On Sufficient and Necessary Conditions for Linearity of the Transverse Poisson Structure

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Abstract

Following our work in [1] we study the possibility of bringing the transverse Poisson structure to a coadjoint orbit (on the dual of a real Lie algebra) to a normal linear form. We conclude that, if the isotropy subgroup of the (singular) point in question is compact, of if the isotropy subalgebra is semisimple, then there is a linear transverse Poisson structure to the corresponding coadjoint orbit.

We proceed to study the relation between two sufficient conditions for linearity (P. Molino's condition in [3] and a new version of our condition in [1]).

Finally we give a necessary condition for linearity of such structures and use it to clarify the situation on $\mathfrak{se}(3)^*$.

2000 *Mathematics Subject Classification:* 53D17, 22E60
Keywords: Poisson manifolds, Lie algebras, coadjoint orbits.

1 Introduction

A real Poisson manifold is a pair $(M, \{,\})$, where M is a real, finite-dimensional, smooth manifold and $\{,\}$ is a Lie algebra structure on $C^\infty(M)$ satisfying the Leibniz identity:

$$\{fg, h\} = f\{g, h\} + \{f, h\}g, \qquad \forall f, g, h \in C^\infty(M).$$

The simplest examples are: (a) symplectic manifolds with their induced Poisson bracket and (b) the dual of any real Lie algebra with the Lie-Poisson structure.

The notion of *symplectic leaf through a point of a Poisson manifold* was introduced by A. Weinstein in [6], together with the notion of *transverse Poisson structure to the symplectic leaf.*

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In the particular case where the starting Poisson manifold is the dual of any Lie algebra $\mathfrak{g}$ with its Lie-Poisson structure, the symplectic leaf through a point $\mu \in \mathfrak{g}^*$ is just the coadjoint orbit of μ. In such a case the following can be taken as a transverse manifold to the coadjoint orbit ([6], [2]):

$$N = \mu + \mathfrak{h}^\circ,$$

where $\mathfrak{h}$ is any supplement of $\mathfrak{g}_\mu$ in $\mathfrak{g}$ ($\mathfrak{g}_\mu$ denotes the isotropy subalgebra of μ) and $\mathfrak{h}^\circ$ stands for the annihilator of $\mathfrak{h}$ in $\mathfrak{g}^*$.

On such a transverse manifold there is a canonically defined Poisson structure ([6]) which is precisely the so-called *transverse Poisson structure to the coadjoint orbit of* μ. Of course, due to the choice involved (choice of the supplement $\mathfrak{h}$ on which N depends), one can obtain different transverse Poisson structures to the coadjoint orbit of μ. Nevertheless, two transverse Poisson structures to the coadjoint orbit of μ will always be Poisson-diffeomorphic. And this brings us to the following question: how can we choose the supplement $\mathfrak{h}$ so that the transverse Poisson structure to the coadjoint orbit of μ is "as simple as possible"?

Around 1984 P. Molino gave a sufficient condition on the supplement $\mathfrak{h}$, so that the transverse Poisson structure on $N = \mu + \mathfrak{h}^\circ$ would be linear. Using Molino's condition, we gave, in [1], a sufficient condition (on the isotropy subalgebra $\mathfrak{g}_\mu$) for the linearity of the transverse Poisson structure.

In section 2 we will use Molino's condition and the condition for linearity that we gave in [1] to conclude that, if: **(a)** $\mathfrak{g}$ is of compact type or **(b)** $\mathfrak{g}_\mu$ is semisimple or **(c)** $\mathfrak{g}$ is semisimple and $\mathfrak{g}_\mu$ is an ideal or **(d)** G_μ (the isotropy subgroup of μ) is compact, then the transverse Poisson structure on a convenient $N = \mu + \mathfrak{h}^\circ$ will be linear.

We will then "relax" the condition for linearity that we produced in [1], and study its relation with Molino's sufficient condition.

We proceed to give an example showing that neither of these two sufficient conditions is necessary.

Finally, in section 3 we will give a necessary condition for linearity of the transverse Poisson structure. We will use it on the Poisson manifold $\mathfrak{s}e(3)^*$ to conclude that the transverse Poisson structure to any of its singular coadjoint orbits is not linear (for any choice of N).

2 On sufficient conditions for linearity of the transverse Poisson structure

Before recalling Molino's result ([3]) and our own's ([1]) we will go through the construction of the transverse Poisson structure to a coadjoint orbit. We refer the reader to [6], [2] and [1] for a more detailed exposition.

Let $\mathfrak{g}$ be a real finite dimensional Lie algebra and consider the Lie-Poisson structure on its dual space: $(M, P) = (\mathfrak{g}^*, L)$. Given $\mu \in \mathfrak{g}^*$ let O_μ denote the symplectic leaf throuhg μ (this is just coadjoint orbit of μ), and $\mathfrak{g}_\mu$ denote the isotropy subalgebra of μ:

$$\mathfrak{g}_\mu = \{\xi \in \mathfrak{g} : \mu \circ ad_\xi = 0\}.$$

Pick any vector subspace $\mathfrak{h}$ such that, as vector spaces:

$$\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{h},$$

(we will refer to such $\mathfrak{h}$ as a *supplement of* $\mathfrak{g}_\mu$), and consider the following transverse manifold to O_μ:

$$N_\mathfrak{h} = \mu + \mathfrak{h}^\circ.$$

On such a manifold build the transverse Poisson structure to O_μ at μ and denote it by $(N_{\mathfrak{h}}, P_{\mathfrak{h}})$. Recall that this Poisson structure depends on the choice of $\mathfrak{h}$, even though different choices of $\mathfrak{h}$ will produce (locally) Poisson-equivalent structures.

We are interested in finding $\mathfrak{h}$ such that the transverse Poisson structure $(N_{\mathfrak{h}}, P_{\mathfrak{h}})$ is as simple as possible, more precisely, we want to consider the problem of finding $\mathfrak{h}$ so that $(N_{\mathfrak{h}}, P_{\mathfrak{h}})$ is linear.

Recall the following theorems:

Theorem 1 ([3]) *Let $\mathfrak{g}$ be any Lie algebra and let $\mu \in \mathfrak{g}^*$ be such that there is a supplement $\mathfrak{h}$ to $\mathfrak{g}_\mu$ satisfying:*

$$[\mathfrak{g}_\mu, \mathfrak{h}] \subset \mathfrak{h}.$$

Then the transverse Poisson structure $(N_{\mathfrak{h}}, P_{\mathfrak{h}})$ to O_μ is linear.

Theorem 2 ([1]) *Let $\mathfrak{g}$ be any Lie algebra and B be an ad-invariant symmetric bilinear form on $\mathfrak{g}$. Let $\mu \in \mathfrak{g}^*$ be such that:*

$$B_{|\mathfrak{g}_\mu \times \mathfrak{g}_\mu}$$

is nondegenerate. Then, taking $\mathfrak{h}$ as the B-orthogonal of $\mathfrak{g}_\mu$, the transverse Poisson structure $(N_{\mathfrak{h}}, P_{\mathfrak{h}})$ to O_μ is linear.

Using these results we can prove the following:

Corollary 1 *Let $\mathfrak{g}$ be a Lie algebra of compact type. Then, for any $\mu \in \mathfrak{g}^*$, there is a linear transverse Poisson structure $(N_{\mathfrak{h}}, P_{\mathfrak{h}})$ to O_μ.*

Proof: on a Lie algebra of compact type there is an invariant symmetric bilinear form B which is positive definite. The restriction of B to any subalgebra will be nondegenerate and theorem 2 can be used . □

Corollary 2 *Let $\mu \in \mathfrak{g}^*$ be such that $\mathfrak{g}_\mu$ is semisimple. Then there is a linear transverse Poisson structure $(N_{\mathfrak{h}}, P_{\mathfrak{h}})$ to O_μ.*

Proof: because $\mathfrak{g}_\mu$ is semisimple the adjoint representation of $\mathfrak{g}_\mu$ on $\mathfrak{g}$ is faithful. Then the Killing form of $\mathfrak{g}$, K, is nondegenerate when restricted to $\mathfrak{g}_\mu$ and theorem 2 can again be used (with $B = K$). □

Corollary 3 *Let $\mathfrak{g}$ be a semisimple Lie algebra, and let $\mu \in \mathfrak{g}^*$ be such that $\mathfrak{g}_\mu$ is an ideal. Then there is a linear transverse Poisson structure $(N_{\mathfrak{h}}, P_{\mathfrak{h}})$ to O_μ.*

Proof: let K denote the Killing form of $\mathfrak{g}$. Then ([5]) taking $\mathfrak{h}$ as the K-orthogonal of $\mathfrak{g}_\mu$ the condition of theorem 1 holds. □

In the next corollary G will denote the connected and simply-connected Lie group with Lie algebra $\mathfrak{g}$, and G_μ will stand for the isotropy subgroup of $\mu \in \mathfrak{g}^*$, i.e.:

$$G_\mu = \{g \in G : Ad_g^*\mu = \mu\}.$$

Corollary 4 *Let $\mu \in \mathfrak{g}^*$ be such that its isotropy subgroup, G_μ, is compact. Then there is a linear transverse Poisson structure $(N_{\mathfrak{h}}, P_{\mathfrak{h}})$ to O_μ.*

Proof: because G_μ is compact, every representation ρ of G_μ on a finite-dimensional vector space V is completely reducible. To prove this, start with any inner product $\langle , \rangle$ on V and use a Haar integral (see for example [4]) - which exists because G_μ is compact - to define a new inner product on V:

$$\langle u, v\rangle_{G_\mu} := \int_{G_\mu} \langle \rho_g(u), \rho_g(v)\rangle dg.$$

This new inner product on V is ρ-invariant because the Haar integral is invariant under right translations on G_μ. This is the same as saying that, with respect to $\langle , \rangle_{G_\mu}$, ρ is orthogonal and therefore, completely reducible.

Now consider the adjoint representation of G_μ on $\mathfrak{g}$:

$$\begin{array}{rcl} Ad^\mu : G_\mu & \longrightarrow & Aut(\mathfrak{g}) \\ g & \longmapsto & d\left(\sigma_g\right)_e \end{array}$$

where σ_g denotes conjugation (by g) on the group G. Clearly $\mathfrak{g}_\mu$ is an Ad^μ-invariant subspace. By complete reducibility of Ad^μ, there is an Ad^μ-invariant supplement, say $\mathfrak{h}$, to $\mathfrak{g}_\mu$. This means that:

$$Ad^\mu(g)(\mathfrak{h}) \subset \mathfrak{h}, \quad \forall g \in G_\mu$$

which in turn implies that:

$$[\mathfrak{g}_\mu, \mathfrak{h}] \subset \mathfrak{h}$$

and theorem 1 can be used. $\square$

This corollary, with the necessary adaptations, was conjectured by R. Loja Fernandes for general Poisson structures. Our proof shows that his conjecture holds in the Lie-Poisson case.

Remark: The condition *"G_μ compact"* cannot be weakened to *"$\mathfrak{g}_\mu$ of compact type"*. In subsection 3.4 we will exhibit an example of $\mathfrak{g}$ and μ such that $\mathfrak{g}_\mu$ is of compact type but there is no supplement $\mathfrak{h}$ such that the transverse Poisson structure $(N_\mathfrak{h}, P_\mathfrak{h})$ is linear.

2.1 A weaker sufficient condition for linearity

We will now prove a slightly more general version of theorem 2 by relaxing the condition of "invariance" of B.

Definition 1 *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}_0$ be any subalgebra of $\mathfrak{g}$. We will say that a symmetric bilinear form B on $\mathfrak{g}$ is* $ad_{\mathfrak{g}_0}$-invariant *if:*

$$B([\xi, \eta], \zeta) + B(\eta, [\xi, \zeta]) = 0, \quad \forall \xi \in \mathfrak{g}_0, \forall \eta, \zeta \in \mathfrak{g}.$$

Clearly ad-invariant bilinear forms on $\mathfrak{g}$ are also $ad_{\mathfrak{g}_0}$-invariant, for any Lie subalgebra $\mathfrak{g}_0$.

It's easy to see that the following results of [1] still hold with the "relaxed invariance of B":

Lemma 1 *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}_0$ be any subalgebra of $\mathfrak{g}$. Let $B : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbf{R}$ be an $ad_{\mathfrak{g}_0}$-invariant symmetric bilinear form on $\mathfrak{g}$ and denote by:*

$$\mathfrak{g}_0^\perp = \{\xi \in \mathfrak{g} : B(\xi, \eta) = 0, \forall \eta \in \mathfrak{g}_0\}$$

the B-orthogonal complement of $\mathfrak{g}_0$. Then:

$$[\mathfrak{g}_0, \mathfrak{g}_0^\perp] \subset \mathfrak{g}_0^\perp.$$

If furthermore $B_{|\mathfrak{g}_0 \times \mathfrak{g}_0}$ is nondegenerate, then: $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_0^\perp$.

This new lemma is all we need to refine theorem 2 as:

Theorem 3 *Let $\mathfrak{g}$ be any Lie algebra and let $\mu \in \mathfrak{g}^*$. Let B be an $ad_{\mathfrak{g}_\mu}$-invariant symmetric bilinear form on $\mathfrak{g}$ such that:*

$$B_{|\mathfrak{g}_\mu \times \mathfrak{g}_\mu}$$

is nondegenerate. Then, taking $\mathfrak{h}$ to be the B-orthogonal of $\mathfrak{g}_\mu$, the transverse Poisson structure $(N_\mathfrak{h}, P_\mathfrak{h})$ to O_μ is linear.

2.2 On the relation between two sufficient conditions for linearity

From what we have just seen, under the conditions of either theorem 1 or theorem 3, the conclusion is that the transverse Poisson structure to O_μ, $\mu \in \mathfrak{g}^*$, is linear on $N_\mathfrak{h} = \mu + \mathfrak{h}^\circ$.

We remark that, if theorem 3 is applicable, then also theorem 1 is applicable (with $\mathfrak{h}$ being the B-orthogonal of $\mathfrak{g}_\mu$), so when studying the relation between theorems 1 and 3 it is enough to see when the condition of theorem 1 implies that of theorem 3. That is precisely the content of the next lemma.

Lemma 2 *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}_\mu$ denote the isotropy subalgebra of $\mu \in \mathfrak{g}^*$. If $\mathfrak{g}_\mu$ is either semisimple or of compact type then the conditions of theorems 1 and 3 are equivalent.*

Proof: We just need to show that if $\mathfrak{g}_\mu$ is either semisimple or of compact type then the existence of $\mathfrak{h}$ from theorem 1 implies the existence of the required $ad_{\mathfrak{g}_\mu}$-invariant symmetric bilinear form.

Let:

$$B_\mu : \mathfrak{g}_\mu \times \mathfrak{g}_\mu \longrightarrow \mathbf{R}$$

be a nondegenerate ad-invariant symmetric bilinear form on $\mathfrak{g}_\mu$ (if $\mathfrak{g}_\mu$ is semisimple take B_μ to be its Killing form, if $\mathfrak{g}_\mu$ is of compact type take a positive definite ad-invariant symmetric bilinear form). Then:

$$B_\mu([X,Y],Z) + B_\mu(Y,[X,Z]) = 0, \quad \forall X,Y,Z \in \mathfrak{g}_\mu.$$

Now use supplement $\mathfrak{h}$ from theorem 1 to define the projection $\pi : \mathfrak{g} \longrightarrow \mathfrak{g}_\mu$ with kernel $\mathfrak{h}$ and let:

$$\begin{array}{rcl} B : \mathfrak{g} \times \mathfrak{g} & \longrightarrow & \mathbf{R} \\ (X,Y) & \longmapsto & B_\mu(\pi(X), \pi(Y)) \end{array}$$

Clearly B is a symmetric bilinear form on $\mathfrak{g}$ which restricts to $\mathfrak{g}_\mu \times \mathfrak{g}_\mu$ as B_μ (nondegenerate). The only thing left to check is that B is $ad_{\mathfrak{g}_\mu}$-invariant. Now, because $\mathfrak{g}_\mu$ is a subalgebra and Molino's condition:

$$[\mathfrak{g}_\mu, \mathfrak{h}] \subset \mathfrak{h},$$

holds, then:

$$\pi[X,Y] = [X, \pi(Y)], \quad \forall X \in \mathfrak{g}_\mu, Y \in \mathfrak{g}.$$

So for $X \in \mathfrak{g}_\mu, Y, Z \in \mathfrak{g}$ we have:

$$\begin{array}{rcl} B([X,Y],Z) + B(Y,[X,Z]) & = & B_\mu(\pi[X,Y], \pi(Z)) + B_\mu(\pi(Y), \pi[X,Z]) \\ & = & B_\mu([X,\pi(Y)], \pi(Z)) + B_\mu(\pi(Y), [X, \pi(Z)]) \end{array}$$

which vanishes by ad-invariance of B_μ. □

2.3 Example: $\mathfrak{so}(4)^*$

In this subsection we show that neither of the conditions of theorems 1 or 3 is necessary for linearity. In fact, the following example shows that the condition of theorem 1 (the weaker of the two conditions) is not necessary.

In [1] we studied the Lie-Poisson structure on $\mathfrak{so}(4)^*$ at its singular points (any point of rank 2). We recall that, on the standard basis for $\mathfrak{so}(4)$ and identifying again elements X_i of the basis with linear coordinates x_i on $\mathfrak{so}(4)^*$, the Poisson matrix for the Lie-Poisson structure is given by:

$$L_x = \begin{pmatrix} . & -x_4 & -x_5 & x_2 & x_3 & . \\ x_4 & . & -x_6 & -x_1 & . & x_3 \\ x_5 & x_6 & . & . & -x_1 & -x_2 \\ -x_2 & x_1 & . & . & -x_6 & x_5 \\ -x_3 & . & x_1 & x_6 & . & -x_4 \\ . & -x_3 & x_2 & -x_5 & x_4 & . \end{pmatrix}.$$

Any point of the form:

$$\mu = (a, b, c, -c, b, -a) \qquad \text{with } a^2 + b^2 + c^2 \neq 0$$

is singular (and has rank 2). A basis for the isotropy subalgebra of such a point is, for example:

$$\{E_1, E_2, E_3, E_4\} = \{X_1 + X_6, X_2 - X_5, X_3 + X_4, cX_4 - bX_5 + aX_6\}.$$

We assume $abc \neq 0$ and choose, for any $\lambda \neq 1$, the following supplement to $\mathfrak{g}_\mu$:

$$\mathfrak{h}_\lambda = <\overbrace{\lambda c(X_1 - X_6) + a(X_4 - X_3)}^{E_5}, \overbrace{b(X_6 - X_1) + a(X_5 + X_2)}^{E_6}> .$$

Then:

1. for any $\lambda \neq 1$ Molino's condition does not hold. In fact:

$$[E_4, E_6] = ac(X_1 - X_6) + bc(X_2 + X_5) + (a^2 + b^2)(X_4 - X_3),$$

which is not in $\mathfrak{h}_\lambda$ (since $\lambda \neq 1$).

2. the transverse Poisson structure on:

$$N_{\mathfrak{h}_\lambda} \simeq \{(a + y_1, b - y_2, c + y_3, -c + y_3 + \lambda c y_4, b + y_2 - b y_4, -a + y_1 + a y_4) : y_1, \ldots, y_4 \in \mathbf{R}\}$$

is given by the **linear** Poisson matrix:

$$P_{\mathfrak{h}_\lambda}(y) = \begin{pmatrix} 0 & -2(2y_3 + \lambda c y_4) & 2(-2y_2 + by_4) & -2(cy_2 + by_3) + (1-\lambda)bcy_4 \\ * & 0 & -2(2y_1 + ay_4) & -2(cy_1 - ay_3) + (\lambda - 1)acy_4 \\ * & * & 0 & 2(by_1 + ay_2) \\ * & * & * & 0 \end{pmatrix},$$

(due to lack of space we only present the upper triangular part of the matrix).

We conclude that Molino's condition is not needed for linearity.

3 Necessary condition for linearity of the transverse Poisson structure

Before presenting a necessary condition for linearity of the transverse Poisson structure we will review the notion of *linear approximation to a Poisson structure at a zero-rank point.* This was introduced by Weinstein in [6] and is the key notion behind linear normal forms.

3.1 The Linear Approximation to a Poisson Structure at a Zero-Rank Point

In this section we will briefly outline Weinstein's construction.

We recall that a Poisson structure on a vector space is linear if the Poisson bracket of linear functions is again linear:

Definition 2 *A Poisson bracket on a real, finite-dimensional, vector space V:*

$$\{,\}^L : C^\infty(V) \times C^\infty(V) \to C^\infty(V),$$

is said to be linear linear *if*

$$\{f, g\}^L \in V^*,$$

for every f, $g \in V^*$.

If $(V, \{,\}^L)$ is a linear Poisson vector space, then $(V^*, \{,\}^L)$ is a (finite-dimensional) Lie subalgebra of $(C^\infty(V), \{,\}^L)$. Conversely, given a real finite dimensional Lie algebra $\mathfrak{g}$, its dual space inherits a linear Poisson structure: the Lie-Poisson structure on $V = \mathfrak{g}^*$.

Being so, linear Poisson structures on a vector space V are in a one-to-one correspondence with Lie algebra structures on V^*. This fact is used in the construction of the *linear approximation to a Poisson structure at a zero rank point.*

Let (M, P) be any Poisson manifold and x_0 a zero-rank point of P. Consider the following subsets of $C^\infty(M)$:

$$\begin{aligned} \mathfrak{m}_{x_0} &= \{f \in C^\infty(M);\, f(x_0) = 0\}, \\ \mathfrak{m}^2_{x_0} &= \{f \in \mathfrak{m}_{x_0};\, df_{x_0} = 0\}. \end{aligned}$$

Because x_0 has rank zero, the subsets $\mathfrak{m}_{x_0}$ and $\mathfrak{m}^2_{x_0}$ are ideals in $(C^\infty(M), \{,\})$. Being so, we can form the quotient:

$$\mathfrak{m}_{x_0}/\mathfrak{m}^2_{x_0}$$

and identify it naturally with the cotangent space, $T^*_{x_0}M$, through the isomorphism:

$$\begin{aligned} I : \mathfrak{m}_{x_0}/\mathfrak{m}^2_{x_0} &\longrightarrow T^*_{x_0}M \\ [f] &\longmapsto df_{x_0}. \end{aligned}$$

Using such an isomorphism and the induced Lie algebra structure on the quotient we obtain a Lie algebra structure on $T^*_{x_0}M$. As remarked above, this amounts to saying that $T_{x_0}M$ has a linear Poisson structure. This structure is known as the *linear approximation to* (M, P) *at* x_0 and is denoted by $(T_{x_0}M, P^o)$ or $(T_{x_0}M, \{,\}^o)$.

In local coordinates, the Poisson tensor P^o is just the first order Taylor series of the original tensor P at the point x_0. Furthermore, one can easily check that:

$$\{df_{x_0}, dg_{x_0}\}^o = d\left(\{f, g\}\right)_{x_0}. \tag{1}$$

3.2 Necessary condition for linearity

We start by proving the following:

Lemma 3 *Let (M,P) and (N,Q) be Poisson manifolds and let $\varphi : M \to N$ be a (local) Poisson diffeomorphism around $x_0 \in M$. Then $d\varphi_{x_0}$ is a Poisson isomorphism between $(T_{x_0}M, P^o)$ and $(T_{\varphi(x_0)}N, Q^o)$.*

Proof: let $y_0 = \varphi(x_0)$ and denote by:

- $x_1, \ldots, x_n$ local coordinates in M;
- $y_i = \varphi_i(x),\ i = 1, \ldots, n$ local coordinates in N;
- $X_i = (dx_i)_{x_0},\ i = 1, \ldots, n$ linear coordinates in $T_{x_0}M$;
- $Y_i = (dy_i)_{y_0},\ i = 1, \ldots, n$ linear coordinates in $T_{y_0}N$.

We will check that $\psi = d\varphi_{x_0} : T_{x_0}M \to T_{\varphi(x_0)}N$ is a Poisson isomorphism, i.e., that:

$$\{Y_i \circ \psi,\, Y_j \circ \psi\}_P^o = \{Y_i,\, Y_j\}_Q^o \circ \psi.$$

First, note that:

$$\begin{aligned} Y_i \circ \psi &= (dy_i)_{y_0}\, (d\varphi)_{x_0} \\ &= (d\varphi_i)_{x_0} \end{aligned}$$

and, due to the hypothesis:

$$\{y_i \circ \varphi,\, y_j \circ \varphi\}_P = \{y_i,\, y_j\}_Q \circ \varphi.$$

Using these and identity (1) we get:

$$\begin{aligned} \{Y_i \circ \Psi,\, Y_j \circ \Psi\}_P^o &= \Big\{(d\varphi_i)_{x_0},\, (d\varphi_j)_{x_0}\Big\}_P^o \\ &= d\left(\{\varphi_i,\, \varphi_j\}_P\right)_{x_0} \\ &= d\Big(\{y_i,\, y_j\}_Q\Big)_{y_0} \circ (d\varphi)_{x_0} \\ &= \Big\{(dy_i)_{y_0},\, (dy_j)_{y_0}\Big\}_Q^o \circ (d\varphi)_{x_0} \\ &= \{Y_i,\, Y_j\}_Q^o \circ \Psi. \end{aligned}$$

This concludes the proof. $\square$

By replacing (N,Q) by a linear Poisson vector space we obtain:

Corollary 5 *If (M,P) is (locally, around a zero-rank point x_0) Poisson equivalent to a linear Poisson structure, then (M,P) is (locally) Poisson equivalent to its linear approximation at x_0.*

Note that, being Poisson equivalent to its linear approximation at the zero-rank point x_0 is the usual definition of being *linearizable at x_0*.

We will now concentrate on *transverse Poisson structures to a coadjoint orbit*, keeping the notation of section 2.

Theorem 4 *If there is a supplement $\mathfrak{h}_o$ to $\mathfrak{g}_\mu$ such that $(N_{\mathfrak{h}_o}, P_{\mathfrak{h}_o})$ is linear, then for every supplement $\mathfrak{h}$ the Poisson structure $(N_{\mathfrak{h}}, P_{\mathfrak{h}})$ is linearizable at μ.*

Proof: pick any supplement $\mathfrak{h}$. Then ([6]) $(N_{\mathfrak{h}}, P_{\mathfrak{h}})$ is locally Poisson-equivalent to $(N_{\mathfrak{h}_o}, P_{\mathfrak{h}_o})$, which is linear. Using the last corollary we get the result. □

Remark: Let us note that the theorem actually implies the following:

- Pick <u>any</u> supplement $\mathfrak{h}$ to $\mathfrak{g}_\mu$ and compute the transverse Poisson structure $P_{\mathfrak{h}}$ on $N_{\mathfrak{h}}$. If there are obstructions to linearizability at μ, then no supplement $\mathfrak{h}_o$ will ever produce a linear transverse Poisson structure (typical obstructions to linearizability are: (a) "non-matching" rank or (b) "non-matching" zero-rank-set, arbitrarily close to μ).

  Furthermore, if obstructions to linearizability occur for supplement $\mathfrak{h}$, then the same obstructions must appear for any other supplement.

- Conversely, if $P_{\mathfrak{h}}$ is linearizable at μ, then the same will happen for <u>any</u> $P_{\mathfrak{h}'}$.

So it is really indifferent which $\mathfrak{h}$ to choose.

3.3 Example: $\mathfrak{s}e(3)^*$

We illustrate theorem 4 with an example on the dual of $\mathfrak{s}e(3)$. This Lie algebra was used in [2] to suggest differences between transverse Poisson structures to coadjoint orbits in the dual of semisimple and nonsemisimple Lie algebras.

Actually such differences are more subtle than what we would expect. Due to the fact that $\mathfrak{s}o(4)$ is semisimple, there is a choice of supplement producing a polynomial transverse Poisson structure ([2]). But also, because $\mathfrak{s}o(4)$ is of compact type, there is a choice of supplement producing a linear transverse Poisson structure ([1]). Nevertheless for some choices of supplement (see again [1]) we obtained **nonpolynomial** transverse structure.

The situation with $\mathfrak{s}e(3)$ seemed different: all attempts to produce polynomial transverse structure were fruitless. But this did not exclude the fact that $\mathfrak{s}e(3)$ could present similar behaviour to that of $\mathfrak{s}o(4)$. What we want to show now, using theorem 4, is that $\mathfrak{s}e(3)$ will never (i.e., for any choice of supplement) produce linearizable structures at μ, let alone linear structures.

It remains open (as far as we know) whether polynomial structures can be obtained in this example.

So let $\mathfrak{g} = \mathfrak{s}e(3) = \mathfrak{s}o(3) \propto \mathbf{R}^3$. A possible basis for this Lie algebra is:

$$\begin{aligned}
X_1 &\cong \left(\begin{pmatrix} . & . & . \\ . & . & -1 \\ . & 1 & . \end{pmatrix}, (0,\,0,\,0) \right), \\
X_2 &\cong \left(\begin{pmatrix} . & . & 1 \\ . & . & . \\ -1 & . & . \end{pmatrix}, (0,\,0,\,0) \right), \\
X_3 &\cong \left(\begin{pmatrix} . & -1 & . \\ 1 & . & . \\ . & . & . \end{pmatrix}, (0,\,0,\,0) \right), \\
X_4 &\cong ((0), (1,\,0,\,0)), \\
X_5 &\cong ((0), (0,\,1,\,0)), \\
X_6 &\cong ((0), (0,\,0,\,1)).
\end{aligned}$$

Using this basis we have computed the Lie-Poisson structure for $\mathfrak{se}(3)^*$ (identifying naturally the elements X_i of the basis with linear coordinates x_i on the dual space $\mathfrak{se}(3)^*$):

$$L_x = \begin{pmatrix} . & x_3 & -x_2 & 0 & x_6 & -x_5 \\ -x_3 & . & x_1 & -x_6 & . & x_4 \\ x_2 & -x_1 & . & x_5 & -x_4 & . \\ . & x_6 & -x_5 & . & . & . \\ -x_6 & . & x_4 & . & . & . \\ x_5 & -x_4 & . & . & . & . \end{pmatrix},$$

(dots stand for zero).

Now we remark that points of rank 4 are regular (therefore the transverse Poisson structure is trivial at such points) and points of rank 2 are of the form:

$$\mu = (a, b, c, 0, 0, 0),$$

with $a^2 + b^2 + c^2 \neq 0$. Such points are all singular.

For an arbitrary point of rank two as above, we have:

$$\mathfrak{g}_\mu = \langle aX_1 + bX_2 + cX_3,\ X_4,\ X_5,\ X_6\rangle.$$

From now on we will assume, without loss of generality, that $c \neq 0$. We choose $\mathfrak{h}$ to be:

$$\mathfrak{h} = \langle X_1,\ X_2\rangle.$$

Then, proceeding with the computations as in [1], we obtain the following:

1. the affine subspace of $\mathfrak{g}^*$ given by:

   $$N_\mathfrak{h} \simeq \{(a, b, c + y_1, y_2, y_3, y_4) : y_1, \ldots, y_4 \in \mathbf{R}\}$$

   is a transverse manifold to the coadjoint orbit of μ;

2. the transverse Poisson structure on $N_\mathfrak{h}$ is given by the matrix:

   $$P_\mathfrak{h}(y) = \begin{pmatrix} 0 & -\dfrac{c(-cy_3 - y_3y_1 + by_4)}{c+y_1} & \dfrac{c(-cy_2 - y_2y_1 + ay_4)}{c+y_1} & \dfrac{c(by_2 - ay_3)}{c+y_1} \\ * & 0 & \dfrac{-y_4^2}{c+y_1} & \dfrac{y_3y_4}{c+y_1} \\ * & * & 0 & -\dfrac{y_2y_4}{c+y_1} \\ * & * & * & 0 \end{pmatrix}$$

   (where, as before, we are only presenting the upper part of the Poisson matrix).

3. the linear approximation to $P_\mathfrak{h}$ at $y = 0$ is given by:

   $$P^o_\mathfrak{h}(y) = \begin{pmatrix} 0 & cy_3 - by_4 & ay_4 - cy_2 & by_2 - ay_3 \\ by_4 - cy_3 & 0 & 0 & 0 \\ cy_2 - ay_4 & 0 & 0 & 0 \\ ay_3 - by_2 & 0 & 0 & 0 \end{pmatrix}.$$

We can easily find obstructions to linearizability of $P_\mathfrak{h}$ at $y = 0$ by non-matching zero--rank sets. In fact the zero-rank set for $P_\mathfrak{h}$ is:

$$S_0(P_\mathfrak{h}) = \{(y_1, 0, 0, 0) : y_1 \in \mathbf{R}\}$$

whereas the same set for $P^0_\mathfrak{h}$ is:

$$S_0(P^0_\mathfrak{h}) = \{(cy_1, ay_4, by_4, cy_4) : y_1, y_4 \in \mathbf{R}\}.$$

And no matter how we choose another supplement $\mathfrak{h}'$, we will always find the same kind of obstructions to linearizability.

3.4 Example with compact-type isotropy subalgebra

In this subsection we present an example of a Lie algebra $\mathfrak{g}$ and of $\mu \in \mathfrak{g}^*$ such that: (a) $\mathfrak{g}_\mu$ is of compact type and (b) the transverse Poisson structure to O_μ is not linear on any $N_\mathfrak{h}$.

To show (b) we will make use, again, of theorem 4.

Let $\mathfrak{g}$ be a real 4-dimensional Lie algebra with basis $\{T_1, T_2, X_1, X_2\}$ and brackets given by:

$$\begin{array}{lll} [T_1, T_2] = 0, & [T_1, X_1] = T_2, & [T_1, X_2] = kX_1 \\ & [T_2, X_1] = 0, & [T_2, X_2] = T_2 \\ & & [X_1, X_2] = T_1 + X_1 \end{array}$$

(k is an arbitrary real number). In other words, the Poisson matrix for the Lie-Poisson structure on $\mathfrak{g}^*$ is given by (again dots stand for zero):

$$L_{(t,x)} = \begin{pmatrix} \cdot & \cdot & t_2 & kx_1 \\ \cdot & \cdot & \cdot & t_2 \\ -t_2 & \cdot & \cdot & t_1 + x_1 \\ -kx_1 & -t_2 & -(t_1 + x_1) & \cdot \end{pmatrix},$$

Take $\mu = (1, 0, 0, 1) \in \mathfrak{g}^*$. Then:

$$\mathfrak{g}_\mu = \langle T_1,\, T_2 \rangle,$$

which is obvioulsy of compact type (for example, $\mathfrak{g}_\mu$ is the Lie algebra of the 2-torus).

Now take the following supplement to $\mathfrak{g}_\mu$:

$$\mathfrak{h} = \langle X_1,\, X_2 \rangle.$$

Then:

$$N_\mathfrak{h} \simeq \{(1 + y_1, y_2, 0, 1) : y_1, y_2 \in \mathbf{R}\},$$

and usual computations (as described in [1]) produce:

$$P_\mathfrak{h}(y) = \begin{pmatrix} 0 & \frac{{y_2}^2}{1+y_1} \\ -\frac{{y_2}^2}{1+y_1} & 0 \end{pmatrix}.$$

Again there are obstructions to linearizability since:

$$P^0_\mathfrak{h}(y) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

which is not Poisson equivalent to $P_\mathfrak{h}$ in any neighbourhood of $y = 0$ (i.e., in any neighbourhood of μ). Theorem 4 then implies that there is no linear transverse Poisson structure to the coadjoint orbit of such μ.

4 Final considerations

As far as we know there is no knowledge of a necessary and sufficient condition (on $\mathfrak{g}$ or on μ) for linearity.

In what concerns "polynomiality", semisimplicity of the Lie algebra $\mathfrak{g}$ is sufficient for the existence of a polynomial transverse ([2]), but obviously it is not necessary (this condition is not necessary even for linearity - see the last example in [1]).

Obstructions to "polynomiality" are harder to get, because only the first nonvanishing homogeneous term in the Taylor series for P is itself a Poisson tensor. As we said in subsection 3.3, we think it is an open question whether there is any polynomial transverse structure on $\mathfrak{se}(3)^*$.

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