Denseness of ergodicity for a class of partially hyperbolic volume-preserving flows

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January 25, 2007

Abstract

Let $\mathcal{P}$ be the set of C^1 partially hyperbolic volume-preserving flows with one dimensional central direction endowed with the C^1-flow topology. We prove that any $X \in \mathcal{P}$ can be approximated by an ergodic C^2 volume-preserving flow. As a consequence ergodicity is dense in $\mathcal{P}$.

MSC 2000: primary 37D30, 37D25; secondary 37A99.
keywords: Dominated splitting; Partial hyperbolicity; Volume-preserving flows; Lyapunov exponents; Stable ergodicity.

1 Introduction

To find the foundations of ergodic theory we must go back to the nineteen century and to the remarkable work of L. Boltzmann. In the context of the dynamic theory of gases he formulated a principle fundamental in statistical physics - *the ergodic hypothesis*. In roughly terms this principle says that *time averages* equal *space averages* at least for typical points.

*Supported by FCT-FSE, SFRH/BPD/20890/2004.
†Partially supported by FCT-POCTI/MAT/61237/2004.

This principle can be formalized by saying that the μ-invariant flow $\varphi_t : M \to M$ must satisfy the following equality

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t \Psi(\varphi_s(x))ds = \int_M \Psi(x)d\mu(x),$$

for μ-a.e. $x \in M$ and any continuous observable $\Psi : M \to \mathbb{R}$. Another equivalent definition of ergodicity says that any φ_t-invariant set, for all t, must have zero or full μ-measure.

A central question is to decide if a given system (flow or diffeomorphism) is ergodic and, even more, if the system is stably ergodic, that is it remains ergodic after small perturbations. It is well known that there are examples of open sets of systems such that

- all the elements of the open set are ergodic (Anosov [1], examples with hyperbolic behavior),
- there are no ergodic systems in the open set (KAM theory, see for instance [16], examples with elliptical behavior).

These examples are far from a complete description of the situation so it is relevant to obtain properties of systems that assure ergodicity or stable ergodicity. In this context *accessibility* and *partial hyperbolicity* have played an important role in the development of this theory.

Based in results related with this central question Pugh-Shub conjectured that, in broad terms, partial hyperbolicity should guarantee denseness of stable ergodicity among conservative systems ([15]). For related results we refer the reader to the survey [11] and to the book [6] and the references there in. We also mention the recent and remarkable results of Burns and Wilkinson [9] and of F. Rodriguez-Hertz, M. Rodriguez-Hertz and Ures [12].

Settling the Pugh and Shub conjecture Bonatti, Matheus, Viana and Wilkinson ([4]) proved that there exists an C^1 open and dense subset, $\mathcal{U}$, of the partially hyperbolic and volume-preserving diffeomorphisms with one dimensional central bundle, such that the C^2 diffeomorphisms of $\mathcal{U}$ are ergodic. As it is unknown whether the C^2 diffeomorphisms are C^1-dense in the set of C^1 volume-preserving diffeomorphisms, their result does not allow to deduce denseness of ergodic systems among the C^1 partially hyperbolic and volume-preserving diffeomorphisms with one dimensional central bundle.

In this paper we obtain the counterpart of their result for the continuous-time setting, considering the space of C^1 partially hyperbolic divergence-free vector fields, $\mathcal{PH}^1_\mu(M)$ (where μ denotes the Lebesgue measure), endowed with a natural topology associated to the flows but weaker than the usual Whitney topology. In this context, combining our result with a theorem of Zuppa ([17]), we obtain that ergodic vector fields are C^1-dense.

Let us now state the fundamental result of this article.

Theorem 1 *Let $X \in \mathcal{PH}^1_\mu(M)$ be a vector field with one dimensional central direction and $\mathcal{V}$ be an arbitrary neighborhood of X in the C^1-flow topology. There exists an C^1 open set $\mathcal{U} \subset \mathcal{V}$ such that any C^2 vector field $Z \in \mathcal{U} \cap \mathcal{PH}^1_\mu(M)$ is ergodic.*

We point out that, if we consider the usual C^1 Whitney topology, by a result of Zuppa ([17]) the subset of C^2 divergence-free vector fields is C^1-dense in the space of C^1 divergence-free vector fields. As the C^1-flow topology is weaker than the Whitney topology (see Section 2 for details) this subset is also dense for the C^1-flow topology. In particular $\mathcal{U} \cap \mathcal{PH}^2_\mu(M)$ is nonempty, where $\mathcal{U}$ is the set given by Theorem 1.

This Theorem has a global formulation. In fact define

$$\mathcal{A} = \bigcup_{X\in\mathcal{P}} \left(\bigcup_{\mathcal{V}\in\mathcal{N}_X} \mathcal{U}(\mathcal{V}) \right),$$

where $\mathcal{N}_X$ denotes the set of all neighbourhoods of X, $\mathcal{P}$ is the subset of vector fields of $\mathcal{PH}^1_\mu(M)$ having unidimensional central direction, and $\mathcal{U}(\mathcal{V})$ is the open set given by the theorem applied to the pair $X, \mathcal{U}$. The set $\mathcal{A}\cap\mathcal{P}$ is open and dense in $\mathcal{P}$ and if $Z \in \mathcal{A}\cap\mathcal{P}$ is of class C^2 then Z is ergodic. In particular ergodicity is dense in $\mathcal{P}$ (for the C^1-flow topology).

One of the steps of the proof of the theorem consists in a continuous-time version of a theorem by Dolgopyat and Wilkinson ([10]). This version uses a perturbation scheme made in the flow context and not in the vector field setting as it could be expected. In this scenario it is natural to consider that vector fields are close if their flows are close, that is to use the flow topology.

As a consequence of the continuous-time version of [10] we obtain the following corollary.

Corollary 1.1 *There exists an open and dense subset $\mathcal{T} \subset \mathcal{PH}^1_\mu(M)$ such that X is transitive, for all $X \in \mathcal{T}$.*

This paper is organized as follows. In Section 2 we introduce some definitions and results. In Section 3 we obtain the main theorem as a consequence of another three theorems. In Sections 4, 5 and 6 we explain how these three results are obtained. Corollary 1.1 is proved in Section 5.

2 Preliminaries and basic results

Let M be a compact, connected and boundaryless smooth Riemannian manifold and let μ denote the Lebesgue measure induced by a fixed volume form on M. Let $r \in \mathbb{N} \cup \{+\infty\}$; a C^r *flow* is a C^r map $\varphi : \mathbb{R} \times M \to M$ such that the map $t \mapsto \varphi_t = \varphi(t, \cdot) \in Diff^r(M)$ is a group homomorphism, where $Diff^r(M)$ denotes the space of C^r diffeomorphisms of M. Let $Diff^r_\mu(M)$ denote the space C^r diffeomorphisms that leave μ invariant. We say that the flow φ is volume-preserving if $\varphi_t \in Diff^r_\mu(M)$, for all $t \in \mathbb{R}$. In the space of C^r flows, $\mathcal{F}^r(M)$, we consider the following topology, τ^r: given $t_0 < t_1$ consider the map

$$\begin{array}{rccc} \psi_{t_0,t_1}: & \mathcal{F}^r(M) & \longrightarrow & C^r([t_0,t_1] \times M, M) \\ & \varphi & \longmapsto & \varphi|_{[t_0,t_1]\times M}. \end{array}$$

The topology τ^r is defined as the smallest topology that makes ψ_{t_0,t_1} continuous, for all $t_0 < t_1$.

It is well known that $(\mathcal{F}^r(M), \tau^r)$ is a Baire space (see [13]).

There exists a natural correspondence between flows and vector fields. Clearly given a C^r vector field $X : M \to TM$ the solution of the equation $x' = X(x)$ gives rise to a C^r flow, φ^X; by the other side given a C^r flow we can define a C^{r-1} vector field by $X^\varphi(x) = \frac{d\varphi_t(x)}{dt}|_{t=0}$. Observe that, by Liouville formula, a (C^2) flow φ is volume-preserving if and only if the corresponding vector field, X^φ, is divergence-free.

Let $\mathfrak{X}^r(M)$ denote the space of C^r vector fields. If we consider the usual C^r (Whitney) topology on this space then map

$$\begin{array}{rccc} \iota: & \mathfrak{X}^r(M) & \longrightarrow & (\mathcal{F}^r(M), \tau^r) \\ & X & \longmapsto & \varphi^X \end{array}$$

is continuous, that is given $X \in \mathfrak{X}^r(M)$, $t_0, t_1 \in \mathbb{R}$ with $t_0 < t_1$ and $\epsilon > 0$ there exists $\delta > 0$ such that if Y is δ-C^r close to X then the maps $\varphi^X|_{[t_0,t_1]\times M}$ and $\varphi^Y|_{[t_0,t_1]\times M}$ are ϵ-C^r close. We note that the map ι is not surjective, unless $r = +\infty$.

In the sequel we consider the space $\mathfrak{X}^1(M)$ endowed with the initial topology induced by the map ι and by the topology τ^1 , which we call *C^1-flow topology* and denote also by τ^1. Observe that τ^1 is weaker than the usual C^1 topology on space of C^1 vector fields; thus τ^1-openness implies C^1-openness and C^1-denseness implies τ^1-denseness.

Given a vector field X we denote by $Sing(X)$ the set of *singularities* of X, say the points $x \in M$ such that $X(x) = \vec{0}$. Let $R := M \setminus Sing(X)$ be the set of *regular* points. Given $x \in R$ we consider its normal bundle $N_x = X(x)^\perp \subset T_xM$ and define the associated *linear Poincaré flow* by $P_t^X(x) := \Pi_{\varphi_t^X(x)} \circ D\varphi_t^X(x)$ where $\Pi_{\varphi_t^X(x)} : T_{\varphi_t^X(x)}M \to N_{\varphi_t^X(x)}$ is the projection along the direction of $X(\varphi_t^X(x))$. An P_t^X-invariant splitting $N = N^1 \oplus ... \oplus N^k$ is called a *ℓ-dominated splitting* for the linear Poincaré flow if there exists $\ell \in \mathbb{N}$ such that, for all $x \in M$ and $0 \leq i < j \leq k$, we have:

$$\frac{\|P_\ell^X(x)|_{N_x^j}\|}{\mathfrak{m}(P_\ell^X(x)|_{N_x^i})} \leq \frac{1}{2},$$

where $\mathfrak{m}(\cdot)$ denotes the co-norm of an operator, that is $\mathfrak{m}(A) = \|A^{-1}\|^{-1}$. We say that the subbundle N_i is *hyperbolic* if there exists $k \in \mathbb{N}$ such that either $\|(P_k^X(x) \cdot u)^{-1}\| \leq 1/2$ (expanding), for all $x \in M$ and any unit vector $u \in N_i(x)$, or $\|P_k^X(x) \cdot u\| \leq 1/2$ (contracting), for all $x \in M$ and any unit vector $u \in N_i(x)$.

Given a vector field X let $\Lambda \subseteq M \setminus Sing(X)$ be an φ_t^X-invariant set. We say that X is (uniformly) *partially hyperbolic for the linear Poincaré flow on* Λ if there exists an P_t^X-invariant dominated splitting $N = N^u \oplus N^c \oplus N^s$ in Λ such that N^u is hyperbolic expanding and N^s is hyperbolic contracting; moreover these two subbundles are not trivial.

By the other hand X is (uniformly) *partially hyperbolic on* Λ if there exists a $D\varphi_t^X$- invariant and a continuous splitting

$$T_xM = E_x^u \oplus E_x^c \oplus \mathbb{R}X(x) \oplus E_x^s,$$

being each subbundle of constant dimension with E_x^s and E_x^u nontrivial, and there exists $\ell \in \mathbb{N}$ such that for all $x \in \Lambda$ one has

- (domination) $\frac{\|D\varphi_\ell^X(x)|_{E_x^c}\|}{\mathfrak{m}(D\varphi_\ell^X(x)|_{E_x^u})} \leq \frac{1}{2}$ and $\frac{\|D\varphi_\ell^X(x)|_{E_x^s}\|}{\mathfrak{m}(D\varphi_\ell^X(x)|_{E_x^c})} \leq \frac{1}{2}$,
- (hyperbolicity) $\|(D\varphi_\ell^X(x) \cdot u)^{-1}\| \leq 1/2$ (expanding) for any unit vector $u \in E_x^u$, and $\|D\varphi_\ell^X(x) \cdot v\| \leq 1/2$ (contracting) for any unit vector $v \in E_x^s$.

We say that X is *partially hyperbolic*, respectively *partially hyperbolic for the linear Poincaré flow*, if X partially hyperbolic on M, respectively partially hyperbolic for the linear Poincaré flow on M. Note that in this case the vector field X does not have singularities. If X is partially hyperbolic on Λ then the diffeomorphism φ_t^X is *partially hyperbolic* on Λ, for all $t \neq 0$ (for the definition of partial hyperbolicity in the diffeomorphisms context see, for example, [6]).

We also observe that, when Λ is compact, the partial hyperbolicity of X on Λ implies the partial hyperbolicity for the linear Poincaré flow of X on Λ. This fact follows easily from the continuity of the splitting and compactness of the set Λ, which guarantee angles between E_x^σ and $\mathbb{R}X(x)$ bounded away from zero, thus allowing to define N_x^σ as the orthogonal projection of E_x^σ onto $\mathbb{R}X(x)^\perp$, $\sigma = u, c, s$.

We denote by $\mathcal{PH}_\mu^k(M)$ the space of partially hyperbolic C^k divergence-free vector fields defined on M, $k \in \mathbb{N}$. This space is an open subset of the space of the C^k divergence-free vector fields for both topologies we considered above. Also the condition of the central subbundle to have dimension equal to one is an open condition (in both topologies).

3 Proof of Theorem 1

The proof of the Theorem 1 follows directly the strategy used by Bonatti, Matheus, Wilkinson and Viana ([4]) to obtain C^1 denseness of ergodicity for C^2 partially hyperbolic and conservative diffeomorphisms having one dimensional central direction. This strategy is based in three previous theorems of Bonatti and Baraviera ([2]), followed by [3], of Dolgopyat and Wilkinson ([10]), and of Burns, Dolgopyat and Pesin ([8]), being the last two adapted to the flow setting.

In this section we present the flow formulation of these results and then deduce the main Theorem. If the reader is not familiar with some notions involved we suggest the previous reading of the Sections 4, 5 and 6.

In the stable ergodic context the first result allows us to remove zero central Lyapunov exponents for flows for $X \in \mathcal{PH}_\mu^1(M)$ and with one dimensional central direction.

Theorem 3.1 *Let $X \in \mathcal{PH}_\mu^1(M)$ be a vector field with one dimensional central direction. Then, for every $\epsilon > 0$, there exists $Y \in \mathcal{PH}_\mu^2(M)$ ϵ-C^1*

close to X, such that

$$\int_M \log \|D\varphi_1^X|_{E_x^c}\| d\mu(x) \neq 0.$$

The second result shows that accessibility is a C^1-open and dense property on the space of conservative partially hyperbolic flows equipped with the C^1-flow topology.

Theorem 3.2 *Let $Y \in \mathcal{PH}_\mu^1(M)$ be a vector field and $\mathcal{W}$ be an arbitrary neighborhood of Y in the C^1-flow topology. There exists an open subset $\mathcal{U}$ of $\mathcal{W}$ such that if $Z \in \mathcal{U} \cap \mathcal{PH}_\mu^1(M)$ then Z has the accessibility property.*

Finally, the third result allows us to obtain ergodicity from the so called *mostly contracting* condition, first introduced by Bonatti and Viana ([5]), and the accessibility property.

Theorem 3.3 *Let $Z \in \mathcal{PH}_\mu^2(M)$. Assume that Z has the accessibility property and that*

$$\int_M \log \|D\varphi_1^Z|_{E_x^c}\| d\mu(x) < 0.$$

Then Z is ergodic.

Let us now explain how one gets the main theorem from the three previous theorems.

Proof: Fix a vector field $X \in \mathcal{PH}_\mu^1(M)$ with unidimensional central direction and choose an arbitrary neighborhood of X in the flow topology, $\mathcal{V}$. Observe that if we take $\mathcal{V}$ small then every $Z \in \mathcal{V}$ is also partially hyperbolic with one dimensional central direction. As $\mathcal{V}$ is also open for the usual C^1-topology, Theorem 3.1 applied to X guarantees that there exists $Y \in \mathcal{V} \cap \mathcal{PH}_\mu^2(M)$ such that

$$I(Y) := \int_M \log \|D\varphi_1^Y|_{E_x^c}\| d\mu(x) \neq 0.$$

We assume that $I(Y) < 0$; otherwise we consider the vector field $-Y$ instead of Y. As the map

$$Z \in \mathcal{PH}_\mu^1(M) \mapsto \int_M \log |det(D\varphi_1^Z|_{E_x^c})| d\mu(x) = I(Z)$$

is continuous for the C^1-flow topology, we can fix an open subset $\mathcal{W}$ such that $Y \in \mathcal{W} \subset \mathcal{V}$ and $I(Z) < 0$, for every $Z \in \mathcal{W} \cap \mathcal{PH}^1_\mu(M)$.

Applying Theorem 3.2 to the pair Y and $\mathcal{W}$ we get an open set $\mathcal{U} \subset \mathcal{W}$ such that every $Z \in \mathcal{U}$ has the accessibility property. Moreover, for $Z \in \mathcal{U} \cap \mathcal{PH}^2_\mu(M)$, we have that $I(Z) < 0$; hence, by Theorem 3.3, Z is ergodic, which ends the proof. $\square$

In Section 4 we prove Theorem 3.1. In Section 5 we explain how to adapt the proof of [10] in order to get Theorem 3.2. Finally, in Section 6 we deduce Theorem 3.3.

4 Proof of Theorem 3.1

In this section we derive Theorem 3.1. In [3], transposing to the vector field scenario a previous result of Baraviera and Bonatti ([2]), we proved that a partially hyperbolic and stably ergodic divergence-free C^1 vector field X can be C^1-perturbed in order to obtain a C^2 vector field whose sum of the central Lyapunov exponents is nonzero. The hypothesis of stably ergodicity was only used to get that the sum of the central Lyapunov exponents is equal to

$$\int_M \log |\det P_1^X|_{N_x^c}| d\mu(x)$$

and then we proved that this integral becomes nonzero after a particular perturbation; hence, without the stable ergodicity assumption what we prove in fact is that

$$\int_M \log |\det P_1^Y|_{N_x^c}| d\mu(x) \neq 0,$$

for a C^2 vector field Y C^1-arbitrary close to X.

The next lemma jointly with Theorem 1 of [3] ends the proof of Theorem 3.1.

Lemma 4.1 *Let $Y \in \mathcal{PH}^1_\mu(M)$ be a vector field with one dimensional central direction. Then one has*

$$\int_M \log |det P_1^Y|_{N_x^c}| d\mu(x) = \int_M \log \|D\varphi_1^Y|_{E_x^c}\| d\mu(x).$$

Proof: As the bundle N_x^c is unidimensional and using the definition of the linear Poincaré flow, we choose an unit vector $v \in N_x^c$ and we get

$$\begin{aligned}\int_M \log|\det P_1^Y|_{N_x^c}|d\mu(x) &= \int_M \log\|P_1^Y(v)\|d\mu(x) = \\ &= \int_M \log\|\Pi_{\varphi_1^Y(x)} \circ D\varphi_1^Y(v)\|d\mu(x).\end{aligned}$$

Now, we write $v = \alpha_x v^c + v^Y$, where v^c is an unit vector of the unidimensional space E_x^c, $v^Y \in \mathbb{R}Y(x)$ and $\alpha \in \mathbb{R}$ is given by $\cos(\gamma_x) = \frac{1}{\alpha_x}$, where $\gamma_x = \measuredangle(E_x^c, N_x^c)$ (partial hyperbolicity implies that this angle is always less than $\frac{\pi}{2}$). Therefore

$$\begin{aligned}&\int_M \log\|\Pi_{\varphi_1^Y(x)} \circ D\varphi_1^Y(v)\|d\mu(x) = \\ &= \int_M \Big(\log(|\alpha_x|) + \log\|\Pi_{\varphi_1^Y(x)} \circ D\varphi_1^Y(v^c)\|\Big)\, d\mu(x) = \\ &= \int_M \left(\log(|\alpha_x|) + \log\|\frac{1}{\alpha_{\varphi_1^Y(x)}} D\varphi_1^Y(v^c)\|\right) d\mu(x) = \\ &= \int_M \Big(\log(|\alpha_x|) - \log|\alpha_{\varphi_1^Y(x)}|\Big)\, d\mu(x) + \int_M \log\|D\varphi_1^Y|_{E_x^c}\|d\mu(x) = \\ &= \int_M \log\|D\varphi_1^Y|_{E_x^c}\|d\mu(x),\end{aligned}$$

where the last equality follows directly from the fact that Y is divergence-free. Thus $|\det(D\varphi_1^Y)| = 1$, which ends the proof of the lemma. $\square$

We note that the one dimensional assumption implies also that

$$\int_M \log\|D\varphi_1^Y|_{E_x^c}\|d\mu(x) = \int_M \log|\det D\varphi_1^Y|_{E_x^c}|d\mu(x),$$

5 Denseness of accessibility on $\mathcal{PH}_\mu^1(M)$

Let $X \in \mathcal{PH}_\mu^1(M)$ be a vector field such that both hyperbolic subbundles are nontrivial; we say that the vector field X has *the accessibility property* if for any p and q in M there exists a C^1 path from p to q whose tangent vector belongs to $E^s \cup E^u$ and vanishes at most finitely many times. This path is called an *us-path* and consists of a finite number of local stable and unstable manifolds, called *legs*. We observe that this definition just means that the diffeomorphism φ_t^X has the accessibility property in the usual sense (see [10]), for all $t \in \mathbb{R} \setminus \{0\}$; also it is easy to verify that if φ_t^X has the accessibility property for some $t \in \mathbb{R} \setminus \{0\}$ then X has the accessibility property. Given a subset $A \subset M$ we say that X is *accessible on* A if for any p and q in A there exists an us-path connecting p to q; X is *accessible modulo* A if for any $p, q \in M$ there exists a sequence of sets $A_1, A_2, ...A_k \subseteq A$ such that there are us-paths from p to a point of A_1, from a point of A_i to A_{i+1}, $1 \leq i \leq k-1$ and, finally, from a point of A_k to q. A *central disk* D centered at $p \in M$ is an embedded disk of $\mathbb{R}^k$, $k = \dim E^c$, such that $T_pD = E^c$ and D is transverse to $E^s \oplus E^u \oplus \mathbb{R}X$.

The proof, given by Dolgopyat and Wilkinson, of denseness of accessibility for partially hyperbolic diffeomorphisms consists in two fundamental steps. First they prove accessibility modulo central disks ([10], Lemma 1.2) and its persistence under small perturbations ([10], Lemma 1.3). Second they establish accessibility on central disks after an adequate perturbation ([10], Lemma 1.1). This lemma is the crucial part of the proof and for a sketch of it we refer to [6] or to [14].

We first observe that Lemmas 1.2 and 1.3 of [10] do not involve any perturbation of the initial diffeomorphism and it is straightforward to verify that they are true for φ_t^X, being X a vector field in $\mathcal{PH}^1_\mu(M)$ and t fixed.

Therefore to transpose their theorem to the continuous-time setting (Theorem 3.2) our strategy could be to begin with a vector field X, use the main Theorem of [10] to approximate its time-one flow by a volume-preserving diffeomorphism, g, having the accessibility property and then return to the vector field setting. However, in general the map g does not embed in a C^1 flow and so this approach does not work directly.

To bypass this difficulty we observe that Lemma 1.1 is proved considering a suitable perturbation of the diffeomorphism f of the form $g = \psi \circ f$, where ψ is a C^∞ volume preserving diffeomorphism C^1 close to the identity. So, we modify the Dolgopyat-Wilkinson's proof just by considering a perturbation of the form $g = \psi \circ \varphi_1^X \circ \psi^{-1}$, being ψ obtained exactly like in their proof, and adapting their bundle perturbation lemma ([10], Lemma 3.4) to this special perturbation. Finally, as $\psi \circ \varphi_t^X \circ \psi^{-1}$

is a flow and g is its time-one map, we consider the vector field

$$Y(p) = \frac{d}{dt}\left(\psi \circ \varphi_t^X \circ \psi^{-1}(p)\right)|_{t=0} = D\psi_{\psi^{-1}(p)}(X(\psi^{-1}(p))),$$

which is C^1 close to X in the flow topology (but not necessarily C^1 close to X in the Whitney topology).

We end this section by proving Corollary 1.1.

Proof: Let us first recall that a vector field X is *transitive* if there exists $x \in M$ whose forward orbit by the flow associated to X is dense in M.

Fix a $X \in \mathcal{PH}^1_\mu(M)$ and let $\mathcal{W}$ be an arbitrary neigbourhood of X. Let $\mathcal{U}_\mathcal{W}$ be the open set given by Theorem 3.2. If $Y \in \mathcal{U}_\mathcal{W} \cap \mathcal{PH}^1_\mu(M)$ then Y has the accessibility property and, as it is divergence-free, its nonwandering set is equal to M. Therefore we can apply a result of Brin (a version for flows of Theorem 1.2 of [7], see the remark immediately after this Theorem) to conclude that Y is transitive. Finally, we take

$$\mathcal{T} = \bigcup_{X \in \mathcal{PH}^1_\mu(M)} \left(\bigcup_{\mathcal{W} \in \mathcal{N}(X)} \mathcal{U}_\mathcal{W} \right),$$

where $\mathcal{N}(X)$ denotes the set of all neighbourhoods of X. $\square$

6 From the mostly contracting and the accessibility conditions to stable ergodicity

Theorem 3.3 is just the flow formulation of a theorem of Burns, Dolgopyat and Pesin (Theorem 4 of [8]). In this section we show how it can be easily deduced from their result.

A μ-invariant vector field $X \in \mathfrak{X}^r(M)$ is said to be *ergodic* with respect to the measure μ if for any measurable set $A \subseteq M$ such that $\varphi_t^X(A) = A$, for all $t \in \mathbb{R}$, one has either $\mu(A) = 1$ or else $\mu(A) = 0$. The vector field X is *stably ergodic* if there exists an open neighbourhood of X, say $\mathcal{U}(X)$, in the C^1-flow topology (τ^1) such that all the vector fields of $\mathcal{U}(X)$ are ergodic. Clearly there are analogous definitions for diffeomorphisms (considering the usual C^1 topology). It is clear that given $X \in \mathfrak{X}^r(M)$ if there exists $t_0 \in \mathbb{R}\backslash\{0\}$ such that the diffeomorphism $\varphi_{t_0}^X$ is ergodic then X is ergodic. Also, as the map

$$
\begin{array}{cccc}
\mathcal{H}: & (\mathfrak{X}^1(M),\tau^1) & \longrightarrow & (Diff^1(M),C^1) \\
 & X & \longmapsto & \varphi_1^X
\end{array}
$$

is continuous, it follows that if φ_1^X is stably ergodic then X is stably ergodic.

Let us fix $Z \in \mathcal{PH}_\mu^2(M)$ such that Z has the accessibility property and that

$$\int_M \log \|D\varphi_1^Z|_{E_x^c}\| d\mu(x) < 0.$$

It follows that $f = \varphi_1^Z$ is a C^2 partially hyperbolic diffeomorphism, the Lebesgue measure μ is f-invariant, f has the accessibility property and satisfies

$$\int_M \log \|Df|_{E_x^c}\| d\mu(x) < 0.$$

It is well known that this last inequality implies the mostly contracting condition that is the existence of a positive measure set whose points have negative central Lyapunov exponent (see for example the comments after Lemma 1 of [8]). These properties of f are exactly the hypotheses of Theorem 4 of [8] which says precisely that f is stably ergodic in $(Diff^2(M), C^1)$. From the previous remark it follows that Z is stably ergodic in $(\mathfrak{X}^2(M), \tau^1)$. In particular Z is ergodic which ends the proof of Theorem 3.3.

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