beta(x)\delta(y) \in I$, whence

$$\delta(xy) = \delta(x)\alpha(y) + \beta(x)\delta(y) \in I.$$

Inasmuch as $xy \in L$ and $\delta(xy) \in I$, we have $xy \in J$. This establishes (1).

Since $L \in \mathfrak{F}$, it follows from (G1) that $(L : \alpha^{-1}(\delta(x))) \in \mathfrak{F}$. Hence by (1), $(J : x) \in \mathfrak{F}$ for all $x \in L$. We conclude from (G2) that $J \in \mathfrak{F}$, as required. □

Since every nonempty family of right ideals of R is trivially invariant with respect to the identity map on R, the following corollary follows immediately from the two previous theorems.

Corollary 3. *Let R be any ring endowed with a derivation map $\delta : R \to R$. Then every hereditary torsion theory on* Mod-R *is differential.*

Remark: The problem of extending derivations to rings of quotients of algebras over fields is a special case of extending Hopf algebra actions to rings of quotients. Let H be a Hopf algebra over a field k acting on a k-algebra A and let $\mathfrak{F}$ be a right Gabriel filter on A with associated ring of quotients $Q_{\mathfrak{F}}(A)$. Denote by $\lambda_h(a) := h \cdot a$ the action of an element $h \in H$ to $a \in A$, which is an additive map. A necessary condition for extending the H-action on A to $Q_{\mathfrak{F}}(A)$ is that H act $\mathfrak{F}$-continuously, i.e., $\mathfrak{F}$ is λ_h-invariant for all $h \in H$ (see [6]). The terminology is justified if A is considered a topological ring whose topology is induced by $\mathfrak{F}$ and interpreting the condition $\lambda_h^{-1}(I) \in \mathfrak{F}$ for any $I \in \mathfrak{F}$ as continuity.

In [6] it is shown that if the Hopf algebra H is pointed, i.e., all simple subcoalgebras are one-dimensional, then H always acts $\mathfrak{F}$-continuously on an algebra A. In the case of a derivation δ of A one might consider the enveloping algebra H of the 1-dimensional Lie algebra which acts as δ on A. Here $H = k[X]$ is a pointed Hopf algebra and hence the action extends to $Q_{\mathfrak{F}}(A)$.

A purely coalgebraic version was given by Rumynin in [7]: a coalgebra C is said to *measure* an algebra A if there exists an action $\cdot : C \otimes A \to A$ such that for all $c \in C$ and $a, b \in A$, $c \cdot (ab) = \sum_{(c)} (c_1 \cdot a)(c_2 \cdot b)$ and $c \cdot 1 = \epsilon(c)1$ where $\Delta(c) = \sum_{(c)} c_1 \otimes c_2 \in C \otimes C$ denotes the comultiplication of C and $\epsilon(c)$ the counit of C. Rumynin proved that if every simple subcoalgebra of C is 1-dimensional and measures A $\mathfrak{F}$-continuously, then C also measures A $\mathfrak{F}$-continuously.

Let α and β be automorphisms on A and $\delta : A \to A$ an (α, β)-derivation. Let C be the 4-dimensional vector space over k with basis $1, g, h$ and x which becomes a coalgebra with comultiplication

$$\Delta(1) = 1 \otimes 1, \ \Delta(g) = g \otimes g, \ \Delta(h) = h \otimes h, \ \Delta(x) = x \otimes g + h \otimes x$$

and counit $\epsilon(1) = \epsilon(g) = \epsilon(h) = 1$ and $\epsilon(x) = 0$. Define the measuring $\cdot : C \otimes A \to A$ by $1 \cdot a = a$, $g \cdot a = \alpha(a)$, $h \cdot a = \beta(a)$ and $x \cdot a = \delta(a)$. The simple subcoalgebras of C are $k1$, kg and kh which are 1-dimensional. If $\mathfrak{F}$ is α and β-invariant, then by [7, Lemma 9], C acts $\mathfrak{F}$-continuously on A, i.e., $\mathfrak{F}$ is δ-invariant. This yields another proof of Theorem 2 for the special case of algebras over fields.

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Departamento de Matemática Pura, Universidade do Porto, Portugal
E-mail address: `clomp@fc.up.pt`

School of Mathematical Sciences, University of KwaZulu-Natal, Pietermaritzburg, South Africa
E-mail address: `vandenberg@ukzn.ac.za`