# On the coding of orbits in discontinuous maps*

*Miguel Mendes*

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**Abstract**

Discontinuous maps provide interesting examples of chaotic behaviour generated not by nonlinearities but by discontinuities themselves. We import the use of a coding map from the context of piecewise isometries in order to identify points with complicated dynamics in other systems. These are proved to be related to those that come arbitrarily close to the discontinuity set.

## Contents



## 1 Introduction

A well known fact in the theory of dynamical systems is that nonlinearities are a fundamental source of chaotic behaviour. Perhaps, not as well known is the fact that discontinuous systems can produce just as intricate phenomena even when the map is linear at continuity points.

Discontinuous maps appear in all sorts of mathematical models. In recent years, special attention has been devoted to digital filters which are electronic components that can be modelled by a piecewise rotation on a rhombhus (see A. C. Davies [1995] and references therein).

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Mathematically, the first discontinuous maps to have been given special focus were interval exchange transformations (IETs). The seminal work of M. Keane ([1975] & [1977]) prompted the works of Katok [1980], Veech [1982], Masur [1982] and many others. Other systems emerged in connection with IETs like, for instance, polygonal billiards (see the survey by E. Gutkin [1996]). But, perhaps, the most interesting generalisation of IETs came with the work of Goetz [1996] in piecewise rotations (PRs) in the plane (see Section 2.1). Naturally, other works followed considering piecewise isometries (PWIs) in any Euclidean space (see Goetz [2000], Buzzi [2000] and Mendes [2007] for instance).

Attention to PRs was mainly drawn by the fascinating pictures provided in the papers by Goetz (see Goetz [1998]). These suggested that the discontinuity line and its preimages had an enormous impact on the dynamics of the map: points that never approach the discontinuity set have periodic motion; but, those whose orbit comes arbitrarily close to the discontinuity line show some form of chaotic dynamics which is successfully described a coding map. Results on the existence of periodic and nonperiodic codings for PRs in any Euclidean space were generalised in Mendes & Nicol [2004]. Later, the stability of periodic points for PWIs in any Euclidean space was established in Mendes [2007]. Points irrationaly coded still lack the much need focus they deserve.

In this paper we borrow these ideas from the theory of piecewise isometries and apply them to more general models such as piecewise linear maps (see examples in Section 2.3). Namely, we use the coding map in exactly the same way as defined in the setting of PWIs and establish results relating the codings of points and the set of all preimages of the discontinuity and the set of all points whose orbit is at distance zero from the discontinuity set (see Section 3).

Section 4 is devoted to obtaining stronger results in the context of PWIs and the Appendix contains all auxiliary results including previous and new.

## 2 Preliminaries and examples

### 2.1 Motivating through an example

Let us start with a very simple - hopefully ellucidating - example of what this paper is about: generating intricate dynamics from trivial maps by aglutinating them in a piecewise fashion.

More precisely let $R_j(z) := e^{i\alpha_j}(z - C_j) + C_j$ (with $j = 0, 1$) be two rotations in the complex plane by angles $\alpha_j$ around distinct centres $C_0$ and $C_1$. We now define a map, which is called the Goetz map since it was first introduced in Goetz [1998], in the following way:

$$G(z) := \begin{cases} R_0(z); & \text{if } \Re e(z) < 0 \\ R_1(z); & \text{if } \Re e(z) \geq 0 \end{cases}.$$

Of course, the dynamics generated by this new map depends heavily on the fact that each of its branches is a rigid rotation and so, we expect that some sort of rotational dynamics be detectable clearly. That is so because the composition of rotations is itself a rotation whose centre is different from the previous two. Therefore, as we iterate the map $G$ we compose $R_0$ and $R_1$ in different combinations so that the resulting maps give rise to cyclic (i.e., periodic) orbits of higher and higher order (i.e., period).

However, not every possible combinaton is admissible and most interestingly, not every admissible sequence generates a periodic orbit. The way to distinguish between these two different types of orbits (cyclic and non-cyclic) is by use of a coding map which assigns a particular sequence of symbols to a point depending on its position and that of its iterates in the phase space.

Namely, let for this particular example, $\chi : \mathbb{C} \to \{0,1\}^{\mathbb{N}}$ be a mapping such that $\chi(z) = w_0 w_1 ...$ if and only if $G^k(z) \in P_{w_k}$ for every $k \in \mathbb{N}_0$ where $P_0 := \{\Re e(z) < 0\}$ and $P_1 := \{\Re e(z) \geq 0\}$. In this fashion if the coding of a point $z$ starts with the block 0110 it means that it started off in $P_0$ then its image $G(z)$ falls in $P_1$, the iterate of which will remain there and the point $G^3(z)$ comes back to $P_0$. Surely, other points will follow this pattern in their first three iterations while others will have other codings. This means that the complex plane can be divided into sets according the codings that each point inside them will possess. And, as we iterate the mapping these sets may be broken into several pieces since different codings may arise from a particular initial word of any given length. If one wants to get an idea of how this process develops one has to picture the preimages of the discontinuity line $\mathcal{D} := \{\Re e(z) = 0\}$ for this set separates points. In Figure 1 we portrait a sequence of preimages of the discontinuity line. For a large number of preimages (say, $n = 500$) we obtain a picture which numerically does not differ from others obtained for larger values of $n$.

If we take the set of all preimages whose numerical approximation is obtained for large $n$ then we may suspect that it has a very intriguing structure resembling that of a fractal set. Indeed, for some particular choices of angles and centres of rotation, it has been shown that there is a self-similarity structure associated to this *exceptional set* (see Goetz [1998a]).

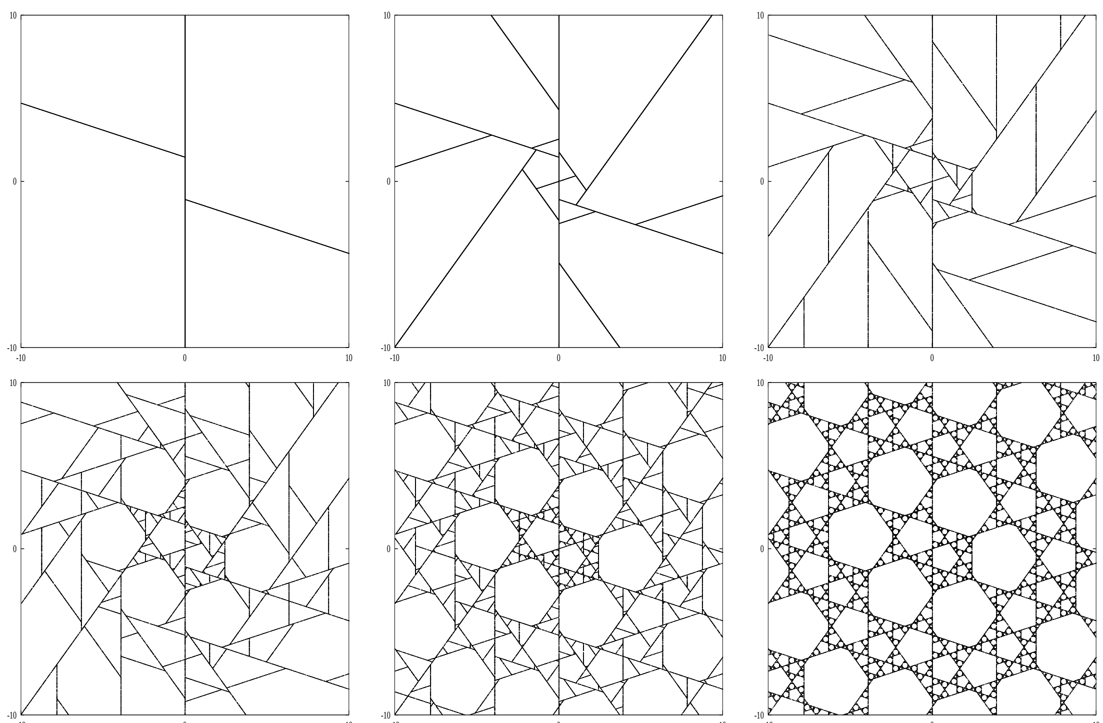

Figure 1: Sequence of preimages of the discontinuity line of a Goetz map whose angles of rotation equal $\alpha := 2\pi/5$ and centres are $C_0 := (-1, tan(\alpha/2)), C_1 := (0.75, -0.75tan(\alpha/2)$: (from left to right downwards) one, four, ten, twenty, fifty and five hundred preimages of the discontinuity line.

In terms of dynamics what is perhaps more interesting to note is that the set of all points whose orbit come close to the discontinuity set yet never actually intersecting it has sensitive dependence on initial conditions in the sense that two nearby points in this set will eventually fall on opposite sides of the discontinuity line. Namely, let $x$ be such a point and consider a ball $B_\varepsilon(x)$ of radius $\varepsilon > 0$ around $x$. Since the orbit of $x$ comes arbitrarily close to the discontinuity then the set $B_\varepsilon(x)$ will chopped up infinitely many times. In fact, it will so in such a manner that what remains of $B_\varepsilon(x)$ with $x$ at infinity is a set of zero measure (in convex settings irrational cells have zero measure). Consequently, points breaking up from $B_\varepsilon(x)$ will be iteraed by different isometries than $x$ will and so will eventually be separated from $x$. This fact is what underpins the statement - with which we tried to engage the reader in the beginning of our Introduction - that the discontinuity set is a source of chaotic dynamics.

It is the aim of this paper to understand the relation between the collection of sets generated by the full set of preimages and the codings generated by maps that constructed in this fashion. Our first motivation was drawn from studying the Goetz map but other cases can have similar properties as we prove in Section 3.

## 2.2 Basic definitions

Consider $\mathbf{X}$ being the standard Euclidean space $\mathbb{R}^n$, or some compact subset of it, endowed with the Euclidean metric $d(\cdot,\cdot)$. Let $\mathcal{P} = \{P_1, \ldots, P_m\}$ be a finite collection of connected open sets with piecewise smooth boundary, i.e, the boundary of each set $P_i$ can be decomposed into a finite number of $(n-1)$-dimensional submanifolds of $\mathbf{X}$. We say that $\mathcal{P}$ is a (topological) *partition* of $\mathbf{X}$ if: $(i)$ $P_i \cap P_j = \emptyset$, for $i \neq j$; and $(ii)$ $\mathbf{X} = \overline{P}_0 \cup \ldots \cup \overline{P}_{m-1}$.

Let $f : \cup_{i=1}^m P_i \to \mathbf{X}$, be such that for every $i$, $f_i := f|_{P_i}$ is at least continuous but $f$ itself does not admit a continuous extension to $\mathbf{X}$. For that reason we say that $\mathcal{D} = \cup_{k=0}^{m-1} \partial P_k$ is the *discontinuity set*. We will require stronger properties from each map $f_i$ later on.

The partition $\mathcal{P}$ gives rise to a way of coding the trajectories. We define $\mathbf{X}^* \subset \mathbf{X}$ to be the set of points whose forward iterates avoid the discontinuity set that is,

$$\mathbf{X}^* = \{x \in \mathbf{X} : \forall n \geq 0, f^n(x) \in \mathcal{P}_{i_n}, \text{ for some } i_n\}$$

Let $\mathcal{A}$ be the alphabet $\{0, \ldots, m-1\}$. We define the *coding map* $\chi : \mathbf{X}^* \to \mathcal{A}^{\mathbb{N}}$ according to the following rule:

$$\chi(x) = w_0 \ldots w_n \ldots, \text{ if and only if } f^n(x) \in P_{w_n}, \text{ where } w_n \in \mathcal{A}.$$

If $x \in \mathbf{X}^*$ is such that its coding $\chi(x)$ is eventually periodic that is, if $\chi(x) = uv \ldots v \ldots$ where $u$ and $v$ are finite words on $\mathcal{A}$ then we call $x$ a *rational* point. Otherwise $x \in \mathbf{X}^*$ is called an *irrational* point. Obviously, every periodic coding $w$ can be written as an infinite adjacent repetition of a finite block $b$ which we represent by $w := [b]$.

The original partition can be refined using the coding map. More precisely, let $K_w$ be the set of all points $x \in \mathbf{X}^*$ with the same coding $w$. Sets of this form will be called *cells*. Thus, the set of all cells, $\mathcal{K}$, is a new partition of $\mathbf{X}$ which is contained in $\mathcal{P}$, in the sense that for each cell $K_w$ there is an atom $P_i$ such that $K_w \subset P_i$.

We shall also denote by $f_w$ the composition of maps $f_{w_n} \circ \ldots \circ f_{w_0}$ corresponding to a given word $w = w_0 \ldots w_n$ .

Finally, we define two important sets in this context: the *exceptional set* and the *singularity set*. The former is the union of all points that will eventually fall on the discontinuity set,

$$\mathcal{E} := \cup_{k=0}^{\infty} f^{-k}(\mathcal{D}) = \mathbf{X} \backslash \mathbf{X}^*$$

and the latter is formed by the set of those points whose trajectory is arbitrarily close to the discontinuity set, that is,

$$\Sigma := \left\{x \in \mathbf{X} : d\left(\mathcal{O}^+(x), \mathcal{D}\right) = 0\right\}$$

where $\mathcal{O}^+(x) = \{f^n(x) ; n \in \Lambda\}$ and $\Lambda$ is a set of the form $\{1, 2, \ldots, N\}$ with $N \in \mathbb{N}$. It turns out that when all $f_i$ are isometries then $\Sigma = \overline{\mathcal{E}}$ (see Goetz [2001]).

**Note:** Under more general conditions, namely, avoiding expansion, we were able to prove that $\Sigma$ is closed (see Appendix). However, we could not prove that it equals $\overline{\mathcal{E}}$.

When each map $f_i$ is smooth, each preimage $f_i^{-k}(\mathcal{D})$ is a finite union of $(n-1)$-dimensional submanifolds and therefore, the total preimage set of the discontinuity, $\cup_{k=0}^{\infty} f^{-k}(\mathcal{D})$, is a zero (Lebesgue) measure set and, consequently, has empty interior. The relevance of this assertion is that, under the latter assumption on $f_i$, the set of points for which the coding map cannot be applied *ad infinitum* is negligible in terms of measure.

### 2.3 Other examples

Noninvertible and/or discontinuous endomorphisms of the plane have been studied in a series of papers (see for instance C. Mira [1996] and references therein) using *critical lines*. Curiously, critical lines are no other than the discontinuity and its preimages or - in the case of continuous but nonivertible maps - lines dividing the plane according to the number of preimages of each point in each connected component.

An interesting example which appeared in C. Mira [1996] is the following. Let $T : \mathbb{R}^2 \circlearrowleft$ be a planar map defined by:

$$T(x, y) = \begin{cases} (y, y - \lambda x) \text{ if } x < 6 \\ (y, y - \gamma x + 6(\gamma - \lambda) + \mu) \text{ if } x > 6 \end{cases} .$$

This is a piecewise linear map with a discontinuity at the vertical line $x = 6$. For particular values of parameters $\lambda, \mu$ and $\gamma$ it is possible to show the existence of an attractor set (see C. Mira [1996] for a more detailed discussion). In Figure 2 we plot a part of the exceptional set inside the attractors for three particular choices of values.

## 3 Rationally vs. irrationally coded orbits

For the time being we assume that each map $f_i$ is a homeomorphism. Let us suppose that $\mathcal{E}$ is not dense, that is $\mathbf{X} \backslash \overline{\mathcal{E}} \neq \emptyset$ and from now on let us assume that $\Sigma$ is closed which is the case for piecewise non-expansive maps . The former assumption alone allows one to conclude that $\mathbf{X} \backslash \overline{\mathcal{E}}$ decomposes into a countable collection of disjoint and open connected sets $\mathcal{C} = \{C_i\}_{i \in \mathbb{N}}$, *i.e.*,

$$\mathbf{X} \backslash \overline{\mathcal{E}} = \cup_{i=0}^{\infty} C_i, \; C_i \cap C_j = \emptyset, i \neq j.$$

This is due to the fact that $\mathbb{R}^n$ can be written as the union of a countable number of open balls $R^n = \cup_{X_i \in \mathbb{Q}^n} B_\delta(X_i)$. Consequently, any open set $O$ can be written as $O = \cup_{i=0}^{\infty} O_i$ where $O_i = B_\delta(X_i) \cap O$. Furthermore, these componentes are interchanged under $f$ as we explain in the following lemma.

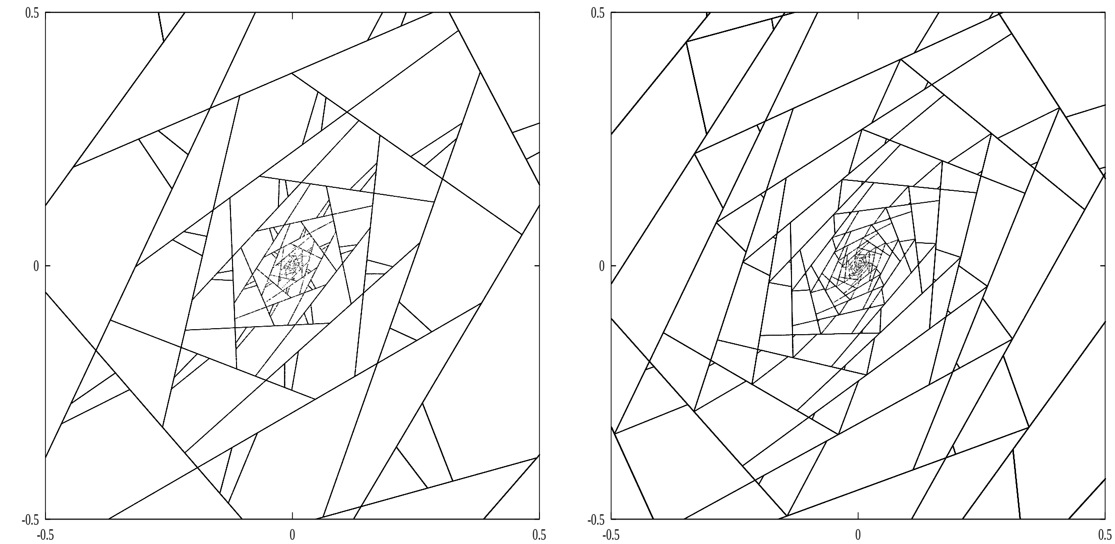


Figure 2: A numerical approximation of the exceptional set for a planar piecewise homeomorphism studied in C. Mira [1996], for different values for parameters: (left) $\lambda = 1.2, \gamma = -0.1, \mu = 12$; (right) $\lambda = 1.12, \gamma = -0.2, \mu = 11$. The point being made here is that the structure of the set of preimages resembles that of the Goetz map (see other figures in this paper).

**Lemma 1** *Let $\{C_i\}_i$ be a collection of disjoint open and connected sets such that $\mathbf{X}\backslash\overline{\mathcal{E}} = \cup_{i=0}^{\infty} C_i$. Then,*

*(i) for every component $C_i$ there exist $j, k \in \mathbb{N}$ such that, $f(C_i) = f_k(C_i) \subset C_j$ . In particular, each component is contained in a cell.*

*(ii) the preimage of each component $C_i$ is either empty or it is made of $r$ $(\leq m)$ components $C_{j_1}, \ldots, C_{j_r} \in \mathcal{C}$, i.e.,*

$$f^{-1}(C_i) = C_{j_1} \cup \ldots \cup C_{j_r} \ .$$

**Proof.** (i) Since $f(\mathcal{E}) \subset \mathcal{E}$ there is a well defined restriction $f_* : \mathbf{X}^* \to \mathbf{X}^*$. Moreover, this map acts continuously and therefore, the image of a connected set in $\mathbf{X}^*$ must be a connected set in $\mathbf{X}^*$. Obviously, this implies that the image of a connected component of $\mathbf{X}\backslash\overline{\mathcal{E}}$ must be another connected component.

(ii) Let $S_k = f_k^{-1}(C_i) \cap P_k$ for $k = 0, \ldots, m-1$. It is clear then that,

$$f_k^{-1}(C_i) = \cup_{i=0}^{m-1} S_k \ .$$

It suffices to show that $S_k = C_{j_k}$ for some $C_{j_k}$. Since $f_k$ is a homeomorphism, $S_k$ is an open connected set. Suppose that $S_k \cap \Sigma \neq \emptyset$ and let $x \in S_k \cap \Sigma$. Thus, there exist $\{x_n\}_n \in \Sigma_0$ such that $x_n \to x$ and since $S_k$ is open we can take $N$ large enough so that $x_p \in S_k$ for $p > N$. By construction, $f(x_p) \in C_i$. However, since $x_p \in \Sigma\backslash\mathcal{D}$ we must have that $f(x_p) \in \Sigma$ which is a contradiction. Therefore, there must exist $C_{j_k} \in \mathcal{C}$ such that $S_k \subset C_{j_k}$ by connectedness of $S_k$. Suppose now that $C_{j_k}\backslash S_k \neq \emptyset$ and take $x \in C_{j_k}\backslash S_k$. By the previous assertion and the definition of $S_k$ we conclude that $f(C_{j_k}) \subset C_i$ and thus $x \in f_k^{-1}(C_i)$. Since all $\{S_k\}_k$ are disjoint we conclude that $S_k = C_{j_k}$ which proves the statement. ■

These conclusions prove the existence of a net of open sets which are interchanged by $f$. In the case of irrational piecewise rotations in planar domains, $\cup_{i=0}^{\infty}\overline{C}_i$ is simply a countable collection of compact circles resembling disk packings. Ashwin and Fu [2001] have studied a family of piecewise rotations on the two-dimensional torus and showed that for a full measure and dense set in the parameter space the disks exhibit no tangencies with each other.

One investigates now when a connected component $C_i$ is rationally coded. Well, if some point $x \in C_i$ does not belong to $\Sigma$ and its $\omega$-limit set is non-empty then the answer is positive. That is the content of Theorem 4. Before that, we adapt a topological lemma which was first used in the study of the symmetries of attractors (see Melbourne *et al.* [1993]).

**Lemma 2** *Let $x \in \mathbf{X}$ be a point such that $\omega(x) \neq \emptyset$. Then, either $\omega(x) \subset \overline{\mathcal{E}}$ or the following are valid:*

*(i) $\omega(x) \subset \overline{C}_0 \cup \ldots \cup \overline{C}_r$ for some $C_0, \ldots, C_r \in \mathcal{C}$;*

*(ii) These components can be ordered so that $f(C_i) \subset C_{i+1(\mathsf{mod}\ r)}$.*

**Proof.** We follow the recurrence argument used in Lemma 2.1 in Melbourne *et al.* [1993]. Take $y \in \omega(x)$ and $\varepsilon > 0$ such that $B_\varepsilon(y) \subset \mathbf{X}\backslash\overline{\mathcal{E}}$. By connectedness of $B_\varepsilon(y)$ we can conclude that $B_\varepsilon(y) \subset C_n$ for some $n \in \mathbb{N}$. From the definition of $\omega$-limit set, there exists a first entrance time $k$ such that $f^k(x) \in B_\varepsilon(y)$. Moreover, there must exist a second entrance time $l > k$ for which $f^l(x) \in B_\varepsilon(y)$. For $r = l - k$ we can conclude that $f^r(B_\varepsilon(y)) \cap B_\varepsilon(y) \neq \emptyset$ since it must contain at least $f^k(x)$.

From continuity of all $f_i$ and connectedness of all $\{C_i\}_{i\in\mathbb{N}}$ it follows that for any $j \in \mathbb{N}$, there exists an $i_j \in \mathbb{N}$ such that $f(C_j) \subset C_{i_j}$. Consequently, the fact that $f^r(C_n) \cap C_n \neq \emptyset$ implies that $f^r(C_n) \subset C_n$ by Lemma 1(i). Take $z = f^k(x)$ and let $D_i = C_j$ where $C_j$ is the connected component visited by $f^i(z)$, for $i = 0, ..., r-1$. This shows that $f(D_i) \subset D_{i+1(\mathsf{mod}\ r)}$. In particular, we have proved that $\omega(z) \subset \overline{D}_0 \cup ... \cup \overline{D}_{r-1}$ and since $\omega(f^k(x)) = \omega(x)$ for every $k \in \mathbb{N}$ assertions $(i)$ and $(ii)$ now follow. ■

This lemma can only be of use for proving the existence of periodic codings if we know that at least one $\omega$-limit set is not contained in $\overline{\mathcal{E}}$. This is proved followingly.

**Lemma 3** *If $x \notin \Sigma$ then $\omega(x) \cap \Sigma = \emptyset$.*

**Proof.** Let $x \in \mathbf{X}\backslash\Sigma$. By definition of $\omega(x)$, for every $y \in \omega(x)$, we conclude that there exists a sequence $\{f^{n_i}(x)\}_{i\in\mathbb{N}}$ such that $f^{n_i}(x) \overset{i\to\infty}{\longrightarrow} y$. Let us suppose that $y \in \Sigma$. Firstly, we show that $y$ cannot be a preimage of the discontinuity. Obviously, $y \notin \mathcal{D}$ for otherwise $d(\mathcal{O}^+(x), \mathcal{D}) = 0$ and so, $x \in \Sigma$. Therefore, for $\delta_1 := d(y, \mathcal{D}) > 0$ we have that $B_{\delta_1}(y) \subset P_{i_1}$, for some $i_1 \in \{0, ..., m-1\}$ since $\mathcal{D}$ is closed. Take $k_1$ large enough so that for all $p > k_1$, we have $f^{n_p}(x) \in B_{\delta_1}(y)$. Consequently, $f^{n_p+1}(x) \overset{p\to\infty}{\longrightarrow} f(y)$ by continuity of $f_{i_1}$ and if $f(y) \in \mathcal{D}$ then obviously, $d(\mathcal{O}^+(x), \mathcal{D}) = 0$, which is a contradiction. Once more, we have that $f(y) \notin \mathcal{D}$.

Analogously, since $f(y) \notin \mathcal{D}$ there is $\delta_2 := d(f(y), \mathcal{D}) > 0$ for which we have that $B_{\delta_2}(f(y)) \subset P_{i_2}$, for some $i_2 \in \{0, ..., m-1\}$. Once again, take $k_2$ large enough so that for all $p > k_2$, we have $f^{n_p+1}(x) \in B_{\delta_2}(f(y))$ which is possible since, as seen before, $f^{n_p+1}(x) \overset{p\to\infty}{\longrightarrow} f(y)$. This implies that $f^{n_p+2}(x) \overset{p\to\infty}{\longrightarrow} f^2(y)$ by continuity of $f_{i_2}$ and if $f^2(y) \in \mathcal{D}$ then obviously, $d(\mathcal{O}^+(x), \mathcal{D}) = 0$, which is, again, a contradiction. In a similar fashion we may prove that for every $n \in \mathbb{N}$, $f^n(y) \notin \mathcal{D}$ and so, $y \notin \cup_{k=0}^{\infty} f^{-k}(\mathcal{D})$.

Consequently, there must exist a sequence of neighbourhoods $V_n$ of each $f^n(y)$, such that $f|_{V_n}$ acts continuously. Therefore, $f^n(y)$ is a point of continuity of $f^n$ and so, the orbit of $y$ is contained in $\omega(x)$ ([1]). Since $\omega(x)$ is closed it follows that $\omega(y) \subseteq \omega(x)$. Consequently, if $y \in \Sigma$, that is, $d(\omega(y), \mathcal{D}) = 0$, then it must be true $\omega(y) \cap \mathcal{D} \neq \emptyset$ because both $\omega(y)$ and $\mathcal{D}$ are closed sets. This latter fact would imply that $\omega(x)$ itself intersected $\mathcal{D}$ and, therefore, $x \in \Sigma$. Hence, $\omega(x) \cap \Sigma = \emptyset$. ■

We can now prove the existence of rationally coded orbits.

**Theorem 4** *If there exists $x \notin \Sigma$ whose $\omega$-limit set is nonempty then $x$ is rationally coded.*

**Proof.** Let $x \notin \Sigma$ be such that $\omega(x) \neq \emptyset$. Then, by Lemma 3 we know that $\omega(x) \cap \Sigma = \emptyset$. Therefore, $\omega(x)$ cannot be contained in $\overline{\mathcal{E}}$ since $\mathcal{E} \subseteq \Sigma$ and by Lemma 2 we conclude that the components containing the orbit of $x$ for sufficiently large iterates are cyclically permuted. ■

A trivial and yet insightful corollary can be drawn.

**Corollary 1** *If $\mathbf{X}$ is compact and $\Sigma \neq \mathbf{X}$ then all points not in $\Sigma$ are rationally coded.*

It should be clear that, although, the dynamics has rational codings outside $\Sigma$ that does not imply that it is trivial. In fact, we could have horsehoes dwelling inside this set and yet, under the coding emap that has been defined, the dynamics is rationally coded and they are not detected. What we want to underline is the fact that these systems have a different type of complicated dynamics which arise naturally from the existence of discontinuities. This dynamics is suitably detected by the coding map defined precisely as it was. In conclusion, Theorem 4 states that any dynamics that is irrationally coded can only be detected within $\Sigma$.

We can be easily tempted into conjecturing that if $\mathbf{X}$ is compact then all points in $\Sigma$ are irrationally coded. However, the best we can prove is the following.

**Theorem 5** *If $\mathbf{X}$ is compact then $\Sigma$ does not contain rational cells.*

**Proof.** Without loss of generality let $K_w$ be a periodic cell. Let also $W := \overline{K}_w$. By rationality and uniqueness of the coding we know that there must exist a continuous return map $f_w : W \to W$. Since $\mathbf{X}$ is compact we conclude that $W$ is also compact. Thus, by the Fixed Point theorem we know that there exists a periodic point, $p$, in $W$. This, in turn, implies that $K_w \cap \Sigma \neq \emptyset$ since the orbit of every periodic point has to be a certain distance apart from the discontinuity set. ■

In fact, the following simple example shows - at least in the context of piecewise isometric systems - that there exist rational cells which intersect both $\Sigma$ (which is equal to $\overline{\mathcal{E}}$ in this case) and its complementary set.

**Example 1** *Consider a piecewise rotation as defined in Section 2.1 having a rotation $R_0$ corresponding to atom $P_0$ with centre $C = (-1, 0)$ and irrational angle $\alpha$. Obviously, there must exist a period one cell $K$ which is an open disc centred in $C$ and radius $1$ plus all points in the boundary*

[1] In general, if $f$ is continuous at $y \in \omega(x)$ then $f(y) \in \omega(x)$. At points $y$ of discontinuity of $f$ that lie in $\omega(x)$, we may have $f(y) \notin \omega(x)$.

*of that disc which will not fall in the origin. Since $\alpha$ is irrational it turns out that the orbit of those points in $K$ which belong to its boundary will be infinitely close to the origin hence they must belong to $\Sigma$ as well. It is also true that the remainder of $K$ does not intersect $\Sigma$.*

Theorem 5 can be restated by asserting that any rational cell intersecting $\Sigma$ must intersect $\mathbf{X}^*\backslash\Sigma$ as well. Thus, another trivial corollary - which can be seen as a generalisation of Theorem 13 - easily follows.

**Corollary 2** *If $\mathbf{X}$ is compact and $\Sigma = \mathbf{X}$ then all points in $\Sigma\backslash\mathcal{E}$ are irrationally coded.*

# 4 A special setting: piecewise isometries

In this section we establish the relationship between connected components and cells for the special case where each induced map $f_i$ is an isometry. Namely, we say that $T$ is a *piecewise isometry* on $\mathbf{X}$ with *partition* $\mathcal{P} = \{P_0, \ldots, P_{m-1}\}$ if

$$T(x) = T_i(x), \text{ if } x \in P_i \ ,$$

where $T_i$ is an isometry defined on $\mathbf{X}$, for every $i = 0, \ldots, m-1$. As defined before, we call $\mathcal{D} := \cup_{k=0}^{m-1} \partial P_k$ the *discontinuity set*, $\mathcal{E} := \cup_{k=0}^{\infty} f^{-k}(\mathcal{D})$ the *exceptional set* and $\mathcal{C}$ is the collection of all the connected components of $\mathbf{X}\backslash\overline{\mathcal{E}}$.

We shall also assume that all atoms in $\mathcal{P}$ are convex sets of $\mathbf{X}$, hence $\mathcal{P}$ is a convex partition of $\mathbf{X}$. This assumption has an implication on the geometrical shape of cells since they must also be convex as each of them can be written as a countable intersection of preimages of atoms and isometries preserve convexity under backward iteration.

Using the fact that $\Sigma = \overline{\mathcal{E}}$ Theorem 4 can be restated in the following form.

**Corollary 3** *If $x \in \mathbf{X}\backslash\overline{\mathcal{E}}$ has nonempty $\omega$-limit set then its coding is rational.*

And a corollary of this result is the following.

**Corollary 4** *If $T$ has no periodic points then every point in $\mathbf{X}\backslash\overline{\mathcal{E}}$ has empty $\omega$-limit set.*

**Proof.** Suppose there exists $x \in \mathbf{X}\backslash\overline{\mathcal{E}}$ such that $\omega(x) \neq \emptyset$. Then, from Corollary 3 we conclude that $x$ has rational coding. By Theorem 14 in appendix the existence of rational points implies that $T$ must have periodic points. ■

This simple corollary can be used to construct examples of piecewise isometries with orbits diverging to infinity in the sense that if such example is to be constructed then one has to avoid periodic points. In the case of piecewise rotations these correspond to the centres of rotation of all rotations in the free group generated by $T_0, \ldots, T_{m-1}$. These are very important cases as we show in Example 2 for we can have cells with nonempty interior which are irrationally coded, somehow contradicting our intuition.

The remainder of this section is devoted to proving results which relate cells and the connected components of $\mathbf{X}\backslash\overline{\mathcal{E}}$.

As we have seen in Lemma 1.(i) for each component $C_i$ there must exist a cell $K_w$ such that $C_i \subset K_w$. The following is a strengthening of that remark in the context of piecewise isometries.

**Proposition 6** *For every component $C_i \in \mathcal{C}$ there exists a cell $K_w \in \mathcal{K}$ such that, $K_w = \overline{C}_i\backslash\mathcal{E}$.*

**Proof.** Let $K_w$ be the cell which Lemma 1.(i) guarantees the existence of. Let us first prove that $K_w \backslash \overline{C}_i = \emptyset$. Suppose there exists $y \in K_w \backslash \overline{C}_i$ and take the convex hull $Y = \mathsf{conv}(\{y\}, C_i)$. Since $y \in K_w$, $C_i \subset K_w$ and $K_w$ is convex we conclude that $Y \subset K_w$ hence all points in $Y$ must have the same coding. By the definition of $\mathcal{C}$, we know that all points in $\partial C_i$ must belong to $\overline{\mathcal{E}}$. Let us choose a point $z \in \partial C_i$ such that there exists an open neighbourhood $N_z$ which is contained in $Y$. This is a simple geometric fact which is sketched in Figure 3.

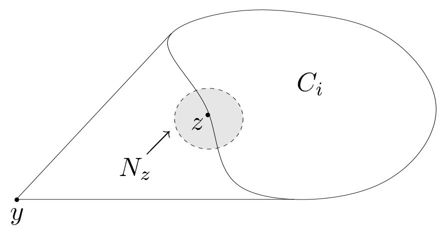


Figure 3: Constructing a neighbourhood $N_z \subset K_w$ whose forward iterates intersect the discontinuity.

Taking into account the fact that $z \in \overline{\mathcal{E}}$ we conclude that there must exist at least one point $\hat{z} \in \mathcal{E} \cap N_z$ (in fact, infinitely many such points) which implies that the orbit of $N_z$ will eventually intersect the discontinuity. Since $N_z$ is an open set this implies that $N_z$ contains points whose iterates will fall in different sides of the discontinuity set. This yields a contradiction for all points in $Y$ must have the same coding. Consequently, $K_w \backslash \overline{C}_i = \emptyset$.

The assertion that $K_w = \overline{C}_i \backslash \mathcal{E}$ is now a trivial consequence of the fact that every point in the boundary of $C_i$ will have the same coding as $C_i$ unless it eventually hits the discontinuity. ■

It should be noted that we cannot conclude whether $K_w$ has rational or irrational coding. Theorem 3 implies that $K_w$ must be rational in the compact setting. However, since we are considering piecewise isometries which are defined in domains that are possibly unbounded this no longer holds. To see this consider the following example.

**Example 2** *In Goetz [1998] it is proved that the orbit of every point in $\mathbb{C}$ of a piecewise rotation of the form,*

$$G(z) = \begin{cases} e^{i\alpha}(z - C_0) + C_0 & \text{if } \Re e(z) \le 0 \\ e^{i\beta}(z - C_1) + C_1 & \text{if } \Re e(z) > 0 \end{cases}$$

*- where $\alpha$ and $\beta$ are incommensurate[2] real numbers and $\Re e(C_0) > 0 > \Re e(C_1)$ - diverges to infinity, meaning that $|G^n(z)| \to \infty$. From Theorem 16 we conclude that those orbits must be irrationally coded. In Figure 4 we depict a numerical simulation of the connected components of $\mathbf{X} \backslash \overline{\mathcal{E}}$ for a map $G$ in this situation.*

Another important implication of Example 2 and Proposition 6 is that Theorem 17 cannot be genralised to spaces with infinite Lebesgue measure because the white components in Figure 4 correspond to cells by Proposition 6 which must be irrationally coded as argued in Example 2.

We now investigate the converse assertion, that is, given a cell $K_w$ is there a connected component of $\mathcal{C}$ that could be related to $K_w$? It turns out that we can only obtain such relation for rational cells.

[2]Two numbers $x$ an $y$ are said to be *incommensurate* if there are no pair of rational numbers $a$ and $b$ such that $ax + by = 0$.

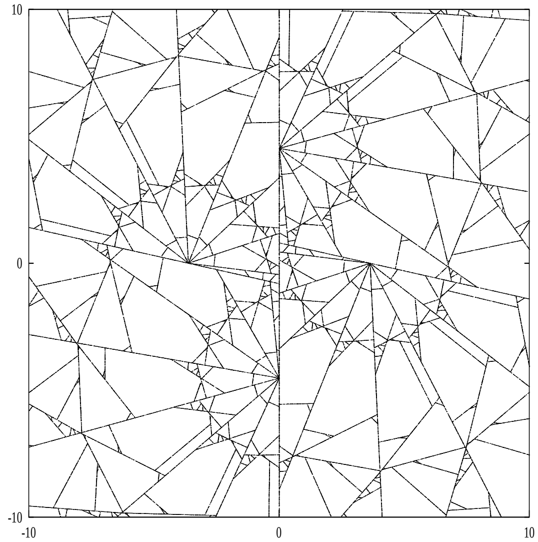


Figure 4: A numerical approximation of the exceptional set for a planar piecewise rotation whose orbits are unbounded as proved in Goetz [1998] .

**Proposition 7** *For all rational cells $K_w \in \mathcal{K}$ there exists a connected component $C_k \in \mathcal{C}$ such that $C_i \subset K_w = \overline{C}_i \backslash \mathcal{E}$. In particular, $K_w$ has nonempty interior.*

**Proof.** It suffices to show that there is a component $C_i$ such that $C_i \subset K_w$ since that would imply that $K_w = \overline{C}_i \backslash \mathcal{E}$ by our previous result and uniqueness of coding.

Let $K_w$ be a periodic cell and take $T_w : K_w \to K_w$ as being the return map on $K_w$. If $T_w$ is a recurrent map then by Theorem 14 (in Appendix) there must exist a periodic point $p \in K_w$. It then follows that the orbit of $p$ must be bounded away from the discontinuity and consequently $p \in C_i$ for some component $C_i \in \mathcal{C}$. Since all points in $C_i$ belong to the same cell we conclude that $C_i \subset K_w$.

Consider now the case when $T_w$ is a nonrecurrent isometry which implies that $K_w$ must be an unbounded convex set and consequently, there must exist an unbounded convex atom, $P_j$, as well such that $K_w \subseteq P_j$. By Lemma 11 there is a halfline $L \subset K_w$ on which $T$ acts as a translation. Consequently, we can choose a point $x \in L$ which is an interior point of $P_j$. By Lemma 12 we know that $d(L, \partial P_j) > 0$, hence the orbit of $x \in L$ is such that it does not approach the discontinuity set. The assertion now follows analogously to the recurrent case.

Finally, assume that $K_w$ is a rational cell and take $K_{w^*}$ as being its corresponding periodic cell which is such that for some positive integer $r$, $\sigma^r(w) = w^*$, where $\sigma$ is the usual shift map. As we have shown above there exists $C_{i^*}$ such that $C_{i^*} \subset K_{w^*}$ . By Lemma 1.(ii) we know that there exists a finite number of components $C_{i_1}, \ldots, C_{i_n}$ $(n \leq r.m)$ such that:

$$T^{-r}(C_{i^*}) = C_{k_1} \cup \ldots \cup C_{k_n} \ .$$

Take $i = k_l$ where $k_l$ is such that $K_w \cap C_{k_j} \neq \emptyset$. Such integer $i$ must exist since $T^r(K_w) \subset K_{w^*}$. Using the fact that $C_i$ is contained in some cell (Lemma 1.(i) again) we can then conclude that $C_k \subset K_w$ by uniqueness of coding. ■

Theorem 15 (in Appendix) is the sharpest result available in the literature concerning codings of points in $\overline{\mathcal{E}}$ and its complement. The following corollary is a refinement of this result in the case when $\mathcal{P}$ is convex. Let us consider the set $\Pi := \cup_{n=0}^{\infty}\overline{C}_n$ which denotes the *packing set* generated by the connected components of $\mathbf{X}\backslash\mathcal{E}$.

**Corollary 5** *Given a piecewise isometry defined on a convex partition $\mathcal{P}$ and compact space $\mathbf{X}$, then the set of points with rational coding is $\Pi\backslash\mathcal{E}$ and the set of points with irrational coding is $\Pi^c\backslash\mathcal{E}$.*

We note that the set of points with irrational coding may not equal $\overline{\mathcal{E}}\backslash\mathcal{E}$ as Example 1 shows. This is best understood by noticing that points in the boundary of the connected components must also belong to the boundary of $\mathcal{E}$.

# 5 Appendix

## 5.1 The singularity set is closed

As was discussed in the introduction the closure of the union of all points that will eventually fall on the discontinuity set,

$$\mathcal{E} := \cup_{k=0}^{\infty} f^{-k}(\mathcal{D})$$

may, in general, be distinct from the set of those points whose trajectory is arbitrarily close to the discontinuity set,

$$\Sigma := \left\{x \in \mathbf{X} : d\left(\mathcal{O}^{+}(x), \mathcal{D}\right) = 0\right\} \ .$$

We have also mentioned earlier on that, in the setting of piecewise isometries, it turns out that (see Goetz [2001])

$$\Sigma = \overline{\mathcal{E}} \ .$$

In general $\Sigma$ is a closed set under the more relaxed assumption on the generating maps $\{f_i\}_i$: let us assume that each induced transformation $f_i$ is a *non-expansive* $\alpha_i$-Hölder map, that is, for all $x, y \in \mathbf{X}$ there are $0 < C_i \le 1$ and $0 < \alpha_i$ such that,

$$d\left(f_i(x), f_i(y)\right) \le C_i d(x,y)^{\alpha_i} \ .$$

The proof is as follows. Let $\{x_n\}_n$ be a sequence of points in $\Sigma$ such that $\lim_{n\to\infty} x_n = x$. Therefore, for each pair of integers $n$ and $k$,

$$\begin{aligned} d\left(f^k(x), f^k(x_n)\right) &\le C_{i_1} d\left(f^{k-1}(x), f^{k-1}(x_n)\right)^{\alpha_{i_1}} \le \\ &\le C_{i_1} C_{i_2}^{\alpha_{i_1}} d\left(f^{k-2}(x), f^{k-2}(x_n)\right)^{\alpha_{i_1}\alpha_{i_2}} \le \dots \\ \dots &\le C_{i_1} C_{i_2}^{\alpha_{i_1}} \dots C_{i_k}^{\alpha_{i_1}\alpha_{i_2}\dots\alpha_{i_k}} d(x, x_n)^{\alpha_{i_1}\alpha_{i_2}\dots\alpha_{i_k}} \ . \end{aligned}$$

By assumption we know that, given sufficiently small $\varepsilon > 0$, we can choose $n$ and $k$ such that,

$$d(x, x_n) < \varepsilon \text{ and } d\left(f^k(x_n), \mathcal{D}\right) < \varepsilon \ .$$

Consequently, by the triangular inequality,

$$\begin{aligned} d\left(f^k(x), \mathcal{D}\right) &\le d\left(f^k(x), f^k(x_n)\right) + d\left(f^k(x_n), \mathcal{D}\right) \le \\ &\le C_{i_1} C_{i_2}^{\alpha_{i_1}} \dots C_{i_k}^{\alpha_{i_1}\alpha_{i_2}\dots\alpha_{i_k}} . \varepsilon^{\alpha_{i_1}\alpha_{i_2}\dots\alpha_{i_k}} + \varepsilon \le \\ &\le \left(C_{i_1} C_{i_2}^{\alpha_{i_1}} \dots C_{i_k}^{\alpha_{i_1}\alpha_{i_2}\dots\alpha_{i_k}} + 1\right) \max\left\{\varepsilon^{\alpha_{i_1}\alpha_{i_2}\dots\alpha_{i_k}}, \varepsilon\right\} \ , \end{aligned}$$

which is arbitrarily small since $C_{i_1} C_{i_2}^{\alpha_{i_1}} \dots C_{i_k}^{\alpha_{i_1}\alpha_{i_2}\dots\alpha_{i_k}}$ is bounded.

## 5.2 Auxiliary lemmata

**Lemma 8** *If $W$ is a linear space then* $\mathsf{proj}_W(\mathsf{conv}A) = \mathsf{conv}(\mathsf{proj}_W A)$, *where* $A \subset \mathbb{R}^n$.

**Proof.** This is a trivial consequence of the fact that $\mathsf{proj}_W$ is a linear map, since $W$ is a linear space. ■

**Lemma 9** $T(\mathsf{conv}(A)) = \mathsf{conv}(T(A))$, *where* $A \subset \mathbb{R}^n$ *and* $T \in \mathbf{E}(n)$.

**Proof.** Every point $y \in \mathsf{conv}(A)$ can be written in the form, $y = a_1x_1 + \ldots + a_px_p$, where $x_1, \ldots, x_p$ and $a_1, \ldots, a_p \in [0,1]$ are such that $\sum_{i=1}^p a_i = 1$. Therefore,

$$\begin{aligned} T.y = R.y + v &= a_1R.x_1 + \ldots + a_pR.x_p + \left(\sum_{i=1}^p a_i\right) v \\ &= \sum_{i=1}^p a_i(R.x_i + v) = a_1T.x_1 + \ldots + a_pT.x_p \end{aligned}$$

which implies that $T.y \in \mathsf{conv}(T(A))$. Note that the argument above can be reversed hence the assertion is proved. ■

**Lemma 10** *For a given recurrent isometry* $T \in \mathbb{E}(n)$ *and* $x \in \mathbb{R}^n$ *there is a* $y \in \mathsf{Fix}(T)$ *such that* $y \in \mathsf{conv}(\mathcal{O}(x))$.

**Proof.** For readability we reproduce an argument used in the proof of Theorem 14 even though the context here is slightly different. We will use the following elementary lemma from Poggiaspalla [2000]: If $T : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry and $S$ is a bounded and invariant set under $T$ then its centre of mass is a fixed point for $T$.

For any isometry $T$, it is true that $T(\mathsf{conv}(S)) = \mathsf{conv}(T(S))$ for every set $S$ by Lemma 9 and therefore, given an invariant set $I$, $\mathsf{conv}(I)$ is a convex set which is also invariant under $T$. Consequently, $J = \mathsf{conv}(\cup_{k=-\infty}^{\infty} T^k(x))$ is a bounded[3], convex and $T$-invariant set.

It now follows that the centre of mass of $J$, $C_J$, is well defined and it is a fixed point for $T$. From a result in Webster [1994] we can also state that $C_J$ can be written as a convex combination of a finite number of points of $\cup_{k=-\infty}^{\infty} T^k(x)$. Furthermore, these points can be taken in $\cup_{k=0}^{\infty} T^k(x)$ simply by iterating under $T$ if necessary which shows that $C_J \in \mathsf{conv}(\cup_{k=0}^{\infty} T^k(x))$. ■

**Lemma 11** *Let $T$ be a nonrecurrent isometry. For any* $x \in \mathbb{R}^n$, $\mathsf{conv}(\mathcal{O}(x))$ *contains a halfline*[4] *$L$ such that $T|_L$ is a translation.*

**Proof.** The case when $T$ is a pure translation is trivial. Suppose $T = R + v$ is a nonrecurrent isometry. Let $\overline{v}$ and $v^{\perp}$ be such that $v = \overline{v} + v^{\perp}$ where $\langle v^{\perp}, \mathsf{Fix}R\rangle = 0$ and $0 \neq \overline{v} \in \mathsf{Fix}R$. It can be easily shown (see proof of Lemma 1 in Mendes [2007]) that $\langle v^{\perp}, \mathsf{Fix}R\rangle = 0$ implies the existence of a vector $u$ such that $(\mathsf{Id} - R).u = v^{\perp}$ which is equivalent to saying that $R.u = u - v^{\perp}$. Let

[3] This stems from the fact that the full orbit of $x$ is diffeomorphic to or contained in a torus $\mathbb{T}^n$ or the union of two tori.

[4] A *halfline* is a set of the form $\{p + \lambda v : \lambda \geq 0\}$ for some $p$ and $v$ in $\mathbb{R}^n$.

$V := \mathsf{Fix}R + u$ and note that $T|_V$ is a translation for given any $y = y^* + u$, where $y^* \in \mathsf{Fix}R$, we have that,

$$T.y = R.(y^* + u) + v = R.y^* + R.u + v = y^* + \left(u - v^\perp\right) + \bar{v} + v^\perp = y + \bar{v} \ .$$

Let $x = \overline{x} + x^\perp \in \mathbb{R}^n$ where $\overline{x} \in \mathsf{Fix}R$ and $x^\perp \in W := (\mathsf{Fix}R)^\perp$. Then,

$$\begin{aligned} T.x &= R.\left(\overline{x} + x^\perp\right) + v = R.\left(\overline{x} + x^\perp + u - u\right) + v = R.(x^\perp - u) + R.u + R.\overline{x} + v = \\ &= R.(x^\perp - u) + \left(u - v^\perp\right) + \overline{x} + \left(\overline{v} + v^\perp\right) = R.(x^\perp - u) + u + (\overline{x} + \overline{v}) \ . \end{aligned}$$

Therefore,

$$\mathsf{proj}_W(T.x) = R.(\mathsf{proj}_W x - u) + u \ . \tag{1}$$

since $\overline{x} + \overline{v} \in \mathsf{Fix}R$.

Let us define the isometry $S$ given by $S(z) := R.(z - u) + u$. We note that given a vector $w \in W$ then,

$$S(w) = R.(w - u) + u = R.w - R.u + u = R.w + v^\perp,$$

since $R$ leaves $W$ invariant by orthogonality. Consequently, $W$ is invariant under $S$ which means that there is a well defined isometry $\widetilde{R}$ denoting the action of $S$ on the linear space $W$. Clearly, $\mathsf{Fix}S = \mathsf{Fix}R + u$ hence $\mathsf{Fix}\widetilde{R} = W \cap \mathsf{Fix}S = \mathsf{proj}_W u$ because $W \cap \mathsf{Fix}R = \left\{\vec{0}\right\}$ by construction.

By Lemma 10 it follows that there is a $y \in \mathsf{Fix}\widetilde{R}$ such that $y \in \mathsf{conv}\left(\mathcal{O}_{\widetilde{R}}(\mathsf{proj}_W x)\right)$ ([5]). As seen before $y = \mathsf{proj}_W u$. Equation (1), can be rewritten as $\mathsf{proj}_W(T.x) = \widetilde{R}(\mathsf{proj}_W x)$ which, when apllied several times over, implies that $\mathsf{proj}_W(T^n.x) = \widetilde{R}^n(\mathsf{proj}_W x)$. This in turn, leads to $\mathcal{O}_{\widetilde{R}}(\mathsf{proj}_W x) = \mathsf{proj}_W(\mathcal{O}(x))$. Consequently, $y \in \mathsf{conv}(\mathsf{proj}_W(\mathcal{O}(x)))$ and therefore, there must exist $z \in \mathsf{conv}(\mathcal{O}(x))$ such that $\mathsf{proj}_W z = y$ taking into account that by Lemma 8 we know that $\mathsf{proj}_W(\mathsf{conv}A) = \mathsf{conv}(\mathsf{proj}_W A)$.

The fact that $\mathsf{proj}_W z = y = \mathsf{proj}_W u$ allows us to conclude that $z \in V$ because $u \in V$ and $W \perp V$. By Lemma 9 it follows that $T(\mathsf{conv}(\mathcal{O}(x))) \subset \mathsf{conv}(\mathcal{O}(x))$ and therefore, $\mathcal{O}(z) \subset \mathsf{conv}(\mathcal{O}(x))$ which obviously implies that $\mathsf{conv}(\mathcal{O}(z)) \subset \mathsf{conv}(\mathcal{O}(x))$.

Finally, it turns out that $L := \mathsf{conv}(\mathcal{O}(z))$ is a halfline with origin at $z$ since $T|_V$ is a translation. ∎

**Lemma 12** *Given an unbounded convex set $P$ and a halfline $L = \{x + \lambda v; \lambda \geq 0\} \subset P$ generated by an interior point $x$ we must have $d(L, \partial P) > 0$.*

**Proof.** By Theorem 18 we know that the closure $\overline{P}$ must contain any halfline of the form $\{y + \lambda v; \lambda \geq 0\}$ for any $y \in \overline{P}$. Thus we can construct a cylinder set enveloping $L$ made up of halflines which are parallel to $L$. This imples that $d(L, \partial\overline{P}) > 0$ which in turn is the same as saying that $d(L, \partial P) > 0$ since $\partial P = \partial\overline{P}$. ∎

[5] Subscript $\widetilde{R}$ means that orbit is obtained under iteration of map $\widetilde{R}$.

### 5.3 Previous results

For self containedness we include full statements of all previous results that are used in this paper. For more information please check references.

**Theorem 13** *(Goetz [2000]) Let* $T : X \to X$ *(*$X$ *is compact) be a piecewise translation map with rationally independent translation vectors. Then every point in* $X$ *has irrational coding.*

**Theorem 14** *(Mendes and Nicol [2004]) There exist recurrent points with ratonal codings if and only if there exist periodic points.*

**Theorem 15** *(Goetz [1996]) Let* $T$ *act on a compact space* $X$ *and suppose that the atoms* $\{P_i\}_i$ *do not have points of full density on their boundaries. Then for every* $x \in X\backslash\overline{\mathcal{E}}$*, its code is rational. For almost every point* $x \in \overline{\mathcal{E}}$*, its code is irrational.*

**Theorem 16** *(Mendes and Nicol [2004]) Suppose* $n$ *is even,* $T \in \mathbf{SE}(\mathcal{P})$ *and the rotations* $\{A_i\}_{i=0}^{m-1}$ *defining* $T$ *are incommensurate. Then, every unbounded orbit is irrationally coded. Furthermore, for almost every* $T \in \mathbf{SE}(\mathcal{P})$ *the corresponding set of rotations* $\{A_i\}_{i=0}^{m-1}$ *defining* $T$ *are incommensurate.*

**Theorem 17** *(Goetz [2000]) If* $X$ *has finite Lebesgue measure, then every cell of positive Lebesgue measure has a rational code.*

**Theorem 18** *(Webster [1994]) Let* $A$ *be a closed unbounded convex set in* $\mathbb{R}^n$*. Then* $A$ *contains a halfline. Moreover, if* $A$ *contains some halfline with direction* $L_0^+$*, then it contains every halfline with direction* $L_0^+$ *whose initial point is in* $A$*.*

# References


[1] Ashwin, P. and Xin-Chu Fu [2001] *Tangencies in invariant disk packings for certain planar piecewise isometries are rare*, Dynamical Systems **16**, 333-345.

[2] Buzzi, J. [2001] *Piecewise isometries have zero topological entropy*, Ergodic Theory Dynam. Systems, **21** (5), 1371-1377.

[3] Davies, A.C. [1995] *Nonlinear oscillations and chaos from digital filter overflow,* Phil. Trans. R. Soc. Lond. **A (**353), 85-99.

[4] Goetz, A. [2001] *Stability of Cells in Non-Hyperbolic Piecewise Affine Maps and in Piecewise Rotations*, Nonlinearity, **14** (2), 205-219.

[5] Goetz, A. [2000] *Dynamics of Piecewise Isometries*, Illinois Journal of Maths, **44** (3), 465-477.

[6] Goetz, A. [1998] *Dynamics of a Piecewise Rotation*, Continuous and Discrete Dynamical Systems, **4** (4), 593-608.

[7] Goetz, A. [1998a] *A self-similar example of a piecewise isometric attractor*, World Scientific, Proceedings of the International Conference: "From Crystals to Chaos", Marseille.

[8] Goetz, A. [1996] *Dynamics of Piecewise Isometries*, PhD Thesis, University of Illinois.

[9] Gutkin, E. [1996] *Billiards in polygons: survey of recent results.* J. of Stat. Phys. **83** (1/2), 1-26.

[10] Katok, A. [1980] *Interval exchange transformations and some special flows are not mixing*, Israel J. of Math. **35** (4), 301-310.

[11] Keane, M. [1977] *Non-ergodic interval exchange transformations*, Israel Jour. of Math., **26** (2), 188-196.

[12] Keane, M. [1975] *Interval exchange transformations*, Math. Z., **141**, 25-31.

[13] Masur, H. [1982] *Interval exchange transformations and measured foliations*, Annals of Math., **115**, 169-200.

[14] Melbourne, I., Dellnitz, M. and Golubitsky, M. [1993] *The structure of symmetric attractors*, Arch. Rational Mech. Anal. **123** (1), 75-98.

[15] Mendes, M. [2007] *Stability of periodic points in piecewise isometries of Euclidean spaces,* Ergo. Th. & Dynam. Sys. **27**, 183-197.

[16] Mendes, M. and Nicol, M. [2004] *Periodicity and Recurrence in Piecewise Rotations of Euclidean Spaces.* Int. J. of Bif. and Chaos, vol. **14**, nr. 7, pp. 2353-2361.

[17] Poggiaspalla, G. [2000], *Une Isométrie par Morceaux sur le Tore*, Université de Marseille.

[18] Veech, W. [1982] *Gauss measures for transformations on the space of interval exchange maps*, Annals of Math., **115**, 201-242.

[19] Webster, R. [1994] *Convexity*, Oxford University Press.

*Miguel Ângelo de Sousa Mendes*

mmendes@fe.up.pt
http://www.fe.up.pt/~mmendes/research/

Centro de Matemática
Universidade do Porto
Rua do Campo Alegre
4619 - 007 PORTO

&

Faculdade de Engenharia
Universidade do Porto
Rua Dr. Roberto Frias s/n
4200 - 465 PORTO