Connection coefficients between orthogonal polynomials and the canonical sequence

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Abstract

We deal with the problem of obtaining closed formulas for the connection coefficients between orthogonal polynomials and the canonical sequence. We use a recurrence relation fulfilled by these coefficients and symbolic computation with the *Mathematica* language. We treat the cases of Bessel, Laguerre and generalized Hermite.

Keywords: connection coefficients; orthogonal polynomials, symbolic computation; *Mathematica*.

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1 Introduction

A basic question is the problem of obtaining connection coefficients between two sequences of functions, in particular, between two orthogonal polynomial sequences. This last problem was already considered by L. Gegenbauer [11] just about the Gegenbauer polynomials, by W. A. Al-Salam concerning the Bessel family [3] and by R. Askey et al. about Laguerre, Gegenbauer and Jacobi cases [4, 5]. The results of those works are based on properties of hypergeometric functions and generating functions of the involving polynomials. In [20] several tables are provided giving the coefficients of some polynomials (Chebyshev of first and second kind, Legendre, Laguerre, Hermite), also giving the coefficients of x^n in terms of those polynomials. In this last case, there is no general expressions, only the values of the first coefficients. More recently, A. Ronveaux et al. [19] present a recursive approach of the problem, with some general assumptions, given results for all classical polynomials. The literature on this subject is extremely vast and a wide variety of methods have been developed using several techniques like, recursion, hypergeometric approach, inversion and other combinatorial formulas, lowering operators, ..., etc, to treat the connection coefficients for continuous, discrete and $q-$ polynomials. See, among others, the publications [1, 2, 6, 7, 10, 26, 27, 12, 13, 14, 15, 16, 17, 18, 28].

Here, we proceed with the simplest method based only on the recurrence relation fulfilled by any orthogonal sequence, which leads to a recurrence relation satisfied by the connection coefficients. But, generally, we do not succeed in resolving it through a compact form. Thus, it is required to guess a closed form for the solution of the recurrence from enough data produced by a symbolic programming language like *Mathematica* [29]. Next, the final goal of the work is to provide a proof of the fact that the closed form is really a solution. For that, we use again the same recurrence relation.

Unfortunately, in general, it is not possible to supply a model easily identifiable, like an analytic object, from the symbolic results: for instance, when they are not factorized (see, at the last section, the cases relating sequences belonging to different families).

This work begins with a section of preliminaries, where we remember basic definitions and results needed in the sequel. Next section is devoted to derive the general recurrence relation satisfied by the connection coefficients, which play a crucial role. After exposing the methodology employed to make the symbolic computations and the proofs, we present closed formulas of the connection coefficients for the families of Bessel, Laguerre and generalized Hermite. In each case, first, we consider the connection coefficients between two sequences belonging to the same family but with different parameters, and after, we consider the connection coefficients between the canonical sequence and the orthogonal one. Finally, we give some tables of coefficients between families with different director polynomials for which we have not found closed formulas.

The sequences of Gegenbauer, Jacobi and a semi-classical type example [9, 23] are exposed in [24]. We note that for Gegenbauer and Jacobi with the canonical family, the methodology does not give a complete answer, but is the start point in the corresponding theoretical proofs. In a forthcoming article [25], we shall present results for other polynomials.

2 Preliminaries

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle u, p\rangle$ the effect of $u \in \mathcal{P}'$ on $p \in \mathcal{P}$. In particular, $\langle u, x^n\rangle := (u)_n \, , n \geqslant 0$ represent the moments of u.

Let $\{P_n\}_{n\geqslant 0}$ be a monic polynomial sequence (MPS) with $\deg P_n = n, \ n \geq 0$ and let $\{u_n\}_{n\geqslant 0}$ be its dual sequence, $u_n \in \mathcal{P}'$, defined by

$$\langle u_n, P_m\rangle = \delta_{n,m} \ , \ n, m \geqslant 0 \ . \tag{1}$$

A form u is said regular [21, 22] if and only if there exist a MPS $\{P_n\}_{n\geqslant 0}$, such that:

$$\begin{aligned} \langle u, P_n P_m\rangle &= 0 \ , \ n \neq m \ , \ n, m \geq 0 \ , & (2)\\ \left\langle u, P_n^2\right\rangle &\neq 0 \ , \ n \geq 0 \ . & (3)\end{aligned}$$

In this case, $\{P_n\}_{n\geqslant 0}$ is said regularly orthogonal with respect to u and is called a monic orthogonal polynomial sequence (MOPS). The orthogonality conditions are given by (2), and (3) corresponds to the regularity conditions.

The sequence $\{P_n\}_{n\geqslant 0}$ is regularly orthogonal with respect to u if and only if [21, 22] there exist two sequences of coefficients $\{\beta_n\}_{n\geqslant 0}$ and $\{\gamma_{n+1}\}_{n\geqslant 0}$, with $\gamma_{n+1} \neq 0, \ n \geqslant 0$, such that, $\{P_n\}_{n\geqslant 0}$ satisfies the following recurrence relation of order 2, with the corresponding initial conditions:

$$P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \ n \geqslant 0 \ , \tag{4}$$

$$P_0(x) = 1, \qquad P_1(x) = x - \beta_0 \ . \tag{5}$$

Futhermore, the recurrence coefficients satisfy:

$$\beta_n = \frac{\langle u, xP_n^2(x)\rangle}{\langle u, P_n^2(x)\rangle}, \quad n \geqslant 0, \tag{6}$$

$$\gamma_{n+1} = \frac{\left\langle u, P_{n+1}^2(x)\right\rangle}{\langle u, P_n^2(x)\rangle}, \quad n \geqslant 0. \tag{7}$$

We remark that, from (5) and (7), the regularity conditions (3) are equivalent to the conditions $\gamma_{n+1} \neq 0,\ n \geqslant 0$.

The canonical sequence $\{X_n\}_{n\geqslant 0}$, $X_n(x) = x^n$, is orthogonal with respect to the Dirac measure δ_0, $\langle \delta_0, f\rangle = f(0)$, defined by the moments $(\delta_0)_n = \delta_{0,n},\ n \geqslant 0$, where δ is the Dirac symbol. This sequence is not regularly orthogonal, since its recurrence coefficients are

$$\beta_n = 0\ ,\ \ \gamma_{n+1} = 0\ ,\ \ n \geqslant 0\ . \tag{8}$$

A form u is said symmetric if and only if $(u)_{2n+1} = 0,\ n \geq 0$. A polynomial sequence, $\{P_n\}_{n\geqslant 0}$, is said symmetric if and only if $P_n(-x) = (-1)^n P_n(x),\ n \geqslant 0$.

Let $\{P_n\}_{n\geqslant 0}$ be a MOPS with respect to u. The following statements are equivalent [8]:

$$a)\ u \text{ is symmetric.}\ \ b)\ \{P_n\}_{n\geqslant 0} \text{ is symmetric.}\ \ c)\ \beta_n = 0,\ n \geqslant 0. \tag{9}$$

3 The general recurrence relation for the connection coefficients

Given two MPS $\{P_n\}_{n\geqslant 0}$ and $\{\tilde{P}_n\}_{n\geqslant 0}$ the coefficients that satisfy the equality

$$P_n(x) = \sum_{\nu=0}^{n} \lambda_{n,\nu} \tilde{P}_\nu(x),\ n \geqslant 0 \tag{10}$$

are called the connection coefficients $\lambda_{n,\nu} := \lambda_{n,\nu}^{P\tilde{P}} := \lambda_{n,\nu}(P \leftarrow \tilde{P})$. It is obvious that these coefficients are unique, because the polynomials are linearly independents.

In the case of others polynomial normalizations, that is, $B_n(x) = k_n P_n(x)$, $k_n \neq 0$ and $\tilde{B}_n(x) = \tilde{k}_n \tilde{P}_n(x)$, $\tilde{k}_n \neq 0$, $n \geq 0$, the corresponding connection coefficients to consider are

$$\lambda_{n,m}^{B\tilde{B}} := k_n \lambda_{n,m}^{P\tilde{P}} \tilde{k}_m^{-1}\ \ ,\ \ \lambda_{n,m}^{BX} := k_n \lambda_{n,m}^{PX}\ \ ,\ \ \lambda_{n,m}^{XB} := \lambda_{n,m}^{XP} k_m^{-1}\ ,\ 0 \leq m \leq n,\ n \geq 0\ .$$

From now on, let us suppose that the sequences are MOPS with respect to the forms u and $\tilde{u}$, and are given by their recurrence coefficients $\{\beta_n\}_{n\geqslant 0}$, $\{\gamma_{n+1}\}_{n\geqslant 0}$ and $\{\tilde{\beta}_n\}_{n\geqslant 0}$, $\{\tilde{\gamma}_{n+1}\}_{n\geqslant 0}$, respectively; let us consider the problem of determining the corresponding connection coefficients.

To find $\lambda_{n,m}$, we multiply both members of (10) by $\tilde{P}_m$ and operating with $\tilde{u}$, we obtain

$$\lambda_{n,m} = \frac{\left\langle \tilde{u}, P_n \tilde{P}_m \right\rangle}{\left\langle \tilde{u}, \tilde{P}_m^2 \right\rangle},\ 0 \leqslant m \leqslant n,\ n \geqslant 0\ . \tag{11}$$

Theorem 3.1 *The connection coefficients, $\lambda_{n,\nu} := \lambda_{n,\nu}(P \leftarrow \tilde{P})$, satisfy the following recurrence relation with the corresponding initial conditions*

$$\lambda_{n+2,m} + \left(\beta_{n+1} - \tilde{\beta}_m\right)\lambda_{n+1,m} + \gamma_{n+1}\lambda_{n,m} = \tilde{\gamma}_{m+1}\lambda_{n+1,m+1} + \lambda_{n+1,m-1} \tag{12}$$
$$0 \leqslant m \leqslant n+1,\ \ n \geqslant 0\ .$$

$$\lambda_{1,0} = \tilde{\beta}_0 - \beta_0\ , \tag{13}$$
$$\lambda_{n,n} = 1,\ \ n \geqslant 0\ , \tag{14}$$
$$\lambda_{n,m} = 0,\ \ n < 0\ \textit{or}\ m < 0\ \textit{or}\ m > n\ . \tag{15}$$

Proof. Taking $\{n \to n+2\}$ in (11) and applying (4), we get

$$\begin{aligned}\lambda_{n+2,m}\left\langle \tilde{u}, \tilde{P}_m^2 \right\rangle &= \left\langle \tilde{u}, P_{n+2}\tilde{P}_m \right\rangle = \left\langle \tilde{u}, \left[(x-\beta_{n+1})\,P_{n+1} - \gamma_{n+1}P_n\right]\tilde{P}_m \right\rangle \\ &= \left\langle \tilde{u}, xP_{n+1}\tilde{P}_m \right\rangle - \beta_{n+1}\left\langle \tilde{u}, P_{n+1}\tilde{P}_m \right\rangle - \gamma_{n+1}\left\langle \tilde{u}, P_n\tilde{P}_m \right\rangle .\end{aligned}$$

Using (11) twice, we obtain

$$\lambda_{n+2,m}\left\langle \tilde{u}, \tilde{P}_m^2 \right\rangle = W_{n,m} - \beta_{n+1}\lambda_{n+1,m}\left\langle \tilde{u}, \tilde{P}_m^2 \right\rangle - \gamma_{n+1}\lambda_{n,m}\left\langle \tilde{u}, \tilde{P}_m^2 \right\rangle , \tag{16}$$

where $W_{n,m} = \left\langle \tilde{u}, xP_{n+1}\tilde{P}_m \right\rangle$.

Taking $\{n \to m-1\}$ in (4), we have $x\tilde{P}_m = \tilde{P}_{m+1} + \tilde{\beta}_m\tilde{P}_m + \tilde{\gamma}_m\tilde{P}_{m-1}$, replacing in $W_{n,m}$, we obtain

$$W_{n,m} = \left\langle \tilde{u}, P_{n+1}\tilde{P}_{m+1} \right\rangle + \tilde{\beta}_m\left\langle \tilde{u}, P_{n+1}\tilde{P}_m \right\rangle + \tilde{\gamma}_m\left\langle \tilde{u}, P_{n+1}\tilde{P}_{m-1} \right\rangle .$$

Applying three times (11), we write

$$W_{n,m} = \lambda_{n+1,m+1}\left\langle \tilde{u}, \tilde{P}_{m+1}^2 \right\rangle + \tilde{\beta}_m\lambda_{n+1,m}\left\langle \tilde{u}, \tilde{P}_m^2 \right\rangle + \tilde{\gamma}_m\lambda_{n+1,m-1}\left\langle \tilde{u}, \tilde{P}_{m-1}^2 \right\rangle .$$

Using (7) for $\{n \to m\}$ and $\{n \to m-1\}$, we get

$$W_{n,m} = \lambda_{n+1,m+1}\tilde{\gamma}_{m+1}\left\langle \tilde{u}, \tilde{P}_m^2 \right\rangle + \tilde{\beta}_m\lambda_{n+1,m}\left\langle \tilde{u}, \tilde{P}_m^2 \right\rangle + \lambda_{n+1,m-1}\left\langle \tilde{u}, \tilde{P}_m^2 \right\rangle .$$

Replacing in (16) and simplifying the factor $\left\langle \tilde{u}, \tilde{P}_m^2 \right\rangle$ in both members, we obtain (12).

In order to prove the initial conditions, let us write (10) for $n = 1$: $P_1(x) = \lambda_{1,0}\tilde{P}_0(x) + \lambda_{1,1}\tilde{P}_1(x)$. Then, from (5), we obtain

$$x - \beta_0 = \lambda_{1,1}x + \left(\lambda_{1,0} + \lambda_{1,1}\tilde{\beta}_0\right) \ ,$$

and we get (13) and (14) for $n = 1$. From (10) and the fact that the polynomials are monic, we get (14), $\forall n \geq 0$. Assuming, as usual, that $P_{-n} = 0, \ n \geqslant 0$, and due to $\deg(P_n) = n$, (15) becomes obvious. ∎

Remark 3.2 *If u and $\tilde{u}$ are symmetric, then, from (11), we have*

$$\lambda_{2n,2m-1} = 0 \ , \ \lambda_{2n+1,2m} = 0 \ , \ 0 \leqslant m \leqslant n, \ n \geqslant 0 \ . \tag{17}$$

4 Methodology

In order to deduce the results given in the next section of examples, we have used the following methodology, which is explaned in more details in [25]:

- From the recurrence coefficients of the two MOPS $\{P_n\}_{n\geq 0}$ and $\{\tilde{P}_n\}_{n\geq 0}$, that is, from $\{\beta_n\}_{n\geq 0}$, $\{\gamma_{n+1}\}_{n\geq 0}$ and $\{\tilde{\beta}_n\}_{n\geq 0}$, $\{\tilde{\gamma}_{n+1}\}_{n\geq 0}$, we compute recursively the corresponding first connection coefficients

$$\{\lambda_{n,m}(P \leftarrow \tilde{P}) : 0 \leq m \leq n, 0 \leq n \leq nmax\} \ ,$$

  with $nmax$ fixed, using the general recurrence relation (12). This is done making symbolic computation in the *Mathematica* language [29]. The relation (12) is implemented using the *Mathematica* function definition that remember values that it find. Often, we have applied the *Mathematica* commands *Simplify*, *FullSimplify*, *Together* and *Factor* to the results given by (12) in order to get the connection coefficients written in a simple convenient form.

- In each case treated here, with exception to those cited below, the careful observation of the *Mathematica* results allows to guess the model to the closed formula for the connection coefficients. This formula can be implemented in *Mathematica* in order to compare the first $nmax$ connection coefficients computed by it with those recursively computed by (12). Of course, this verification does not constitute a mathematical proof. The demonstration of the closed formula correspond to show that the model is a solution of the general recurrence relation (12), that we write as follows

$$\lambda_{n+2,m} = \tilde{\gamma}_{m+1}\lambda_{n+1,m+1} + \lambda_{n+1,m-1} - \left(\beta_{n+1} - \tilde{\beta}_m\right)\lambda_{n+1,m} - \gamma_{n+1}\lambda_{n,m} \ , \tag{18}$$

for $\forall n \geq 0$ and $\forall m : \ 0 \leq m \leq n$. This could also be done, in principle, directly via *Mathematica*, though that depends on the simplifying capabilities of certain commands like *FullSimplify* and *FunctionExpand* with respect to expressions involving products and ratios of Gamma function's values.

In general, we have used the following procedure.

Procedure 4.1 *Find A, B, C and D such that*

$$\tilde{\gamma}_{m+1}\lambda_{n+1,m+1} = A\ \lambda_{n+2,m} \quad , \quad \lambda_{n+1,m-1} = B\ \lambda_{n+2,m} \ ,$$
$$-\left(\beta_{n+1} - \tilde{\beta}_m\right)\lambda_{n+1,m} = C\ \lambda_{n+2,m} \quad , \quad -\gamma_{n+1}\lambda_{n,m} = D\ \lambda_{n+2,m} \ .$$

Thus, (18) is equivalent to $\lambda_{n+2,m} = (A+B+C+D)\lambda_{n+2,m}$, and we show that $A+B+C+D=1$ with Mathematica.

In the case $\{P_n\}_{n\geq 0}$ and $\{\tilde{P}_n\}_{n\geq 0}$ are symmetric, the relation (18) is equivalent to

$$\lambda_{2n+2,2m} = \tilde{\gamma}_{2m+1}\lambda_{2n+1,2m+1} + \lambda_{2n+1,2m-1} - \gamma_{2n+1}\lambda_{2n,2m} \ , \tag{19}$$
$$\lambda_{2n+3,2m+1} = \tilde{\gamma}_{2m+2}\lambda_{2n+2,2m+2} + \lambda_{2n+2,2m} - \gamma_{2n+2}\lambda_{2n+1,2m+1} \ , \tag{20}$$

and we have to show that the models to $\lambda_{2n,2m}$ and $\lambda_{2n+1,2m+1}$ are solutions of these equations, $\forall n \geq 0$ and $\forall m : \ 0 \leq m \leq n$. Thus the preceding procedure becomes the following.

Procedure 4.2 *Find A_1, B_1, C_1 and A_2, B_2, C_2 such that*

$$\tilde{\gamma}_{2m+1}\lambda_{2n+1,2m+1} = A_1\ \lambda_{2n+2,2m} \quad , \quad \tilde{\gamma}_{2m+2}\lambda_{2n+2,2m+2} = A_2\ \lambda_{2n+3,2m+1} \ ,$$
$$\lambda_{2n+1,2m-1} = B_1\ \lambda_{2n+2,2m} \quad , \quad \lambda_{2n+2,2m} = B_2\ \lambda_{2n+3,2m+1} \ ,$$
$$-\gamma_{2n+1}\lambda_{2n,2m} = C_1\ \lambda_{2n+2,2m} \quad , \quad -\gamma_{2n+2}\lambda_{2n+1,2m+1} = C_2\ \lambda_{2n+3,2m+1} \ .$$

Thus, (19) and (20) are equivalent to

$$\lambda_{2n+2,2m} = (A_1+B_1+C_1)\lambda_{2n+2,2m} \ , \ \lambda_{2n+3,2m+1} = (A_2+B_2+C_2)\lambda_{2n+3,2m+1} \ ,$$

and we show that $A_1+B_1+C_1=1$ and $A_2+B_2+C_2=1$ with Mathematica.

- The success of this methodology depends on the possibility of writing the connection coefficients computed recursively in a form that allows us to infer the model for the closed formulas. To achieve it, we need to factorize the numerators and the denominators of the connection coefficients so that we can write them with the same "appearance" than the recurrence coefficients. For that, the above cited *Mathematica* commands play an important rôle.

- Exceptions to this methodology.

  In the last section 5.4, we consider classical sequences belonging to different families. We do not achieve to guess the model, because the results are not products of simple elements neither ratio of products of simple elements, they are sums of elements that can not be factorized. In spite of this, fixing the values of the parameters, it is always possible to produce tables for the first *nmax* symbolic connection coefficients.

5 Examples

5.1 Bessel polynomials

Let us express a monic Bessel polynomial sequence $\{P_n(\alpha;.)\}_{n\geq 0}$ with parameter α in terms of other monic Bessel polynomial sequence $\{P_n(\tilde{\alpha};.)\}_{n\geq 0}$ with parameter $\tilde{\alpha}$. For that purpose, we need to recall the Bessel recurrence coefficients (see, for example, [8, 21, 22]),

$$\beta_0(\alpha) = -\frac{1}{\alpha} \ , \ \beta_{n+1}(\alpha) = \frac{1-\alpha}{(n+\alpha)(n+\alpha+1)} \ , \tag{21}$$

$$\gamma_{n+1}(\alpha) = -\frac{(n+1)(n+2\alpha-1)}{(2n+2\alpha-1)(n+\alpha)^2(2n+2\alpha+1)} \ , \tag{22}$$

$$\tilde{\beta}_n = \beta_n(\tilde{\alpha}),\ \tilde{\gamma}_{n+1} = \gamma_{n+1}(\tilde{\alpha}) \ , \tag{23}$$

for $n \geq 0$, with the regularity conditions $\alpha, \tilde{\alpha} \neq -\frac{n}{2},\ n \geq 0$.

In this case, the general recurrence relation (12) produces the results of the table 1. The observation of these results allows to suppose that the connection coefficients are given by the following closed formula

$$\lambda_{n,m} = (-1)^{n+m} \binom{n}{m} \frac{\prod_{\nu=0}^{n-1-m}(\frac{\nu}{2}+\alpha-\tilde{\alpha})}{\prod_{\nu=n-1+m}^{2(n-1)}(\frac{\nu}{2}+\alpha)\prod_{\nu=2m}^{n-1+m}(\frac{\nu}{2}+\tilde{\alpha})} \ ,$$

for $0 \leq m \leq n-1,\ n \geq 1$.

This formula can be written in terms of the Gamma function following to the fact that

$$\Gamma(1) = 1 \ , \ \Gamma(a+1) = a\Gamma(a) \ , \ \frac{\Gamma(a+n)}{\Gamma(a)} = \prod_{\nu=0}^{n-1}(a+\nu) \ , \ n \geq 1 \ . \tag{24}$$

Thus, we have the following result.

Table 1: P represents the monic Bessel polynomials.

n	m	$\lambda_{n,m} := \lambda_{n,m}(P(\alpha;-) \leftarrow P(\tilde{\alpha};-))$
5	0,	$-\frac{4(2\alpha-2\tilde{\alpha}+1)(2\alpha-2\tilde{\alpha}+3)(\alpha-\tilde{\alpha})(\alpha-\tilde{\alpha}+1)(\alpha-\tilde{\alpha}+2)}{(\alpha+2)(\alpha+3)(\alpha+4)(2\alpha+5)(2\alpha+7)\tilde{\alpha}(\tilde{\alpha}+1)(\tilde{\alpha}+2)(2\tilde{\alpha}+1)(2\tilde{\alpha}+3)} \quad \vert$
	1,	$\frac{20(2\alpha-2\tilde{\alpha}+1)(2\alpha-2\tilde{\alpha}+3)(\alpha-\tilde{\alpha})(\alpha-\tilde{\alpha}+1)}{(\alpha+3)(\alpha+4)(2\alpha+5)(2\alpha+7)(\tilde{\alpha}+1)(\tilde{\alpha}+2)(2\tilde{\alpha}+3)(2\tilde{\alpha}+5)} \quad \vert$
	2,	$-\frac{20(2\alpha-2\tilde{\alpha}+1)(\alpha-\tilde{\alpha})(\alpha-\tilde{\alpha}+1)}{(\alpha+3)(\alpha+4)(2\alpha+7)(\tilde{\alpha}+2)(\tilde{\alpha}+3)(2\tilde{\alpha}+5)} \quad \vert$
	3,	$\frac{20(2\alpha-2\tilde{\alpha}+1)(\alpha-\tilde{\alpha})}{(\alpha+4)(2\alpha+7)(\tilde{\alpha}+3)(2\tilde{\alpha}+7)} \quad \vert$
	4,5	$-\frac{5(\alpha-\tilde{\alpha})}{(\alpha+4)(\tilde{\alpha}+4)} \quad \vert \quad 1$
6	0,	$\frac{8(2\alpha-2\tilde{\alpha}+1)(2\alpha-2\tilde{\alpha}+3)(2\alpha-2\tilde{\alpha}+5)(\alpha-\tilde{\alpha})(\alpha-\tilde{\alpha}+1)(\alpha-\tilde{\alpha}+2)}{(\alpha+3)(\alpha+4)(\alpha+5)(2\alpha+5)(2\alpha+7)(2\alpha+9)\tilde{\alpha}(\tilde{\alpha}+1)(\tilde{\alpha}+2)(2\tilde{\alpha}+1)(2\tilde{\alpha}+3)(2\tilde{\alpha}+5)} \quad \vert$
	1,	$-\frac{24(2\alpha-2\tilde{\alpha}+1)(2\alpha-2\tilde{\alpha}+3)(\alpha-\tilde{\alpha})(\alpha-\tilde{\alpha}+1)(\alpha-\tilde{\alpha}+2)}{(\alpha+3)(\alpha+4)(\alpha+5)(2\alpha+7)(2\alpha+9)(\tilde{\alpha}+1)(\tilde{\alpha}+2)(\tilde{\alpha}+3)(2\tilde{\alpha}+3)(2\tilde{\alpha}+5)} \quad \vert$
	2,	$\frac{60(2\alpha-2\tilde{\alpha}+1)(2\alpha-2\tilde{\alpha}+3)(\alpha-\tilde{\alpha})(\alpha-\tilde{\alpha}+1)}{(\alpha+4)(\alpha+5)(2\alpha+7)(2\alpha+9)(\tilde{\alpha}+2)(\tilde{\alpha}+3)(2\tilde{\alpha}+5)(2\tilde{\alpha}+7)} \quad \vert$
	3,	$-\frac{40(2\alpha-2\tilde{\alpha}+1)(\alpha-\tilde{\alpha})(\alpha-\tilde{\alpha}+1)}{(\alpha+4)(\alpha+5)(2\alpha+9)(\tilde{\alpha}+3)(\tilde{\alpha}+4)(2\tilde{\alpha}+7)} \quad \vert$
	4,5,6	$\frac{30(2\alpha-2\tilde{\alpha}+1)(\alpha-\tilde{\alpha})}{(\alpha+5)(2\alpha+9)(\tilde{\alpha}+4)(2\tilde{\alpha}+9)} \quad \vert \quad -\frac{6(\alpha-\tilde{\alpha})}{(\alpha+5)(\tilde{\alpha}+5)} \quad \vert \quad 1$

Proposition 5.1 *(see, also, [3])*
The $\lambda_{n,m} := \lambda_{n,m}(P(\alpha;-) \leftarrow P(\tilde{\alpha};-))$, where P denotes the Bessel polynomials, are given by

$$\lambda_{n,m} = (-1)^{n+m} 2^{n-m} \binom{n}{m} \frac{\Gamma(2(\alpha-\tilde{\alpha})+n-m)\Gamma(2\alpha+n+m-1)\Gamma(2(\tilde{\alpha}+m))}{\Gamma(2(\alpha-\tilde{\alpha}))\Gamma(2(\alpha+n)-1)\Gamma(2\tilde{\alpha}+n+m)}\ , \tag{25}$$

for $0 \leq m \leq n,\ n \geq 0$.

Proof. Let us use the procedure 4.1.

Taking $\{n \to n+2\}$, $\{n \to n+1, m \to m+1\}$, $\{n \to n+1, m \to m-1\}$ and $\{n \to n+1\}$ in (25), we get, respectively

$$\begin{aligned} \lambda_{n+2,m} &= (-1)^{n+m} 2^{n-m+2} \binom{n+2}{m} \\ &\quad \frac{\Gamma(2(\alpha-\tilde{\alpha})+n-m+2)\Gamma(2\alpha+n+m+1)\Gamma(2(\tilde{\alpha}+m))}{\Gamma(2(\alpha-\tilde{\alpha}))\Gamma(2(\alpha+n)+3)\Gamma(2\tilde{\alpha}+n+m+2)}\ , \end{aligned} \tag{26}$$

$$\begin{aligned} \lambda_{n+1,m+1} &= (-1)^{n+m} 2^{n-m} \binom{n+1}{m+1} \\ &\quad \frac{\Gamma(2(\alpha-\tilde{\alpha})+n-m)\Gamma(2\alpha+n+m+1)\Gamma(2(\tilde{\alpha}+m)+2)}{\Gamma(2(\alpha-\tilde{\alpha}))\Gamma(2(\alpha+n)+1)\Gamma(2\tilde{\alpha}+n+m+2)}\ , \end{aligned} \tag{27}$$

$$\begin{aligned} \lambda_{n+1,m-1} &= (-1)^{n+m} 2^{n-m+2} \binom{n+1}{m-1} \\ &\quad \frac{\Gamma(2(\alpha-\tilde{\alpha})+n-m+2)\Gamma(2\alpha+n+m-1)\Gamma(2(\tilde{\alpha}+m)-2)}{\Gamma(2(\alpha-\tilde{\alpha}))\Gamma(2(\alpha+n)+1)\Gamma(2\tilde{\alpha}+n+m)}\ , \end{aligned} \tag{28}$$

$$\begin{aligned} \lambda_{n+1,m} &= (-1)^{n+m+1} 2^{n-m+1} \binom{n+1}{m} \\ &\quad \frac{\Gamma(2(\alpha-\tilde{\alpha})+n-m+1)\Gamma(2\alpha+n+m)\Gamma(2(\tilde{\alpha}+m))}{\Gamma(2(\alpha-\tilde{\alpha}))\Gamma(2(\alpha+n)+1)\Gamma(2\tilde{\alpha}+n+m+1)}\ . \end{aligned} \tag{29}$$

From (26) and (27), noting that $\binom{n+1}{m+1} = \frac{(n-m+1)(n-m+2)}{(n+2)(m+1)} \binom{n+2}{m}$ and taking the properties (24) of the Gamma function into account, we can write

$$\begin{aligned} \lambda_{n+1,m+1} &= \frac{(n-m+1)(n-m+2)}{(n+2)(m+1)} (2(\alpha+n)+1)(\alpha+n+1) \\ &\quad \frac{(\tilde{\alpha}+m)(2(\tilde{\alpha}+m)+1)}{(2(\alpha-\tilde{\alpha})+n-m)(2(\alpha-\tilde{\alpha})+n-m+1)} \lambda_{n+2,m}\ . \end{aligned} \tag{30}$$

Hence and (23), we get $\tilde{\gamma}_{m+1}\lambda_{n+1,m+1} = A\lambda_{n+2,m}$, where

$$A = -\frac{(n-m+1)(n-m+2)}{(n+2)}(2(\alpha+n)+1)(\alpha+n+1) \tag{31}$$
$$\frac{(2\tilde{\alpha}+m-1)}{(\tilde{\alpha}+m)(2(\tilde{\alpha}+m)-1)(2(\alpha-\tilde{\alpha})+n-m)(2(\alpha-\tilde{\alpha})+n-m+1)}\,.$$

Due to (26) and (28), using (24) and noting that $\binom{n+1}{m-1} = \frac{m}{n+2}\binom{n+2}{m}$, we get $\lambda_{n+1,m-1} = B\lambda_{n+2,m}$, where

$$B = \frac{m}{(n+2)}\frac{(2(\alpha+n)+1)(\alpha+n+1)}{(2\alpha+n+m-1)(2\alpha+n+m)} \tag{32}$$
$$\frac{(2\tilde{\alpha}+n+m)(2\tilde{\alpha}+n+m+1)}{(\tilde{\alpha}+m-1)(2(\tilde{\alpha}+m)-1)}\,.$$

By virtue of (26) and (29), using (24), noting that $\binom{n+1}{m} = \frac{n-m+2}{n+2}\binom{n+2}{m}$, and also from (21) and (23), we obtain $-(\beta_{n+1} - \tilde{\beta}_m)\lambda_{n+1,m} = C\lambda_{n+2,m}$, where

$$C = \frac{(n-m+2)}{(n+2)}\left\{\frac{1-\alpha}{(n+\alpha)(n+\alpha+1)} - \frac{1-\tilde{\alpha}}{(m+\tilde{\alpha}-1)(m+\tilde{\alpha})}\right\}$$
$$\frac{(2(\alpha+n)+1)(\alpha+n+1)(2\tilde{\alpha}+n+m+1)}{(2\alpha+n+m)(2(\alpha-\tilde{\alpha})+n-m+1)}\,. \tag{33}$$

On account of (25) and (26), using (24) and noting that $\binom{n}{m} = \frac{(n-m+1)(n-m+2)}{(n+1)(n+2)}\binom{n+2}{m}$, we get

$$\lambda_{n,m} = \frac{(n-m+1)(n-m+2)}{(n+1)(n+2)} \tag{34}$$
$$\frac{(2(\alpha+n)-1)(\alpha+n)(2(\alpha+n)+1)(\alpha+n+1)}{(2\alpha+n+m-1)(2\alpha+n+m)}$$
$$\frac{(2\tilde{\alpha}+n+m)(2\tilde{\alpha}+n+m+1)}{(2(\alpha-\tilde{\alpha})+n-m)(2(\alpha-\tilde{\alpha})+n-m+1)}\lambda_{n+2,m}\,. \tag{35}$$

From this and (22), we obtain $-\gamma_{n+1}\lambda_{n,m} = D\lambda_{n+2,m}$, where

$$D = \frac{(n-m+1)(n-m+2)}{(n+2)}\frac{(\alpha+n+1)(n+2\alpha-1)}{(\alpha+n)(2\alpha+n+m-1)(2\alpha+n+m)}$$
$$\frac{(2\tilde{\alpha}+n+m)(2\tilde{\alpha}+n+m+1)}{(2(\alpha-\tilde{\alpha})+n-m)(2(\alpha-\tilde{\alpha})+n-m+1)}\,. \tag{36}$$

It can be shown in *Mathematica* [29], using, for example, the command *Together* or *Simplify*, that $A + B + C + D = 1$, as desired. ∎

Next, we give, the connection coefficients corresponding to the canonical sequence and the Bessel polynomials.

Proposition 5.2 *(see, also, [3])*
The $\lambda_{n,m} := \lambda_{n,m}(X \leftarrow P(\tilde{\alpha}; -))$, *where* P *denotes the Bessel polynomials, are given by*

$$\lambda_{n,m} = (-1)^{n+m} 2^{n-m} \binom{n}{m} \frac{\Gamma(2(\tilde{\alpha} + m)}{\Gamma(2\tilde{\alpha} + n + m)}, \; 0 \leq m \leq n, \; n \geq 0 \; . \tag{37}$$

Proof. In this case, we must consider (8) and $\tilde{\beta}_n, \; \tilde{\gamma}_{n+1}, n \geq 0$ given by (23) and afterward follow the same method of the preceding proof. ■

5.2 Laguerre polynomials

We recall the Laguerre recurrence coefficients (see, for example, [8, 21, 22]),

$$\beta_n(\alpha) = 2n + \alpha + 1 \; , \; \gamma_{n+1}(\alpha) = (n+1)(n + \alpha + 1), \; n \geq 0 \; ,$$

with the regularity condition $\alpha \neq -n, \; n \geq 0$. Following the same method, we achieve to the corresponding connection coefficients.

Proposition 5.3 *(see, also, [4, 5])*
Denoting by P *the Laguerre polynomials, it holds:*

- *The* $\lambda_{n,m} := \lambda_{n,m}(P(\alpha; -) \leftarrow P(\tilde{\alpha}; -))$ *are given by*

$$\lambda_{n,m} = (-1)^{n+m} \binom{n}{m} \frac{\Gamma(\alpha - \tilde{\alpha} + n - m)}{\Gamma(\alpha - \tilde{\alpha})}, \; 0 \leq m \leq n, \; n \geq 0.$$

- *The* $\lambda_{n,m} := \lambda_{n,m}(X \leftarrow P(\tilde{\alpha}; -))$ *are given by*

$$\lambda_{n,m} = \binom{n}{m} \frac{\Gamma(\tilde{\alpha} + n + 1)}{\Gamma(\tilde{\alpha} + m + 1)}, \; 0 \leq m \leq n, \; n \geq 0.$$

5.3 Generalized Hermite polynomials

Let us consider the case of the generalized Hermite sequence $\{P_n(\mu; .)\}_{n \geq 0}$. The recurrence coefficients are [8]

$$\beta_n = 0 \quad , \quad \gamma_{n+1} := \gamma_{n+1}(\mu) = \frac{1}{2} \left(n + 1 + \mu(1 + (-1)^n) \right), \; n \geq 0. \tag{38}$$

We remark that this is a symmetric, semi-classical of class 1 sequence. When $\mu = 0$, we recover the classical Hermite sequence.

Proposition 5.4 *Denoting by P the generalized Hermite polynomials, it holds, for $0 \le m \le n$ and $n \ge 0$:*

1. *The $\lambda_{n,m} := \lambda_{n,m}(P(\mu;-) \leftarrow P(\tilde{\mu};-))$ are given by*

$$\lambda_{2n,2m} = (-1)^{n+m}\binom{n}{m}\frac{\Gamma(\mu-\tilde{\mu}+n-m)}{\Gamma(\mu-\tilde{\mu})}\ , \tag{39}$$

$$\lambda_{2n+1,2m+1} = \lambda_{2n,2m}\ . \tag{40}$$

2. *The $\lambda_{n,m} := \lambda_{n,m}(X \leftarrow P(\tilde{\mu};-))$ are given by*

$$\lambda_{2n,2m} = \binom{n}{m}\frac{\Gamma(\tilde{\mu}+n+1/2)}{\Gamma(\tilde{\mu}+m+1/2)}\ ,$$

$$\lambda_{2n+1,2m+1} = \frac{\tilde{\mu}+n+1/2}{\tilde{\mu}+m+1/2}\ \lambda_{2n,2m}\ .$$

3. *The $\lambda_{n,m} := \lambda_{n,m}(P(\mu;-) \leftarrow X)$ are given by*

$$\lambda_{2n,2m} = (-1)^{n+m}\binom{n}{m}\frac{\Gamma(\mu+n+1/2)}{\Gamma(\mu+m+1/2)}\ ,$$

$$\lambda_{2n+1,2m+1} = \frac{\mu+n+1/2}{\mu+m+1/2}\ \lambda_{2n,2m}\ .$$

4. *In all the cases,*

$$\lambda_{2n,2m-1} = 0 = \lambda_{2n+1,2m}\ .$$

Proof. We begin by showing that the first assertion holds. The proofs of the statements 2 and 3 are analogous. The statement 4 is the same as (17).

From (40), the formulas of the procedure 4.2 become

$$\begin{aligned}
\tilde{\gamma}_{2m+1}\lambda_{2n,2m} &= A_1\ \lambda_{2n+2,2m} \quad , \quad \tilde{\gamma}_{2m+2}\lambda_{2n+2,2m+2} = A_2\ \lambda_{2n+2,2m}\ ,\\
\lambda_{2n,2m-2} &= B_1\ \lambda_{2n+2,2m} \quad , \quad \lambda_{2n+2,2m} = B_2\ \lambda_{2n+2,2m}\ ,\\
-\gamma_{2n+1}\lambda_{2n,2m} &= C_1\ \lambda_{2n+2,2m} \quad , \quad -\gamma_{2n+2}\lambda_{2n,2m} = C_2\ \lambda_{2n+2,2m}\ .
\end{aligned}$$

It is easy to deduce that

$$\begin{aligned}
A_1 &= -\frac{n-m+1}{n+1}\frac{(m+\tilde{\mu}+\frac{1}{2})}{(\mu-\tilde{\mu}+n-m)} \quad , \quad A_2 = -\frac{n-m+1}{\mu-\tilde{\mu}+n-m}\ ,\\
B_1 &= \frac{m}{n+1} \quad , \quad B_2 = 1\ ,\\
C_1 &= \frac{n-m+1}{n+1}\frac{(n+\mu+\frac{1}{2})}{(\mu-\tilde{\mu}+n-m)} \quad , \quad C_2 = \frac{n-m+1}{\mu-\tilde{\mu}+n-m}\ .
\end{aligned}$$

Now, we can verify that $A_1+B_1+C_1 = 1$ and $A_2+B_2+C_2 = 1$. ■

5.4 Miscellaneous examples

In this section, we give some tables of connection coefficients between sequences belonging to different classical families with specific values of parameters up to a fixed value of degree. More precisely, we consider the Hermite polynomials in terms of the Legendre ones and vice-versa. Also, we present the Laguerre polynomials with null parameter in terms of the Bessel ones with parameter equal to 1, and vice-versa. Of course, we could give many other examples.

Table 2: P and Q represent the monic Hermite and the monic Legendre polynomials.

n	$\lambda_{n,m}(P \leftarrow Q),\ m = 0, \ldots, n$									
0	1									
1	0	1								
2	$-\frac{1}{6}$	0	1							
3	0	$-\frac{9}{10}$	0	1						
4	$-\frac{1}{20}$	0	$-\frac{15}{7}$	0	1					
5	0	$\frac{33}{28}$	0	$-\frac{35}{9}$	0	1				
6	$\frac{29}{56}$	0	$\frac{155}{28}$	0	$-\frac{135}{22}$	0	1			
7	0	$-\frac{37}{24}$	0	$\frac{2065}{132}$	0	$-\frac{231}{26}$	0	1		
8	$-\frac{335}{144}$	0	$-\frac{1115}{66}$	0	$\frac{9975}{286}$	0	$-\frac{182}{15}$	0	1	
9	0	$-\frac{117}{176}$	0	$-\frac{20195}{286}$	0	$\frac{8757}{130}$	0	$-\frac{270}{17}$	0	1

Table 3: Q and P represent the monic Legendre and the monic Hermite polynomials .

n	$\lambda_{n,m}(Q \leftarrow P),\ m = 0, \ldots, n$									
0	1									
1	0	1								
2	$\frac{1}{6}$	0	1							
3	0	$\frac{9}{10}$	0	1						
4	$\frac{57}{140}$	0	$\frac{15}{7}$	0	1					
5	0	$\frac{65}{28}$	0	$\frac{35}{9}$	0	1				
6	$\frac{1955}{1848}$	0	$\frac{335}{44}$	0	$\frac{135}{22}$	0	1			
7	0	$\frac{2135}{264}$	0	$\frac{10815}{572}$	0	$\frac{231}{26}$	0	1		
8	$\frac{77791}{20592}$	0	$\frac{9877}{286}$	0	$\frac{1029}{26}$	0	$\frac{182}{15}$	0	1	
9	0	$\frac{1411263}{38896}$	0	$\frac{48153}{442}$	0	$\frac{12573}{170}$	0	$\frac{270}{17}$	0	1

Table 4: $P(0;-)$ and $Q(1;-)$ represent the monic Laguerre and the monic Bessel polynomials with parameters 0 and 1, respectively.

n	$\lambda_{n,m}(P(0;-) \leftarrow Q(1;-)),\ m=0,\ldots,n$							
0	1							
1	-2	1						
2	$\frac{20}{3}$	-5	1					
3	$-\frac{91}{3}$	$\frac{138}{5}$	-10	1				
4	$\frac{2602}{15}$	$-\frac{2668}{15}$	$\frac{620}{7}$	-17	1			
5	$-\frac{53552}{45}$	$\frac{27862}{21}$	$-\frac{17105}{21}$	$\frac{2030}{9}$	-26	1		
6	$\frac{2993008}{315}$	$-\frac{394921}{35}$	$\frac{508145}{63}$	$-\frac{25832}{9}$	$\frac{5352}{11}$	-37	1	
7	$-\frac{5444449}{63}$	$\frac{2912906}{27}$	$-\frac{3903026}{45}$	$\frac{3687845}{99}$	$-\frac{272545}{33}$	$\frac{12110}{13}$	-50	1

Table 5: $Q(1;-)$ and $P(0;-)$ represent the monic Bessel and the monic Laguerre polynomials with parameters 1 and 0, respectively.

n	$\lambda_{n,m}(Q(1;-) \leftarrow P(0;-)),\ m=0,\ldots,n$							
0	1							
1	2	1						
2	$\frac{10}{3}$	5	1					
3	$\frac{127}{15}$	$\frac{112}{5}$	10	1				
4	$\frac{3251}{105}$	$\frac{2432}{21}$	$\frac{570}{7}$	17	1			
5	$\frac{138826}{945}$	$\frac{44381}{63}$	$\frac{6085}{9}$	$\frac{1948}{9}$	26	1		
6	$\frac{8853202}{10395}$	$\frac{2458121}{495}$	$\frac{597350}{99}$	$\frac{86044}{33}$	$\frac{5230}{11}$	37	1	
7	$\frac{157259497}{27027}$	$\frac{154013984}{3861}$	$\frac{41903552}{715}$	$\frac{13682690}{429}$	$\frac{23435}{3}$	$\frac{11940}{13}$	50	1

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