ON THE IRREDUCIBILITY OF PSEUDOVARIETIES OF SEMIGROUPS

J. ALMEIDA AND O. KLÍMA

ABSTRACT. We show that, for every pseudovariety of groups H, the pseudovariety $\bar{\mathsf{H}}$, consisting of all finite semigroups all of whose subgroups lie in H, is irreducible for join and the Mal'cev and semidirect products. The proof involves a Rees matrix construction which motivates the study of iterated Mal'cev products with the pseudovariety of bands. We further provide a strict infinite filtration for $\bar{\mathsf{H}}$ using such iterated Mal'cev products, in which the decidability of each level depends only on the decidability of H.

1. INTRODUCTION

Since the establishment of Eilenberg's correspondence between varieties of regular languages and pseudovarieties of semigroups [4], the theory of finite semigroups has evolved mostly in the direction of their classification in pseudovarieties. The most recent account on this topic is [11], which contains a wealth of results, centered on the Krohn-Rhodes complexity theory, but not limited to it. The typical problem, motivated by the origins of this research area, consists in determining whether the membership problem for a given pseudovariety is decidable. The difficulty lies in the fact that, very often, pseudovarieties are given by generators, rather than by characteristic structural properties of their members. The generators are often obtained by applying some natural algebraic construction to members of given pseudovarieties. For instance, the direct and semidirect products of semigroups lead respectively to the join and semidirect product of pseudovarieties, while the existence of a congruence whose idempotent classes lie in a given pseudovariety and whose quotient lies in another pseudovariety leads to the Mal'cev product. The interest in such operators on pseudovarieties is that they allow to decompose, in the pseudovariety sense, complicated finite semigroups in terms of simpler ones. For example, the Krohn-Rhodes decomposition theory concerns building arbitrary finite semigroups from finite simple groups and finite aperiodic semigroups using semidirect products.

Thus, a key ingredient in the theory of pseudovarieties of semigroups is to break them up, when possible, into simpler pseudovarieties using natural operators. There are two ways in which this might be achieved: through a finite decomposition, or through an iterated decomposition, providing a

2010 *Mathematics Subject Classification.* Primary 20M07; Secondary 20M05, 20M35.
Key words and phrases. pseudovariety, profinite semigroup, encoding, Malcev product, semidirect product, join, irreducibility, pseudoword, Krohn-Rhodes complexity.
Corresponding author: J. Almeida.

filtration of the pseudovariety in terms of subpseudovarieties which admit finite decompositions, as in the Krohn-Rhodes complexity theory.

In this paper, we improve on earlier results of Margolis, Sapir and Weil [7] and Rhodes and Steinberg [10, 11] concerning the pseudovarieties of the form $\bar{\mathsf{H}}$, consisting of all finite semigroups all of whose subgroups lie in a given pseudovariety of groups H. In [7], Koryakov's embedding approach [5] is improved to show that $\bar{\mathsf{H}}$ is finitely indecomposable in terms of Mal'cev and semidirect products and joins, provided H is closed under semidirect product. In [10], a stronger form of join indecomposability (called finite join irreducibility) is established for $\bar{\mathsf{H}}$ in case H contains some non-nilpotent group, the technique being the construction of so-called Kovács-Newman semigroups. In both works, it is proposed as an open problem to determine whether every pseudovariety of the form $\bar{\mathsf{H}}$ is join indecomposable. This also appears in [11] as Problem 47.

We give an affirmative solution to those problems. Our approach is similar to that of [7] but uses a Rees matrix-like construction to obtain an improved embedding of free pro-$\bar{\mathsf{H}}$ semigroups, which does not require that H be closed under semidirect product. This allows us to use the arguments of [7] to show that $\bar{\mathsf{H}}$ is indecomposable, in the stronger sense, with respect to both join and the Mal'cev and semidirect products. In fact, we show that every pseudovariety that is closed under our construction is join indecomposable in the weaker sense. The construction leads to a new operator at the level of pseudovarieties, which we call the *bullet*. We have not found any pseudovarieties other than those of the form $\bar{\mathsf{H}}$ which are closed under bullet. We do give two types of sufficient conditions for a pseudovariety to be closed under bullet, involving only Mal'cev products or both Mal'cev and semidirect products. For the sufficient condition involving only Mal'cev products, we are able to show that the pseudovarieties of the form $\bar{\mathsf{H}}$ are indeed the only ones that satisfy it, which, as a sub-product, provides a decidable strict filtration for $\bar{\mathsf{H}}$ in case H is decidable.

For the special case of the pseudovariety G of all finite groups, we show that there is no proper subpseudovariety of the pseudovariety of all finite semigroups which is closed under the bullet operator and contains G. In particular, the bullet operator is powerful enough to increase the Krohn-Rhodes complexity, provided we are allowed to use arbitrarily general finite groups.

2. Preliminaries

We assume familiarity with the basic theory of pseudovarieties of semigroups, including the role played by free profinite semigroups, in particular through Reiterman's theorem [9], defining pseudovarieties by pseudoidentities. The reader is referred to [1, 2, 11] for a few alternative introductions to this subject.

For a pseudovariety V, $\overline{\Omega}_A\mathsf{V}$ and $\overline{\Omega}_n\mathsf{V}$ denote the pro-V semigroups freely generated respectively by the set A and a set of cardinality n. Elements of such semigroups will be called *pseudowords*.

We adopt the usual conventions for semigroup pseudoidentities such as that $u = 1$ and $u = 0$ are, respectively, abbreviations of the pseudoidentities

$ux = x = xu$ and $ux = u = xu$, where x is a variable that does not occur in u.

For the reader's convenience, the following is a catalog of pseudovarieties of finite semigroups that play a role in this paper. For each of them, besides a, sometimes incomplete, verbal description, a well-known definition in terms of pseudoidentities is also provided.

- I: trivial ($[\![x = y]\!]$).
- Sl: semilattices ($[\![x^2 = x, xy = yx]\!]$).
- LZ: left zero ($[\![xy = x]\!]$).
- RZ: right zero ($[\![xy = y]\!]$).
- RB: rectangular bands ($[\![xyx = x]\!]$).
- MD_1: right regular bands ($[\![x^2 = x, xyx = yx]\!]$).
- $\mathsf{B}_{m,n}$: Burnside condition ($[\![x^{m+n} = x^m]\!]$).
- $\mathsf{B}_{\infty,n}$: same as $\mathsf{B}_{\omega,n}$ ($[\![x^{\omega+n} = x^\omega]\!]$).
- B: bands ($[\![x^2 = x]\!] = \mathsf{B}_{1,1}$).
- A: aperiodic ($[\![x^{\omega+1} = x^\omega]\!] = \mathsf{B}_{\omega,1}$).
- G: groups ($[\![x^\omega = 1]\!] = \mathsf{B}_{0,\omega}$).
- S: all ($[\![x = x]\!] = \mathsf{B}_{\omega,\omega}$).
- N_2: null (nilpotency index at most 2, $[\![xy = 0]\!]$).
- N_n: nilpotency index n ($[\![x_1 \cdots x_n = 0]\!]$).
- N: nilpotent ($[\![x^\omega = 0]\!]$).
- D: definite ($[\![xy^\omega = y^\omega]\!]$).
- K_n: reverse definite of index n ($[\![x_1 \cdots x_n y = x_1 \cdots x_n]\!]$).
- K: reverse definite ($[\![x^\omega y = x^\omega]\!]$).
- LI: locally trivial ($[\![x^\omega y x^\omega = x^\omega]\!]$).
- Ab: Abelian groups ($[\![x^\omega = 1, xy = yx]\!]$).
- J: $\mathcal{J}$-trivial ($[\![(xy)^\omega x = (xy)^\omega = y(xy)^\omega]\!]$).
- Com: commutative ($[\![xy = yx]\!]$).
- IE: only one idempotent ($[\![x^\omega = y^\omega]\!]$).

Let V be a pseudovariety of semigroups. A homomorphism $\varphi : S \to T$ between two finite semigroups S and T is said to be a V-*homomorphism* if $\varphi^{-1}(e) \in \mathsf{V}$ for every idempotent $e \in T$.

We also need the following operators on pseudovarieties of semigroups:

- *Mal'cev product*: $\mathsf{V} \circledcirc \mathsf{W}$ is the pseudovariety generated by all finite semigroups S for which there is a V-homomorphism $\varphi : S \to T$ into a semigroup $T \in \mathsf{W}$;
- *semidirect product*: $\mathsf{V} * \mathsf{W}$ is the pseudovariety generated by all semidirect products of the form $S * T$ with $S \in \mathsf{V}$ and $T \in \mathsf{W}$;
- *bar*: for a pseudovariety H (of groups), $\bar{\mathsf{H}}$ is the pseudovariety consisting of all finite semigroups all of whose subgroups belong to H;
- *localization*: for a pseudovariety V, LV consists of all finite semigroups S such that, for every idempotent e in S, the local subsemigroup eSe belongs to V.

It is well known that $\mathsf{V} * \mathsf{W}$ is also generated by the wreath products of the form $S \circ T$ with $S \in \mathsf{V}$ and $T \in \mathsf{W}$, a fact which may be used to deduce that the semidirect product of pseudovarieties is associative. In contrast, the Mal'cev product is not associative, satisfying only the following inclusion, which may be proper:

(1) $$\mathsf{U} \circledast (\mathsf{V} \circledast \mathsf{W}) \subseteq (\mathsf{U} \circledast \mathsf{V}) \circledast \mathsf{W}.$$

For pseudovarieties V and W, denote by $\mathsf{V} \circledast_n \mathsf{W}$ the iterated Mal'cev product on the right with W, defined recursively by

$$\mathsf{V} \circledast_0 \mathsf{W} = \mathsf{V} \text{ and } \mathsf{V} \circledast_{n+1} \mathsf{W} = (\mathsf{V} \circledast_n \mathsf{W}) \circledast \mathsf{W}.$$

We further let $\mathsf{V} \circledast_\infty \mathsf{W} = \bigcup_{n\geq 0}(\mathsf{V} \circledast_n \mathsf{W})$.

The *Basis Theorem* for the Mal'cev product of pseudovarieties [8] states that, if $\mathsf{V} = [\![u_i(x_1,\ldots,x_{n_i}) = v_i(x_1,\ldots,x_{n_i}) : i \in I]\!]$ then $\mathsf{V} \circledast \mathsf{W}$ is defined by the pseudoidentities of the form $u_i(w_1,\ldots,w_{n_i}) = v_i(w_1,\ldots,w_{n_i})$ where the w_j are pseudowords such that W satisfies the pseudoidentities $w_1^2 = w_1 = \cdots = w_{n_i}$ $(i \in I)$.

We say that a pseudovariety V is a *Mal'cev idempotent* if $\mathsf{V} \circledast \mathsf{V} = \mathsf{V}$. There are many Mal'cev idempotents. In particular, it is routine to verify that the following pseudovarieties are Mal'cev idempotents: B, Sl, LZ, RZ, RB, N, K, J, and A. A pseudovariety of groups is a Mal'cev idempotent if and only if it is closed under extensions. The intersection of any nonempty family of Mal'cev idempotents is a Mal'cev idempotent and so, for every pseudovariety, there is a smallest Mal'cev idempotent containing it.

3. A Rees matrix extension construction

Let S and T be semigroups and $f : S^1 \to T^1$ be a function. The set

$$M(S,T,f) = S \uplus S^1 \times T^1 \times S^1$$

is endowed with the multiplication defined by the following formulas for all $s \in S$, $s_i, s_i' \in S^1$, and $t, t' \in T^1$:

$$\begin{aligned} s_1 \cdot s_2 &= s_1 s_2 \\ s \cdot (s_1,t,s_2) &= (ss_1,t,s_2) \\ (s_1,t,s_2) \cdot s &= (s_1,t,s_2 s) \\ (s_1,t,s_2) \cdot (s_1',t',s_2') &= (s_1, tf(s_2 s_1')t', s_2'). \end{aligned}$$

The following lemma contains some preliminary observations about this algebraic structure.

Lemma 3.1. *The set $M(S,T,f)$ is a semigroup for the above operation. All its subgroups are isomorphic to subgroups of either S or T.*

Proof. From the definition, it is clear that S is a subsemigroup and that, its complement, the subset $R = S^1 \times T^1 \times S^1$ constitutes a Rees matrix subsemigroup. Moreover, the formulas indicate that S acts both on the left and on the right of R, respectively by left multiplication on the first component and right multiplication on the third component. Therefore, the two actions commute. Hence, the only case of the associativity law that remains to be considered is

$$\big((s_1,t,s_2)\cdot s\big)\cdot(s_1',t',s_2') = (s_1,t,s_2)\cdot\big(s\cdot(s_1',t',s_2')\big),$$

and it is easily checked that both sides are equal to $(s_1, tf(s_2 s s_1')t', s_2')$.

Let H be a subgroup of $M(S,T,f)$. Since R is an ideal, H must be contained in either S or R, and it suffices to consider the latter case. Let

(s_1, t, s_2) be an idempotent of H. Then $tf(s_2s_1)t = t$ and $e = tf(s_2s_1)$ is an idempotent of T^1. Let G be the maximal subgroup of T^1 containing e. Consider the mapping $\varphi : H \to T^1$ which sends each element (s_1, t', s_2) to $t'f(s_2s_1)$. It is routine to check that φ is an injective homomorphism which takes its values in G. □

Given two pseudovarieties of semigroups U and V, we denote by $\mathsf{U} \bullet \mathsf{V}$ the pseudovariety generated by all semigroups of the form $M(S, T, f)$, with $S \in \mathsf{U}$ and $T \in \mathsf{V}$. For lack of a better name, we call *bullet* this operation on pseudovarieties. We say that V is a *bullet idempotent* if $\mathsf{V} \bullet \mathsf{V} = \mathsf{V}$. The following embedding theorem is the core of our irreducibility results.

Theorem 3.2. *Let V be a pseudovariety which is a bullet idempotent. Let A and B be finite sets and suppose that $\theta : B \to \overline{\Omega}_A\mathsf{V}$ is an injective function. Then the unique continuous homomorphism $\psi : \overline{\Omega}_B\mathsf{V} \to \overline{\Omega}_{A\uplus\{z\}}\mathsf{V}$ such that $\psi(b) = \theta(b)z$ is injective.*

Proof. Arguing by contradiction, let $u, v \in \overline{\Omega}_B\mathsf{V}$ be such that $\psi(u) = \psi(v)$ and $u \neq v$. Let $\tau : \overline{\Omega}_B\mathsf{V} \to T$ be a continuous homomorphism into a semigroup T from V such that $\tau(u) \neq \tau(v)$. Let $\sigma : \overline{\Omega}_A\mathsf{V} \to S$ be a continuous homomorphism into a semigroup S from V such that the mapping $\sigma \circ \theta$ is injective. Let $f : S^1 \to T^1$ be the function defined by $f(\sigma(\theta(b))) = \tau(b)$, and $f(s) = 1$ for all other $s \in S^1$. Since V is a bullet idempotent, there is a unique continuous homomorphism $\varphi : \overline{\Omega}_{A\uplus\{z\}}\mathsf{V} \to M(S, T, f)$ such that $\varphi(a) = \sigma(a)$ for $a \in A$ and $\varphi(z) = (1, 1, 1)$. We claim that the equality

$$\varphi(z\psi(w)) = (1, \tau(w), 1) \tag{2}$$

holds for every $w \in \overline{\Omega}_B\mathsf{V}$. This will complete the proof since it contradicts the initial assumptions $\psi(u) = \psi(v)$ and $\tau(u) \neq \tau(v)$.

Since the mappings φ, ψ and τ are continuous, it suffices to prove (2) in case $w \in B^+$, which we establish by induction on $|w|$. Suppose first that $w = b \in B$. Then we have

$$\begin{aligned}\varphi(z\psi(b)) &= \varphi(z\theta(b)z) = \varphi(z)\varphi(\theta(b))\varphi(z)\\ &= (1, 1, 1)\sigma(\theta(b))(1, 1, 1) = (1, f(\sigma(\theta(b))), 1) = (1, \tau(b), 1).\end{aligned}$$

Suppose next that $b \in B$ and $w \in B^+$ satisfies (2). Then we may compute

$$\begin{aligned}\varphi(z\psi(bw)) &= \varphi(z\theta(b)z\psi(w)) = \varphi(z)\varphi(\theta(b))\varphi(z\psi(w))\\ &= (1, 1, 1)\sigma(\theta(b))(1, \tau(w), 1) = (1, f(\sigma(\theta(b)))\tau(w), 1)\\ &= (1, \tau(b)\tau(w), 1) = (1, \tau(bw), 1),\end{aligned}$$

which completes the induction step and establishes the claim. □

Recall that a pseudovariety V is *monoidal* if it contains the monoid S^1 whenever it contains the semigroup S.

Lemma 3.3. *Every bullet idempotent is monoidal.*

Proof. Let T be an arbitrary semigroup from V and consider the associated semigroup $U = M(\{1\}, T, f)$, where $f : \{1\} \to T^1$ maps 1 to 1. Then $\{1\} \times T^1 \times \{1\}$ is a subsemigroup of U which is isomorphic with T^1. Hence, T^1 belongs to $\mathsf{V} \bullet \mathsf{V}$ and, therefore, also to V. □

We say that a pseudovariety V has *finite index* if it satisfies a pseudoidentity of the form $x^m = x^{m+\omega}$ with m a positive integer. In this case, the smallest such m is called the *index* of V. If there is no such m, then V is said to have *infinite index*.

Lemma 3.4. *Every bullet idempotent contains* LI *and, in particular, it has infinite index.*

Proof. Let V be a bullet idempotent. As a pseudovariety, it must contain the trivial semigroup $I = \{1\}$ and, therefore, also the two-element semilattice $S = M(I, I, f)$. For the constant mapping f with value 0, the subsemigroup of $M(S, S, f)$ given by $S \times \{0\} \times S$ is a 2×2 rectangular band, while the subsemigroup $\{1\} \times S \times \{1\}$ is a two-element null semigroup. Hence V contains $\mathsf{RB} = \mathsf{LZ} \vee \mathsf{RZ}$ and N_2.

Let A be a finite alphabet and let $K_n = \overline{\Omega}_A \mathsf{K}_n$. It is well known that K_n may be represented by the set $A^{\leq n}$ of nonempty words of length at most n, with multiplication given by concatenation followed by truncation to the prefix of length n if the resulting word has length greater than n. By induction on n, we prove that $K_n \in \mathsf{V}$. The case of $n = 1$ follows from the above since $\mathsf{K}_1 = \mathsf{LZ}$. Assuming that $K_n \in \mathsf{V}$, consider the semigroup $U = M(\overline{\Omega}_A \mathsf{N}_2, K_n, f)$, where $f : (\overline{\Omega}_A \mathsf{N}_2)^1 \to \mathsf{K}_n$ maps each free generator $a \in A$ of $\overline{\Omega}_A \mathsf{N}_2$ to the corresponding free generator of K_n. Noting that $\overline{\Omega}_A \mathsf{N}_2 = A \uplus \{0\}$, the set $T = A \times K_n \times \{1\}$ is a subsemigroup of U and the mapping $T \to K_{n+1}$ which sends each triple $(a, w, 1) \in T$ to aw is an onto homomorphism, we deduce that $K_{n+1} \in \mathsf{V}$.

The preceding paragraph entails that K is contained in V. Dually, so is D and, therefore so is $\mathsf{LI} = \mathsf{K} \vee \mathsf{D}$. □

We adopt the same terminology as in [11, Definition 6.1.5] for various irreducibility notions in a lattice. In particular, we say that

- a pseudovariety V is *strictly finite join irreducible* (*sfji*) if $\mathsf{V} = \mathsf{U} \vee \mathsf{W}$ implies $\mathsf{V} = \mathsf{U}$ or $\mathsf{V} = \mathsf{W}$;
- a pseudovariety V is *finite join irreducible* (*fji*) if $\mathsf{V} \subseteq \mathsf{U} \vee \mathsf{W}$ implies $\mathsf{V} \subseteq \mathsf{U}$ or $\mathsf{V} \subseteq \mathsf{W}$.

Theorem 3.5. *Every bullet idempotent is sfji.*

Proof. Let V be a bullet idempotent and let U and W be pseudovarieties of semigroups such that $\mathsf{U} \vee \mathsf{W} = \mathsf{V}$. We claim that V must be contained (and therefore be equal) to at least one of U and W. Otherwise, by Reiterman's theorem, there are pseudoidentities $u_1 = u_2$ and $v_1 = v_2$ such that U satisfies $u_1 = u_2$, W satisfies $v_1 = v_2$, and V fails both pseudoidentities. We may assume that each of the pseudoidentities $u_1 = u_2$ and $v_1 = v_2$ involves the minimum possible number of variables so that, in particular, every pseudoidentity which is obtained from them by identifying variables is valid in V. Without loss of generality, we further assume that the number of variables n involved in $u_1 = u_2$ is at least the number of variables involved in $v_1 = v_2$.

Consider first the case where $n \geq 2$ and let $B = \{x_1, \ldots, x_n\}$ be the set of variables involved in the pseudoidentity $u_1 = u_2$. We may assume that the pseudoidentity $v_1 = v_2$ is written on a disjoint set C of variables and we let $A = B \cup C$. We further consider a new variable z. Since V contains all finite rectangular bands by Lemma 3.4, we deduce that the mapping $\theta : B \to \overline{\Omega}_A \mathsf{V}$ sending x_i to v_i for $i = 1, 2$ and fixing all other x_j is injective. By Theorem 3.2, the unique continuous homomorphism $\psi : \overline{\Omega}_B \mathsf{V} \to \overline{\Omega}_{A\cup\{z\}} \mathsf{V}$ such that $\psi(x_i) = \theta(x_i)z$ is also injective. Since the pseudoidentity $u_1 = u_2$ fails in V, it follows that so does the pseudoidentity

$$u_1(v_1z, v_2z, x_3z, \ldots, x_nz) = u_2(v_1z, v_2z, x_3z, \ldots, x_nz). \tag{3}$$

Note the pseudoidentity (3) is valid in U, being an obvious consequence of $u_1 = u_2$. It is also valid in W since this pseudovariety satisfies $v_1 = v_2$, so that W satisfies (3) if and only if it satisfies the pseudoidentity

$$u_1(v_1z, v_1z, x_3z, \ldots, x_nz) = u_2(v_1z, v_1z, x_3z, \ldots, x_nz),$$

which, by the assumption on the minimality of the number n, is valid in V, whence also in W. Hence $\mathsf{U} \vee \mathsf{W}$ satisfies the pseudoidentity (3), while V does not, which contradicts the assumption that $\mathsf{V} = \mathsf{U} \vee \mathsf{W}$.

It remains to consider the case where $n = 1$. In this case, the pseudoidentity $u_1 = u_2$ is of the form $x^\alpha = x^\beta$ for some unary pseudowords x^α, x^β. Then $x^\alpha y^\beta = x^\beta y^\alpha$ is still a pseudoidentity valid in U. It fails in V since V is monoidal by Lemma 3.3. We may then apply basically the same argument as in the case $n \geq 2$ to the pseudoidentity $x^\alpha y^\beta = x^\beta y^\alpha$ playing the role of $u_1 = u_2$. The only difference in the argument concerns the verification that the pseudoidentity (3) holds in W, which is now trivial since, in the presence of $v_1 = v_2$, (3) is equivalent to $(v_1z)^\alpha(v_1z)^\beta = (v_1z)^\beta(v_1z)^\alpha$. □

In view of Lemma 3.1, Theorem 3.5 applies to the pseudovarieties of the form $\bar{\mathsf{H}}$, which gives an affirmative solution to the first part of [11, Problem 47]. However, we prove in Section 4 that in fact $\bar{\mathsf{H}}$ is fji, which provides an affirmative answer also to the second part of [11, Problem 47].

4. An improved finite join irreducibility result

In this section we prove that every pseudovariety of the form $\bar{\mathsf{H}}$ is irreducible not only for the join (fji) but also enjoys the analogous properties for both the Mal'cev and semidirect products. This improves the main results of [7].

Theorem 4.1. *Let H be a pseudovariety of groups. If $\bar{\mathsf{H}}$ is contained in $\mathsf{V} \circledm \mathsf{W}$, then it is contained in at least one of V and W.*

Proof. Suppose first that $\bar{\mathsf{H}} \subseteq \mathsf{V} \circledm \mathsf{W}$ and that $\bar{\mathsf{H}}$ is contained in neither V nor W. By Reiterman's theorem, there are pseudoidentities $u = v$ and $w = t$ which fail in $\bar{\mathsf{H}}$ but hold respectively in V and W. By [5, Theorem 1], we may assume that $u, v \in \overline{\Omega}_2\mathsf{S}$. Let A be a finite set such that $w, t \in \overline{\Omega}_A\mathsf{S}$ and choose $z \notin A$. Note that the pseudoidentity

$$u\big((wz)^\omega(tz)^\omega, (wz)^\omega(tzwz)^\omega(tz)^\omega\big) = v\big((wz)^\omega(tz)^\omega, (wz)^\omega(tzwz)^\omega(tz)^\omega\big) \tag{4}$$

holds in $\mathsf{V} \circledm \mathsf{W}$. Indeed, this pseudovariety is generated by all finite semigroups S for which there exists a homomorphism $f : S \to T$ into a semigroup from W such that $f^{-1}(e) \in \mathsf{V}$ for every idempotent e from T. For a continuous homomorphism $\varphi : \overline{\Omega}_A\mathsf{S} \to S$, since W satisfies the pseudoidentity $w = t$ and $T \in \mathsf{W}$, $f(\varphi((wz)^\omega(tz)^\omega))$ and $f(\varphi((wz)^\omega(tzwz)^\omega(tz)^\omega))$ are the same idempotent e. Thus, $\varphi((wz)^\omega(tz)^\omega)$ and $\varphi((wz)^\omega(tzwz)^\omega(tz)^\omega)$ are both elements of the subsemigroup $f^{-1}(e)$ of S, which in turn belongs to V. Hence φ maps both sides of the pseudoidentity (4) to the same element of S.[1]

It remains to show that the pseudoidentity (4) fails in $\bar{\mathsf{H}}$. Indeed, by [7, Proposition 3.2], since the pseudovariety $\bar{\mathsf{H}}$ is monoidal and it fails the pseudoidentity $u = v$, it also fails the pseudoidentity

$$u(x^\omega y^\omega, x^\omega(yx)^\omega y^\omega) = v(x^\omega y^\omega, x^\omega(yx)^\omega y^\omega). \tag{5}$$

Let $p : \overline{\Omega}_{\{x,y\}}\mathsf{S} \to \overline{\Omega}_{\{x,y\}}\bar{\mathsf{H}}$ and $q : \overline{\Omega}_{A\cup\{z\}}\mathsf{S} \to \overline{\Omega}_{A\cup\{z\}}\bar{\mathsf{H}}$ be the natural continuous homomorphisms. To apply Theorem 3.2, let $\theta : \{x, y\} \to \overline{\Omega}_A\bar{\mathsf{H}}$ be the function that maps x to $q(w)$ and y to $q(t)$. Note that θ is injective because the pseudoidentity $w = t$ fails in $\bar{\mathsf{H}}$. Let $\psi : \overline{\Omega}_{\{x,y\}}\bar{\mathsf{H}} \to \overline{\Omega}_{A\cup\{z\}}\bar{\mathsf{H}}$ be the resulting injective continuous homomorphism defined in Theorem 3.2. Since, for $s \in \overline{\Omega}_{\{x,y\}}\mathsf{S}$,

$$\psi\Big(p\big(s(x^\omega y^\omega, x^\omega(yx)^\omega y^\omega)\big)\Big) = q\Big(s\big((wz)^\omega(tz)^\omega, (wz)^\omega(tzwz)^\omega(tz)^\omega\big)\Big)$$

and ψ is injective, from the fact that the pseudoidentity (5) fails in $\bar{\mathsf{H}}$ it follows that so does (4). $\square$

From Theorem 4.1, one may adapt the arguments used in [7, proofs of Corollaries 3.3 and 3.4] to deduce the following results. The adaptation consists in dropping the hypothesis that the pseudovariety of groups H is closed under extensions, and noting that in all cases the arguments yield finite irreducibility rather than just strict finite irreducibility. The short proofs are included for the sake of completeness.

Corollary 4.2. *Let* H *be an arbitrary pseudovariety of groups. If* $\bar{\mathsf{H}}$ *is contained in a semidirect product* $\mathsf{V} * \mathsf{W}$*, then it is contained in at least one of the factors* V *and* W*.*

Proof. By [11, Corollary 4.1.32], we have $\mathsf{V} * \mathsf{W} \subseteq \mathsf{LV} ⓜ \mathsf{W}$. From the hypothesis and Theorem 4.1, we deduce that $\bar{\mathsf{H}}$ is contained in at least one of the Mal'cev factors LV and W. In the former case, since $\bar{\mathsf{H}}$ is monoidal, it follows that it is contained in V. $\square$

Corollary 4.3. *If* H *is an arbitrary pseudovariety of groups, then* $\bar{\mathsf{H}}$ *is fji.*

Proof. It suffices to note that $\mathsf{V} \vee \mathsf{W} \subseteq \mathsf{V} * \mathsf{W}$ and apply Corollary 4.2. $\square$

The special case of Corollary 4.3 where H is a pseudovariety of groups containing some non-nilpotent group is part of [11, Corollary 7.4.23], which is based on the construction of so-called *Kovács-Newman semigroups*. Corollary 4.3 solves [11, Problem 47]. The strict version of that problem, as well the Mal'cev and semidirect products counterparts had already been proposed in [7]. From strict finite join irreducibility of $\bar{\mathsf{H}}$, it follows that $\bar{\mathsf{H}}$ contains no maximal proper subpseudovariety. For pseudovarieties of groups H containing Ab, this had previously been proved by Margolis [6].

[1] This is basically the easy part of the proof of the Pin and Weil Basis Theorem for Mal'cev products [8].

5. Further remarks on the bullet operator

Theorem 3.5 motivates a deeper understanding of the bullet operator. In this section, we present a number of results that are meant to shed some light on this operator. We start with a proposition which relates the bullet operator with other operators on pseudovarieties of semigroups.

Proposition 5.1. *Let* V *and* W *be pseudovarieties of semigroups, with* W *monoidal. Then* $\mathsf{V} \bullet \mathsf{W}$ *is contained in each of the following pseudovarieties:*
(1) $(\mathsf{V} \vee ((\mathsf{N}_2 ⓜ \mathsf{W}) ⓜ \mathsf{RB})) ⓜ \mathsf{SI}$;
(2) $(\mathsf{V} \vee \mathsf{LZ} \vee (\mathsf{W} * \mathsf{MD}_1)) ⓜ \mathsf{SI}$.

Proof. Given $S \in \mathsf{V}$, $T \in \mathsf{W}$, and an arbitrary function $f : S^1 \to T^1$, consider the semigroup $M(S,T,f)$. The mapping $\varphi : M(S,T,f) \to \{0,1\}$ that sends all elements of S to 1 and all remaining elements to 0 is a homomorphism into the 2-element semilattice under the usual multiplication. To prove the proposition, it suffices therefore to show that the subsemigroup $R = S^1 \times T^1 \times S^1$ of $M(S,T,f)$ belongs to both the pseudovarieties $(\mathsf{N}_2 ⓜ \mathsf{W}) ⓜ \mathsf{RB}$ and $\mathsf{LZ} \vee (\mathsf{W} * \mathsf{MD}_1)$.

The mapping $\psi : R \to S^1 \times S^1$ that sends (s_1, t, s_2) to (s_1, s_2) is a homomorphism into the rectangular band $S^1 \times S^1$. Thus, to establish (1), it remains to show that, for each pair $(s_1, s_2) \in S^1 \times S^1$, the subsemigroup $R_{s_1,s_2} = \{s_1\} \times T^1 \times \{s_2\}$ of R belongs to $\mathsf{N}_2 ⓜ \mathsf{W}$. Indeed, consider the mapping $\theta : R_{s_1,s_2} \to T^1 \times T^1$ that sends the triple (s_1, t, s_2) to the pair (tu, ut), where $u = f(s_2 s_1)$. Since W is monoidal, θ takes its values in a semigroup which is a member of W. We claim that, for each idempotent $(e, f) \in T^1 \times T^1$, the subsemigroup $\theta^{-1}(e, f)$ of R_{s_1,s_2} is null. Let $t, t' \in T^1$ be elements such that $(s_1, t, s_2), (s_1, t', s_2) \in \theta^{-1}(e, f)$. Then we have the equalities $e = tu = t'u$ and $f = ut = ut'$. It follows that

$$(s_1, t, s_2)(s_1, t, s_2) = (s_1, t', s_2)(s_1, t, s_2) = (s_1, t', s_2)(s_1, t', s_2),$$

and so the semigroup R_{s_1,s_2} satisfies the identities $x^2 = xy = y^2$, which implies that it is null.

To prove that $R \in \mathsf{LZ} \vee (\mathsf{W} * \mathsf{MD}_1)$, we adapt the proof of [1, Proposition 10.6.4]. Consider the semigroups $LZ(S^1)$ and $RZ(S^1)$, which are the set S^1, respectively under left zero and right zero multiplications. We will also be considering the monoid $M = RZ(S^1)^1$, whose identity element we denote e. Let $(T^1)^M * M$ be the semidirect product (in fact the wreath product $T^1 \circ M$), where the left action of $s \in M$ on a function $g : M \to T^1$ is given by $[w]^s g = [w \odot s]g$, the symbol $\odot$ denoting multiplication in M. For $s \in S^1$ and $t \in T^1$, let $g_{s,t} : M \to T^1$ be defined by

$$[w]g_{s,t} = \begin{cases} t & \text{if } w = e \\ f(ws)t & \text{otherwise.} \end{cases}$$

This allows us to define a mapping

$$\begin{aligned} \lambda : R &\to LZ(S^1) \times ((T^1)^M * M) \\ (s_1, t, s_2) &\mapsto (s_1, g_{s_1,t}, s_2). \end{aligned}$$

Since $[e]g_{s_1,t} = t$, the mapping λ is injective. To conclude the proof, it suffices to show that λ is a homomorphism. Indeed, given $(s_1, t, s_2), (s_1', t', s_2') \in R$, we have

$$\lambda\big((s_1,t,s_2)(s_1',t',s_2')\big) = (s_1, g_{s_1,tf(s_2s_1')t'}, s_2')$$
$$\lambda(s_1,t,s_2)\lambda(s_1',t',s_2') = (s_1,h,s_2'),$$

where $h = g_{s_1,t} \cdot {}^{s_2}g_{s_1',t'}$ is the function given by

$$[w]h = \begin{cases} t\cdot[s_2]g_{s_1',t'} = tf(s_2s_1')t' & \text{if } w=e \\ f(ws_1)t\cdot[w\odot s_2]g_{s_1',t'} = f(ws_1)tf(s_2s_1')t' & \text{otherwise,} \end{cases}$$

which shows that $h = g_{s_1,tf(s_2s_1')t'}$. □

The upper bounds for the bullet operation of Proposition 5.1 yield sufficient conditions for a pseudovariety to be a bullet idempotent.

Corollary 5.2. *Let* V *be a pseudovariety such that either* $\mathsf{N}_2 ⓜ \mathsf{V} = \mathsf{V} ⓜ \mathsf{B} = \mathsf{V}$ *or* $\mathsf{V} * \mathsf{A} = \mathsf{V} ⓜ \mathsf{SI} = \mathsf{V}$. *Then* V *is a bullet idempotent.* □

In view of Proposition 5.1, the following auxiliary results are of interest.

Lemma 5.3. *The following conditions hold for an arbitrary pseudovariety* V *of semigroups:*
(a) $\mathsf{V} ⓜ \mathsf{W} = \mathsf{V}$ *for all* $\mathsf{W} \in \{\mathsf{SI}, \mathsf{RZ}, \mathsf{LZ}\}$*;*
(b) $\mathsf{V} ⓜ \mathsf{W} = \mathsf{V}$ *for all* $\mathsf{W} \in \{\mathsf{SI}, \mathsf{RB}\}$*;*
(c) $\mathsf{V} ⓜ \mathsf{B} = \mathsf{V}$.

Proof. The implications (c) ⇒ (b) ⇒ (a) are obvious. Suppose that V satisfies the closure properties (a). We claim that V also satisfies (c). Consider a V-homomorphism $\varphi : S \to T$ into a band T. It is well known that the relation $\mathcal{J}$ is a congruence on T, the quotient $T/\mathcal{J}$ being a semilattice and each congruence class being a rectangular band. Hence, for the natural quotient homomorphism $\eta : T \to T/\mathcal{J}$, the composite $\eta\circ\varphi$ is a ($\mathsf{V} ⓜ \mathsf{RB}$)-homomorphism into a finite semilattice, which shows that $S \in (\mathsf{V} ⓜ \mathsf{RB}) ⓜ \mathsf{SI} = \mathsf{V}$. We leave it to the reader to verify that $\mathsf{V} ⓜ \mathsf{RB} \subseteq (\mathsf{V} ⓜ \mathsf{LZ}) ⓜ \mathsf{RZ}$, which shows that (a) ⇒ (c). □

The following lemma may be compared with Lemma 3.4, although neither of them seems to imply the other one.

Lemma 5.4. *We have* $\mathsf{K}_{n+1} \subseteq (\mathsf{N}_2 ⓜ \mathsf{K}_n) ⓜ \mathsf{LZ}$.

Proof. We use the Basis Theorem for Mal'cev products to compute the two Mal'cev products of the statement of the lemma. First, $\mathsf{N}_2 ⓜ \mathsf{K}_n$ is defined by all identities of the form

$$a_1\cdots a_n x a_1\cdots a_n y = a_1\cdots a_n z a_1\cdots a_n t \tag{6}$$

where the a_i are distinct variables and each of x,y,z,t may be either a variable or 1. Hence, $(\mathsf{N}_2 ⓜ \mathsf{K}_n) ⓜ \mathsf{LZ}$ is defined by the pseudoidentities obtained from (6) by replacing each variable s by bx_s, where x_s is either the variable s or 1, and b is a new variable, independent of s. Note that both sides of such an identity start with the word $ba_1\cdots ba_nb$, whose length is at least $n+1$, so that the identity holds in K_{n+1}. □

Proposition 5.5. *The pseudovariety* A *is the smallest pseudovariety* V *that satisfies the equations* $\mathsf{N}_2 ⓜ \mathsf{V} = \mathsf{V} ⓜ \mathsf{B} = \mathsf{V}$.

Proof. Since A is a Mal'cev idempotent and contains both N_2 and B, it certainly satisfies the equations in question. Suppose next that V is a solution of the equations. By Lemma 5.4, the pseudovariety K is contained in V, whence so is the iterated Mal'cev product $\mathsf{W} = \mathsf{K} ⓜ_\infty \mathsf{B}$. By [11, Corollary 2.4.13], we have

$$\begin{aligned} \mathsf{W} * \mathsf{MD}_1 &\subseteq ([\![xy = xz]\!] ⓜ \mathsf{W}) ⓜ \mathsf{MD}_1 \\ &\subseteq ([\![xy = xz]\!] ⓜ (\mathsf{K} ⓜ_\infty \mathsf{B})) ⓜ \mathsf{MD}_1 \\ &\subseteq (([\![xy = xz]\!] ⓜ \mathsf{K}) ⓜ_\infty \mathsf{B}) ⓜ \mathsf{B} \\ &\subseteq \mathsf{K} ⓜ_\infty \mathsf{B} = \mathsf{W} \end{aligned}$$

since $[\![xy = xz]\!] \subseteq \mathsf{K}$ and K is a Mal'cev idempotent. As A is the closure under semidirect product of MD_1 by the Krohn-Rhodes Decomposition Theorem [11, Theorem 4.1.30], we deduce that $\mathsf{A} \subseteq \mathsf{W} * \mathsf{A} = \mathsf{W} \subseteq \mathsf{V}$. $\square$

A continuous homomorphism of the free profinite semigroup $\overline{\Omega}_A\mathsf{S}$ into $\overline{\Omega}_B\mathsf{S}$ is also called a *substitution* over A. The following result is an immediate application of the Basis Theorem for Mal'cev products.

Lemma 5.6. *Let $u = v$ be a pseudoidentity over the finite alphabet A. Then the pseudovariety $[\![u = v]\!] ⓜ \mathsf{B}$ is defined by all pseudoidentities of the form $\varphi(u) = \varphi(v)$, where φ is a substitution over A such that each of the pseudoidentities $\varphi(a_1) = \varphi(a_2)$ $(a_1, a_2 \in A)$ holds in B. In particular, if the pseudovariety V is defined by a semigroup pseudoidentity in a single variable, then $\mathsf{V} ⓜ \mathsf{B} = \mathsf{V}$.* $\square$

For a finite semigroup S and a subsemigroup T, the maximum of the numbers n such that there is a strict $\mathcal{J}$-chain $t_1 >_\mathcal{J} t_2 >_\mathcal{J} \cdots >_\mathcal{J} t_n$ of elements of T, where the order $>_\mathcal{J}$ is that of S, is called the *$\mathcal{J}$-height of T in S* and we denote it $\mathsf{h}^S_\mathcal{J}(T)$. We also write $\mathsf{h}_\mathcal{J}(S)$ instead of $\mathsf{h}^S_\mathcal{J}(S)$.

Lemma 5.7. *Let S be a finite semigroup, let T be a subsemigroup of S, and let $\psi : \overline{\Omega}_B\mathsf{S} \to T$ be a continuous homomorphism, where $|B| \geq 2$. Suppose that $f : \overline{\Omega}_A\mathsf{S} \to \overline{\Omega}_B\mathsf{S}$ is a continuous homomorphism such that $c(f(a)) = B$ for every $a \in A$ and, moreover, the pseudovariety B satisfies every pseudoidentity of the form $f(a_1) = f(a_2)$ with $a_1, a_2 \in A$. Then either the subsemigroup $U = \operatorname{Im}(\psi \circ f)$ is a group or it satisfies the inequality $\mathsf{h}^S_\mathcal{J}(U) < \mathsf{h}^S_\mathcal{J}(T)$.*

Proof. Let $w_1, \ldots, w_r \in \overline{\Omega}_A\mathsf{S}$ be such that $\psi(f(w_1)) >_\mathcal{J} \cdots >_\mathcal{J} \psi(f(w_r))$ where $r = \mathsf{h}^S_\mathcal{J}(U)$. Suppose that $r = \mathsf{h}^S_\mathcal{J}(T)$. For each $a \in A$ and $b \in B$, since $b \in c(f(a))$, we have $\psi(b) \geq_\mathcal{J} \psi(f(a)) \geq_\mathcal{J} \psi(f(w_1))$. The assumption that $r = \mathsf{h}^S_\mathcal{J}(T)$ then implies that $\psi(b)$, $\psi(f(a))$, and $\psi(f(w_1))$ all lie in the same $\mathcal{J}$-class of S. Since all $f(a)$ start with the same letter from B and they all end with the same letter from B, we deduce that all $\psi(f(a))$ lie in the same $\mathcal{H}$-class of S. Moreover, since $c(f(a)) = B$, and $|B| \geq 2$, it follows that all $\psi(f(a))$ lie in the same subgroup of S. Hence U is a group. $\square$

For a pseudovariety H of groups and a positive integer n, we let

$$\mathsf{H}'_n = (\mathsf{N}_n \cap \mathsf{Com}) \vee \mathsf{H},$$
$$\mathsf{H}' = \mathsf{H}'_\infty = \bigcup_{n\geq 1} \mathsf{H}'_n = (\mathsf{N} \cap \mathsf{Com}) \vee \mathsf{H}.$$

The following result shows that the only pseudovarieties for which one may use the first sufficient condition of Corollary 5.2 to establish that they are bullet idempotents are those of the form $\bar{\mathsf{H}}$.

Theorem 5.8. *Let* V *be a pseudovariety and let* $\mathsf{H} = \mathsf{V} \cap \mathsf{G}$. *If* $m \in \{1, 2, \ldots\} \cup \{\infty\}$ *is the index of* V*, then the following equalities hold:*

$$\mathsf{V} \,ⓜ_\infty\, \mathsf{B} = \mathsf{H}'_m \,ⓜ_\infty\, \mathsf{B} = \mathsf{B}_{m,\omega} \cap \bar{\mathsf{H}}.$$

Proof. From the hypothesis that V has index m, it is easy to deduce that V contains $\mathsf{N}_m \cap \mathsf{Com}$. Since $\bar{\mathsf{H}}$ is a Mal'cev idempotent and $\mathsf{B}_{m,\omega} \,ⓜ\, \mathsf{B} = \mathsf{B}_{m,\omega}$ by Lemma 5.6, it suffices to establish the inclusion

$$\mathsf{B}_{m,\omega} \cap \bar{\mathsf{H}} \subseteq \mathsf{H}'_m \,ⓜ_\infty\, \mathsf{B}. \tag{7}$$

To establish (7), let $S \in \mathsf{B}_{m,\omega} \cap \bar{\mathsf{H}}$ and let $n = \mathsf{h}_{\mathcal{J}}(S)$. Iterating the Basis Theorem for Mal'cev products, we obtain that the pseudovariety $\mathsf{H}'_m \,ⓜ_n\, \mathsf{B}$ is defined by the pseudoidentities of the form $g(u) = g(v)$, where

- H'_m satisfies $u = v$;
- $A_0 = A$;
- each $f_i : \overline{\Omega}_{A_{i-1}}\mathsf{S} \to \overline{\Omega}_{A_i}\mathsf{S}$ is a continuous homomorphism such that B satisfies the pseudoidentity $f_i(x) = f_i(y)$ for all $x, y \in A_{i-1}$;
- $g = f_n \circ \cdots \circ f_1$.

Additionally, we may assume that $c(f_i(x)) = A_i$ for every $x \in A_{i-1}$ ($i = 1, \ldots, n$). We claim that S satisfies every such pseudoidentity $g(u) = g(v)$, which will therefore establish that S belongs to $\mathsf{H}'_m \,ⓜ_n\, \mathsf{B}$, thereby proving (7). To prove the claim, consider an arbitrary continuous homomorphism $\varphi : \overline{\Omega}_{A_n}\mathsf{S} \to S$.

Suppose first that $\varphi \circ g$ takes all its values in some subgroup of S. Since $S \in \bar{\mathsf{H}}$, that subgroup satisfies the pseudoidentity $u = v$, whence $\varphi(g(u)) = \varphi(g(v))$. Thus, we may assume that $\mathrm{Im}(\varphi \circ g)$ is not a group.

Suppose next that A_i is a singleton set for some i. If both u and v are not words of length less than m, then the pseudoidentity $g(u) = g(v)$ is equivalent in S to a pseudoidentity which is obtained by substituting in $u = v$ each variable x by some w^{α_x}, where y^{α_x} is a suitable infinite unary pseudoword. Hence, $\varphi \circ g$ takes all its values in the same subgroup, a case which has already been excluded. If $u = v$ is an identity in which both sides have length less than m then, since it is valid in $\mathsf{N}_m \cap \mathsf{Com}$, every variable appears the same number of times in both u and v, and so $g(u) = g(v)$ is a trivial pseudoidentity. Hence, we may assume that $|A_i| \geq 2$ for $i = 1, \ldots, n$.

For $i = 1, \ldots, n$, let $T_i = \mathrm{Im}(\varphi \circ f_n \circ \cdots \circ f_i)$. Then the inclusions $\mathrm{Im}(\varphi \circ g) = T_1 \subseteq \cdots \subseteq T_n \subseteq S$ hold and none of the subsemigroups T_i is a group. By Lemma 5.7, we obtain the inequalities

$$1 \leq \mathsf{h}^S_{\mathcal{J}}(T_1) < \cdots < \mathsf{h}^S_{\mathcal{J}}(T_n) < \mathsf{h}_{\mathcal{J}}(S) = n,$$

which is absurd. The claim therefore holds in all cases. $\square$

In particular, for an arbitrary pseudovariety of groups H and an index $m \in \{1, 2, \ldots\} \cup \{\infty\}$, Theorem 5.8 gives a filtration of $\mathsf{B}_{m,\omega} \cap \bar{\mathsf{H}}$, namely

$$\mathsf{H}'_m \subseteq \mathsf{H}'_m \textcircled{m} \mathsf{B} \subseteq \mathsf{H}'_m \textcircled{m}_2 \mathsf{B} \subseteq \cdots \subseteq \mathsf{H}'_m \textcircled{m}_n \mathsf{B} \subseteq \mathsf{H}'_m \textcircled{m}_{n+1} \mathsf{B} \subseteq \cdots, \tag{8}$$

in the sense that the union of the chain is $\mathsf{B}_{m,\omega} \cap \bar{\mathsf{H}}$. More generally, we have the following result, where we say that a filtration is *decidable* if all its terms are decidable.

Corollary 5.9. *Let H be a pseudovariety of groups, let m be a positive integer or ∞, and let V be a pseudovariety such that $\mathsf{H}'_m \subseteq \mathsf{V} \subseteq \mathsf{B}_{m,\omega} \cap \bar{\mathsf{H}}$. Then the chain*

$$\mathsf{V} \subseteq \mathsf{V} \textcircled{m} \mathsf{B} \subseteq \mathsf{V} \textcircled{m}_2 \mathsf{B} \subseteq \cdots \subseteq \mathsf{V} \textcircled{m}_n \mathsf{B} \subseteq \mathsf{V} \textcircled{m}_{n+1} \mathsf{B} \subseteq \cdots \tag{9}$$

is a filtration of $\mathsf{B}_{m,\omega} \cap \bar{\mathsf{H}}$. The filtration is decidable if and only if so is V. If $m = \infty$ and $\mathsf{V} \subsetneqq \bar{\mathsf{H}}$, then the filtration is strict.

Proof. Theorem 5.8 gives that (9) is indeed a filtration for $\mathsf{B}_{m,\omega} \cap \bar{\mathsf{H}}$. For the decidability statement, it suffices to recall the well-known fact that the operator $\_ \textcircled{m} \mathsf{W}$ preserves decidability whenever W is locally finite and the semigroups $\overline{\Omega}_n \mathsf{W}$ are computable.

Suppose now that $m = \infty$ and $\mathsf{V} \subsetneqq \bar{\mathsf{H}}$. If $\mathsf{V} \textcircled{m}_n \mathsf{B} = \mathsf{V} \textcircled{m}_{n+1} \mathsf{B}$ for some $n \geq 0$, then the equality $\mathsf{V} \textcircled{m}_n \mathsf{B} = \mathsf{V} \textcircled{m}_{n+k} \mathsf{B}$ holds for every $k \geq 1$, and so $\mathsf{V} \textcircled{m}_n \mathsf{B} = \bar{\mathsf{H}}$, again by Theorem 5.8. Now, if we take $n \geq 0$ to be minimum so that $\mathsf{V} \textcircled{m}_n \mathsf{B} = \bar{\mathsf{H}}$, then $n > 0$ by the hypothesis that V is a proper subpseudovariety of $\bar{\mathsf{H}}$, while $n > 0$ is impossible by Theorem 4.1. Hence the filtration (9) is strict. $\square$

Corollary 5.9 suggests the question whether the decidability of H entails that of H'. Using the methods of [3] or [12], it is easy to show that H' is decidable if H-pointlike subsets of finite semigroups are computable. We do not know if this property always holds for a decidable pseudovariety of groups H.

In contrast, it is very easy to show that $\mathsf{N} \vee \mathsf{H}$ is decidable if and only if the pseudovariety of groups H is decidable. Indeed, as observed in [1, Section 9.1], given a semigroup $S \in \mathsf{IE}$, with minimum ideal K, the mapping $S \to (S/K) \times K$ that sends $s \in S$ to $(s/K, s^{\omega+1})$ is an injective homomorphism. Hence the pseudovariety $\mathsf{N} \vee \mathsf{H} = \mathsf{IE} \cap \bar{\mathsf{H}}$ consists of all finite nilpotent extensions of groups from H, whence it is decidable if and only if so is H. Since, if H satisfies no nontrivial identities valid in Ab, then $\mathsf{H}' = \mathsf{N} \vee \mathsf{H}$ by the methods of [1, Section 9.1], it follows trivially that H' is also decidable in such a case.

6. The bullet operator and the Krohn-Rhodes complexity

Still motivated by the search of further applications of Theorem 3.5, it is natural to ask how the bullet operator behaves with respect to the Krohn-Rhodes complexity. We refer the reader to [11, Chapter 4] for a recent presentation of the Krohn-Rhodes decomposition theory, and in particular to Section 4.12 in that book.

The *complexity pseudovarieties* are defined recursively by $\mathsf{C}_0 = \mathsf{A}$, and $\mathsf{C}_{n+1} = \mathsf{C}_n * \mathsf{G} * \mathsf{A}$ $(n \geq 0)$. The *complexity* of a finite semigroup S is the smallest $n \geq 0$ such that $S \in \mathsf{C}_n$ and is denoted $c(S)$. It remains an open problem whether there is an algorithm to compute $c(S)$ from a given finite semigroup S.

For a positive integer n, denote by T_n the full transformation semigroup of the set $[n] = \{1, \ldots, n\}$, and by S_n the full permutation group of $[n]$.

Proposition 6.1. *There is a function $f : S_n \to T_{n-1}$ such that T_n is a homomorphic image of $M(S_n, T_{n-1}, f)$.*

Proof. Let $e \in T_n$ be the (idempotent) mapping which is the identity on the set $[n-1]$ and maps n to $n-1$. We define the mapping $f : S_n \to T_{n-1}$ by putting $f(\sigma) = e\sigma e|_{[n-1]}$. For $t \in T_{n-1}$, let $\bar{t} \in T_n$ be the extension of t to $[n]$ which fixes n. Finally, consider the mapping $\varphi : M(S_n, T_{n-1}, f) \to T_n$ defined by $\varphi(\sigma) = \sigma$ for each $\sigma \in S_n$, and $\varphi(\sigma, t, \tau) = \sigma e \bar{t} e \tau$ for each triple $(\sigma, t, \tau) \in S_n \times T_{n-1} \times S_n$. We claim that φ is an onto homomorphism.

Taking into account the way the multiplication is defined in the semigroup $M(S_n, T_{n-1}, f)$, the only case requiring some calculation to verify that φ is a homomorphism is that of a product of two triples $(\sigma_1, t_1, \tau_1), (\sigma_2, t_2, \tau_2) \in S_n \times T_{n-1} \times S_n$:

$$\begin{aligned} \varphi\big((\sigma_1, t_1, \tau_1)(\sigma_2, t_2, \tau_2)\big) &= \sigma_1 e\, \overline{t_1\, (e\tau_1\sigma_2 e)|_{[n-1]}\, t_2}\, e\tau_2 \\ \varphi(\sigma_1, t_1, \tau_1)\, \varphi(\sigma_2, t_2, \tau_2) &= \sigma_1\, e\bar{t}_1 e\, \tau_1\sigma_2\, e\bar{t}_2 e\, \tau_2. \end{aligned}$$

Thus, to prove that φ is a homomorphism, it suffices to show that every $i \in [n]$ has the same image under the mappings $e\, \overline{t_1\, (e\tau_1\sigma_2 e)|_{[n-1]}\, t_2}\, e$ and $e\bar{t}_1 e\, \tau_1\sigma_2\, e\bar{t}_2 e$, which amounts to a straightforward calculation.

To prove that φ is onto, since it is well know that T_n is generated by S_n together with any idempotent of rank $n-1$, it suffices to observe that e, which is such an idempotent, belongs to the image of φ: indeed we have $e = \varphi(1_n, 1_{n-1}, 1_n)$, where 1_k stands for the identity mapping on the set $[k]$. $\square$

Since every finite semigroup embeds in some T_n, we deduce the following result.

Corollary 6.2. *The only solution V of the equation $\mathsf{G} \bullet \mathsf{V} = \mathsf{V}$ is the pseudovariety S.* $\square$

In particular, the only complexity pseudovariety C_n which is a bullet idempotent is $\mathsf{C}_0 = \mathsf{A}$.

More generally, for an arbitrary pseudovariety of groups H, the pseudovariety $\bar{\mathsf{H}}$ is a solution of the equation $\mathsf{H} \bullet \mathsf{V} = \mathsf{V}$ by Lemma 3.1. We do not know whether it is the smallest solution.

Consider the sequence of pseudovarieties $(\tilde{\mathsf{H}}_n)_n$ defined recursively by $\tilde{\mathsf{H}}_0 = \mathsf{H}$ and $\tilde{\mathsf{H}}_{n+1} = (\tilde{\mathsf{H}}_n * \mathsf{MD}_1) Ⓜ \mathsf{SI}$. By Corollary 5.2, the sequence defines a filtration of the bullet idempotent $\tilde{\mathsf{H}}_\infty = \bigcup_{n \geq 0} \tilde{\mathsf{H}}_n$. Since $\bar{\mathsf{H}} * \mathsf{A} = \bar{\mathsf{H}} Ⓜ \mathsf{SI} = \bar{\mathsf{H}}$, the inclusion $\tilde{\mathsf{H}}_\infty \subseteq \bar{\mathsf{H}}$ always holds. By the Krohn-Rhodes Decomposition Theorem [11, Theorem 4.1.30], we have $\tilde{\mathsf{I}}_\infty = \mathsf{A}$. On the other hand, Corollary 6.2 yields $\tilde{\mathsf{G}}_\infty = \mathsf{S}$. We do not know whether in general $\tilde{\mathsf{H}}_\infty = \bar{\mathsf{H}}$. By Theorem 4.1 and Corollary 4.2, if the equality holds then the filtration $(\tilde{\mathsf{H}}_n)_n$ is strict. While the operator _ Ⓜ SI preserves decidability, we do not know if this is also the case for the operator _ $* \mathsf{MD}_1$. In particular, we do not whether the decidability of H entails that of the filtration $(\tilde{\mathsf{H}}_n)_n$.

Note that $c(T_n) = n - 1$ (see [11, Theorem 4.12.31]) and $c(S_n) = 1$ for $n \geq 2$. Thus, by Proposition 6.1, the complexity of $M(S, T, f)$ may be the sum of the complexities of S and T. Here is a more precise result, albeit somewhat more restricted, in the spirit of [11, Theorem 4.12.32].

Theorem 6.3. *Let G be a finite group, T a finite monoid, and $f : G \to T$ a mapping such that $f(1) = 1$, for every $g \in G$ there are some $h_1, h_2 \in G$ such that $f(h_1^{-1}) = f(h_1 g) = f(h_2^{-1}) = f(gh_2) = 1$, and $\langle \mathrm{Im}(f)\rangle = T$. Then $c\big(M(G, T, f)\big) = c(T) + c(G)$.*

Proof. Let $M = M(G, T, f)$. In case G is the trivial group $\{1\}$, the assumption on the mapping f entails that T is a trivial group. Hence M is a two-element semilattice and $c(M) = c(T) = 0$. From hereon, we assume that G is a nontrivial group, i.e., that $c(G) = 1$.

Let $e = (1, 1, 1) \in M$. To establish that $c(M) = c(T) + 1$, we verify that the following properties hold:

(a) $M = \langle G \cup \{e\}\rangle$,
(b) $MeM \subseteq \langle E(M)\rangle$,
(c) eMe is isomorphic with T.

By [11, Proposition 4.12.23], it follows that $c(M) = c(eMe) + 1 = c(T) + c(G)$.

(a) For $g \in G$, we have $(1, f(g), 1) = ege$. Since T is generated by the set $\mathrm{Im}(f)$ and $f(1) = 1$, it follows that $\{1\} \times T \times \{1\}$ is contained in $\langle G \cup \{e\}\rangle$. By the definition of the operation on M, we deduce $G \times T \times G \subseteq \langle G \cup \{e\}\rangle$, which proves (a).

(b) For $g \in G$, choose $h_1, h_2 \in G$ such that $f(h_1^{-1}) = f(h_1 g) = f(h_2^{-1}) = f(gh_2) = 1$. Then the elements $(g, 1, 1) = (g, 1, h_1)(h_1^{-1}, 1, 1)$ and $(1, 1, g) = (1, 1, h_2^{-1})(h_2, 1, g)$ are products of idempotents, whence so is $(1, f(g), 1) = (1, 1, 1)(g, 1, 1)$. Since $\mathrm{Im}(f)$ generates T and $f(1) = 1$, it follows that $\{1\} \times T \times \{1\} \subseteq \langle E(M)\rangle$. Hence, for $g_1, g_2 \in G$ and $t \in T$, $(g_1, t, g_2) = (g_1, 1, 1)(1, t, 1)(1, 1, g_2)$ is also a product of idempotents.

(c) Since the mapping $(1, t, 1) \mapsto t$ is an isomorphism $\{1\} \times T \times \{1\} \to T$, it suffices to note that $eMe = \{1\} \times T \times \{1\}$. □

As an example, suppose that $n \geq 3$ and consider the mapping $f : S_n \to T_{n-1}$ which sends the cycles $(1, 2)$ and $(1, 2, \dots, n-1)$ to themselves, $(1, 2, \dots, n)$ to the idempotent that fixes all points of $[n-1]$ and maps n to $n-1$, and every other element to the identity on $[n-1]$. It is well known that $\langle \mathrm{Im}(f)\rangle = T_{n-1}$ and it is easy to check that f satisfies the hypothesis of Theorem 6.3. Hence, $c(M(S_n, T_{n-1}, f)) = c(T_{n-1}) + c(S_n) = n - 1$.

Acknowledgments. Both authors were partially supported by the Ministry of Education of the Czech Republic under the project MSM 0021622409 and by the Grant 201/09/1313 of the Grant Agency of the Czech Republic.

The first author also acknowledges partial funding by the European Regional Development Fund, through the programme COMPETE, and by the Portuguese Government through FCT – *Fundação para a Ciência e a Tecnologia*, under the project PEst-C/MAT/UI0144/2011. This work has been carried out during the first author's sabbatical visit to the Department of

Mathematics and Statistics of Masaryk University, whose hospitality and support is hereby gratefully acknowledged.

References

1. J. Almeida, *Finite semigroups and universal algebra*, World Scientific, Singapore, 1995, English translation.
2. ______, *Profinite semigroups and applications*, Structural Theory of Automata, Semigroups and Universal Algebra (New York) (V. B. Kudryavtsev and I. G. Rosenberg, eds.), Springer, 2005, pp. 1–45.
3. J. Almeida, A. Azevedo, and M. Zeitoun, *Pseudovariety joins involving $\mathcal{J}$-trivial semigroups*, Int. J. Algebra Comput. **9** (1999), 99–112.
4. S. Eilenberg, *Automata, languages and machines*, vol. B, Academic Press, New York, 1976.
5. I. O. Koryakov, *Embeddings of pseudofree semigroups*, Russian Math. (Iz. VUZ) **39** (1995), 53–59.
6. S. Margolis, *On maximal pseudo-varieties of finite monoids and semigroups*, Russian Math. (Iz. VUZ) **39** (1995), 60–64.
7. S. Margolis, M. Sapir, and P. Weil, *Irreducibility of certain pseudovarieties*, Comm. Algebra **26** (1998), 779–792.
8. J.-E. Pin and P. Weil, *Profinite semigroups, Mal'cev products and identities*, J. Algebra **182** (1996), 604–626.
9. J. Reiterman, *The Birkhoff theorem for finite algebras*, Algebra Universalis **14** (1982), 1–10.
10. J. Rhodes and B. Steinberg, *Join irreducible pseudovarieties, group mapping, and Kovács-Newman semigroups*, LATIN 2004: Theoretical informatics, Lecture Notes in Comput. Sci., vol. 2976, Springer, Berlin, 2004, pp. 279–291.
11. ______, *The q-theory of finite semigroups*, Springer Monographs in Mathematics, Springer, 2009.
12. B. Steinberg, *On pointlike sets and joins of pseudovarieties*, Int. J. Algebra Comput. **8** (1998), 203–231.

(Permanent address) CMUP, Dep. Matemática, Faculdade de Ciências, Universidade do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal. Tel.: +351 220402132; Fax: +351 220402108

E-mail address: `jalmeida@fc.up.pt`

Dept. of Mathematics and Statistics, Masaryk University, Kotlářská 2, 611 37 Brno, Czech Republic

E-mail address: `klima@math.muni.cz`