Coupled Cell Networks: Hopf bifurcation and Interior Symmetry

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We consider interior symmetric coupled cell networks where a group of permutations of a subset of cells partially preserves the network structure. In this setup, the full analogue of the Equivariant Hopf Theorem for networks with symmetries was obtained by Antoneli, Dias and Paiva (Hopf Bifurcation in Coupled Cell Networks with Interior Symmetries, *SIAM J. Appl. Dynam. Sys.* (2007) to appear). In this note we present an alternative proof of this result using center manifold reduction.

Keywords: Hopf bifurcation; center manifold reduction; coupled cell systems.

1. Introduction

Coupled cell systems are networks of dynamical systems (the cells) that are coupled together. Relevant aspects in the study of the dynamics of these systems can be encoded by a directed graph (*coupled cell network*): the nodes represent the cells and the edges indicate which cells are coupled and if the couplings are of the same type. We consider a special class of non-symmetric networks – the *interior symmetric* coupled cell networks. These networks admit a subset $\mathcal{S}$ of the cells such that the cells in $\mathcal{S}$ together with all the edges directed to them form a subnetwork which possesses a non-trivial symmetry group $\Sigma_{\mathcal{S}}$. Here, we follow the theory of Stewart *et al.*[2,4,6]

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The local synchrony-breaking bifurcations in a coupled cell system occur when a synchronous state loses stability and bifurcates to a state with less synchrony. Such bifurcations can be considered to be a generalisation of symmetry-breaking bifurcations in symmetric coupled cell systems. See Golubitsky *et al.*[3] An analogue of the Equivariant Hopf Theorem for coupled cell systems with interior symmetries was obtained by Golubitsky *et al.*[2] proving the existence of states whose linearizations on certain subsets of cells, near bifurcation, are superpositions of synchronous states with states having *spatial symmetries*. Antoneli *et al.*[1] extended this result obtaining states whose linearizations on certain subsets of cells, near bifurcation, are superpositions of synchronous states with states having *spatio-temporal symmetries*, that is, corresponding to *interiorly* $\mathbf{C}$-*axial* subgroups of $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$. The proof of this result uses a modification of the Lyapunov-Schmidt reduction to arrive at a situation where the proof of the Standard Hopf Bifurcation Theorem can be applied. In this note, we present an alternative proof using center manifold reduction. This approach can be useful in the development of normal form theory aiming at the study of the stability of such periodic solutions.

In Section 2 we recall the definition of interior symmetry and the structure of coupled cell systems associated with interior symmetric networks. In Section 3 we state the Interior Symmetry-Breaking Hopf Bifurcation Theorem and prove it using Center Manifold Reduction.

2. Coupled Cell Networks with Interior Symmetry

Given a coupled cell network $\mathcal{G}$, the associated coupled cell systems are dynamical systems compatible with the architecture of $\mathcal{G}$. More specifically, each cell c is equipped with a phase space P_c, and the total phase space of the network is the cartesian product $P = \prod_c P_c$. Call the set of edges directed to a cell c by the *input set* of c. A vector field f is called *admissible* if its component f_c for cell c depends only on variables associated with the input set of c (*domain condition*), and if its components for cells c, d that have isomorphic input sets are identical up to a suitable permutation of the relevant variables (*pull-back condition*). See Ref. 4 for the formal definitions of coupled cell network and admissible vector fields.

Consider a subset $\mathcal{S}$ of the set of cells of $\mathcal{G}$ and let $\mathcal{G}_{\mathcal{S}}$ be the sub-network of $\mathcal{G}$ formed by the cells in $\mathcal{G}$ and the edges that are directed to cells in $\mathcal{S}$. By Ref. 1 (Proposition 3.3), the group of interior symmetries of $\mathcal{G}$ (on the subset $\mathcal{S}$) can be canonically identified with the group of symmetries of $\mathcal{G}_{\mathcal{S}}$. See Ref. 2 for the original definition of interior symmetry and Ref. 1 for the details about its identification with the group of symmetries of $\mathcal{G}_{\mathcal{S}}$.

Suppose that $\mathcal{G}$ admits a non-trivial group of interior symmetries $\Sigma_{\mathcal{S}}$ on a subset of cells $\mathcal{S}$. We can decompose the phase space P as a cartesian product $P = P_{\mathcal{S}} \times P_{\mathcal{C}\setminus\mathcal{S}}$ where $P_{\mathcal{S}} = \prod_{s\in\mathcal{S}} P_s$ and $P_{\mathcal{C}\setminus\mathcal{S}} = \prod_{c\in\mathcal{C}\setminus\mathcal{S}} P_c$. For any $x \in P$ we write $x = (x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}})$ where $x_{\mathcal{S}} \in P_{\mathcal{S}}$ and $x_{\mathcal{C}\setminus\mathcal{S}} \in P_{\mathcal{C}\setminus\mathcal{S}}$ and we can take the action of $\Sigma_{\mathcal{S}}$ on P given by:

$$\sigma\left(x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}}\right) = \left(\sigma x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}}\right) \quad (\sigma \in \Sigma_{\mathcal{S}})\ . \tag{1}$$

Here $\Sigma_{\mathcal{S}}$ acts on $x_{\mathcal{S}}$ by permuting the coordinates corresponding to the cells in $\mathcal{S}$. For a subgroup $K \subseteq \Sigma_{\mathcal{S}}$ define

$$\mathrm{Fix}_P(K) = \{(x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}}) : \sigma x_{\mathcal{S}} = x_{\mathcal{S}}\ \ \forall\, \sigma \in K\}.$$

By Ref. 2 (Proposition 1) the subspace $\mathrm{Fix}_P(\Sigma_{\mathcal{S}})$ is flow-invariant under any admissible vector field on P. Since $\mathrm{Fix}_P(\Sigma_{\mathcal{S}})$ is $\Sigma_{\mathcal{S}}$-invariant and $\Sigma_{\mathcal{S}}$ acts trivially on the cells in $\mathcal{C} \setminus \mathcal{S}$ we have that $P_{\mathcal{C}\setminus\mathcal{S}} \subset \mathrm{Fix}_P(\Sigma_{\mathcal{S}})$. The action of the group $\Sigma_{\mathcal{S}}$ decomposes the set $\mathcal{S}$ as

$$\mathcal{S} = \mathcal{S}_1 \cup \ldots \cup \mathcal{S}_k,$$

where the sets $\mathcal{S}_i$ $(i = 1, \ldots, k)$ are the orbits of the $\Sigma_{\mathcal{S}}$-action. Let

$$W = \left\{ x \in P\ :\ x_c = 0\ \ \forall\, c \in \mathcal{C} \setminus \mathcal{S}\ \text{ and } \sum_{s\in\mathcal{S}_i} x_s = 0\ \text{ for } 1 \leqslant i \leqslant k \right\}. \tag{2}$$

Since W is a $\Sigma_{\mathcal{S}}$-invariant subspace of $P_{\mathcal{S}}$ and $W \cap \mathrm{Fix}_P(\Sigma_{\mathcal{S}}) = \{0\}$ we can decompose the phase space P as a direct sum of $\Sigma_{\mathcal{S}}$-invariant subspaces:

$$P = W \oplus \mathrm{Fix}_P(\Sigma_{\mathcal{S}}). \tag{3}$$

Consider a 1-parameter family of coupled cell systems

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, \lambda) \tag{4}$$

with interior symmetry group $\Sigma_{\mathcal{S}}$ on $\mathcal{S}$. Let $U = \mathrm{Fix}_P(\Sigma_{\mathcal{S}})$. We can choose coordinates (w, u) with $w \in W$ and $u \in U$ adapted to the decomposition (3) and write any admissible vector field f as

$$f(w, u, \lambda) = \begin{bmatrix} f_W(w, u, \lambda) \\ f_U(w, u, \lambda) \end{bmatrix} + \begin{bmatrix} 0 \\ h(w, u, \lambda) \end{bmatrix}, \tag{5}$$

where $f_U, h\ :\ P \times \mathbf{R} \to U$, $f_W\ :\ P \times \mathbf{R} \to W$ and $\tilde{f}(w, u, \lambda) = (f_W(w, u, \lambda), f_U(w, u, \lambda))$ is the $\Sigma_{\mathcal{S}}$-*equivariant part* of f. That is,

$$\begin{bmatrix} \sigma f_W(w, u, \lambda) \\ f_U(w, u, \lambda) \end{bmatrix} = \begin{bmatrix} f_W(\sigma w, u, \lambda) \\ f_U(\sigma w, u, \lambda) \end{bmatrix} \qquad (\forall\, \sigma \in \Sigma_{\mathcal{S}})\,, \tag{6}$$

since $\Sigma_{\mathcal{S}}$ acts trivially on U.

In the linear case, we may choose a basis of P adapted to the decomposition (3) and then a $\mathcal{G}$-admissible linear vector field L can be written as

$$L = \begin{bmatrix} A & 0 \\ C & B \end{bmatrix} \tag{7}$$

where $B = L|_U : U \to U$, $C : W \to U$ and $A : W \to W$ satisfies, by (6),

$$A\,\sigma = \sigma A \qquad (\forall\, \sigma \in \Sigma_{\mathcal{S}})\,.$$

The spectral properties of L in (7) are given in Ref. 2 (Lemma 1, p. 399).

Consider a 1-parameter family of coupled cell systems (4), with interior symmetry group $\Sigma_{\mathcal{S}}$ on $\mathcal{S}$, undergoing a codimension-one synchrony-breaking bifurcation at a synchronous equilibrium $x_0 \in \mathrm{Fix}_P(\Sigma_{\mathcal{S}})$ when $\lambda = \lambda_0$. Let $L = (df)_{(x_0,\lambda_0)}$ be written as in (7). By Ref. 1, f undergoes a *codimension-one interior symmetry-breaking Hopf bifurcation* if the following conditions hold:

(a) All the critical eigenvalues μ of L come from the $\Sigma_{\mathcal{S}}$-equivariant sub-block A of L.
(b) The critical eigenvalues μ extend uniquely and smoothly to eigenvalues $\mu(\lambda)$ of $(\mathrm{d}f)_{(x_0,\lambda)}$ for λ near λ_0.
(c) The *eigenvalue crossing condition*: if $\sigma(\lambda) = \mathrm{Re}(\mu(\lambda))$ then $\sigma'(\lambda_0) \neq 0$.
(d) The matrix A is non-singular and (after rescaling time if necessary) all the critical eigenvalues have the form $\pm i$ and the associated center subspace is given by $E_i(A) = \{x \in W : (A^2 + 1)x = 0\}$.

Assume that L as in (7) has $\pm i$ as eigenvalues that come only from the sub-block A of L and that they are the only critical eigenvalues of L. Consider $A^c = A|_{E_i(A)}$. As A has $\pm i$ as eigenvalues there is a natural action of $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$ on P, where $\mathbf{S}^1$ acts on $E_i(A)$ by $\exp(s(A^c)^t)$ and trivially on $P \setminus E_i(A)$. The action of $\Sigma_{\mathcal{S}}$ on P is given by (1).

Now suppose the family (4) undergoes a codimension-one interior symmetry-breaking Hopf bifurcation at the equilibrium x_0 when $\lambda = \lambda_0$. Then the center subspace $E^c(A) \equiv E_i(A)$ of the $\Sigma_{\mathcal{S}}$-equivariant sub-block A of the linearization $L = (\mathrm{d}f)_{(x_0,\lambda_0)}$ of f at (x_0, λ_0) is a $\Sigma_{\mathcal{S}}$-invariant subspace of W. Therefore, the action of the circle group $\mathbf{S}^1$ defined by $\exp(s(A^c)^{\mathrm{t}})$ commutes with the action of $\Sigma_{\mathcal{S}}$. Thus $E^c(A)$ is a $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$-invariant subspace and so there is a well-defined action of $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$ on $E^c(A)$ (and W). Following Ref. 1 (Definition 4.6), an isotropy subgroup $\Delta \subseteq \Sigma_{\mathcal{S}} \times \mathbf{S}^1$ is called *interiorly* $\mathbf{C}$-*axial (on* $E^c(A)$*)* if

$$\dim_{\mathbf{R}} \mathrm{Fix}_{E^c(A)}(\Delta) = 2.$$

Definition 2.1. We say that f is in *interior normal form (to all orders)* near λ_0, if $\tilde{f}(\cdot,\lambda)$ is in normal form (to all orders) near λ_0, that is, $\tilde{f}$ commutes with the action of $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$ on P defined above, for λ near λ_0. $\diamondsuit$

3. Interior Symmetry-breaking Hopf Bifurcation Theorem

In this section we prove using center manifold reduction approach the following result:

Theorem 3.1 (Antoneli *et al.*[1]). *Let $\mathcal{G}$ be a coupled cell network admitting a non-trivial group of interior symmetries $\Sigma_{\mathcal{S}}$ relative to a subset $\mathcal{S}$ of cells and fix a phase space P. Consider a smooth 1-parameter family of $\mathcal{G}$-admissible vector fields $f : P \times \mathbf{R} \to P$ is on P. Suppose that (4) undergoes a codimension-one interior symmetry-breaking Hopf bifurcation at an equilibrium point $x_0 \in \mathrm{Fix}_P(\Sigma_{\mathcal{S}})$ when $\lambda = 0$ and that f is in interior normal form (to all orders) near $\lambda = 0$. Let $L = (df)_{(x_0,0)}$ be written as in (7) and $\Delta \subset \Sigma_{\mathcal{S}} \times \mathbf{S}^1$ be an interiorly $\mathbf{C}$-axial subgroup (on $E^c(A)$). Then generically there exists a family of small amplitude periodic solutions of (4) bifurcating from $(x_0, 0)$ and having period near 2π. Moreover, to lowest order in the bifurcation parameter λ, the solution $x(t)$ is of the form*

$$x(t) \approx w(t) + u(t), \tag{8}$$

where $w(t) = \exp(tL)w_0$ ($w_0 \in \mathrm{Fix}_W(\Delta)$) has exact spatio-temporal symmetry Δ on the cells in $\mathcal{S}$ and $u(t) = \exp(tL)u_0$ ($u_0 \in \mathrm{Fix}_P(\Sigma_{\mathcal{S}})$) is synchronous on the $\Sigma_{\mathcal{S}}$-orbits of cells in $\mathcal{S}$.

The proof of the above theorem uses the following lemma:

Lemma 3.1. *Consider $L = (df)_{(x_0,0)}$ in the conditions of Theorem 3.1 and written as in (7). Let $\Delta \subset \Sigma_{\mathcal{S}} \times \mathbf{S}^1$ be an isotropy subgroup for the action of $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$ as defined in the previous section. Then $\dim(E_i(A)) = \dim(E_i(L))$ and $\dim(\mathrm{Fix}_{E_i(A)}(\Delta)) = \dim(\mathrm{Fix}_P(\Delta) \cap E_i(L))$.*

Proof. Consider $x = (w, u) \in P$ where $w \in W, u \in \mathrm{Fix}_P(\Sigma_S)$. Assume $\dim W = k$ and $\dim \mathrm{Fix}_P(\Sigma_S) = l$. As

$$(L^2 + I_{k+l})x = 0 \quad \Longleftrightarrow \quad \begin{bmatrix} A^2 + I_k & 0 \\ CA + BC & B^2 + I_l \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and B do not have $\pm i$ as eigenvalues, we get

$$E_i(L) = \{(w, -(B^2 + I_l)^{-1}(CA + BC)w), w \in E_i(A)\}.$$

In particular, it follows that $\dim(E_i(A)) = \dim(E_i(L))$. As $\mathrm{Fix}_P(\Delta) = \mathrm{Fix}_W(\Delta) \oplus \mathrm{Fix}_P(\Sigma_S)$, we have

$$\mathrm{Fix}_P(\Delta) \cap E_i(L) = \{(v, -(B^2 + I_l)^{-1}(CA + BC)v), v \in \mathrm{Fix}_{E_i(A)}(\Delta)\}$$

and so $\dim(\mathrm{Fix}_P(\Delta) \cap E_i(L)) = \dim(\mathrm{Fix}_{E_i(A)}(\Delta))$. □

Proof of Theorem 3.1. Consider f written in the coordinates (w, u) adapted to the decomposition (3) as in (5). Thus $\tilde{f}(w, u, \lambda) = (f_W(w, u, \lambda), f_U(w, u, \lambda)$ is Σ_S-equivariant. By hypothesis, a codimension-one interior symmetry-breaking Hopf bifurcation occurs at an equilibrium point $x_0 \in \mathrm{Fix}_P(\Sigma_S)$ when $\lambda = 0$. Since f is in interior normal form near $\lambda = 0$, $\tilde{f}$ is $\Sigma_S \times \mathbf{S}^1$-equivariant and so $\tilde{f}(\mathrm{Fix}_P(\Delta) \times \mathbf{R}) \subseteq \mathrm{Fix}_P(\Delta)$ for every $\Delta \subseteq \Sigma_S \times \mathbf{S}^1$. As $h : P \times \mathbf{R} \to P_{C \setminus S}$ and $P_{C \setminus S} \subseteq \mathrm{Fix}_P(\Delta)$ we have

$$f(\mathrm{Fix}_P(\Delta) \times \mathbf{R}) \subseteq \mathrm{Fix}_P(\Delta). \tag{9}$$

In our case, $E^c(L) = E_i(L)$ since the only critical eigenvalues of L are $\pm i$ and these come only from the sub-block A of L. Then, under the condition (9), Ref. 5 (Lemma 4.12) grants that a center manifold reduction $f^c : E_i(L) \to E_i(L)$ can be chosen so that $f^c(E_i(L) \cap \mathrm{Fix}_P(\Delta)) \subseteq E_i(L) \cap \mathrm{Fix}_P(\Delta)$. By hypothesis $\dim(\mathrm{Fix}_{E_i(A)}(\Delta)) = 2$. Then, by Lemma 3.1, it follows that $\dim(E_i(L) \cap \mathrm{Fix}_P(\Delta)) = 2$. Finally, the standard Hopf theorem gives the result. □

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