PROJECTIONS OF PERIODIC FUNCTIONS AND MODE INTERACTIONS
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Abstract. We study solutions of bifurcation problems with periodic boundary conditions, with periods in an $n+1$-dimensional lattice and their projection into n-dimensional space through integration of the last variable. We show that generically the projection of a single mode solution is a mode interaction. This can be applied to the study of black-eye patterns.

1. Introduction

The aim of this article is to obtain understanding of models for the formation of periodic patterns in a thin layer. It is motivated by results found in reaction-diffusion experiments in the Turing instability regime, [23]. Typically, a reaction occurs in a thin layer of gel, fed by diffusion from one or two faces with chemicals contained in stirred tanks. The first reaction that provided Turing instabilities was in experiments on the chlorite-iodide-malonic-acid (CIMA) reaction, [3].

In experiments decribed before, the pattern itself and its observed state can occur in different dimensions, see [14, 19, 11]. This happens for instance when an experiment is done in a 3-dimensional medium but the patterns are only observed on its surface, a 2-dimensional object, see [14, Section 4]. The interpretation of this 2-dimensional outcome is subject to discussion: the black-eye pattern observed by [19] has been explained both as a mode interaction in 2 dimensions in [12], and described as a projection of a fully 3-dimensional pattern in [11]. The main challenge is to choose one of these descriptions. Our result indicates that these descriptions may coincide.

What is believed to be the first evidence of projection on the CIMA reaction can be found in [24, Chapter 13]. The author gives details about the geometry of the formation of wave patterns in malonic-acid reaction performed on sufficiently thin layers. In [14] the authors conducted experiments on the CIMA reaction and aimed at describing experimental observations of spontaneous symmetry breaking phenomena associated with steady-state instabilities. In [14, Section 4], the authors highlight the natural environment we must consider when we carry out CIMA reactions, in particular they state that all of their observations were based on projection of 3-dimensional structures. Moreover, the regions where Turing patterns are observed are associated by projection to a body-centred cubic lattice. More discussion on this can be found in [8, 15, 2, 22].

Here, we study the relation between patterns in a $(n+1)$-dimensional space and their projection in a n-dimensional space. We achieve this by an algebraic approach to the relevant symmetry-related objects that contribute to the description of patterns.

2. Bifurcation Problems with Euclidean Symmetry

This section contains a rigorous formulation of the setting of the article and a statement of its main results. For this definitions and basic results that are used in the article are stated here. More information on crystallographic groups may be found in Miller [17], Armstrong [1] chapters 24 to 26, Senechal [21], the International Tables for Crystallography (ITC) volume A [13]. Results on equivariant bifurcation theory can be found in Golubitsky and Stewart's book [9]. For mode interactions we refer the reader to Castro [4, 5, 6] and [10, Ch XIX and XX].

Let $E(n+1)$ be the *Euclidean group* of all isometries on $\mathbb{R}^{n+1}$, that may be described as the semi-direct sum $E(n+1) \cong \mathbb{R}^{n+1} \dot{+} O(n+1)$. Its elements are ordered pairs, (v,δ), where $v \in \mathbb{R}^{n+1}$ is a translation and δ

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is an element of the orthogonal group $O(n+1)$. The group operation is $(v_1,\delta_1)\cdot(v_2,\delta_2) = (v_1+\delta_1 v_2, \delta_1\delta_2)$. Elements $(v,\delta) \in E(n+1)$ act on functions $f:\mathbb{R}^{n+1}\to\mathbb{R}$ by $((v,\delta)\cdot f)(x) = f(\delta^{-1}(x-v))$.

Consider a one parameter family of partial differential equations

$$\frac{\partial u}{\partial t}(x,t) = \mathcal{F}(u(x,t),\lambda) \tag{1}$$

where $\mathcal{F}:\mathcal{X}\times\mathbb{R}\longrightarrow\mathcal{Y}$ is an operator between suitable function spaces $\mathcal{X}$ and $\mathcal{Y}$ and $\lambda\in\mathbb{R}$ is a bifurcation parameter. The function $u:\mathbb{R}^{n+1}\times\mathbb{R}\to\mathbb{R}$ in $\mathcal{X}$ is a function of a spatial variable $x\in\mathbb{R}^{n+1}$ and of time t.

Suppose that $\mathcal{F}$ is *equivariant* under the Euclidean group $E(n+1)$:

$$\gamma\cdot\mathcal{F}(u,\lambda) = \mathcal{F}(\gamma\cdot u,\lambda), \quad \text{for all} \quad \gamma\in E(n+1).$$

Equilibria of (1) are time independent solutions $u(x)$ that satisfy $\mathcal{F}(u(x),\lambda) = 0$.

We give a brief description of a standard method to use symmetries to study the way steady-states in (1) bifurcate from the trivial solution $\mathcal{F}(0,\lambda)\equiv 0$. Details may be found in [7, 9]. The first step is to restrict the problem to a subspace $\mathcal{X}_{\mathcal{L}}$ of functions periodic under an $(n+1)$-*dimensional lattice*, $\mathcal{L}\subset\mathbb{R}^{n+1}$, a set generated over the integers by $n+1$ linearly independent elements $l_1,\dots,l_{n+1}\in\mathbb{R}^{n+1}$.

The restricted problem is equivariant under the action of the group $\Gamma_{\mathcal{L}}$, the largest group constructed from $E(n+1)$ that leaves the space $\mathcal{X}_{\mathcal{L}}$ invariant. Translations map $\mathcal{X}_{\mathcal{L}}$ into itself, the action of translations on $\mathcal{X}_{\mathcal{L}}$ is that of the torus $\mathbb{T}^{n+1} = \mathbb{R}^{n+1}/\mathcal{L}$. The subgroup of $O(n+1)$ that maps $\mathcal{L}$ to itself, and thus leaves $\mathcal{X}_{\mathcal{L}}$ invariant, forms the *holohedry* of $\mathcal{L}$ (called *point symmetry of the lattice* in [13]) and is denoted by $H_{\mathcal{L}}$. The holohedry is always finite, see [21, Sec. 2.4.2]. The group $\Gamma_{\mathcal{L}}$ is thus compact and can be written as a semi-direct product

$$\Gamma_{\mathcal{L}} = \mathbb{R}^{n+1}/\mathcal{L}\dot{+}H_{\mathcal{L}} = \mathbb{T}^{n+1}\dot{+}H_{\mathcal{L}}.$$

The next step is to analise the bifurcation of solutions to the restricted problem. A steady-state bifurcation at $\lambda = 0$ occurs when the linearisation of $\mathcal{F}(0,0)$ has a non-trivial kernel $V\subset\mathcal{X}_{\mathcal{L}}$. The kernel is always $\Gamma_{\mathcal{L}}$-invariant in the sense that if $f\in V$ then $\gamma\cdot f\in V$ for all $\gamma\in\Gamma_{\mathcal{L}}$. Since $\Gamma_{\mathcal{L}}$ is compact, we expect V to be finite dimensional. This simplifies the problem considerably, as the study of bifurcating solutions is reduced to a $\Gamma_{\mathcal{L}}$-equivariant bifurcation problem defined in a finite dimensional space V. Solution branches can then be obtained using the Equivariant Branching Lemma [9, Lemma 1.31].

Let Γ be a symmetry group acting on a vector space V. The *orbit* of $v\in V$ under this action is the set $\Gamma\cdot v = \{\gamma v:\ \gamma\in\Gamma\}$. The *isotropy subgroup* Σ_v of $v\in V$ is $\Sigma_v = \{\gamma\in\Gamma:\ \gamma v = v\}$. We say that a vector subspace $W\subset V$ is Γ-*invariant* if $\gamma\in\Gamma$ and $v\in W$ implies $\gamma v\in W$. A Γ-invariant subspace $W\subset V$ is Γ-*irreducible* if W contains no proper non-trivial Γ-invariant subspace, hence a problem defined in W cannot be reduced further.

Consider a Γ-equivariant bifurcation problem :

$$\frac{dz}{dt} = g(z,\lambda) \qquad \text{with} \qquad g(0,\lambda) = 0 \tag{2}$$

with $g:V\times\mathbb{R}\longrightarrow V$ satisfying $g(\gamma z,\lambda) = \gamma g(z,\lambda)$ for all $\gamma\in\Gamma$ and all $(z,\lambda)\in V\times\mathbb{R}$.

Definition 1. The bifurcation problem (2) has:

- a *single mode bifurcation* at $\lambda = 0$ if the kernel of $(dg)_{0,0}$ is non-trivial and Γ-irreducible;
- an *r-mode interaction* at $\lambda = 0$ if the kernel of $(dg)_{0,0}$ ca be decomposed as the direct sum of $r > 1$ non-trivial components that are Γ-irreducible.

A characterisation of $\Gamma_{\mathcal{L}}$-irreducible subspaces of $\mathcal{X}_{\mathcal{L}}$ is given by Dionne and Golubitsky [7], as follows.

Let $\langle\cdot,\cdot\rangle$ be the usual inner product in $\mathbb{R}^{n+1}$. We assume that all the functions in $\mathcal{X}_{\mathcal{L}}$ admit a Fourier expansion in terms of the *waves*

$$\omega_k(x,y) = exp(2\pi i\langle k,(x,y)\rangle),$$

where k is a *wave vector* in the dual lattice, $\mathcal{L}^* = \{k\in\mathbb{R}^{n+1};\ \langle k,l_i\rangle\in\mathbb{Z},\ i = 1,\cdots,\ n+1\}$, of $\mathcal{L}$ and with the notation $(x,y)\in\mathbb{R}^{n+1}$, $x\in\mathbb{R}^n$, $y\in\mathbb{R}$.

Subspaces of $\mathcal{X}_{\mathcal{L}}$ that are $\Gamma_{\mathcal{L}}$-ireducible must be, in particular, $\mathbb{T}^{n+1}$-invariant. Given $\ell\in\mathbb{T}^{n+1}$ and $k\in\mathcal{L}^*$ we have $\ell\cdot\omega_k(x,y) = \omega_k(-\ell)\omega_k(x,y)$. Hence, the two-dimensional subspaces

$$V_k = \{Re(z\omega_k(x,y));\ z\in\mathbb{C}\} \tag{3}$$

are $\mathbb{T}^{n+1}$-invariant and $\mathbb{T}^{n+1}$-irreducible. Morever, V_k and $V_{k'}$ are distinct $\mathbb{T}^{n+1}$ representations if $k \neq \pm k'$.

The action of $\gamma \in H_{\mathcal{L}}$ on $\omega_k(x,y)$ satisfies

$$\gamma\omega_k(x,y) = exp(2\pi i\langle k, \gamma^{-1}(x,y)\rangle) = exp(2\pi i\langle \gamma k, (x,y)\rangle) = \omega_{\gamma k}(x,y) \tag{4}$$

by the orthogonality of γ. Then, we have:

Proposition 1 (Dionne and Golubitsky [7])**.** *The space $V = V_{k_1} \oplus \ldots \oplus V_{k_s} \subset \mathcal{L}$ is $\Gamma_{\mathcal{L}}$-irreducible if and only if $\{\pm k_1, \ldots, \pm k_s\}$ is an $H_{\mathcal{L}}$-orbit in $\mathcal{L}^*$. In particular, $2s$ divides the order of $H_{\mathcal{L}}$.*

We say that V is *generated by the orbit* of k. Since $H_{\mathcal{L}} \subset O(n+1)$, it follows that all the k_j have the same norm.

2.1. **Projections of $\mathcal{L}$-periodic Functions.**

Definition 2. For $f \in \mathcal{X}_{\mathcal{L}}$ and $y_0 > 0$, *the projection operator* Π_{y_0} is given by:

$$\Pi_{y_0}(f)(x) = \int_0^{y_0} f(x,y)dy.$$

The region $\{(x,y) \in \mathbb{R}^{n+1} : \ 0 \leq y \leq y_0\}$ is called the *band of projection*.

Hence, we are interested in describing the effect of the projection Π_{y_0} on $\Gamma_{\mathcal{L}}$-irreducible subspaces of $\mathcal{X}_{\mathcal{L}}$.

Pinho and Labouriau [16, 20] have characterised the group of symmetries of projected functions. From their results it follows:

Proposition 2. *All functions in $\Pi_{y_0}(\mathcal{X}_{\mathcal{L}})$ are invariant under the action of the translation $v \in \mathbb{R}^n$ if and only if one of the following conditions holds:*

I) $(v,0) \in \mathcal{L}$;
II) $(0,y_0) \in \mathcal{L}$ *and* $(v,y_1) \in \mathcal{L}$, *for some* $y_1 \in \mathbb{R}$.

We denote by $\Pi_{y_0}(\mathcal{L}) = \widetilde{\mathcal{L}}$ the set of all translations under which the functions in $\Pi_{y_0}(\mathcal{X}_{\mathcal{L}})$ are invariant.

The action of $\Gamma_{\mathcal{L}}$ on $\mathcal{X}_{\mathcal{L}}$ induces a similar action of a group on $\Pi_{y_0}(\mathcal{X}_{\mathcal{L}})$. Translations act as the torus $\mathbb{R}^n/\widetilde{\mathcal{L}}$. To see this, let $g = \Pi_{y_0}(f)$. If $(v,0) \in \mathcal{L}$ then $g(x-v) = \int_0^{y_0} f(x-v, y-0)dy \in \Pi_{y_0}(\mathcal{X}_{\mathcal{L}})$. If $(0,y_0)$ and $(v,y_1) \in \mathcal{L}$, since $f(x,y+y_0) = f(x,y)$ then,

$$\int_0^{y_0} f(x-v, y-y_1)dy = \int_{y_1}^{y_0+y_1} f(x-v,z)dz = g(x-v) \in \Pi_{y_0}(\mathcal{X}_{\mathcal{L}}).$$

Given $\alpha \in O(n)$, define $\alpha_\pm \in O(n+1)$ by:

$$\alpha_+ := \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \alpha_- := \begin{pmatrix} \alpha & 0 \\ 0 & -1 \end{pmatrix}.$$

If either α_+ or $\alpha_- \in H_{\mathcal{L}}$, then α belongs to the holohedry of $\widetilde{\mathcal{L}}$. For $g = \Pi_{y_0}(f)$ we get $\Pi_{y_0}(\alpha_+ \cdot f) = \alpha \cdot g$. If, in addition $(0,y_0) \in \mathcal{L}$, then

$$\Pi_{y_0}(\alpha_- \cdot f)(x) = \int_0^{y_0} f(\alpha^{-1}x, -y)dy = \int_{-y_0}^{0} f(\alpha^{-1}x, z)dz = \alpha \cdot g(x)$$

because we are integrating over a period. We will denote by $\widetilde{J} = \Pi_{y_0}(H_{\mathcal{L}})$ the group of *induced orthogonal symmetries*. It follows that $\alpha \in \widetilde{J}$ if and only if one of the conditions below holds:

I) $\alpha_+ \in H_{\mathcal{L}}$;
II) $(0,y_0) \in \mathcal{L}$ and $\alpha_- \in H_{\mathcal{L}}$.

2.2. **Outline of the article and statement of results.** In Section 3 we obtain necessary and sufficient conditions under which the image of $\mathcal{X}_{\mathcal{L}}$ by the projection Π_{y_0} is the whole function space $\mathcal{X}_{\widetilde{\mathcal{L}}}$, showing that this holds for almost all y_0.

The main result appears in Section 5, where we obtain conditions under which a single mode bifurcation in $\mathcal{X}_{\mathcal{L}}$ is projected into a mode interaction in $\mathcal{X}_{\widetilde{\mathcal{L}}}$, and show that this situation is rather common. For this some algebraic results are developed in Section 4, that contains a description of the action of $H_{\mathcal{L}}$ on the dual lattice $\mathcal{L}^*$ and its behaviour under projection. The results of Sections 4 and 5 are illustrated by the projection from the simple cubic lattice in $\mathbb{R}^3$, in two different positions.

3. Projection of the space of $\mathcal{L}$-periodic functions

Given an $(n+1)$-dimensional lattice $\mathcal{L}$ and its projection $\widetilde{\mathcal{L}} = \Pi_{y_0}(\mathcal{L})$, the projection of the space $\mathcal{X}_{\mathcal{L}}$ of all $\mathcal{L}$-periodic functions does not necessarily coincide with the space of all functions with period in $\widetilde{\mathcal{L}}$. In this section we characterise the situation when this is true.

Denote by $P : \mathbb{R}^{n+1} \to \mathbb{R}^n$ the projection $P(x,y) = x$ and by $\{y = 0\}$ the space $\{(x,y) \in \mathbb{R}^{n+1};\ y = 0\}$.

Lemma 3. *Let $\Gamma = \mathcal{L}$ be an (n+1)-dimensional lattice with $\widetilde{\mathcal{L}} = \Pi_{y_0}(\mathcal{L})$. Then $\widetilde{\mathcal{L}}^* \subset P(\mathcal{L}^*)$.*

Proof. We analyse the two cases of Proposition 2 . If $(0,y_0) \in \mathcal{L}$, then $\widetilde{\mathcal{L}} = P(\mathcal{L})$. If $\tilde{k} \in \widetilde{\mathcal{L}}^*$ and $(v,z) \in \mathcal{L}$ then $\langle \tilde{k}, v\rangle = \langle (\tilde{k}, z), (v,0)\rangle \in \mathbb{Z}$ because $v \in \widetilde{\mathcal{L}}$ and hence $(\tilde{k}, 0) \in \mathcal{L}^*$, therefore $\tilde{k} \in P(\mathcal{L}^*)$, as we wanted.

If $(0,y_0) \notin \mathcal{L}$ then $\widetilde{\mathcal{L}} = P(\mathcal{L} \cap \{y = 0\})$. If $(\tilde{k}, z) \in \mathcal{L}^*$ then for every $v \in \widetilde{\mathcal{L}}$ we have $(v,0) \in \mathcal{L}$ and thus $\langle (\tilde{k}, z), (v,0)\rangle = \langle \tilde{k}, v\rangle \in \mathbb{Z}$, hence $\tilde{k} \in \widetilde{\mathcal{L}}^*$, therefore $\widetilde{\mathcal{L}}^* = P(\mathcal{L}^*)$. □

Theorem 4. *Let $\mathcal{L} \subset \mathbb{R}^{n+1}$ be a lattice and $\widetilde{\mathcal{L}} = \Pi_{y_0}(\mathcal{L})$. The space $\Pi_{y_0}(\mathcal{X}_{\mathcal{L}})$ is equal to $\mathcal{X}_{\widetilde{\mathcal{L}}}$ if and only if for each $\tilde{k} \in \widetilde{\mathcal{L}}^*$ there exists $z \in \mathbb{R}$ such that $(\tilde{k}, z) \in \mathcal{L}^*$ and $zy_0 \notin \mathbb{Z} \setminus \{0\}$.*

Proof. Suppose that $\Pi_{y_0}(\mathcal{X}_{\mathcal{L}}) = \mathcal{X}_{\widetilde{\mathcal{L}}}$. Then, for all $\omega_{\tilde{k}} \in \mathcal{X}_{\widetilde{\mathcal{L}}}$, there exists a non-zero function $f \in \mathcal{X}_{\mathcal{L}}$ such that

$$\omega_{\tilde{k}}(x) = \Pi_{y_0}(f)(x) \neq 0.$$

By Lemma 3 we know that $\widetilde{\mathcal{L}} \subset P(\mathcal{L})$. Since $f \in \mathcal{X}_{\mathcal{L}}$ admits a Fourier expansion in terms of the waves ω_k then, without loss of generality we may take $f = c\, \omega_{(\tilde{k},z)}$, for some $(\tilde{k}, z) \in \mathcal{L}^*$. Moreover since $\Pi_{y_0}(f)(x) \neq 0$, then $\int_0^{y_0} w_z(y)dy \neq 0$. Therefore, $zy_0 \notin \mathbb{Z} \setminus \{0\}$.

Conversely, it is clear from Proposition 2 that $\Pi_{y_0}(\mathcal{X}_{\mathcal{L}}) \subseteq \mathcal{X}_{\widetilde{\mathcal{L}}}$. To get the other inclusion, we want to show that all the wave functions $\omega_{\tilde{k}} \in \mathcal{X}_{\widetilde{\mathcal{L}}}$ are in $\Pi_{y_0}(\mathcal{X}_{\mathcal{L}})$. Suppose that for each $\tilde{k} \in \widetilde{\mathcal{L}}^*$ there exists $z \in \mathbb{R}$ such that $(\tilde{k}, z) \in \mathcal{L}^*$ and $zy_0 \notin \mathbb{Z} \setminus \{0\}$. Then,

$$\Pi_{y_0}(\omega_{(\tilde{k},z)}) = c_0 \omega_{\tilde{k}}, \quad \text{with} \quad c_0 = \int_0^{y_0} w_z(y)dy.$$

Since $zy_0 \notin \mathbb{Z} \setminus \{0\}$, then $c_0 \neq 0$. Therefore $\Pi_{y_0}(\mathcal{X}_{\mathcal{L}}) = \mathcal{X}_{\widetilde{\mathcal{L}}}$. □

From Theorem 4 it follows:

Corollary 5. *Let $\mathcal{L} \subset \mathbb{R}^{n+1}$ be a lattice and $\widetilde{\mathcal{L}} = \Pi_{y_0}(\mathcal{L})$. Then for almost all values of y_0 we have $\Pi_{y_0}(\mathcal{X}_{\mathcal{L}}) = \mathcal{X}_{\widetilde{\mathcal{L}}}$.*

Proof. By Lemma 3, for each $\tilde{k} \in \widetilde{\mathcal{L}}^*$ there exists $z \in \mathbb{R}$ such that $(\tilde{k}, z) \in \mathcal{L}^*$. Consider the discrete set

$$K = \bigcup_{\substack{k_2 \neq 0; \\ (k_1,k_2) \in \mathcal{L}^*}} \mathbb{Z} \cdot \frac{1}{k_2}$$

then the complement of K is a dense set in $\mathbb{R}$.

For $y_0 \in \mathbb{R} \setminus K$, we have $\int_0^{y_0} w_{k_2}(y)dy \neq 0$. This implies that, for all $y_0 \notin K$, we have $\Pi_{y_0}(\mathcal{X}_{\mathcal{L}}) = \mathcal{X}_{\widetilde{\mathcal{L}}}$. □

4. Projecting $H_{\mathcal{L}}$-orbits in $\mathcal{L}^*$

We want to describe the effect of Π_{y_0} on irreducible subspaces of $\mathcal{X}_{\mathcal{L}}$. Proposition 1 tells us that this is done studying the orbit of elements k of $\mathcal{L}^*$ under the holohedry $H_{\mathcal{L}}$. The projection of the irreducible subspace generated by this orbit can be decomposed into irreducible subspaces under the action of symmetries of $\widetilde{\mathcal{L}} = \Pi_{y_0}(\mathcal{L})$ that lie in the group $\widetilde{J} = \Pi_{y_0}(H_{\mathcal{L}})$ that was defined in 2.1 above. This is equivalent to decomposing $P(H_{\mathcal{L}}\cdot k)$ into $\widetilde{J}$-orbits in $\widetilde{\mathcal{L}}^*$. To do this, we decompose $H_{\mathcal{L}}$ in subsets such that the orbit of k under each of these subsets is projected into exactly one $\widetilde{J}$-orbit.

Define the subset $\widetilde{J}^{\uparrow}$ of $H_{\mathcal{L}}$ as

$$\widetilde{J}^{\uparrow} = \{\alpha_+ \in H_{\mathcal{L}};\ \alpha \in \widetilde{J}\} \cup \{\alpha_- \in H_{\mathcal{L}};\ \alpha \in \widetilde{J}\}. \tag{5}$$

Then for every $k = (\tilde{k}, z) \in \mathcal{L}^*$, the projection $P(\widetilde{J}^{\uparrow}\cdot k) \subset \widetilde{J}\cdot\tilde{k}$. However, the set of $\delta \in H_{\mathcal{L}}$ such that $P(\delta k) \in \widetilde{J}\cdot\tilde{k}$ is in general larger than $\widetilde{J}^{\uparrow}$. Next, we describe the subset of $\delta \in H_{\mathcal{L}}$ such that the projection by P of δk lies in $\widetilde{J}\cdot\tilde{k}$.

Given $k = (\tilde{k}, z) \in \mathcal{L}^*$ and $\alpha \in \widetilde{J}$, it is convenient to define J_k^{α}, the subset of $H_{\mathcal{L}}$ given by:

$$J_k^{\alpha} = \{\delta \in H_{\mathcal{L}};\ P(\delta(\tilde{k}, z)) = \alpha\tilde{k}\}.$$

Denote the union of the sets J_k^{α} by

$$S_k = \bigcup_{\alpha\in\widetilde{J}} J_k^{\alpha}.$$

Then the projection of $S_k\cdot k$ satisfies $P(S_k\cdot k) = \widetilde{J}\cdot P(k)$. We show in Theorem 7 that $H_{\mathcal{L}}k$ is decomposed into orbits given by cosets δS_k and that $P(\delta S_k\cdot k) = \widetilde{J}\cdot P(\delta k)$. The main problem is that S_k is not necessarily a group. We start with some properties of the sets we have defined. From now on for $v = (x, y) \in \mathbb{R}^{n+1}$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}$ we use the notation $v_{|2} = y$.

Proposition 6. *The following properties hold for $k = (\tilde{k}, z) \in \mathcal{L}^*$:*

1. *If $\delta \in J_k^{Id_n}$, then the last coordinate, $\delta k_{|2}$, of δk is equal to $\pm z$.*
2. *For $\alpha \in \widetilde{J}$ then $J_k^{\alpha} = \gamma\cdot J_k^{Id_n}$ where $\gamma \in H_{\mathcal{L}}$ is either $\gamma = \alpha_+$ or $\gamma = \alpha_-$.*
3. *Let Σ_k be the isotropy subgroup of k in $H_{\mathcal{L}}$. Then either $J_k^{Id_n} = \Sigma_k$ or it is the disjoint union $\Sigma_k \uplus \beta_-\Sigma_k$, for some $\beta_- \in J_k^{Id_n}$.*
4. *The set S_k satisfies $S_k = \bigcup_{\alpha\in\widetilde{J}} J_k^{\alpha} = \widetilde{J}^{\uparrow} J_k^{Id_n}$. Moreover, $S_k\cdot k = \widetilde{J}^{\uparrow}\cdot k$.*
5. *If $J_k^{Id_n}$ is a subgroup of $H_{\mathcal{L}}$ and if $\widetilde{J}^{\uparrow} J_k^{Id_n} = J_k^{Id_n}\widetilde{J}^{\uparrow}$, then S_k is a subgroup of $H_{\mathcal{L}}$*

Proof. For $\delta \in J_k^{Id_n}$, we write $\delta k = (\tilde{k}, \delta k_{|2})$. Then by orthogonality of $H_{\mathcal{L}}$:

$$\|(\tilde{k}, z)\| = \|\delta k\| = \|(\tilde{k}, \delta(\tilde{k}, z)_{|2}\|$$

Therefore $|\delta k_{|2}| = |z|$. This proves item 1.

To prove item 2, let $\gamma = \alpha_+$ or $\gamma = \alpha_-$, depending on whether either α_+ or α_- is in $H_{\mathcal{L}}$. Let $\phi : J_k^{Id_n} \to J_k^{\alpha}$ given by $\delta \mapsto \phi(\delta) = \gamma\delta$. We show that the map ϕ is injective and onto.

In fact, if $\phi(\delta_1) = \phi(\delta_2)$, for some $\delta_1,\ \delta_2 \in J_k^{Id_n}$, then $\gamma\delta_1 = \gamma\delta_2$, for $\gamma = \alpha_+$ or $\gamma = \alpha_-$, implying that $\delta_1 = \delta_2$. Thus ϕ is injective.

Now consider $\rho \in J_k^{\alpha}$, then there exist $\alpha_{\pm}^{-1}\rho \in H_{\mathcal{L}}$ and $\tilde{k}_2$, such that

$$\alpha_{\pm}^{-1}\rho(\tilde{k}, z) = \alpha_{\pm}^{-1}(\alpha\tilde{k}, \tilde{k}_2) = (\tilde{k}, \pm\tilde{k}_2)$$

that is, $\alpha_{\pm}^{-1}\rho \in J_{\tilde{k}}^{Id_n}$ and $\phi(\alpha_{\pm}^{-1}\rho) = \rho$.

The items 3 and 5 are now immediate. □

Note that item 3 implies that, in general, if $J_k^{Id_n} \neq \Sigma_k$ then $J_k^{Id_n}$ is not a group, as the next example will show.

Example 1. Let $\mathcal{L}_1$ be the simple 3-dimensional cubic lattice generated by $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, with dual lattice $\mathcal{L}_1^*$ generated by the same vectors.

The holohedry $H_{\mathcal{L}_1}$ has 24 rotations, they are generated by: the identity Id_3; three rotations of order 4: R_x: rotation about $(1,0,0)$; R_y: rotation about $(0,1,0)$ and R_z: rotation about $(0,0,1)$. The remaining rotations are denoted by R_v as a rotation about the axis v: $R_{(1,1,1)}$, $R_{(1,-1,-1)}$, $R_{(1,-1,1)}$, $R_{(1,1,1)}$ of order 3, and $R_{(1,0,1)}$, $R_{(1,0,-1)}$, $R_{(1,1,0)}$, $R_{(1,-1,0)}$, $R_{(0,1,1)}$ and $R_{(0,1,-1)}$of order 2.

The remaining elements of $H_{\mathcal{L}_1}$ can be obtained by multiplying these matrices by $-Id_3$.

TABLE 1. Relation of the set S_k for all $k \in \mathcal{L}_1^* \smallsetminus \{(0,0,0)\}$ and the group $H_{\mathcal{L}}$ for Example 1. Note that $|S_{(a,b,b)}| = 32$, hence $S_{(a,b,b)}$ is not a subgroup of $H_{\mathcal{L}}$, since $|H_{\mathcal{L}}| = 48$.

k with $abc \neq 0$ $a \neq b \neq c \neq a$	Σ_k	$J_k^{Id_2}$	S_k
$(a,0,0)$	$\Sigma_{(a,0,0)} = J^{Id_2}_{(a,0,0)}$	$J^{Id_2}_{(a,0,0)} = \{Id_3, R_x, R_x^2, R_x^3, -R_y^2, -R_z^2, -R_{(0,1,1)}, -R_{(0,1,-1)}\}$	$\lvert S_{(a,0,0)}\rvert = 32$
(a,b,b)	$\Sigma_{(a,b,b)} = \{Id_3, -R_{(0,1,-1)}\}$	$J^{Id_2}_{(a,b,b)} = \{Id_3, -R_z^2, R_x^3, -R_{(0,1,-1)}\}$	$\lvert S_{(a,b,b)}\rvert = 32$
(a,a,b)	$\Sigma_{(a,a,b)} = \{Id_3, -R_{(1,-1,0)}\}$	$J^{Id_2}_{(a,a,b)} = \{Id_3, -R_z^2, R_{(1,1,0)}, -R_{(1,-1,0)}\}$	$S_{(a,a,b)} = \widetilde{J}^{\uparrow}$
$(a,a,0)$	$\Sigma_{(a,a,0)} = J^{Id_2}_{(a,a,0)}$	$J^{Id_2}_{(a,a,0)} = \{Id_3, -R_z^2, R_{(1,1,0)}, -R_{(1,-1,0)}\}$	$S_{(a,a,0)} = \widetilde{J}^{\uparrow}$
$(a,b,0)$	$\Sigma_{(a,b,0)} = J^{Id_2}_{(a,b,0)}$	$J^{I_2}_{(a,b,0)} = \{I_3, -R_z^2\}$	$S_{(a,b,0)} = \widetilde{J}^{\uparrow}$
(a,b,c)	$\Sigma_{(a,b,c)} = \{I_3\}$	$J^{I_2}_{(a,b,c)} = \{I_3, -R_z^2\}$	$S_{(a,b,c)} = \widetilde{J}^{\uparrow}$
(a,a,a)	$\Sigma_{(a,a,a)} = \{Id_3, R_{(1,1,1)}, R^2_{(1,1,1)}, -R_{(1,0,-1)}, -R_{(0,1,-1)}, -R_{(1,-1,0)}\}$	$J^{Id_2}_{(a,a,a)} = \Sigma_{(a,a,a)} \cup \{R_x^3, R_y, -R_z^2, -R_{(1,-1,-1)}, -R^2_{(1,-1,1)}, -R_{(1,1,0)}\}$	$S_{(a,a,a)} = H_{\mathcal{L}_1}$

For any y_0, the projected lattice $\Pi_{y_0}(\mathcal{L}_1) = \widetilde{\mathcal{L}}_1$ is generated by the vectors $(1,0)$, $(0,1)$, its dual, $\widetilde{\mathcal{L}}_1^*$ is generated by the same vectors.

The subgroup $\widetilde{J}^{\uparrow}$ has order 16 and is given by $\{\pm Id_3, \pm R_x^2, \pm R_y^2, \pm R_z, \pm R_z^2, \pm R_z^3, \pm R_{(1,1,0)}, \pm R_{(1,-1,0)}\}$.

The orbit of an element $(a,b,c) \in \mathcal{L}_1^*$ by $H_{\mathcal{L}_1}$ is given by:

$$\{(\pm a, \pm b, \pm c), (\pm b, \pm a, \pm c)\} \cup \{(\pm a, \pm c, \pm b), (\pm c, \pm a, \pm b)\} \cup \{(\pm b, \pm c, \pm a), (\pm c, \pm b, \pm a)\} \tag{6}$$

and the projection $P(H_{\mathcal{L}_1}(a,b,c))$ is given by:

$$\{(\pm a, \pm b), (\pm b, \pm a)\} \cup \{(\pm a, \pm c),\ (\pm c, \pm a)\} \cup \{(\pm c, \pm b),\ (\pm b, \pm c)\}$$

On the other hand, for any $y_0 \in \mathbb{R}$, the group $\widetilde{J} = \Pi_{y_0}(H_{\mathcal{L}_1})$ is the dihedral group of symmetries of the square, D_4, generated by:

$$\gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \kappa = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

For every $\alpha \in \widetilde{J}$ both α_+ and $\alpha_- \in H_{\mathcal{L}}$. The orbit $\widetilde{J}(a,b)$ is $\{(\pm a, \pm b),\ (\pm b, \pm a)\}$.

In Table 1 we describe, for each $k \in \mathcal{L}_1^* \setminus \{(0,0,0)\}$, the subgroup Σ_k and the set $J_k^{Id_2}$ as well as the way the set S_k and the group $H_{\mathcal{L}}$ are related. Some special cases are worth mentioning. If $b \neq 0 \neq a$ then $J_{(a,b,b)}^{Id_2}$ is not a group, because it contains the order 4 element R_x^3 but not its powers. In the other cases where $J_k^{Id_2} \neq \Sigma_k$ we have that β_- has order 2 and $J_k^{Id_2}$ is a group.

The only cases where $S_k \neq \widetilde{J}^{\uparrow}$ are for $k = (a,a,a)$, where S_k is the whole group $H_{\mathcal{L}_1}$ and for $k = (a,b,b)$, $0 \neq a \neq b$. For the second case, note that $\delta \in H_{\mathcal{L}}$ is not in $S_{(a,b,b)}$ if $\delta(a,b,b) = (\pm b, \pm b, \pm a)$. There are 8 rotations with this property: R_y and R_y^3; the order 2 rotations $R_{(1,0,1)}$ and $R_{(1,0,-1)}$; and one and only one rotation of order 3 around each of the four axes of order 3. Multiplying them by $-Id_3$ yelds 16 elements not in $S_{(a,b,b)}$, hence $|S_{(a,b,b)}| = 32$ and thus $S_{(a,b,b)}$ is not a subgroup of $H_{\mathcal{L}}$.

Example 2. Consider now the lattice $\mathcal{L}_2$ that has generators $(1,0,0), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right), \left(\frac{1}{2}, \frac{\sqrt{3}}{6}, \frac{\sqrt{6}}{6}\right)$ given by $\mathcal{L}_2 = \frac{1}{\sqrt{2}} A \mathcal{L}_1$ where A is:

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{pmatrix}.$$

The lattice $\mathcal{L}_2$ has holohedry $H_{\mathcal{L}_2} = A H_{\mathcal{L}_1} A^{-1}$.

If $y_0 \neq n\sqrt{6}/2$ with $n \in \mathbb{Z} \setminus \{0\}$, then $(0,0,y_0) \notin \mathcal{L}_2$. In this case $\widetilde{\mathcal{L}}_2$ is generated by $(1,0)$ and $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $\widetilde{J}$ is isomorphic to the dihedral group D_3. However, the holohedry of $\widetilde{\mathcal{L}}_2$ is larger than $\widetilde{J}$. It contains rotations of order 6 and is isomorphic to D_6.

For $y_0 = n\sqrt{6}/2$ with $n \in \mathbb{Z} \setminus \{0\}$, then $\widetilde{\mathcal{L}}_2$ is generated by $(1,0)$ and $(\frac{1}{2}, \frac{\sqrt{3}}{6})$ and $\widetilde{J}$ is isomorphic to the dihedral group D_6, coinciding with the holohedry of $\widetilde{\mathcal{L}}_2$.

The dual lattice $\mathcal{L}_2^* = \sqrt{2} A \mathcal{L}_1^*$ is generated by $(0,0,\sqrt{6})$, $(1, -\frac{\sqrt{3}}{3}, -\frac{\sqrt{6}}{3})$, and $(0, \frac{2\sqrt{3}}{3}, -\frac{\sqrt{6}}{3})$.

For $k = (2,0,0) \in \mathcal{L}_2^*$, the set $J_{(2,0,0)}^{Id_2} = \Sigma_{(2,0,0)}$ is a subgroup of $H_{\mathcal{L}_2}$. The $\widetilde{J}$-orbit of $P(2,0,0) = (2,0)$ contains 6 elements, independently of y_0, and is

$$\widetilde{J} \cdot (2,0) = \{(\pm 2, 0), (\pm 1, \pm\sqrt{3})\}.$$

We claim that $S_{(2,0,0)}$ is not a group. To see this we compute

$$AR_yA^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{6} & -\frac{2\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{2} & \frac{1}{6} & -\frac{\sqrt{2}}{3} \\ 0 & \frac{2\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix} \qquad AR_y^2A^{-1} = \begin{pmatrix} 0 & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{6}}{3} \\ -\frac{\sqrt{3}}{3} & -\frac{2}{3} & \frac{\sqrt{2}}{3} \\ -\frac{\sqrt{6}}{3} & \frac{\sqrt{2}}{3} & -\frac{1}{3} \end{pmatrix} \qquad AR_y^3A^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{6} & \frac{1}{6} & \frac{2\sqrt{2}}{3} \\ -\frac{\sqrt{6}}{3} & -\frac{\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix}.$$

Then $AR_yA^{-1} \in S_{(2,0,0)}$, since $P(AR_yA^{-1}(2,0,0)) = (1, -\sqrt{3})$ and yet the powers of AR_yA^{-1} are not in $S_{(2,0,0)}$, establishing the claim.

We can now prove the main result of this section:

Theorem 7. *Consider an $(n+1)$-dimensional lattice $\mathcal{L}$ with holohedry $H_{\mathcal{L}}$, and let $\widetilde{J} = \Pi_{y_0}(H_{\mathcal{L}})$. Let $k \in \mathcal{L}^*$. Then there are $k_1 = k, k_2 = \delta_2 k, \ldots, k_r = \delta_r k$ such that $H_{\mathcal{L}} \cdot k$ is the disjoint union*

$$H_{\mathcal{L}} \cdot k = (S_{k_1} \cdot k_1) \cup (S_{k_2} \cdot k_2) \cup \ldots \cup (S_{k_r} \cdot k_r)$$

and therefore the projection $P(H_{\mathcal{L}} k)$ is a disjoint union of $\widetilde{J}$-orbits.

Proof. Given $\alpha \in \widetilde{J}$ let $\alpha_* \in \widetilde{J}^{\uparrow} \subset H_{\mathcal{L}}$ stand for either α_+ or α_-, as appropriate. We claim that for any $\delta \in H_{\mathcal{L}}$ any $\alpha \in \widetilde{J}$ and $k \in \mathcal{L}^*$, if $\alpha_* \delta \in S_k$ then $\delta \in S_k$.

If $\alpha_*\delta \in S_k$ then there exists $\beta \in \widetilde{J}$ such that $\alpha_*\delta k = \beta_* k$, by items 1 and 2 of Proposition 6. Hence, $\delta k = \alpha_*^{-1}\beta_* k$ and since $\alpha_*^{-1}\beta_* \in \widetilde{J}^{\uparrow}$ this implies that $\delta \in S_k$, as we had claimed.

Now consider $H_{\mathcal{L}} = S_k \cup S_k^c$ and $\delta_2 \in S_k^c$, where S_k^c is the complement of S_k. Then for any $\alpha_* \in \widetilde{J}^{\uparrow}$ we have $\alpha_*\delta_2 \in S_k^c$. In particular, $\widetilde{J}^{\uparrow} \cdot (\delta_2 k) \cap S_k \cdot k = \varnothing$. On the other hand, by Proposition 6 item 4, the set $\widetilde{J}^{\uparrow} \cdot (\delta_2 k) = S_{\delta_2 k} \cdot (\delta_2 k) \subset H_{\mathcal{L}} k$. Thus, we can write

$$H_{\mathcal{L}} \cdot k = S_k \cdot k \cup S_{\delta_2 k} \cdot (\delta_2 k) \cup (S_k \cup S_{\delta_2 k}\delta_1)^c k.$$

Since $H_{\mathcal{L}}$ is a finite group, we can repeat the process to obtain:

$$H_{\mathcal{L}} \cdot k = S_k \cdot k \cup S_{\delta_2 k} \cdot (\delta_2 k) \cup \ldots \cup S_{\delta_r k} \cdot (\delta_r k). \tag{7}$$

Then, for $u_1 = P(k)$ and $= u_i P(\delta_i k)$, we have the disjoint union

$$P(H_{\mathcal{L}} \cdot k) = \bigcup_{i=1}^{r} \widetilde{J} \cdot u_i \tag{8}$$

and the result follows. □

We are interested in the case when there are several components in (8). The next corollary provides a condition for this, in terms of the size of the different sets that are used in this section.

Corollary 8. *For $k \in \mathcal{L}^*$, the projection $P(H_{\mathcal{L}} \cdot k)$ in (8) contains more than one $\widetilde{J}$-orbit if and only if*

$$\frac{|H_{\mathcal{L}}|}{|\widetilde{J}^{\uparrow}|} > \frac{|\Sigma_k|}{|\widetilde{J}^{\uparrow} \cap \Sigma_k|}. \tag{9}$$

Proof. We know that the cardinal number of the orbit of k by $H_{\mathcal{L}}$ is the index

$$|H_{\mathcal{L}} k| = \frac{|H_{\mathcal{L}}|}{|\Sigma_k|}. \tag{10}$$

The elements of $H_{\mathcal{L}} \cdot k$ that are projected into $\widetilde{J}^{\uparrow} \cdot P(k)$ are those in $S_k \cdot k$, and this set coincides with $\widetilde{J}^{\uparrow} \cdot k$ by item 4 of Proposition 6. Since the isotropy subgroup of k in $\widetilde{J}^{\uparrow}$ is $\widetilde{J}^{\uparrow} \cap \Sigma_k$, then the number of different elements in $H_{\mathcal{L}} \cdot k$ whose projection lies in $\widetilde{J} \cdot P(k)$ is $|\widetilde{J}^{\uparrow}|/|\widetilde{J}^{\uparrow} \cap \Sigma_k|$. Therefore, there are points in $H_{\mathcal{L}} \cdot k$ whose projection does not lie in $\widetilde{J} \cdot P(k)$ if and only if

$$\frac{|H_{\mathcal{L}}|}{|\Sigma_k|} > \frac{|\widetilde{J}^{\uparrow}|}{|\widetilde{J}^{\uparrow} \cap \Sigma_k|}$$

and this condition is equivalent to (9). □

5. $\Gamma_{\mathcal{L}}$-Irreducible decomposition

We are ready to give a decomposition of the projection of $\Gamma_{\mathcal{L}}$-irreducible subspaces of $\mathcal{X}_{\mathcal{L}}$. As in the previous section, we use the notation $\widetilde{J} = \Pi_{y_0}(H_{\mathcal{L}})$. The group $\widetilde{\Gamma}_{\widetilde{\mathcal{L}}} = \mathbb{R}^n/\widetilde{\mathcal{L}} \dotplus \widetilde{J}$ is contained in $\Gamma_{\widetilde{\mathcal{L}}}$, but does not necessarily coincide with it, since the inclusion $\widetilde{J} \subset H_{\widetilde{\mathcal{L}}}$ may be strict. This is the case in Example 2 above, other examples appear in [18].

Theorem 9. *Let V be the $\Gamma_{\mathcal{L}}$-irreducible subspace of $\mathcal{X}_{\mathcal{L}}$ generated by the orbit of $k \in \mathcal{L}^*$. The projection $\Pi_{y_0}(V)$ is a $\widetilde{\Gamma}_{\widetilde{\mathcal{L}}}$-invariant subspace of $\mathcal{X}_{\widetilde{\mathcal{L}}}$. If, moreover, condition (9) holds:*

$$\frac{|H_{\mathcal{L}}|}{|\widetilde{J}^{\uparrow}|} > \frac{|\Sigma_k|}{|\widetilde{J}^{\uparrow} \cap \Sigma_k|}$$

then $\Pi_{y_0}(V)$ is the sum of more than one $\widetilde{\Gamma}_{\widetilde{\mathcal{L}}}$-irreducible subspace, for almost all values of $y_0 \in \mathbb{R}$.

Proof. First note that $\Pi_{y_0}\left(\omega_{(k_1,k_2)}\right) = c_0\omega_{k_1}$ with $c_0 = \int_0^{y_0} e^{2\pi i k_2 y} dy$ and thus, $c_0 = 0$ if and only if $k_2 y_0 \in \mathbb{Z} \smallsetminus \{0\}$. Therefore, $\omega_{k_1}(x) \in \Pi_{y_0}(V)$ if and only if $k_1 \in P\left(H_{\mathcal{L}} \cdot k \smallsetminus Z_{y_0}\right)$ where

$$Z_{y_0} = \{(x, ny_0) \in \mathbb{R}^{n+1} : \ n \in \mathbb{Z} \smallsetminus \{0\}\}.$$

If $H_{\mathcal{L}} \cdot k \subset Z_{y_0}$ then it follows that $\Pi_{y_0}(V) = \{0\}$, a subspace that is trivially $\widetilde{\Gamma}_{\widetilde{\mathcal{L}}}$-invariant.

TABLE 2. For the simple cubic lattice $\mathcal{L}_1$ in Example 1, we know that $|H_{\mathcal{L}}|/|\widetilde{J}^{\uparrow}| = 3$ and for each $k \in \mathcal{L}_1^*$ this table gives the numbers $|\Sigma_k|$ and $|\widetilde{J}^{\uparrow} \cap \Sigma_k|$ and the number N of non-trivial $\widetilde{J}$-irreducible components of the projection of the $\Gamma_{\mathcal{L}_1}$-irreducible subspace $V_k \in \mathcal{X}_{\mathcal{L}_1}$ generated by the orbit of k. Here $a \neq b \neq c \neq a$ with $abc \neq 0$.

k	$(a,0,0)$	(a,b,b)	(a,a,b)	$(a,a,0)$	$(a,b,0)$	(a,b,c)	(a,a,a)
$\lvert\Sigma_k\rvert$	8	2	2	4	2	1	6
$\lvert\widetilde{J}^{\uparrow} \cap \Sigma_k\rvert$	4	1	2	4	2	1	2
N	2	2	2	2	3	3	1

Suppose then that $\hat{k} \in H_{\mathcal{L}} \cdot k \setminus Z_{y_0} \neq \varnothing$ and hence $\omega_{k_1}(x) \in \Pi_{y_0}(V)$. If $\alpha \in \widetilde{J}$ then either α_+ or $\alpha_- \in H_{\mathcal{L}}$, let α_* be the appropriate one. Then $\alpha_* \hat{k} \notin Z_{y_0}$, implying that $\omega_{\alpha k_1}(x) \in \Pi_{y_0}(V)$. Since from (4) we know that $\alpha\omega_{k_1}(x) = \omega_{\alpha k_1}(x)$, this shows that $\Pi_{y_0}(V)$ is $\widetilde{\Gamma}_{\widetilde{\mathcal{L}}}$-invariant.

When condition (9) holds, by Corollary 8 we know that $P(H_{\mathcal{L}} \cdot k)$ contains at least two distinct $\widetilde{J}$-orbits, say $\widetilde{J} \cdot P(u_1, z_1)$ and $\widetilde{J} \cdot P(u_2, z_2)$. The set $K = \dfrac{1}{z_1}\mathbb{Z} \cup \dfrac{1}{z_2}\mathbb{Z}$ is a discrete subset of $\mathbb{R}$ and for all $y_0 \in \mathbb{R}$ such that $y_0 \notin K$, the projection $\Pi_{y_0}(V)$ has at least two irreducible components. □

Corollary 10. *Suppose that V is a $\Gamma_{\mathcal{L}}$-irreducible subspace of $\mathcal{X}_{\mathcal{L}}$ generated by the orbit of $k \in \mathcal{L}^*$. If there exist (u_1, z_1) and (u_2, z_2) in $H_{\mathcal{L}} \cdot k$ with $z_1 \neq \pm z_2$, then for almost all $y_0 \in \mathbb{R}$ the projection of a single mode is a mode interaction.*

Proof. If $z_1 \neq \pm z_2$, then by item 2 of Proposition 6, the $\widetilde{J}$-orbits of u_1 and u_2 are distinct. Since $K = \dfrac{1}{z_1}\mathbb{Z} \cup \dfrac{1}{z_2}\mathbb{Z}$ is a discrete subset of $\mathbb{R}$, for all $y_0 \notin K$ the projection $\Pi_{y_0}(V)$ has at least two $\widetilde{\Gamma}_{\widetilde{\mathcal{L}}}$-irreduclble components. Hence, we have an interaction of at least two modes. □

We return to the examples of Section 4.

Example 1. For the simple cubic lattice $\mathcal{L}_1$, it is always true that $\widetilde{\mathcal{L}_1} = P(\widetilde{\mathcal{L}} \cap \{y = 0\})$, for any y_0. Hence, the projection of a $\Gamma_{\mathcal{L}_1}$-irreducible subspace of $\mathcal{X}_{\mathcal{L}_1}$ is never the trivial subspace. The number of $\widetilde{J}$-irreducible components of $\Pi_{y_0}(V_k)$ for each $k \in \mathcal{L}_1^*$ is shown in Table 2. For instance, when $k = (a,b,0)$, $a \neq b \in \mathbb{Z} \setminus \{0\}$ the projection $\widetilde{V} = \Pi_{y_0}(V)$ is the sum of three $\widetilde{\Gamma}_{\widetilde{\mathcal{L}_1}}$-irreducible subspaces, generated by the orbits of $(a,0)$, of $(0,b)$ and of (a,b). For $k = (a,0,0)$, $a \neq 0$, the projection $P(V_k)$ has the two irreducible components generated by the orbits of $(a,0)$ and of $(0,0)$, one of the components consists of constant functions. The only case when the projection is irreducible is when V is generated by the orbit of $k = (a,a,a)$, $a \neq 0$, where the only irreducible component is generated by the orbit of (a,a).

Example 2 is more typical, because the projection depends on y_0.

Example 2. The holohedry $H_{\mathcal{L}_2}$ of the rotated cubic lattice $\mathcal{L}_2$ has order 48. The subgroup $\widetilde{J}^{\uparrow}$ has order 12 if $y_0 = n\sqrt{6}/2$ with $n \in \mathbb{Z} \setminus \{0\}$, otherwise it has order 6. Tables 3 and 4 give all the information necessary to apply Theorem 9 to this lattice. For each k there is at most a discrete set of values of y_0 for which the projection of a single mode is a single mode. Generically the projection is a mode interaction.

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TABLE 3. For the rotated cubic lattice $\mathcal{L}_2$ in Example 2, and $y_0 = m\sqrt{6}/2$, $m \in \mathbb{Z} \smallsetminus \{0\}$, we have that $|H_{\mathcal{L}}|/|\widetilde{J}^{\uparrow}| = 4$. For each $k \in \mathcal{L}_1^*$ this table gives the transformed vector in $\mathcal{L}_2^*$, the numbers $|\Sigma_k|$, the set $\widetilde{J}^{\uparrow} \cap \Sigma_k$, the number $|\Sigma_k|/|\Sigma_k \cap \widetilde{J}^{\uparrow}|$ and the restriction on y_0. The only case where $\Pi_{y_0}(V_k)$ has only one irreducible component is $k = \left(a, \frac{\sqrt{3}a}{3}, \frac{\sqrt{6}a}{3}\right)$, but a small change in y_0 destroys this situation, see Table 4. Here $a \neq b \neq c \neq a$ with $abc \neq 0$, $n \in \mathbb{Z} \smallsetminus \{0\}$.

$k \in \mathcal{L}_1^*$	$\sqrt{2}Bk$	$\lvert\Sigma_k\rvert$	$\Sigma_k \cap \widetilde{J}^{\uparrow}$	$\lvert\Sigma_k\rvert/\lvert\Sigma_k \cap \widetilde{J}^{\uparrow}\rvert$	restriction
$(a,0,0)$	$\left(a, \frac{\sqrt{3}a}{3}, \frac{\sqrt{6}a}{3}\right)$	8	$\{Id_3, -R_{(0,1,1)}\}$	4	-
(a,b,b)	$\left(a-b, \sqrt{3}\left(\frac{a}{3}+b\right), \frac{\sqrt{6}a}{3}\right)$	2	$\mathbb{1}$	2	$y_0 \neq \frac{\sqrt{6}}{2a}n$
$(0,a,a)$	$\left(-a, \sqrt{3}a, 0\right)$	4	$\{Id_3, R_{(0,1,1)}\}$	2	-
$(a,b,0)$	$\left(a-b, \frac{\sqrt{3}}{3}(a+b), \frac{\sqrt{6}}{3}(a+b)\right)$	2	$\mathbb{1}$	2	$y_0 \neq \frac{\sqrt{6}}{2(a+b)}n$
(a,b,c)	$\left(a-b, \frac{\sqrt{3}}{3}(a+b+2c), \frac{\sqrt{6}}{3}(a+b-c)\right)$	1	$\mathbb{1}$	1	$y_0 \neq \frac{\sqrt{6}}{2(a+b-c)}n$
(a,a,a)	$\left(0, \frac{4\sqrt{3}a}{3}, \frac{\sqrt{6}a}{3}\right)$	6	$\{Id_3, R_{(1,-1,0)}\}$	3	$y_0 \neq \frac{\sqrt{6}}{2a}n$
$(a,b,a+b)$	$\left(a-b, \frac{2\sqrt{3}}{3}(a+b), 0\right)$	1	$\mathbb{1}$	1	-
$(a,0,a)$	$\left(a, \frac{2\sqrt{3}a}{3}, 0\right)$	4	$\{Id_3, R_{(1,0,1)}\}$	2	-
$(a,a,2a)$	$\left(0, \frac{4\sqrt{3}a}{3}, 0\right)$	2	$\{Id_3, -R_{(1,-1,0)}\}$	1	-

TABLE 4. For the rotated cubic lattice $\mathcal{L}_2$ in Example 2, and $y_0 \neq m\sqrt{6}/2$, $m \in \mathbb{Z} \smallsetminus \{0\}$, we have that $|H_{\mathcal{L}}|/|\widetilde{J}^{\uparrow}| = 8$. For each $k \in \mathcal{L}_1^*$ this table gives the transformed vector in $\mathcal{L}_2^*$, the numbers $|\Sigma_k|$, the set $\widetilde{J}^{\uparrow} \cap \Sigma_k$, the number $|\Sigma_k|/|\Sigma_k \cap \widetilde{J}^{\uparrow}|$ and the restriction on y_0. The projection $\Pi_{y_0}(V_k)$ has always more than one irreducible component as long as the restriction on y_0 holds. Here $a \neq b \neq c \neq a$ with $abc \neq 0$, $n \in \mathbb{Z} \smallsetminus \{0\}$.

$k \in \mathcal{L}_1^*$	$\sqrt{2}Bk$	$\lvert\Sigma_k\rvert$	$\Sigma_k \cap \widetilde{J}^{\uparrow}$	$\lvert\Sigma_k\rvert/\lvert\Sigma_k \cap \widetilde{J}^{\uparrow}\rvert$	restriction
$(a,0,0)$	$\left(a, \frac{\sqrt{3}a}{3}, \frac{\sqrt{6}a}{3}\right)$	8	$\{Id_3, -R_{(0,1,1)}\}$	4	$y_0 \neq \frac{\sqrt{6}}{2a}n$
(a,b,b)	$\left(a-b, \sqrt{3}\left(\frac{a}{3}+b\right), \frac{\sqrt{6}a}{3}\right)$	2	$\mathbb{1}$	2	$y_0 \neq \frac{\sqrt{6}}{2a}n$
$(0,a,a)$	$\left(-a, \sqrt{3}a, 0\right)$	4	$\mathbb{1}$	4	-
$(a,b,0)$	$\left(a-b, \frac{\sqrt{3}}{3}(a+b), \frac{\sqrt{6}}{3}(a+b)\right)$	2	$\mathbb{1}$	2	$y_0 \neq \frac{\sqrt{6}}{2(a+b)}n$
(a,b,c)	$\left(a-b, \frac{\sqrt{3}}{3}(a+b+2c), \frac{\sqrt{6}}{3}(a+b-c)\right)$	1	$\mathbb{1}$	1	$y_0 \neq \frac{\sqrt{6}}{2(a+b-c)}n$
(a,a,a)	$\left(0, \frac{4\sqrt{3}a}{3}, \frac{\sqrt{6}a}{3}\right)$	6	$\mathbb{1}$	6	$y_0 \neq \frac{\sqrt{6}}{2a}n$
$(a,b,a+b)$	$\left(a-b, \frac{2\sqrt{3}}{3}(a+b), 0\right)$	1	$\mathbb{1}$	1	-
$(a,0,a)$	$\left(a, \frac{2\sqrt{3}a}{3}, 0\right)$	4	$\mathbb{1}$	4	-
$(a,a,2a)$	$\left(0, \frac{4\sqrt{3}a}{3}, 0\right)$	2	$\{Id_3, -R_{(1,-1,0)}\}$	1	-

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