A $\mathbb{Z}_n$-SYMMETRIC LOCAL BUT NOT GLOBAL ATTRACTOR

B. ALARCÓN, S.B.S.D. CASTRO, AND I.S. LABOURIAU

Abstract. There are many tools for studying local dynamics. An important problem is how this information can be used to obtain global information. For instance, if a map has a unique fixed point which is a local attractor, when can one guarantee that it is a global attractor?

One attempt at getting tools for global dynamics was made through the Discrete Markus-Yamabe conjecture. A counter-example (Szlenk's example) to this conjecture in dimension 2 was presented in [A. Cima, A. Gasull and F. Mañosas, The Discrete Markus-Yamabe Problem *Nonlinear Analysis*, 35, 343-354, 1999]. In the present article we show that Szlenk's example has symmetry $\mathbb{Z}_4$. Based on this example we construct, for any natural $n \geq 3$, planar maps whose symmetry group is $\mathbb{Z}_n$ having a local attractor that is not a global attractor. The same construction can be applied to obtain examples that are also dissipative. The symmetry of these maps forces them to have rational rotation numbers, leading to the new question of whether $\mathbb{Z}_n$-symmetry implies rational rotation number.

1. Introduction

At the end of the 19^{th} century, Lyapunov [10] related the local stability of an equilibrium point to the eigenvalues of the Jacobian matrix of the vector field at that point. This led to the Markus-Yamabe Conjecture [12] in the 1960's, and fifteen years later to a version for maps of the original conjecture, using the relation between stability of fixed points and the eigenvalues of the Jacobian matrix of the map at that point [11]. In the 1990's, this was named, by analogy, the Discrete Markus-Yamabe Conjecture and remains unproven. It may be stated as follows:

Discrete Markus-Yamabe Conjecture: Let F be a C^1 map from $\mathbb{R}^m$ to itself such that $F(0) = 0$. If all the eigenvalues of the Jacobian matrix at every point have modulus less than one, then the origin is a global attractor.

It is known that the original conjecture holds for $m = 2$ and is, in this case, equivalent to the injectivity of the vector field [9], [7]. It is false for $m > 2$ [3], [5]. On the other hand, the Discrete Markus-Yamabe Conjecture holds, for all m, if the Jacobian matrix of the map is triangular and, additionally for $m = 2$, for polynomial maps [6]. It is false in higher dimensions, also for polynomial maps [5]. This striking difference between the discrete and continuous versions encouraged the study of the dynamics of continuous and injective maps of the plane that satisfy the hypotheses of the Discrete Markus-Yamabe Conjecture. This is now known as the Discrete Markus-Yamabe Problem. From the results in [1], it follows that the Discrete Markus-Yamabe Problem is true for $m = 2$ for dissipative maps, by introducing as an extra condition the existence of an invariant ray (a continuous curve without self-intersections connecting the origin to infinity). An invariant ray can be, for instance an axis of symmetry.

In the presence of symmetry, that is, when the map is equivariant, the ultimate question can be stated as follows:

Equivariant Discrete Markus-Yamabe Problem: Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a dissipative C^1 equivariant planar map such that $f(0) = 0$. Assume that all eigenvalues of the Jacobian matrix at every point have modulus less than one. Is the origin a global attractor?

The Equivariant Discrete Markus-Yamabe Problem is false if the reflection is not a group element. In fact, the example constructed by Szlenk and reported in [6] satisfies all the hypotheses of the Discrete Markus-Yamabe Problem, is equivariant (as we show here) under the standard action of $\mathbb{Z}_4$, but the origin is not a global attractor. Indeed, there is an orbit of period 4 and the rotation number defined in [13] is $\frac{1}{4}$.

We use Szlenk's example to construct differentiable maps on the plane with symmetry group $\mathbb{Z}_n$ for all $n \geq 2$. Each example has an attracting fixed point at the origin and a periodic orbit of minimal period n which prevents local dynamics to extend globally.

Moreover, the symmetry of those examples implies that the rotation number is rational. Implications of this fact are discussed in the final section.

1.1. **Equivariant Planar Maps.** Let Γ be a compact Lie group acting on $\mathbb{R}^2$, that is, a group which has the structure of a compact $C^\infty-$differentiable manifold such that the map $\Gamma \times \Gamma \to \Gamma$, $(\gamma, \delta) \mapsto \gamma\delta^{-1}$ is of class C^∞. The reference for the folllowing definitions and results is Golubitsky *et al.* [8], to which we refer the reader interested in further detail.

Our concern is about groups acting linearly on $\mathbb{R}^2$ (see Chapter XII of Golubitsky *et al.* [8] for details) and more particularly about the action of $\mathbb{Z}_n$, $n \geq 2$ on $\mathbb{R}^2$. Identifying $\mathbb{R}^2 \simeq \mathbb{C}$, the finite group $\mathbb{Z}_n$ is generated by one element R_n, the rotation by $2\pi/n$ around the origin, with action given by

$$R_n \cdot z = \mathrm{e}^{2\pi i/n} z.$$

We are interested in maps that reflect the symmetries associated to the action of a given Γ on $\mathbb{R}^2$.

Definition 1.1. *We say that* $\gamma \in GL(2)$ *is a* symmetry of a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ *if* $f(\gamma x) = \gamma f(x)$. *We define the* symmetry group *of* f *as the biggest closed subgroup of* $GL(2)$ *containing all the symmetries of* f.

Definition 1.2. *We say that* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *is* $\Gamma-$ equivariant *or that* f commutes *with* Γ *if*

$$f(\gamma x) = \gamma f(x) \quad \textit{for all} \quad \gamma \in \Gamma.$$

Proposition 1.3. *Every map* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *is equivariant under the action of its symmetry group.*

An important space in the study of equivariant dynamics is the following.

Definition 1.4. *Let* Σ *be a subgroup of* Γ. *The* fixed-point subspace *of* Σ *is*

$$\mathit{Fix}(\Sigma) = \{p \in \mathbb{R}^2 : \sigma p = p \ \textit{ for all } \ \sigma \in \Sigma\}.$$

If Σ *is generated by a single element* $\sigma \in \Gamma$, *we write* $\mathrm{Fix}\langle\sigma\rangle$.

We note that, for each subgroup Σ of Γ, $\mathrm{Fix}(\Sigma)$ is invariant by the dynamics of a $\Gamma-$ equivariant map ([8], XIII, Lemma 2.1). In fact, we have for $p \in \mathrm{Fix}(\Sigma)$

$$f(p) = f(\sigma p) = \sigma f(p),$$

showing that $f(p) \in \mathrm{Fix}(\Sigma)$.

When f is $\Gamma-$ equivariant, we can use the symmetry to generalize information obtained for a particular point. This is achieved through the *group orbit* of a point, which is defined to be

$$\Gamma x = \{\gamma x : \quad \gamma \in \Gamma\}.$$

Lemma 1.5. *Let* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *be* $\Gamma-$*equivariant and let* p *be a fixed point of* f. *Then all points in the group orbit of* p *are fixed points of* f.

Proof. If $f(p) = p$ it follows that $f(\gamma p) = \gamma f(p) = \gamma p$, showing that γp is a fixed point of f for all $\gamma \in \Gamma$. $\square$

Since most of our results depend on the existence of a unique fixed point for f, the group actions we are concerned with are such that $\mathrm{Fix}(\Gamma) = \{0\}$.

2. Example with periodic points

The next examples refer to a local attractor, examples with a local repellor may be obtained considering f^{-1}.

Theorem 2.1. *For each $n \geq 2$ there exists $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that:*

a) *f is a differentiable homeomorphism;*
b) *f has symmetry group $\mathbb{Z}_n$;*
c) *$Fix(f) = \{0\}$;*
d) *The origin is a local attractor;*
e) *There exists a periodic orbit of minimal period n.*

Before proceeding, it is useful to establish some concepts that will be used in the proofs to come. Let $S_{1,n} \subset \mathbb{R}^2$ be the open sector

$$S_{1,n} = \{(x, y) = (r\cos\theta, r\sin\theta) :\ 0 < \theta < 2\pi/n\}$$

and define $S_{j,n}$, $j = 2, \cdots, n$ recursively by $S_{j,n} = R_n(S_{j-1,n})$. Then $\mathbb{R}^2 = \bigcup_{j=1}^{n} \overline{S_{j,n}}$, where $\overline{A}$ is the closure of A. Moreover, $S_{1,n} = R_n(S_{n,n})$. Then each $\overline{S_{j,n}}$ is a *fundamental domain* for the action of $\mathbb{Z}_n$, in particular if $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is $\mathbb{Z}_n$-equivariant then f is completely determined by its restriction to $\overline{S_{j,n}}$.

A *line ray* is a half line through the origin, of the form $\{t(\alpha, \beta) : t \geq 0\}$, with $0 \neq (\alpha, \beta) \in \mathbb{R}^2$.

Our construction of the map f in Theorem 2.1 benefits from a closer look at a simpler known example with $\mathbb{Z}_4$ symmetry and satisfying Theorem 2.1. This was obtained by Szlenk and is presented in [6]. The next Proposition establishes the relevant properties of this example that will be used in the construction of other $\mathbb{Z}_n$-equivariant maps.

Proposition 2.2 (Szlenk's example)**.** *Let $F_4 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by*

$$F_4(x, y) = \left(-\frac{ky^3}{1 + x^2 + y^2}, \frac{kx^3}{1 + x^2 + y^2}\right) \qquad \text{for} \quad 1 < k < \frac{2}{\sqrt{3}}.$$

The map F_4 has the following properties:

1) *F_4 is a differentiable homeomorphism.*
2) *$Fix(F_4) = \{0\}$.*

3) $F_4^4(P) = P$ *for* $P = \big((k-1)^{-1/2}, 0\big)$, *with* $F_4^j(P) = R_4^j(P) \neq P$ *for* $j = 2, 3$.
4) 0 *is a local attractor.*
5) F_4 *is* $\mathbb{Z}_4$*-equivariant.*
6) *The restriction of* F_4 *to any line ray is a homeomorphism onto another line ray.*
7) $F_4\left(\overline{S_{j,4}}\right) = \overline{S_{j+1,4}}$ *for* $j = 1, \cdots, 4 \pmod 4$ *with* $F_4\left(\partial S_{j,4}\right) = \partial S_{j+1,4}$.
8) *The curve* $F_4(\cos\theta, \sin\theta)$ *goes across each line ray and is transverse to line rays at all points* $\theta \neq \frac{m\pi}{2}$ *for* $m = 0, 1, 2, 3$.

Proof. Statements 1), 3) and 4) are proved in [6]. Statement 2) is proved in [2]. Note that the periodic orbit of P of statement 3) lies in the boundary of the sectors $\bigcup_j \partial S_{j,4}$.

Concerning 5) note that R_4, the generator of $\mathbb{Z}_4$, acts on the plane as $R_4(x, y) = (-y, x)$. In order to prove that $F_4(x, y)$ is $\mathbb{Z}_4$-equivariant we compute

$$F_4(R_4(x, y)) = (-\frac{kx^3}{1 + x^2 + y^2}, -\frac{ky^3}{1 + x^2 + y^2})$$

and

$$R_4F_4(x, y) = R_4(-\frac{ky^3}{1 + x^2 + y^2}, \frac{kx^3}{1 + x^2 + y^2}) = (\frac{-kx^3}{1 + x^2 + y^2}, \frac{-ky^3}{1 + x^2 + y^2}).$$

Observing that these are equal establishes statement 5).

The behaviour of F_4 on line rays described in 6) is easier to understand if we write (x, y) in polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$ yielding:

$$F_4(r\cos\theta, r\sin\theta) = \frac{kr^3}{1 + r^2}\left(-\sin^3\theta, \cos^3\theta\right) \ . \tag{1}$$

From this expression it follows that for each fixed θ, the line ray through $(\cos\theta, \sin\theta)$ is mapped into the line ray through $(-\sin^3\theta, \cos^3\theta)$. The mapping is a bijection, since $r^3/(1+r^2)$ is a monotonically increasing bijection from $[0, +\infty)$ into itself.

The behaviour of F_4 on sectors and their boundary is the essence of 7). From the definition of the sectors we have

$$S_{j+1,4} = R_4\left(S_{j,4}\right)$$

and therefore, by $\mathbb{Z}_4$-equivariance,

$$F_4\left(S_{j+1,4}\right) = F_4\left(R_4\left(S_{j,4}\right)\right) = R_4\left(F_4\left(S_{j,4}\right)\right) \ .$$

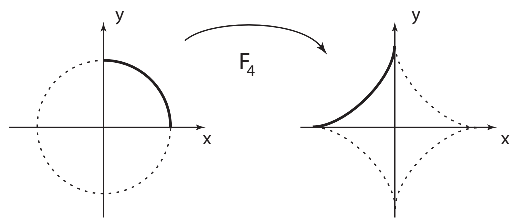

FIGURE 1. Szlenk's example F_4 maps a quarter of the unit circle into a quarter of the astroid $\frac{k}{2}(-\sin^3\theta, \cos^3\theta)$.

It then suffices to show that $F_4\left(\overline{S_{1,4}}\right) = \overline{S_{2,4}}$. The sectors $S_{1,4}$ and $S_{2,4}$ have the simple forms

$$S_{1,4} = \{(x,y): \quad x > 0, \quad y > 0\} \qquad S_{2,4} = \{(x,y): \quad x < 0, \quad y > 0\}.$$

From the expression of F_4 it is immediate that if $x > 0$ and $y > 0$ then the first coordinate of $F_4(x,y)$ is negative and the second is positive and thus $F_4(S_{1,4}) \subset S_{2,4}$. It remains to show the equality, which we delay until after the proof of 8).

The expression (1) in polar coordinates shows that the circle $(\cos\theta, \sin\theta)$, $0 \leq \theta \leq 2\pi$ is mapped by F_4 into the curve $\gamma(\theta) = \frac{k}{2}(-\sin^3\theta, \cos^3\theta)$ known as the *astroid* (Figure 1). The arc $\gamma(\theta)$, $0 \leq \theta \leq \pi/2$ joins $(0, \frac{k}{2})$ to $(-\frac{k}{2}, 0)$. Since for $\theta \in (0, \pi/2)$ the functions $\cos^3\theta$ and $-\sin^3\theta$ are both monotonically decreasing with strictly negative derivatives, then the $0 \leq \theta \leq \pi/2$ arc of the astroid has no self intersections and the restriction of F_4 to the quarter of a circle $0 \leq \theta \leq \pi/2$ is a bijection into this arc (Figure 1).

Moreover, the determinant of the matrix with rows $\gamma(\theta)$ and $\gamma'(\theta)$ is

$$\det\begin{pmatrix} \gamma(\theta) \\ \gamma'(\theta) \end{pmatrix} = \frac{3k^2}{4}\sin^2\theta\cos^2\theta$$

showing that the arc of the astroid is transverse at each point $\gamma(\theta)$, $0 < \theta < \pi/2$ to the line ray through it. Transversality fails at the end points of the arc, but the line rays still go across the astroid at the cusp points — this is assertion 8).

Thus, F_4 induces a bijection between line rays in $S_{1,4}$ and line rays in $S_{2,4}$ and using the radial property 6) it follows that $F_4(S_{1,4}) = S_{2,4}$. The behaviour on the boundary of $S_{1,4}$ also follows either from the radial property or from a simple direct calculation, concluding the proof of 7). $\square$

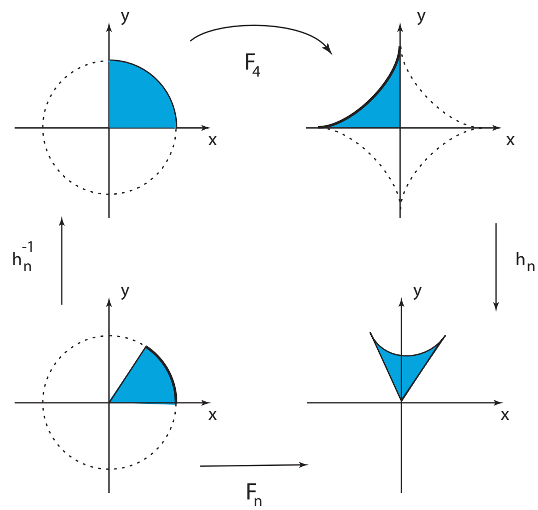

FIGURE 2. Construction of the $\mathbb{Z}_n$-equivariant example F_n in a fundamental domain of the $\mathbb{Z}_n$-action, shown here for $n = 6$.

Proof of Theorem 2.1. For $n \geq 2$, the map

$$h_n\left(r\cos\theta, r\sin\theta\right) = \left(r\cos\frac{4\theta}{n}, r\sin\frac{4\theta}{n}\right) \tag{2}$$

is a local diffeomorphism at all points in $\mathbb{R}^2\backslash\{0\}$, is continuous at 0 and $h_n(S_{1,4}) = S_{1,n}$, $h_n(S_{2,4}) = S_{2,n}$ with $|h_n(x,y)| = |(x,y)|$. Moreover, the restriction of h_n to $\overline{S_{1,4}}$ is a bijection onto $\overline{S_{1,n}}$ and h_n maps each line ray through the origin into another line ray through the origin.

Similar properties hold for the inverse

$$h_n^{-1}\left(r\cos\theta, r\sin\theta\right) = \left(r\cos\frac{n\theta}{4}, r\sin\frac{n\theta}{4}\right)$$

with $h_n^{-1}(S_{1,n}) = S_{1,4}$.

Let $F_n : \overline{S_{1,n}} \longrightarrow \overline{S_{2,n}}$ be defined by (see Figure 2)

$$F_n(x,y) = h_n \circ F_4 \circ h_n^{-1}(x,y)\ . \tag{3}$$

We extend F_n to a $\mathbb{Z}_n$-equivariant map $F_n : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ recursively, as follows.

Suppose for $1 \leq j \leq n-1$ the map F_n is already defined in $S_{j,n}$ with $F_n(S_{j,n}) = S_{j+1,n}$. If $(x,y) \in S_{j+1,n}$ we have $R_n^{-1}(x,y) \in S_{j,n}$ and thus $F_n \circ R_n^{-1}(x,y)$ is well defined, with $F_n \circ R_n^{-1}(x,y) \in S_{j+1,n}$. Define

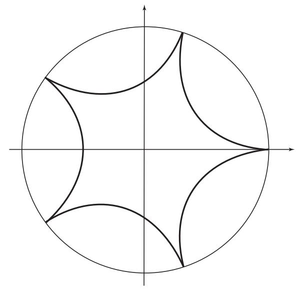

FIGURE 3. Image of the circle $(\sin\theta, \cos\theta)$ by the $\mathbb{Z}_n$-equivariant example F_n, shown here for $n = 5$.

$F_n(x, y)$ for $(x, y) \in S_{j+1,n}$ as $F_n(x, y) = R_n \circ F_n \circ R_n^{-1}(x, y) \in S_{j+2,n}$. Finally, for $(x, y) \in S_{n-1,n}$ we obtain $F_n(x, y) \in S_{1,n}$.

The following properties of F_n now hold by construction, using Proposition 2.2:

- F_n is $\mathbb{Z}_n$-equivariant.
- $Fix(F_n) = \{0\}$.
- The origin is a local attractor.
- $F_n^n(P) = P$ for $P = \big((k-1)^{-1/2}, 0\big)$, with $F_n^j(P) \neq P$ for $j = 2, \ldots, n-1$. Note that all $F_n^j(P)$ lie on the boundaries $\partial S_{j,n}$ of the sectors $S_{j,n}$.
- F_n maps each line ray through the origin onto another line ray through the origin.

Since h_n maps line rays to line rays, to see that F_n is a homeomorphism it is sufficient to observe that $\gamma_n(\theta) = F_n(\cos\theta, \sin\theta)$, $0 \leq \theta \leq 2\pi$ is a simple closed curve that meets each line ray only once and does not go through the origin (Figure 3). This is true because away from the origin both h_n and h_n^{-1} are differentiable with non-singular derivatives. Since h_n and h_n^{-1} map line rays into line rays, it follows from assertion 8) of Proposition 2.2 that γ_n is transverse to line rays except at the cusp points $\gamma_n(\theta)$, $\theta = \frac{2m\pi}{n}$, $m = 0, 1, \ldots, n-1$ where the line ray goes across it.

It remains to show that F_n is everywhere differentiable in $\mathbb{R}^2$. This is done in Lemma 2.3 below.

□

Lemma 2.3. *F_n is everywhere differentiable in $\mathbb{R}^2$.*

Proof. First we show that $DF_4(0, 0) = (0)$ (zero matrix) implies that F_n is differentiable at the origin with $DF_n(0, 0) = (0)$. That $DF_4(0, 0) = (0)$ means that for every $\varepsilon > 0$ there is a $\delta > 0$ such that, for every

$X \in \mathbb{R}^2$, if $|X| < \delta$ then

$$|F_4(X) - F_4(0,0) - DF_4(0,0)X| = |F_4(X)| < \varepsilon\,|X|\ .$$

Since h_n and h_n^{-1} preserve the norm, we have that if $Y = h_n(X)$ then $|Y| = |X|$ and furthermore, for any Y such that $|Y| < \delta$ we obtain

$$|F_n(Y)| = \left|h_n\left(F_4\left(h_n^{-1}(Y)\right)\right)\right| = |h_n\left(F_4(X)\right)| = |F_4(X)| < \varepsilon\,|X| = \varepsilon\,|Y|$$

proving our claim, since $F_n(0,0) = (0,0)$.

Recall that in (3) and in the text thereafter the map F_n is made up by gluing different functions on sectors: in $S_{1,n}$ the expression of F_n is given by $h_n \circ F_4 \circ h_n^{-1}$ and in $S_{2,n}$ by $R_n \circ h_n \circ F_4 \circ h_n^{-1} \circ R_n^{-1}$. Both expressions define differentiable functions away from the origin since both h_n and h_n^{-1} are of class C^1 in $\mathbb{R}^2\backslash\{(0,0)\}$. We have already shown that F_n is differentiable at the origin. It remains to prove that the derivatives of the two functions coincide at the common boundary of $\partial S_{1,n}$ and $\partial S_{2,n}$. At the remaining boundaries the result follows from the $\mathbb{Z}_n$-equivariance of F_n.

Since we are working away from the origin, we may use polar coordinates. The expressions for h_n, R_n and their inverses take the simple forms below, where we use $\widehat{f}$ to indicate the expression of f using polar coordinates in both source and target:

$$\widehat{h}_n(r,\theta) = \left(r, \frac{4\theta}{n}\right) \qquad \widehat{h}_n^{-1}(r,\theta) = \left(r, \frac{n\theta}{4}\right)$$

$$\widehat{R}_n(r,\theta) = \left(r, \theta + \frac{2\pi}{n}\right) \qquad \widehat{R}_n^{-1}(r,\theta) = \left(r, \theta - \frac{2\pi}{n}\right)\ .$$

Let $\widehat{F}_4(r,\theta) = (R_4(r,\theta), \Phi_4(r,\theta))$ be the expression of F_4 in polar coordinates. From (1) we get:

(4)
$$R_4(r,\theta) = \frac{kr^3}{1+r^2}\sqrt{\cos^6\theta + \sin^6\theta} = \frac{kr^3}{1+r^2}\sqrt{1 - 3\cos^2\theta + 3\cos^4\theta}$$

(5)
$$\Phi_4(r,\theta) = \begin{cases} \arctan\left(-\dfrac{\cos^3\theta}{\sin^3\theta}\right) & \text{if} \quad \theta \neq k\pi \\[2ex] \operatorname{arccot}\left(-\dfrac{\sin^3\theta}{\cos^3\theta}\right) & \text{if} \quad \theta \neq \frac{\pi}{2} + k\pi\ . \end{cases}$$

The derivative $D\widehat{F}_4(r,\theta)$ of $\widehat{F}_4$ is thus,

(6)

$$\begin{pmatrix} kr^2\dfrac{3+r^2}{(1+r^2)^2}\sqrt{\cos^6\theta+\sin^6\theta} & \dfrac{kr^3}{1+r^2}\dfrac{3\sin\theta\cos\theta\left(\sin^4\theta-\cos^4\theta\right)}{\sqrt{\cos^6\theta+\sin^6\theta}} \\ & \\ 0 & \dfrac{3\sin^2\theta\cos^2\theta}{\cos^6\theta+\sin^6\theta} \end{pmatrix}$$

where the two alternative forms for $\Phi_4(r,\theta)$ yield the same expression for the derivative.

Note that the Jacobian matrix of $\widehat{h}_n$ is constant and the same is true for its inverse. The derivatives of both $\widehat{R}_n$ and of $\widehat{R}_n^{-1}$ are the identity. Let $(r, 2\pi/n)$ be the polar coordinates of a point ξ in $(\partial S_{1,n}\cap\partial S_{2,n})\setminus\{0\}$. In order to show that the derivatives at ξ of $\widehat{h}_n\circ\widehat{F}_4\circ\widehat{h}_n^{-1}$ and of $\widehat{R}_n\circ\widehat{h}_n\circ\widehat{F}_4\circ\widehat{h}_n^{-1}\circ\widehat{R}_n^{-1}$ coincide, we only need to show that $D\widehat{F}_4$ at $\widehat{h}_n^{-1}(r,2\pi/n)=(r,\pi/2)$ equals $D\widehat{F}_4$ at $\widehat{h}_n^{-1}(\widehat{R}_n^{-1}(r,2\pi/n))=(r,0)$. More precisely, for any (r,θ)

$$D\widehat{h}_n(r,\theta)=A_n=\begin{pmatrix}1&0\\0&\frac{4}{n}\end{pmatrix}\qquad D\widehat{h}_n^{-1}(r,\theta)=B_n=\begin{pmatrix}1&0\\0&\frac{n}{4}\end{pmatrix}$$

and thus

$$\begin{aligned} & D\left(\widehat{R}_n\circ\widehat{h}_n\circ\widehat{F}_4\circ\widehat{h}_n^{-1}\circ\widehat{R}_n^{-1}\right)(\xi) \\ = \;& D\widehat{R}_n(\widehat{h}_n(\widehat{F}_4((r,0)))D\widehat{h}_n(\widehat{F}_4((r,0))D\widehat{F}_4(r,0)D\widehat{h}_n^{-1}(r,0)D\widehat{R}_n^{-1}(r,2\pi/n) \\ = \;& Id\cdot A_n\cdot D\widehat{F}_4(r,0)\cdot B_n\cdot Id \\ = \;& A_n\cdot D\widehat{F}_4(r,0)\cdot B_n \end{aligned}$$

and

$$\begin{aligned} & D\left(\widehat{h}_n\circ\widehat{F}_4\circ\widehat{h}_n^{-1}\right)(\xi) \\ = \;& D\widehat{h}_n(\widehat{F}_4((r,\pi/2))D\widehat{F}_4(r,\pi/2)D\widehat{h}_n^{-1}(r,2\pi/n) \\ = \;& A_n\cdot D\widehat{F}_4(r,\pi/2)\cdot B_n\ . \end{aligned}$$

From (6) it follows that

$$D\widehat{F}_4(r,\pi/2)=D\widehat{F}_4(r,0)=\begin{pmatrix} kr^2\dfrac{3+r^2}{(1+r^2)^2} & 0 \\ & \\ 0 & 0\end{pmatrix}$$

completing our proof. □

The construction in the proof of Theorem 2.1 only works because Szlenk's example F_4 has the special properties 6, 7 and 8 of Proposition 2.2. For instance, identifying $\mathbb{R}^2 \sim \mathbb{C}$ the map $f(z) = \overline{z}^3$ is $\mathbb{Z}_4$-equivariant, but does not have the properties above and $h_5 \circ f \circ h_5^{-1}(z) = f(z)$.

Alarcón *et al.* [1, Theorem 4.4] construct, starting from F_4, an example having the additional property that ∞ is a repelllor. The new example, $G(x, y)$, is of the form

$$G(x, y) = \phi(|F_4(x, y)|)F_4(x, y)$$

where $\phi : [0, \infty) \longrightarrow [0, \infty)$ is described in [1, Lemma 4.6].

Then G has all the properties of Proposition 2.2. Therefore, applying to G the construction of Theorem 2.1 we obtain the following:

Corollary 2.4. *For each $n \geq 2$ there exists a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying properties a)–e) of Theorem 2.1 and, moreover, for which ∞ is a repellor.*

A very interesting problem in Dynamical Systems is to describe the global dynamics with hypotheses based on local properties of the system. The Markus-Yamabe Conjecture is an example but not the only one. For instance, Alarcón *et al.* [1] prove the existence of a global attractor arising from a unique local attractor, using the theory of free homeomorphisms of the plane. Recently, Ortega and Ruiz del Portal in [13], have studied the global behavior of an orientation preserving homeomorphism introducing techniques based on the theory of prime ends. They define the rotation number for some orientation preserving homeomorphisms of $\mathbb{R}^2$ and show how this number gives information about the global dynamics of the system.

The theory of prime ends was introduced by Carathéodory in order to study the complicated shape of the boundary of a simply connected open subset of $\mathbb{R}^2$. When such a subset U is non empty and proper, U is homeomorphic to the open unit disk and Carathéodory's compactification associates the boundary of U with the space of prime ends $\mathbb{P}$, which is homeomorphic to $\mathbb{S}^1$. In that way, $U \cup \mathbb{P}$ is homeomorphic to the closed unit disk and if f is an orientation preserving homeomorphism with $f(U) = U$, then f induces an orientation preserving homeomorphism $\tilde{f}$ in $\mathbb{P}$. Since the space of prime ends is homeomorphic to the unit circle, the rotation number of $\tilde{f}$ is well defined and the rotation number of f is defined to be equal to the rotation number of $\tilde{f}$.

The points in $\partial_{\mathbb{S}^2}U$, the boundary of U in the one point compactification of the plane, that play an important role in the dynamics are accessible points. A point $\alpha \in \partial_{\mathbb{S}^2}U$ is *accessible* from U if there exists an arc ξ such that p is an end point of ξ and $\xi \setminus \{\alpha\} \subset U$. Then α determines a prime end $p(\alpha) \in \mathbb{P}$, which may not be unique, such that $\xi \setminus \{p\} \cup \{p(\alpha)\}$ is an arc in $U \cup \mathbb{P}$.

Accessible points are dense in $\partial_{\mathbb{S}^2}U$, but for instance, in the case of fractal boundaries there exist points which are not accessible from U. On the contrary, when the boundary is well behaved, for instance an embedded curve of $\mathbb{R}^2$, accessible points define a unique prime end. That means that accessible periodic points of f are periodic points of $\tilde{f}$ with the same period. Consequently the rotation number of f is 1 divided the period. See [14] and [4] for more details and definitions.

Proposition 2.5. *The examples F_n in Theorem 2.1 have rotation number $1/n$.*

Proof. By construction of the maps in Theorem 2.1, the basin of attraction of the origin

$$U_n = \bigcup_{j=0}^{n-1} R_n^j \left(h_n(U) \cap S_{1,n}\right)$$

is invariant by the map F_n and is a non empty and proper simply connected open set. Moreover, as the periodic point P is hyperbolic, the boundary of U is an embedded curve of $\mathbb{R}^2$ in a neighborhood of P. In addition, P is an accessible point from U_n, thus the rotation number of F_n is $\frac{1}{n}$. □

The fact that the symmetry forces the maps in Theorem 2.1 to have a rational rotation number seems to point out at a connection between symmetry and rotation number. It raises the question: for orientation preserving homeomorphisms of the plane with a non global asymptotically stable fixed point, does $\mathbb{Z}_n-$equivariance imply a rational rotation number?

The question is relevant because the rotation number gives strong information about the global dynamics of the system. For instance, consider a dissipative orientation preserving $\mathbb{Z}_n-$equivariant homeomorphism f of the plane with an asymptotically stable fixed point p. If the question has an affirmative answer, then Proposition 2 of [13] implies that p is a global attractor under f if and only if f has no other periodic point.

Acknowledgements. The research of all authors at Centro de Matemática da Universidade do Porto (CMUP) had financial support from the European Regional Development Fund through the programme COMPETE and from the Portuguese Government through the Fundação para a Ciência e a Tecnologia (FCT) under the project PEst-C/MAT/UI0144/2011. B. Alarcón was also supported from Programa Nacional de Movilidad de Recursos Humanos of the Plan Nacional de I+D+I 2008-2011 of the Ministerio de Educación (Spain) and grant MICINN-08-MTM2008-06065 of the Ministerio de Ciencia e Innovación (Spain).

References

[1] B. Alarcón, V. Guíñez and C. Gutierrez. Planar Embeddings with a globally attracting fixed point. *Nonlinear Anal.*, 69:(1), 140-150, 2008.

[2] B. Alarcón, C. Gutierrez and J. Martínez-Alfaro. Planar maps whose second iterate has a unique fixed point. *J. Difference Equ. Appl.*, 14:(4), 421-428, 2008.

[3] J. Bernat and J. Llibre, Counterexample to Kalman and Markus-Yamabe conjectures in dimension 4, Discrete of Continuous, Discrete and Impulsive Systems, 2, 337-379, 1996.

[4] Cartwright, M. L., Littlewood, J. E.: Some fixed point theorems. Ann. of Math. 54, 1–37 (1951)

[5] A. Cima, A. van den Essen, A. Gasull, E.-M. G. M. Hubbers and F. Maï¿½osas, A polynomial counterexample to the Markus-Yamabe conjecture, Advances in Mathematics, 131 (2), 453- 457, 1997.

[6] A. Cima, A. Gasull and F. Mañosas. The Discrete Markus-Yamabe Problem, *Nonlinear Analysis*, 35, 343-354, 1999.

[7] R. Fessler, A solution to the two dimensional Global Asymptotic Jacobian Stability Conjecture, Annales Polonici Mathematici, 62, 45-75, 1995.

[8] M. Golubitsky, I. Stewart and D.G. Schaeffer. Singularities and Groups in Bifurcation Theory Vol. 2. *Applied Mathematical Sciences*, 69, Springer Verlag, 1985.

[9] C. Gutierrez, A solution to the bidimensional Global Asymptotic Stability Conjecture, Ann. Inst. H. Poincarï¿½. Anal. Non Linï¿½aire, 12 (6), 627-672, 1995.

[10] J. LaSalle and S. Lefschetz, Stability by Liapunov's direct method with applications, Academic Press, 1961.

[11] J. LaSalle, The stability of dynamical systems, CBMS-NSF Regional Conference Series in Applied Math., vol. 25, 1976.

[12] L. Markus and H. Yamabe, Global stability criteria for differential systems, Osaka Math. Journal, 12, 305-317, 1960.

[13] R. Ortega and F. R. Ruiz del Portal, Attractors with vanishing rotation number, J. European Math. Soc., 13(6), 1569-1590, 2011.

[14] Ch. Pommerenke, Boundary Behaviour of Conformal Maps. Grundlehren Math. Wiss. 299, Springer (1991).

B. Alarcón — CMUP; Rua do Campo Alegre, 687; 4169-007 Porto; Portugal — permanent address: Departament of Mathematics. University of Oviedo; Calvo Sotelo s/n; 33007 Oviedo; Spain
E-mail address: alarconbegona@uniovi.es

S.B.S.D. Castro — A CMUP and FEP.UP; Rua Dr. Roberto Frias; 4200-464 Porto; Portugal
E-mail address: sdcastro@fep.up.pt

I.S. Labouriau — CMUP and FCUP; Rua do Campo Alegre, 687; 4169-007 Porto; Portugal
E-mail address: islabour@fc.up.pt