

# Graded geometry and Poisson reduction

A.S. Cattaneo\* and M. Zambon†

\**Institut für Mathematik, Universität Zürich-Irchel, Winterthurerstr. 190, CH-8057 Zürich, Switzerland*

†*Universidade do Porto, Departamentos de Matemática Pura, Rua do Campo Alegre, 687, 4169-007 Porto, Portugal*

**Abstract.** The main result of [2] extends the Marsden-Ratiu reduction theorem [4] in Poisson geometry, and is proven by means of graded geometry. In this note we provide the background material about graded geometry necessary for the proof in [2]. Further, we provide an alternative algebraic proof for the main result.

**Keywords:** Poisson manifold, reduction, graded manifold, coisotropic submanifold.

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## 1. INTRODUCTION

Many geometric structures on an ordinary manifold may be rephrased as the data of a Poisson-self-commuting function on an associated super symplectic manifold (with a refinement in the grading—a *graded* symplectic manifold). This is the case, e.g., of Poisson and Courant structures.

The problem of reduction of such structures may be equivalently rephrased in the associated super version. One then has to perform some form of symplectic reduction – the most general being that of presymplectic submanifolds – in the super setting, and further one has to find conditions for the associated function to descend and to be still self-commuting on the quotient. In the special case of coisotropic submanifolds the first condition implies the second, but in general this is not the case. Some rather general sufficient, but not necessary, conditions may be worked out in a reduction-by-stages framework.

Once these results have been obtained, they can be translated back into the ordinary diffeogeometric language. Namely, one gets a general reduction theory for Poisson manifolds in terms of vector bundles on submanifolds satisfying certain conditions. Once the result is known, it can also be proved directly without reference to supergeometry, see Section 8.

All this generalizes the known reduction procedures in Poisson geometry, in particular the celebrated Marsden–Ratiu reduction [4].

In this note we concentrate on the main results of this approach referring to [2] for more details and complete proofs. The classical proof of Section 8 is new. The case of reduction of Courant algebroids and of generalized complex structures will be treated in [1].

## 2. POISSON MANIFOLDS

**Definition 1**  $M$  is a Poisson manifold if  $C^\infty(M)$  is endowed with a Lie bracket  $\{\bullet, \bullet\}_M$  satisfying

$$\{f, gh\}_M = \{f, g\}_M h + g\{f, h\}_M. \quad (1)$$

Let  $M$  be a smooth manifold. The Lie bracket of vector fields on  $M$  extends to a bracket – called *Schouten bracket* – on all multivector fields on  $M$ . The Schouten bracket, together with the wedge product, endows the set of multivector fields  $A := \Gamma(\wedge^\bullet TM)$  with the structure of a *Gerstenhaber algebra* (also called *graded Poisson algebra of degree 1*). This means that  $A$  is a graded commutative algebra, that  $A[1]$  (defined by  $A[1]_i = A_{i+1}$ ) is a graded Lie algebra (see [6]), and that the two structures are compatible in the sense that the adjoint action of a homogeneous element  $X \in A$  is a graded derivation of degree  $\deg(X) - 1$ :

$$[X, Y \wedge Z] = [X, Y] \wedge Z + (-1)^{(\deg(X)-1) \cdot \deg(Y)} Y \wedge [X, Z]. \quad (2)$$

The Schouten bracket between a vector field and a function is  $[X, f] = X(f)$ , and the Schouten bracket of two vector fields is the usual Lie bracket. This determines the Schouten bracket on the whole of  $\Gamma(\wedge^\bullet TM)$  by virtue of (2).

The Gerstenhaber algebra of multivector fields is relevant for us because it allows us to describe a Poisson manifold  $M$  as a manifold with a bivector field  $\pi \in \Gamma(\wedge^2 TM)$  satisfying the Schouten-bracket relation  $[\pi, \pi] = 0$ . The connection to Def. 1 is established as follows: the bracket  $\{\bullet, \bullet\}_M$  is encoded by a bivector field  $\pi$  due to (1), the correspondence being  $\{f, g\}_M = \pi(df, dg)$ . The fact that  $\{\bullet, \bullet\}_M$  satisfies the Jacobi identity is equivalent to  $[\pi, \pi] = 0$ .

## 3. GRADED MANIFOLDS AND THE PROBLEM

Ordinary manifolds are modeled on open subsets of  $\mathbb{R}^n$ . We start describing the local model for a *graded* manifold.

**Definition 2** Let  $U \subset \mathbb{R}^n$  open subset and  $V = \bigoplus_{i \neq 0} V_i$  a finite dimensional  $\mathbb{Z}$ -graded vector space.

The local model for a graded manifold consists of the pair

- $U$  (the “body”)
- $C^\infty(U) \otimes S^\bullet(V^*)$  (the graded commutative algebra of “functions”).

Notice that here  $S^\bullet(V^*)$  denotes the *graded* symmetric algebra over  $V^*$ , so its homogeneous elements anticommute if they both have odd degree.

**Definition 3** A graded manifold consists of a pair as follows:

- a topological space  $M$  (the “body”)

- a sheaf  $\mathcal{O}_M$  over  $M$  of graded commutative algebras, locally isomorphic to the above local model (the sheaf of “functions”).

We use the notation  $C(\mathcal{M})$  for  $\mathcal{O}_M(M)$ , the space of “functions on  $\mathcal{M}$ ”.

**Remark 4** Graded manifolds as defined above are sometimes referred to as  $\mathbb{Z}$ -graded manifolds, because their algebras of functions are  $\mathbb{Z}$ -graded. A special case are  $\mathbb{Z}_{\geq 0}$ -graded manifolds, whose algebras of functions are concentrated in non-negative degrees (so in the local model  $V$  is concentrated in negative degrees and  $V^*$  in positive degrees). An example is provided by ordinary manifolds (the case  $V = \{0\}$ ), for which all functions have degree zero.

A graded vector bundle  $E = \bigoplus_{i \neq 0} E_i \rightarrow M$  can be viewed as a graded manifold with body  $M$  and functions  $\Gamma(S^\bullet E^*)$ . Recall that  $E[1]$  is defined by  $(E[1])_i = E_{i+1}$ .

**Example 5**  $T^*[1]M$  is a graded manifold with body  $M$  and functions

$$\Gamma(S^\bullet(T[-1]M)) = \Gamma(\wedge^\bullet TM) = \{\text{multivector fields on } M\}.$$

Explicitly, we can choose coordinates  $x_j$  on  $M$ , giving rise coordinates  $p_j$  on the fibers of  $T^*M$ ; assigning degree 1 to them we obtain coordinates  $\xi_j$  of fibers of  $T^*[1]M$ . In these coordinates, a vector field  $a_i(x)\partial_{x_i}$  corresponds to the degree 1 function  $a_i(x)\xi_i$  on  $T^*[1]M$ .

Exactly as usual cotangent bundles,  $T^*[1]M$  has a symplectic form  $\omega = -dx_j \wedge d\xi_j$ , which gives rise to a Poisson bracket of degree  $-1$  on  $C(T^*[1]M)$  determined by  $\{\xi_j, x_k\} = \delta_{jk}$ ,  $\{\xi_j, \xi_k\} = 0$ ,  $\{x_j, x_k\} = 0$  and the Leibniz rule. But this is just the Schouten bracket on multivector fields! Hence we see that the degree 1 graded Poisson algebra structure on the functions on  $T^*[1]M$  coincides with the one defined in Section 2. Summarizing, we obtain the following bijective correspondences (this is Prop. 4.1 of [5]):

**Proposition 6** Poisson bracket  $\{\bullet, \bullet\}_M$  on  $M \leftrightarrow$   
bivector field  $\pi \in \Gamma(\wedge^2 TM)$  satisfying  $[\pi, \pi] = 0 \leftrightarrow$   
degree 2 function  $\mathcal{S}$  on  $T^*[1]M$  satisfying  $\{\mathcal{S}, \mathcal{S}\} = 0$ .

Hence a Poisson structure on a manifold  $M$  can equivalently be regarded as a very simple kind of structure – indeed, just a function – on  $T^*[1]M$ .

**Remark 7** A degree  $n$  graded symplectic manifold is a  $\mathbb{Z}_{\geq 0}$ -graded manifold endowed with a non-degenerate, closed 2-form whose corresponding Poisson bracket has degree  $-n$ . The degrees of coordinates on the graded manifold lie between 0 and  $n$  (Lemma 2.4 of [5]). A choice of degree  $n+1$  self-commuting function determines a geometric structure by the so-called derived bracket construction.

For  $n = 1$  this geometric structure is the one of a Poisson manifold (Prop. 6). All degree 1 symplectic manifolds are of the form  $T^*[1]X$  for some  $X$  (Prop. 3.1 of [5]).

For  $n = 2$  the geometric structure is a so-called Courant algebroid [5]. Some of the constructions for the case  $n = 2$  carried out in this note for  $n = 1$  are considered in [1].

In the rest of this note we want to consider the following reduction problem:

Let  $M$  be a Poisson manifold. Specify geometric data on  $M$  out of which one can construct canonically a new Poisson manifold.

In virtue of Prop. 6 the problem becomes: specify geometric data on the pair  $(T^*[1]M, \mathcal{S})$  which allow us to construct canonically a new degree 1 symplectic manifold – which will be of the form  $T^*[1]X$  for some manifold  $X$ , see Remark 7 – and a self-commuting degree 2 function on it.

A common way in ordinary symplectic geometry to construct new symplectic manifolds is to take a submanifold  $N$  which is presymplectic (i.e.  $\ker(\iota^*\omega) = TN^\omega \cap TN$  has constant rank; the special case where  $TN^\omega \subset TN$  is called coisotropic) and to consider the quotient  $N/\ker(\iota^*\omega)$ , which is automatically symplectic if smooth.

This suggests to consider presymplectic submanifolds  $\mathcal{C}$  of  $T^*[1]M$  so that the function  $\mathcal{S}$  descends to the quotient of  $\mathcal{C}$  by its characteristic distribution and is self-commuting there. In the next sections we will carry this out, and in Thm 14 we will give an answer to the above reduction problem.

## 4. GRADED SUBMANIFOLDS

In the rest of this note we will consider graded manifolds of degree 1, i.e. graded manifolds whose algebra of functions is generated by functions of degree 0 and 1. They are always of the form  $\mathcal{M} = W[1]$  where  $W \rightarrow M$  is a vector bundle.

We define graded submanifolds of  $\mathcal{M}$  in terms of coordinate functions on  $\mathcal{M}$ . To this aim recall that functions  $x_i$  of degree zero ( $i \leq \dim(M)$ ) and  $\xi_j$  of degree one ( $j \leq \text{rk}(W)$ ) defined over an open subset  $U \subset M$  are called *coordinates* if the  $x_i$  are usual coordinates on  $U$  and there is an isomorphism of graded commutative algebras from  $\mathcal{O}_M(U)$  to the local model  $C^\infty(U) \otimes S^\bullet(V^*)$  so that under this isomorphism the  $\xi_j$  correspond a basis of  $V^*$ . Here  $V$  is a vector space concentrated in degree  $-1$  with  $\dim(V) = \text{rk}(W)$ .

**Definition 8** A graded submanifold  $\mathcal{C}$  of  $\mathcal{M}$  is given by a homogeneous graded ideal  $\mathcal{I} \subset C(\mathcal{M}) := \mathcal{O}_M(M)$  satisfying the following “smoothness” property. In a neighborhood  $U$  of any point  $x \in M$  satisfying  $\mathcal{I}_0(x) = 0$  there exist coordinates  $x_i$  and  $\xi_j$  so that  $\mathcal{I}(U)$  is generated by  $x_{\dim_0(\mathcal{C})+1}, \dots, x_{\dim(M)}$  and  $\xi_{\dim_1(\mathcal{C})+1}, \dots, \xi_{\text{rk}(W)}$ . Further, we require that the vanishing set of  $\mathcal{I}_0$  be closed in  $M$ . The integers  $\dim_i(\mathcal{C})$  are called the dimensions of  $\mathcal{C}$  in degree  $i$  ( $i = 0, 1$ ).

In concrete terms, we have  $\mathcal{I}_0 = Z(C)$  for some closed submanifold  $C$  of  $M$  and  $\mathcal{I}_1 = \tilde{\Gamma}(E)$  for some vector subbundle  $E \rightarrow C$  of  $W^* \rightarrow M$ . Here and in the sequel we use the notation  $\tilde{\Gamma}(\bullet)$  to denote sections of a vector bundle which restrict to sections of the subbundle  $\bullet$ , so  $\tilde{\Gamma}(E) = \{X \in \Gamma(W^*) : X|_C \subset E\}$ .

Since  $C(\mathcal{M})/\mathcal{I}$  is canonically isomorphic to  $C(E^\circ[1])$  we write  $\mathcal{C} = E^\circ[1]$ . Here  $E^\circ \subset W|_C$  denotes the annihilator of  $E$ .

## 5. COISOTROPIC SUBMANIFOLDS

In the previous section we saw that a submanifold  $\mathcal{C}$  of  $\mathcal{M} := T^*[1]M$  is of the form  $E^\circ[1]$  for some vector subbundle  $E \rightarrow C$  of  $TM \rightarrow M$ . Denote by  $\mathcal{I}$  the ideal defining  $\mathcal{C}$  and by  $\mathcal{N}(\mathcal{I})$  the Poisson normalizer of  $\mathcal{I}$ , i.e. the set of functions  $\phi \in C(\mathcal{M})$  satisfying  $\{\phi, \mathcal{I}\} \subset \mathcal{I}$ . One computes

$$\mathcal{N}(\mathcal{I})_0 = \{f \in C^\infty(M) : df|_C \subset E^\circ\} =: C_E^\infty(M), \quad (3)$$

$$\mathcal{N}(\mathcal{I})_1 = \{X \in \tilde{\Gamma}(TC) : [X, \tilde{\Gamma}(E)] \subset \tilde{\Gamma}(E)\}. \quad (4)$$

**Definition 9** *The submanifold  $\mathcal{C}$  is coisotropic if  $\{\mathcal{I}, \mathcal{I}\} \subset \mathcal{I}$  (i.e.  $\mathcal{I} \subset \mathcal{N}(\mathcal{I})$ ).*

**Proposition 10** *A coisotropic submanifold  $\mathcal{C}$  of  $T^*[1]M$  corresponds to a submanifold  $C$  of  $M$  endowed with an integrable distribution  $E$ . The coisotropic quotient of  $\mathcal{C}$  is smooth iff  $\underline{C} = C/E$  is smooth, and in that case the coisotropic quotient is canonically symplectomorphic to  $T^*[1]\underline{C}$ .*

*If  $M$  further has a Poisson structure  $\pi$ , the induced function  $\mathcal{I}$  on  $T^*[1]M$  descends to a degree 2 self-commuting function on  $T^*[1]\underline{C}$  (which hence corresponds to a Poisson structure on  $\underline{C}$ ) iff  $\sharp N^*C \subset E$  and  $C_E^\infty(M)$  is closed under the Poisson bracket. Here  $\sharp : T^*M \rightarrow TM$  denotes contraction with  $\pi$  and  $N^*C := \{\xi \in T^*M|_C : \langle \xi, TC \rangle = 0\}$ .*

**Proof** By degree reasons  $\{\mathcal{I}_0, \mathcal{I}_0\}$  always vanishes. If  $X \in \mathcal{I}_1 = \tilde{\Gamma}(E)$  and  $f \in \mathcal{I}_0 = Z(C)$  we have  $\{f, X\} = -X(f)$ . So  $\{\mathcal{I}_0, \mathcal{I}_1\} \subset \mathcal{I}_0$  is equivalent to  $E \subset TC$ . If  $X, Y \in \mathcal{I}_1$  then  $\{X, Y\} = [X, Y]$ , so  $\{\mathcal{I}_1, \mathcal{I}_1\} \subset \mathcal{I}_1$  is equivalent to the involutivity of the distribution  $E$  on  $C$ .

In this case, since by construction  $\mathcal{I}$  is a Poisson ideal in the Poisson algebra  $\mathcal{N}(\mathcal{I})$ , the Poisson bracket descends making  $\mathcal{N}(\mathcal{I})/\mathcal{I}$  into a graded Poisson algebra. In degree 0 by eq. (3) it consists of the  $E$ -invariant functions on  $C$ , so let us assume that the quotient  $\underline{C}$  of  $C$  by the foliation integrating  $E$  be a smooth manifold (so that  $C \rightarrow \underline{C}$  is a submersion). In degree 1 by eq. (4)  $\mathcal{N}(\mathcal{I})/\mathcal{I}$  consists of vector fields on  $C$  which are projectable w.r.t. the projection  $C \rightarrow \underline{C}$ , modulo vector fields lying in the kernel of the projection. In other words  $(\mathcal{N}(\mathcal{I})/\mathcal{I})_1$  is isomorphic to the space of vector fields on  $\underline{C}$ . We conclude that  $\mathcal{N}(\mathcal{I})/\mathcal{I}$  is the graded Poisson algebra on a graded symplectic manifold iff  $\underline{C}$  is smooth, and in that case it is the Poisson algebra of functions on  $T^*[1]\underline{C}$ .

Further, if  $M$  is a Poisson manifold, the function  $\mathcal{I}$  induces a function  $\underline{\mathcal{I}}$  on  $T^*[1]\underline{C}$  iff  $\mathcal{I} \in \mathcal{N}(\mathcal{I})$ . In that case, by the way we defined the bracket on  $\mathcal{N}(\mathcal{I})/\mathcal{I}$ , it is clear that  $\underline{\mathcal{I}}$  commutes with itself. Hence we obtain a reduced Poisson structure on  $\underline{C}$ . We spell out what it means for  $\mathcal{I}$  to lie in  $\mathcal{N}(\mathcal{I})$ . Since for any function  $f$  on  $M$  we have  $\{\mathcal{I}, f\} = [\pi, f] = \sharp df$ ,  $\{\mathcal{I}, \mathcal{I}_0\} \subset \mathcal{I}_1$  is equivalent to  $\sharp N^*C \subset E$ . Notice that in particular  $C$  is a coisotropic submanifold of  $M$ . Further, for any vector field  $X$  on  $M$ ,  $\{\mathcal{I}, X\} = [\pi, X] = -\mathcal{L}_X \pi$ , so  $\{\mathcal{I}, \mathcal{I}_1\} \subset \mathcal{I}_2$  is equivalent to  $(\mathcal{L}_X \pi)|_C \in \Gamma(E \wedge TM|_C)$  for any  $X \in \tilde{\Gamma}(E)$ , which using eq. (13) below is equivalent to  $C_E^\infty(M)$  being closed under the Poisson bracket of  $M$ .  $\square$

The result obtained in the coisotropic case is the simplest statement concerning Poisson reduction. In order to obtain more interesting results we need to allow for more general submanifolds  $S$ , namely, presymplectic submanifolds.

## 6. PRESYMPLECTIC SUBMANIFOLDS

Let  $M$  be a smooth manifold. We consider again a submanifold  $\mathcal{C} = E^\circ[1]$  of  $\mathcal{M} := T^*[1]M$ , and denote by  $\mathcal{I}$  its vanishing ideal. To define presymplectic submanifolds we need the following

**Definition 11** Let  $A_{ij}$  be a matrix with entries in  $C(\mathcal{C}) := C(\mathcal{M})/\mathcal{I}$ .

We say that  $A$  has constant rank along  $\mathcal{C}$  iff, switching rows and adding  $C(\mathcal{C})$ -multiples of a row to another row, the matrix  $A$  can be brought to the form  $\begin{pmatrix} \star \\ 0 \end{pmatrix}$  where the degree zero part of the rows of  $\star$  are linearly independent at every point of the body of  $\mathcal{C}$ .

**Definition 12** A submanifold  $\mathcal{C}$  is presymplectic iff  $\mathcal{I}$  is generated by homogeneous functions  $\phi_i$  for which the matrix  $\{\phi_i, \phi_j\} \bmod \mathcal{I}$  has constant rank along  $\mathcal{C}$ .

Translating in terms of classical geometry we obtain

**Lemma 13**  $\mathcal{C}$  is a graded presymplectic submanifold iff  $TC \cap E$  is a constant rank, involutive distribution on  $C$ .

The quotient of  $\mathcal{C}$  by its characteristic distribution, defined as  $\mathcal{N}(\mathcal{I})/\mathcal{N}(\mathcal{I}) \cap \mathcal{I}$ , is smooth iff the quotient  $\underline{C} := C/(TC \cap E)$  is smooth. In this case it is isomorphic to  $T^*[1]\underline{C}$  as a graded symplectic manifold.

Now we assume that  $M$  is a Poisson manifold and address the issue of when the function  $\mathcal{I}$  induces a function  $\underline{\mathcal{I}}$  on the quotient  $\underline{C} := T^*[1]\underline{C}$ .  $\mathcal{I}$  descends iff its image under the map  $C(\mathcal{M}) \rightarrow C(\mathcal{M})/\mathcal{I}$  lies in  $\mathcal{N}(\mathcal{I})/\mathcal{N}(\mathcal{I}) \cap \mathcal{I}$ , i.e. iff  $\mathcal{I}$  lies in  $\mathcal{N}(\mathcal{I}) + \mathcal{I}$ .

When  $\mathcal{I}$  descends,  $\underline{\mathcal{I}}$  might not commute with itself. The reason is that the Poisson bracket on  $\mathcal{N}(\mathcal{I})/\mathcal{N}(\mathcal{I}) \cap \mathcal{I}$  is computed lifting to elements of  $\mathcal{N}(\mathcal{I})$  (and not to arbitrary elements of  $\mathcal{N}(\mathcal{I}) + \mathcal{I}$ ).

It is clear that if  $\mathcal{I}$  lies in  $\mathcal{N}(\mathcal{I})$  then the induced function on  $\underline{C}$  still commutes with itself. It turns out that it suffices to require that

$$\{\mathcal{I}, \mathcal{I}_0\} \subset \mathcal{I}_1 \tag{5}$$

(or equivalently  $\sharp TC^\circ \subset E$ ); this conditions leads to the statement of Prop. 3.6 of [2] and Prop. 4.1 of [3], which is a mild improvement of that of [4]. We do not state it here because in Thm. 14 we will state a yet better result.

## 7. REDUCTION IN STAGES AND THE THEOREM

To derive a condition weaker than (5) we perform reduction in stages, as follows. We imbed the presymplectic submanifold  $\mathcal{C}$  in a larger coisotropic submanifold  $\mathcal{A}$  of  $\mathcal{M}$ . We assume that the quotient  $\underline{\mathcal{C}}$  of  $\mathcal{C}$  by its characteristic distribution  $T\mathcal{C} \cap T\mathcal{C}^\omega$  is smooth. Locally the quotient can be realized in two stages: first take the image  $\bar{\mathcal{C}}$  of  $\mathcal{C}$  under the projection  $\mathcal{A} \rightarrow \bar{\mathcal{A}} := \mathcal{A}/T\mathcal{A}^\omega$ ; assuming that  $T\mathcal{C} \cap T\mathcal{A}^\omega$  has constant rank,  $\bar{\mathcal{C}}$  is a presymplectic submanifold. Then take the presymplectic quotient of  $\bar{\mathcal{C}}$ . It

will be (locally) symplectomorphic to  $\underline{\mathcal{C}}$ . Now assume that

$$\mathcal{S} \text{ descends to } \underline{\mathcal{C}} \quad (6)$$

$$\mathcal{S} \text{ descends to a function } \bar{\mathcal{S}} \text{ on } \bar{\mathcal{A}} \quad (7)$$

$$\bar{\mathcal{S}} \text{ satisfies condition (5), i.e. } \{\bar{\mathcal{S}}, (\mathcal{I}_{\bar{\mathcal{C}}})_0\} \subset (\mathcal{I}_{\bar{\mathcal{C}}})_1. \quad (8)$$

Then the reasoning of the previous section implies that the function  $\underline{\mathcal{S}}$  on  $\underline{\mathcal{C}}$  commutes with itself. Since we are ultimately interested in a quotient of  $\mathcal{C}$ , it is clear that condition (7) can be weakened.

The geometric procedure described above is carried out in algebraic terms in [2]. Writing  $\mathcal{A} = D^\circ[1]$  for a subbundle  $D \rightarrow A$  of  $TM$ , we obtain

**Theorem 14** *Let  $C$  be a submanifold of the Poisson manifold  $(M, \pi)$  and  $E \subset TM|_C$  a subbundle such that  $F := TC \cap E$  is a constant rank, involutive distribution on  $C$ .*

*Let  $D|_C$  be a subbundle of  $TM|_C$  with  $F \subset D|_C \subset E$  and*

$$\sharp E^\circ \subset TC + D|_C. \quad (9)$$

*Let  $A$  be a submanifold containing  $C$  such that  $TA|_C = TC + D|_C$ , and assume that  $D|_C$  can be extended to an integrable distribution  $D$  on  $A$  such that*

$$(\mathcal{L}_{X_i} \pi)|_C \subset E \wedge TM|_C \quad (10)$$

*where  $\{X_i\}$  is an extension from  $A$  to  $M$  of a (local) frame of sections of  $D$ .*

*Then  $\underline{C} := C/F$  inherits a Poisson manifold structure.*

## 8. AN ALGEBRAIC PROOF

Without the graded geometric interpretation it would have been hard to derive Thm. 14. Once the statement is known, however, it is easy to give an alternative algebraic proof. We will do so in this section.

The following algebraic statement generalizes slightly Prop. A.1 in [3] and reduces to an obvious one when  $\mathcal{B} = \mathcal{D}$ , for in that case  $\mathcal{B} \cap \mathcal{I}$  a Poisson ideal in the Poisson subalgebra  $\mathcal{B}$ .

**Proposition 15** *Let  $\mathcal{P}$  be a Poisson algebra,  $\mathcal{B} \subset \mathcal{D}$  multiplicative subalgebras of  $\mathcal{P}$  and  $\mathcal{I}$  a multiplicative ideal of  $\mathcal{P}$ . Assume that*

$$\{\mathcal{B}, \mathcal{B}\} \subset \mathcal{D} \cap (\mathcal{I} + \mathcal{B}) \quad (11)$$

and

$$\{\mathcal{B}, \mathcal{I} \cap \mathcal{D}\} \subset \mathcal{I}. \quad (12)$$

*Then there is an induced Poisson algebra structure on  $\frac{\mathcal{B}}{\mathcal{B} \cap \mathcal{I}}$ , whose bracket is determined by the commutative diagram*

$$\begin{array}{ccc} \mathcal{B} \times \mathcal{B} & \xrightarrow{\{\cdot, \cdot\}} & \mathcal{D} \cap (\mathcal{I} + \mathcal{B}) \\ \downarrow & & \downarrow \\ \frac{\mathcal{B}}{\mathcal{B} \cap \mathcal{I}} \times \frac{\mathcal{B}}{\mathcal{B} \cap \mathcal{I}} & \longrightarrow & \frac{\mathcal{B}}{\mathcal{B} \cap \mathcal{I}} \end{array}$$

**Proof** The above diagram is well-defined because of (11) and  $\{\mathcal{B}, \mathcal{I} \cap \mathcal{B}\} \subset \mathcal{I}$  (which holds by (12)). The induced bilinear operation on  $\frac{\mathcal{B}}{\mathcal{B} \cap \mathcal{I}}$  satisfies the Leibniz rule (1) because the Poisson bracket on  $\mathcal{P}$  does. To check the Jacobi identity consider  $f, g, h \in \frac{\mathcal{B}}{\mathcal{B} \cap \mathcal{I}}$  and lifts  $\tilde{f}, \tilde{g}, \tilde{h}$  and  $\widetilde{\{g, h\}}$  to elements of  $\mathcal{B}$ . Since  $\{g, h\}$  and  $\{\tilde{g}, \tilde{h}\}$  are lifts of the same element, using again (11) we see that their difference  $\Delta$  lies in  $\mathcal{I} \cap [\mathcal{D} \cap (\mathcal{I} + \mathcal{B})] = \mathcal{I} \cap \mathcal{D}$ . Hence

$$\{f, \{g, h\}\} = \{\tilde{f}, \widetilde{\{g, h\}}\} \text{ mod } \mathcal{I} = \{\tilde{f}, \{\tilde{g}, \tilde{h}\}\} + \{\tilde{f}, \Delta\} \text{ mod } \mathcal{I}$$

where  $\{\tilde{f}, \Delta\}$  lies in  $\mathcal{I}$  by (12). Taking the cyclic sum over  $f, g, h$  we see that the Jacobi identity on  $\frac{\mathcal{B}}{\mathcal{B} \cap \mathcal{I}}$  follows from the one on  $\mathcal{P}$ .  $\square$

**Lemma 16** *Theorem 14 follows from Prop. 15 setting  $(\mathcal{P}, \{\bullet, \bullet\}) = (C^\infty(M), \{\bullet, \bullet\}_M)$ ,  $\mathcal{D} = C^\infty(M)_{D|_C}$ ,  $\mathcal{B} = C_E^\infty(M) \cap C_D^\infty(M)$  and  $\mathcal{I} = \{f \in C^\infty(M) : f|_C = 0\}$ . Here we use the notation introduced for  $C_E^\infty(M)$  at the beginning of Section 5.*

**Proof** Condition (12) is satisfied because of requirement (9). Now we check condition (11) in two steps.

First, using (9) (i.e.  $\sharp E^\circ \subset TA|_C$ ), requirement (10) is equivalent to  $\{\mathcal{B}, \mathcal{B}\}_M \subset \mathcal{D}$ . To see this apply to  $f, g \in \mathcal{B}$  and  $X \in \tilde{\Gamma}(D)$  the identity

$$X\{f, g\}_M = (\mathcal{L}_X \Pi)(df, dg) + \Pi(d(Xf), dg) + \Pi(df, d(Xg)). \quad (13)$$

Second, we have  $\{\mathcal{B}, \mathcal{B}\}_M \subset \mathcal{I} + \mathcal{B}$ . Indeed the bracket of two elements of  $\mathcal{B}$  annihilates  $D|_C$  by the above, so its restriction to  $C$  annihilates  $D|_C \cap TC = F$ . So it suffices to show: any function  $h$  in  $C_F^\infty(C)$  can be extended to a function in  $\mathcal{B}$ . Using fact that locally  $C/F$  embeds naturally into  $A/D$  we can extend  $h$  to a function in  $C_D^\infty(A)$ . Choosing a complement of  $E \cap TA|_C = D|_C$  in  $E$ , we can extend to an element of  $\mathcal{B}$ .

Hence the assumptions of Prop. 15 are satisfied, and therefore  $\frac{\mathcal{B}}{\mathcal{B} \cap \mathcal{I}}$  is a Poisson algebra. It is clear that  $\frac{\mathcal{B}}{\mathcal{B} \cap \mathcal{I}} \subset C_F^\infty(C)$ . Equality holds because, as shown above, any function in  $C_F^\infty(C)$  can be extended to a function in  $\mathcal{B}$ .  $\square$

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