Construction of Symmetric Heteroclinic Cycles

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Abstract

This paper studies sufficient conditions for the existence of persistent heteroclinic cycles in symmetric problems. If Γ is a compact Lie group acting linearly on $\mathbf{R}^n$, we present conditions on the shape of the isotropy lattice and the action of Γ on certain fixed-point spaces that guarantee the existence of a Γ-equivariant vector field with a persistent heteroclinic cycle. We use and establish properties of polar actions of Lie groups to construct the heteroclinic cycle. Examples that illustrate the application of the main result are provided.

1 Introduction

The existence of heteroclinic or homoclinic cycles in systems with symmetry is no longer a surprising feature. A good survey of the literature up until 1997 can be found in Krupa [10]. There are however issues concerning such cycles which still justify attention.

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One issue, which we shall not address, concerns the dynamics near such a cycle, especially when the cycle is part of a more complicated object such as a heteroclinic network. This has been done, for instance, by Aguiar *et al.* [1].

Another, that of the asymptotic stability of the cycle, has been addressed by Krupa and Melbourne [12, 13]. These authors provide a systematic classification of certain cycles forced by symmetry and necessary and/or sufficient conditions for the asymptotic stability of an existing cycle, existence conditions not being their concern. They deal both with discrete and continuous symmetry groups obtaining results which are more complete in the discrete case. The necessary and sufficient conditions are obtained for simple heteroclinic cycles in $\mathbf{R}^4$ — those with heteroclinic connections contained in 2-dimensional fixed-point spaces.

A third issue concerns bifurcation from a cycle. This is considered by Chossat *et al.* [3] for homoclinic cycles. Again the existence of such a cycle is assumed without further comments.

It becomes apparent that the existence of the objects whose stability and bifurcation draw considerable attention should be addressed. The existence problem has been studied in a systematic way in a series of articles by Sottocornola [15, 16, 17]. These articles provide a classification of homoclinic cycles in spaces of dimension less than or equal to 5. The case of heteroclinic cycles is not addressed and the homoclinic cycles classified are simple — all equilibria are on coordinate axes and the invariant planes and hyperplanes are the coordinate ones. The classification is complete in dimension 4 but the techniques used do not extend to higher dimensions (see [17]). The existence problem is extended to the case of continuous symmetry groups by Ashwin and Montaldi [2] where group theoretic conditions are provided that guarantee the existence of robust homoclinic cycles among relative equilibria.

Traditionally, the existence of heteroclinic (and homoclinic) cycles has been proved in very much an *ad-hoc* basis. There are several examples of cycles arising in problems equivariant under the action of specific symmetry groups. Similarities among some of these constructions may be found *a posteriori* and the systematization of these methods is the purpose of this paper. We provide general conditions for the existence of such cycles.

Our main result shows the existence of equivariant vector fields possessing a persistent heteroclinic cycle under some hypotheses on the action of the symmetry group. These hypotheses concern

- the shape of the isotropy lattice (this was already mentioned by Melbourne *et al.* [14] and is a recurring feature in many examples);

- the action on fixed-point spaces which we ask to be polar with regular orbits of codimension two – this guarantees the existence of 2-dimensional Cartan subspaces inside the fixed-point space;

- the action on the Cartan subspaces of the group acting on the corresponding fixed-point spaces.

These hypotheses are not very restrictive. The abundance of polar groups is illustrated in Dadok [4]. The geometry of the isotropy lattice is a standard assumption and the further hypotheses on the group action are satisfied by a wide variety of examples, some of which are presented in section 6.

This paper is organized as follows: in the next section we introduce preliminary concepts and results. Section 3 is devoted to the statement of the main result together with an additional definition and some comments on the hypotheses. Sections 4 and 5 contain the proof of the main theorem, with all the technical results concerning polar actions grouped in section 5. Section 4 is divided in two subsections: the first showing that the dynamics inside a fixed-point space exhibits heteroclinic connections and the second proving that this behaviour remains true when considering all relevant fixed-point spaces simultaneously. In section 6 we illustrate the application of our main result by proving the existence of cycles of various types in two examples, one of which is well-known in the literature. The paper finishes with a discussion of the scope of our results.

2 Definitions and preliminary results

Let f be a vector field on $\mathbf{R}^n$. A compact set A, invariant for the flow of f, is called here an *invariant saddle* if

$$A \subset \overline{W^s(A) \backslash A}, \overline{W^u(A) \backslash A}.$$

Note that A need not be an equilibrium of f, and in case it is, it does not have to be a saddle in the usual sense. Let $A_i, A_j \in \mathbf{R}^n$ be two invariant saddles and $W^u(A_i) \cap W^s(A_j) \neq \emptyset$. A trajectory in $W^u(A_i) \cap W^s(A_j)$ is called a *heteroclinic connection* from A_i to A_j. We want to think of a heteroclinic cycle as a sequence of heteroclinic connections, using the definition of Field [5]:

Definition 2.1 ([5], 6.7) Suppose that $\mathcal{A} = \{A_i : i = 0, ..., n-1\}$ is a finite ordered set of mutually disjoint invariant saddles. We say that there is a *heteroclinic cycle* associated to $\mathcal{A}$ if

$$W^u(A_i) \cap W^s(A_{i+1}) \neq \emptyset, \ i \geq 0 \ (\text{mod } n).$$

We refer to the set $C \subset \mathbf{R}^n$ defined by

$$C = \cup_{i=0}^{n-1} A_i \cup [W^u(A_i) \cap W^s(A_{i+1})],$$

as the (maximal) heteroclinic cycle determined by $\mathcal{A}$.

We shall be particularly interested in heteroclinic cycles arising in problems with symmetry. Let Γ be a compact Lie group acting linearly on $\mathbf{R}^n$, as a smooth representation of a subgroup of $\mathbf{O}(n)$ and suppose $f : \mathbf{R}^n \to \mathbf{R}^n$ satisfies $f(\gamma x) = \gamma f(x)\ \forall \gamma \in \Gamma$. In this case we say that f is Γ-*equivariant* or that f *commutes with* Γ. For $\Sigma \subset \Gamma$ a subgroup of Γ, we define the fixed-point subspace

$$\text{Fix}\ \Sigma = \{x \in \mathbf{R}^n : \sigma \cdot x = x\ \forall \sigma \in \Sigma\}.$$

This subspace is a vector subspace of $\mathbf{R}^n$ and is invariant by f and by the flow of f.

We say that a heteroclinic cycle is *persistent* if each connection is either a transverse intersection or it takes place inside a fixed-point subspace, where it is transverse.

The isotropy subgroup of $x \in \mathbf{R}^n$ is the subgroup of Γ defined by

$$\Gamma_x = \{\gamma \in \Gamma : \gamma \cdot x = x\}.$$

Suppose that C is a heteroclinic cycle and $\mathcal{A}$ is the associated set of invariant saddles. Let Γ_C denote the isotropy subgroup of C, that is

$$\Gamma_C = \{\gamma \in \Gamma : x \in C \Rightarrow \gamma \cdot x \in C\}.$$

The group Γ_C acts on the set of invariant saddles $\mathcal{A}$. If this action is transitive, *i.e.,* for any given $A_i, A_j \in \mathcal{A}$, there is $\gamma \in \Gamma_C$ such that $\gamma(A_i) = (A_j)$, then C is called a *homoclinic cycle* (Field [5]).

Given a vector subspace $V \subset \mathbf{R}^n$, we will denote by $N_\Gamma(V)$ the subgroup that acts on V defined by

$$N_\Gamma(V) = \{\gamma \in \Gamma : \gamma \cdot x \in V \text{ for all } x \in V\}$$

and by $Z_\Gamma(V)$ the subgroup of Γ whose elements fix the points in V, defined by

$$Z_\Gamma(V) = \{\gamma \in \Gamma : \gamma \cdot x = x \text{ for all } x \in V\}.$$

If $V = \text{Fix}(\Sigma)$ for some isotropy subgroup Σ then $N_\Gamma(V) = N(\Sigma)$, the normalizer in Γ of Σ, and $Z_\Gamma(V) = \Sigma$. Thus, $N_\Gamma(V)/Z_\Gamma(V)$ represents the *effective action* of Γ on V.

Let Γ be a compact Lie group, and $\rho : \Gamma \longrightarrow \mathbf{O}(n)$ a smooth representation on the real vector space $\mathbf{R}^n$ with inner product $\langle .,. \rangle$. For $x \in \mathbf{R}^n$ define the Γ-*orbit* of x as $\Gamma(x) = \{\gamma \cdot x : \gamma \in \Gamma\}$ and denote by $T_x\Gamma(x)$ the subspace of $\mathbf{R}^n$ of vectors tangent to $\Gamma(x)$; let $N_x = (T_x\Gamma(x))^\perp$ be its orthogonal complement, that we call a *cross-section*, because it meets every Γ-orbit in $\mathbf{R}^n$ (Lemma 1 of Dadok [4]). To obtain a cross-section of minimal dimension we choose x on an orbit of maximal dimension; such points are called *regular*. If a point is not regular, we call it *singular*.

We are interested in representations, called *polar*, for which the minimal dimensional cross-sections are all Γ-conjugate. In this case, the minimal cross-sections are called *Cartan subspaces*. Another characterization is that all orbits under polar actions intersect a Cartan subspace orthogonally (Dadok [4, Proposition 1]). Section 5 presents the statement and proof of new results on polar actions needed for our construction.

3 Statement of the main result

In this section we describe some classes of symmetric systems where persistent heteroclinic networks are to be found.

Definition 3.1 A *heteroclinic cycle of isotropy subgroups* of a compact Lie group Γ is a sequence of distinct conjugacy classes of isotropy subgroups T_i, with $i = 1, ..., k$, $k \geq 2$ and Σ_j, with $j = 1, ..., k$, where the T_i and the Σ_j are related as follows:

$$\begin{tikzcd}
T_1 & & T_2 & & \cdots & & T_k & & T_{k+1} \\
& \Sigma_1 \arrow[ul] \arrow[ur] & & \Sigma_2 \arrow[ul] \arrow[ur] & \cdots & \Sigma_{k-1} \arrow[ul] \arrow[ur] & & \Sigma_k \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

with the classes T_{k+1} and T_1 equal.

We assume:
(H1) $\mathrm{Fix}(\Gamma) = \{0\}$ and on each $\mathrm{Fix}(\Sigma_j)$ the action of $N_\Gamma(\Sigma_j)/\Sigma_j = G_j$ is polar and the regular orbits for this action have codimension two on $\mathrm{Fix}(\Sigma_j)$.

Suppose that Γ is a compact Lie group with a heteroclinic cycle of isotropy subgroups satisfying (H1). We may conclude that for each j there exists $V_j \subset \mathrm{Fix}(\Sigma_j)$ such that V_j is a 2-dimensional vector subspace intersecting all G_j-orbits. It follows, because the G_j action is effective that V_j intersects all Γ-orbits lying in $\mathrm{Fix}(\Sigma_j)$.

Let $l_{1j} = \mathrm{Fix}(T_j) \cap V_j$ and $l_{2j} = \mathrm{Fix}(T_{j+1}) \cap V_j$. In order to construct heteroclinic cycles we will need two additional hypotheses on the group action. They are:

(H2) $\dim l_{ij} = 1$ for $i = 1, 2$, $j = 1, \ldots, k+1$.

(H3) For $G_j = N_\Gamma(\Sigma_j)/\Sigma_j$, we have $N_{G_j}(V_j)/Z_{G_j}(V_j) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ and each l_{ij} is the fixed-point subspace of one of the two subgroups $\mathbf{Z}_2$.

Note that in (H2) we are only ruling out the possibility that $\dim \mathrm{Fix}(T_j) = 2$, with V_j conjugate to $\mathrm{Fix}(T_j)$. As $\dim l_{ij} = 1$, then $N_{G_j}(l_{ij})$ can only act as $\mathbf{Z}_2$ or $\mathbf{1}$. That the action of $N_{G_j}(l_{ij})$ on l_{ij} is nontrivial is guaranteed by (H3).

Theorem 3.2 *Let Γ be a compact Lie group acting linearly on $\mathbf{R}^n$ with a heteroclinic cycle of isotropy subgroups satisfying* (H1–H3). *Then there are smooth Γ-equivariant vector fields on $\mathbf{R}^n$ with a persistent heteroclinic cycle.*

In the next section we will prove theorem 3.2. The proof constructs a vector field f and is based on the fact that the spaces $\mathrm{Fix}(\Sigma_j)$ are invariant by any Γ-equivariant vector field. The proof takes the following steps:

1. We work on each subspace $\mathrm{Fix}(\Sigma_j)$ and use the normal form of degree three of an arbitrary $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-equivariant vector field, on $V_j \subset \mathrm{Fix}(\Sigma_j)$, denoted by f_{V_j}. We establish conditions satisfied in an open set of normal forms so that the equilibria of f_{V_j} have the desired stability and we use Poincaré-Bendixson-like arguments to show the existence of saddle-sink connections on $V_j \subset \mathrm{Fix}(\Sigma_j)$ (section 4.1).

2. We show that the connections in V_j may be lifted to connections between relative equilibria of f in $\mathrm{Fix}(\Sigma_j)$ (section 5).

3. We show how the connections in each V_j give rise to a cycle in $\mathbf{R}^n$ (section 4.2).

4 Proof of theorem 3.2

We start by studying the dynamics of Γ-equivariant vector fields in the invariant spaces $\mathrm{Fix}(\Sigma_j)$.

4.1 Dynamics inside a fixed-point subspace

Let $X = \mathrm{Fix}(\Sigma_j)$, for arbitrary $j = 1, \ldots, k$, where Γ has the effective action of $G = N(\Sigma_j)/\Sigma_j$. For simplicity of notation in this subsection we denote by $V \subset X$ the Cartan subspace V_j and let $l_1 = l_{1j}$, $l_2 = l_{2j}$, and $T_1 = T_j, T_2 = T_{j+1}$. From (H1) we know that V is 2-dimensional.

For a Cartan subspace $V \subset X$, let $P : X \longrightarrow V$, $Q : X \longrightarrow V^\perp$, $Q = I - P$ be orthogonal projections. Given a smooth G-equivariant vector field f on X (the restriction of a Γ-equivariant vector field on $\mathbf{R}^n$) define a vector field f_V on V by the projection $f_V = P\left(f|_V\right)$ and write $f_T(x) = Q\left(f|_V(x)\right)$ so that for $x \in V$ we have $f(x) = f_V(x) + f_T(x)$. Note that f_V and f_T are as differentiable as f is. It can be shown that, since the action of G is polar, both f_V and f_T are G-equivariant and that, at regular points, f_T is tangent to G-orbits. We postpone this proof to section 5 (Theorem 5.1).

Since $N_G(V)/Z_G(V)$ acts on V as $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ (cf. (H3)), the vector field f_V is $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-equivariant (Theorem 5.1).

Considering coordinates $x_1 \in l_1$ and $x_2 \in l_2$, the normal form of degree 3 for f_V on V is [6, Lemma X, 1.1]:

$$\begin{cases} \dot{x_1} = x_1(a_1\lambda + b_1 x_1^2 + d_1 x_2^2) \\ \dot{x_2} = x_2(a_2\lambda + c_2 x_1^2 + b_2 x_2^2) \end{cases} \tag{4.1}$$

and if the normal form is non-degenerate, then higher order terms are not important for the problem [7, Theorem XIV, 4.2]. It is shown in [14, Proposition 2.6] that if

$$a_1, a_2 > 0 \qquad b_1, b_2 < 0 \qquad C \equiv \frac{d_1 a_2}{b_2 a_1} + \frac{c_2 a_1}{b_1 a_2} > -2 \tag{4.2}$$

then, for $\lambda > 0$, all trajectories starting within a circle of radius $O(\sqrt{\lambda})$ stay bounded near the origin.

Note that, under the conditions (4.2) and for $\lambda > 0$, the origin is a source and equations (4.1) have equilibria $A_1^{\pm} = (\pm\sqrt{-a_1\lambda/b_1}, 0)$ and $A_2^{\pm} = (0, \pm\sqrt{-a_2\lambda/b_2})$ on the axes. Forcing $A_1^{\pm}$ to be saddles and $A_2^{\pm}$ to be sinks is equivalent to choosing

$$a_2 - c_2\frac{a_1}{b_1} > 0 \ \text{ and } \ a_1 - d_1\frac{a_2}{b_2} < 0. \tag{4.3}$$

If we reverse both inequalities of (4.3), the equilibria $A_1^{\pm}$ become sinks and $A_2^{\pm}$ become saddles. A direct calculation using Cramer's rule shows that for a vector field of the form (4.1) satisfying (4.2) and (4.3) and for $\lambda > 0$ there are no equilibria outside the axes. The same is true if all the inequalities in (4.3) are reversed.

Lemma 4.1 *Consider a vector field of the form (4.1) satisfying (4.2) and (4.3). Then for $\lambda > 0$ there is a saddle-sink connection from A_1^+ to A_2^+.*

Proof The ω-limit set of the unstable manifold of A_1^+, $\omega(W^u(A_1^+))$, cannot contain the origin since this is a source. The y axis is a fixed-point subspace, therefore flow invariant, so A_2^+ cannot be an equilibrium surrounded by a periodic orbit.

There are no equilibria on the sector $x > 0$, $y > 0$, so we conclude that $\omega(W^u(A_1^+))$ cannot contain any periodic orbit. The only alternative, since there are no invariant lines in the sector, is that $\omega(W^u(A_1^+))$ contains A_2^+. We conclude that there exists a saddle-sink connection from A_1^+ to A_2^+. ∎

A point $x \in V$ in the f_V-trajectory connecting A_1^+ to A_2^+ must be regular for the $N_G(V)/Z_G(V)$-action — otherwise it would lie in a fixed-point subspace of V and, since these are flow-invariant, there could be no connection. We will show (Theorem 5.3) that the f-trajectory of the same point x connects points in $G(A_1^+)$ to points in $G(A_2^+)$. This proof uses different techniques and appears in section 5 below.

4.2 Cycling the fixed-point subspaces

We return to the study of Γ-equivariant vector fields f in $\mathbf{R}^n$. So far, we have obtained conditions on the coefficients of each f_{V_j} for the existence of trajectories connecting two group orbits. It remains to see that these conditions may be satisfied consistently in all the Cartan subspaces V_j, thus closing the cycle.

To do this, we want to identify the two lines $l_{2j} \subset V_j$ and $l_{1(j+1)} \subset V_{j+1}$ (figure 1) and define a projected vector field in $V_j \cup V_{j+1}$. This amounts to choosing a representative of the equivalence class of Cartan subspaces under the Γ action and identifying the lines where they meet each representative of $\mathrm{Fix}(T_j)$, dealing more carefully with isotropy classes.

Lemma 4.2 *Consider the following portion of the heteroclinic cycle of isotropy subgroups of Γ, satisfying (H1–H3)*

$$\begin{tikzcd} & T & \\ \Sigma_1 \arrow[ur] & & \Sigma_2 \arrow[ul] \end{tikzcd}$$

and let $\widehat{\Sigma}_j$, $j = 1, 2$ be representatives of Σ_j with 2-dimensional Cartan subspaces V_j for the action of G_j on $\mathrm{Fix}(\widehat{\Sigma}_j)$. Then there are two representatives $\widehat{T}$ and $\widetilde{T}$ of T with $\widehat{\Sigma}_1 < \widehat{T}$, $\widehat{\Sigma}_2 < \widetilde{T}$ and $\gamma\widehat{T}\gamma^{-1} = \widetilde{T}$ for some $\gamma \in \Gamma$ such that $l = V_1 \cap \mathrm{Fix}(\widehat{T})$ is a line satisfying $\gamma \cdot l = V_2 \cap \mathrm{Fix}(\widetilde{T})$.

If f is a Γ-equivariant vector field then for any $x \in l$ we have $f_{V_1}(x) = f_{V_2}(\gamma \cdot x)$ in suitably chosen coordinates.

Proof Let $\widehat{\Sigma}_1$ and $\widehat{\Sigma}_2$ be representatives of Σ_1 and Σ_2 and let $\overline{T}$, $\overline{\overline{T}}$ be two representatives of T with $\widehat{\Sigma}_1 < \overline{T}$ and $\widehat{\Sigma}_2 < \overline{\overline{T}}$, and $\gamma_1 \in \Gamma$ such that $\gamma_1\overline{T}\gamma_1^{-1} = \overline{\overline{T}}$. Then $\mathrm{Fix}(\overline{T}) \subset \mathrm{Fix}(\widehat{\Sigma}_1)$ and $\mathrm{Fix}(\overline{\overline{T}}) \subset \mathrm{Fix}(\widehat{\Sigma}_2)$.

Let V_j be Cartan subspaces in $\mathrm{Fix}(\widehat{\Sigma}_j)$, $j = 1, 2$, respectively. Then either $\mathrm{Fix}(\overline{T}) \cap V_1$ is a line l (in which case $\overline{T} = \widehat{T}$) or we may take another representative $\widehat{T} = \gamma_2^{-1}\overline{T}\gamma_2$ of T, with $\gamma_2 \in N_\Gamma(\widehat{\Sigma}_1)$ for which this is true, since V_1 is a Cartan subspace and thus meets every $N_\Gamma(\widehat{\Sigma}_1)$-orbit. It follows that $\widehat{\Sigma}_1 < \widehat{T}$ and that $\delta\widehat{T}\delta^{-1} = \overline{\overline{T}}$ for $\delta = \gamma_1 \cdot \gamma_2$.

If $x \in \mathrm{Fix}(\widehat{T}) \cap V_1$ then $\delta \cdot x \in \mathrm{Fix}(\overline{\overline{T}}) \subset \mathrm{Fix}(\widehat{\Sigma}_2)$. Again if $\delta \cdot x \notin V_2$ then for some $\gamma_3 \in N_\Gamma(\widehat{\Sigma}_2)$ we have $\gamma_3 \cdot \delta \cdot x \in V_2$ and we may take $\widetilde{T} = \gamma_3\overline{\overline{T}}\gamma_3^{-1}$ for the second representative of T, with $\gamma = \gamma_3 \cdot \gamma_2 \cdot \gamma_1$. Thus we have $\gamma \cdot x \in \mathrm{Fix}(\widetilde{T}) \cap V_2$ and if $x \in l$, $x \neq 0$, it follows by linearity that $\gamma \cdot l = \mathrm{Fix}(\widetilde{T}) \cap V_2$.

Each one of the lines l and $\gamma \cdot l$ is a fixed-point subspace for an isotropy subgroup of $N_{G_j}(V_j)/Z_{G_j}(V_j) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$, $j = 1, 2$. Thus $f_{V_1}(x) \in l$ and $f_{V_2}(\gamma \cdot x) \in \gamma \cdot l$, for any

$x \in l$, by equivariance of f_{V_j} and invariance of fixed-point subspaces. It remains to see that $f_{V_1}(x) = f_{V_2}(\gamma \cdot x)$.

Choose orthonormal bases (b_1, b_2^j, u^j, v^j) for $\mathrm{Fix}(\Sigma_j)$, $j = 1, 2$ (where u^j and v^j each spans a subspace that may have dimension higher than one) as follows:

- (b_1, b_2^1) is an orthonormal basis for V_1
  $(\gamma \cdot b_1, b_2^2)$ is an orthonormal basis for V_2
- (b_1, u^1) is an orthonormal basis for $\mathrm{Fix}(\widehat{T})$
  $(\gamma \cdot b_1, u^2)$ is an orthonormal basis for $\mathrm{Fix}(\widetilde{T})$
- $v^1 \in (V_1 + \mathrm{Fix}(\widehat{T}))^\perp \subset V_1^\perp$
  $v^2 \in (V_2 + \mathrm{Fix}(\widetilde{T}))^\perp \subset V_2^\perp$.

For $x \in l$, we have $f(x) \in \mathrm{Fix}(\widehat{T})$ with coordinates $f(x) = (\alpha, 0, \beta, 0)$ in the first basis above and $f(\gamma \cdot x) = \gamma \cdot f(x) = (\alpha, 0, \beta, 0) \in \mathrm{Fix}(\widetilde{T})$ in the second basis. Note that $P_j(f(x)) = f_{V_j}(x)$ and so, for $x \in l$, $P_1(f(x)) = (\alpha, 0) = P_2(f(\gamma \cdot x))$. ∎

It follows from this lemma that if $V_1 \cap V_2 = l$, i.e. if γ acts as the identity on l, then f_{V_1} and f_{V_2} define a consistent vector field on $V_1 \cup V_2$ (figure 1). It follows that the projected vector fields f_{V_j} define a vector field on the set $(V_1 \cup \ldots \cup V_k)/(l_{2j} = l_{1(j+1)})$ (mod $k + 1$). Theorem 5.1 shows that the projection does not depend on the choice of Cartan subspace.

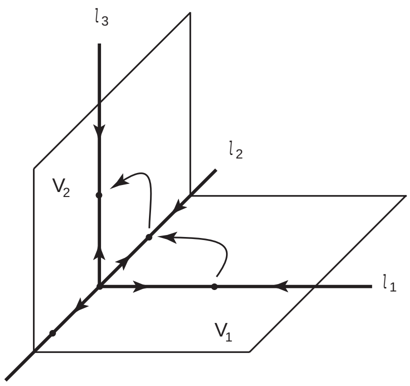

Figure 1: Connection between the equilibria of f_{V_j} on $V_1 \cup V_2$.

To guarantee the necessary stability for the existence of saddle-sink connections we have chosen coefficients inside each V_j. To finish the proof we show that this choice may be made globally for the vector field f.

Choosing coordinates (x_j, x_{j+1}) on V_j, the degree 3 normal form (4.1) yields:
-on V_1:

$$\begin{cases} \dot{x_1} = x_1(a_1\lambda + b_1x_1^2 + d_1x_2^2) \\ \dot{x_2} = x_2(a_2\lambda + c_2x_1^2 + b_2x_2^2) \end{cases}$$

- on V_2:

$$\begin{cases} \dot{x_2} = x_2(a_2\lambda + b_2x_2^2 + d_2x_3^2) \\ \dot{x_3} = x_3(a_3\lambda + c_3x_2^2 + b_3x_3^2) \end{cases}$$

...
- on V_n:

$$\begin{cases} \dot{x_n} = x_n(a_n\lambda + b_nx_n^2 + d_nx_1^2) \\ \dot{x_1} = x_1(a_1\lambda + c_{n+1}x_n^2 + b_1x_1^2) \end{cases}$$

Suppose $\lambda > 0$ and $a_j > 0, b_j < 0, j = 1, \ldots, n$. These inequalities imply the existence of equilibria

$$A_j^{\pm} = (0, \ldots, 0, \pm\sqrt{-a_j\lambda/b_j}, 0, \ldots, 0)$$

on the x_j axis $j = 1, \ldots, n$. On each Cartan subspace V_i the equilibria are a saddle on x_i axis and a sink on the x_{i+1} axis. This is achieved with the following conditions on the coefficients:

$$a_{i+1} - c_{i+1}\frac{a_i}{b_i} > 0 \qquad a_i - d_i\frac{a_{i+1}}{b_{i+1}} < 0,$$

respectively, where $a_{n+1} = a_1$, $b_{n+1} = b_1$.

These conditions are satisfied if in addition we choose $c_{j+1} > 0$ $(j = 1, \ldots, n)$ and

$$d_1 < \frac{a_1b_2}{a_2} < 0, \quad d_2 < \frac{a_2b_3}{a_3} < 0, \ldots, d_n < \frac{a_nb_1}{a_1} < 0.$$

Looking now at the boundedness condition (4.2), using $d_1 < \frac{a_1b_2}{a_2} < 0$ we may conclude that $0 < \frac{d_1a_2}{b_2a_1} < 1$. Therefore $\frac{c_2a_1}{b_1a_2} > -2$ implies $C > -2$, so, by choosing a_1, b_1, a_2, c_2 conveniently, these inequalities are satisfied. Similar inequalities for the remaining Cartan subspaces $(V_j, j = 2, \ldots, n)$ are satisfied by choosing $a_j, b_j, a_{j+1}, c_{j+1}$ conveniently.

Summarizing, sufficient conditions for the existence of the connections are:

$$\lambda > 0, \quad a_j > 0, \quad b_j < 0, \quad c_{j+1} > 0, \quad d_j < \frac{a_jb_{j+1}}{a_{j+1}} \text{ and } \frac{c_{j+1}a_j}{b_ja_{j+1}} > -2,$$

which are all compatible in an open subset of the space of coefficients.

We conclude that there is an open set in the space of coefficients for which on each V_j we have exactly five equilibria (including the origin) with the stability type chosen above. Moreover, there is a disc on each V_j containing the origin and attracting the flow. We have already claimed that each connection for the f_{V_j}-flow inside V_j may be lifted inside $\mathrm{Fix}(\Sigma_j)$ to a connection of relative equilibria for the flow of f. By taking Γ-orbits we obtain the desired persistent cycle (see remark 5.4 below) completing the proof of Theorem 3.2.

5 Polar actions

In this section we prove the results on polar actions that have been used in the proof of Theorem 3.2. For this, let G be a compact Lie group with a polar action on the finite-dimensional space X and $V \subset X$ a Cartan subspace for this action. Recall that for $x \in V$ we have defined $f_V(x) = P\left(f|_V(x)\right)$ and $f_T(x) = Q\left(f|_V(x)\right)$, with $f(x) = f_V(x) + f_T(x)$, where $P : X \longrightarrow V$, $Q : X \longrightarrow V^{\perp}$, $P + Q = I$ are orthogonal projections, so f_V defines a vector field in V and f_T is a section of the normal bundle of V that is tangent to G-orbits at regular points.

Theorem 5.1 *Let G be a compact Lie group with a polar action on the finite-dimensional vector space X with a Cartan subspace $V \subset X$ and let f be a smooth G-equivariant vector field on X. Then f_V and f_T are G-equivariant on V and if $x \in V$ is regular for the effective action of G on V then $f_T(x) \in T_xG(x)$. This decomposition does not depend on the choice of Cartan subspace.*

Proof If $x \in V$ is regular then $(T_xG(x))^{\perp} = V$ and thus $f_T(x) = Qf(x)$ is tangent to $G(x)$. For a regular point $x \in V$ and $\gamma \in G$ such that $\gamma \cdot x \in V$ we have

$$f(\gamma \cdot x) = \gamma \cdot f(x) = \gamma \cdot Pf(x) + \gamma \cdot Qf(x).$$

Since $Qf(x) \in T_xG(x)$ then $\gamma \cdot Qf(x) \in T_{\gamma \cdot x}G(\gamma \cdot x)$ and, by orthogonality of γ, it follows that $\gamma \cdot Pf(x) \in V$. Therefore

$$\gamma \cdot f_V(x) = \gamma \cdot Pf(x) = Pf(\gamma \cdot x) = f_V(\gamma \cdot x)$$

and

$$\gamma \cdot f_T(x) = \gamma \cdot Qf(x) = Qf(\gamma \cdot x) = f_T(\gamma \cdot x)$$

showing that f_T and f_V are G-equivariant at regular points.

Since regular points form an open and dense subset of X they are also dense in V and thus the result can be extended by continuity to singular points in V.

If V_1 and V_2 are two Cartan subspaces of X, then there is an element $\gamma \in G$ such that $V_2 = \gamma \cdot V_1$. The expression of f_{V_1} in coordinates with respect to a basis $\{b_1, b_2\}$ of V_1 is the same as that of f_{V_2} with respect to the basis $\{\gamma \cdot b_1, \gamma \cdot b_2\}$ of V_2, i.e., the decomposition $f = f_V + f_T$ is independent of the choice of Cartan subspace. ∎

The next step is to relate the limit behaviour of the flow $\varphi_f(t,x)$ of f to the flow $\varphi_{f_V}(t,x)$ of f_V. We apply the following result:

Theorem 5.2 (Krupa, [10], Theorems 2.1 and 2.2) *Let G be a compact group acting on the finite-dimensional vector space X and f a G-equivariant vector field on X. For each $x_0 \in X$ there exist a G-invariant neighborhood U of the orbit $G(x_0)$ in X, and smooth and G-equivariant vector fields f_K and f_N such that*

$$f(u) = f_K(u) + f_N(u)$$

for all u in U, satisfying $f_K(u) \in T_u G(u)$ for all u in U and with $f_N(x) \in N_x$ for all $x \in G(x_0)$. Moreover, there exists a smooth curve of group elements $\gamma(t)$ and a trajectory $y(t)$ of the restriction of f_N to the space N_{x_0} such that $\gamma(t)y(t) = u(t)$ for all $t \geq 0$ such that both $y(t)$ and $u(t)$ remain in the neighbourhood U.

Even though in general f_N is not necessarily normal to group orbits other than $G(x_0)$, this is true, however, at regular points x_0 if the action of G is polar. Then, on the Cartan subspace $V = N_{x_0}$ the two decompositions coincide, with $f_N = f_V$ and $f_K = f_T$. Note that if $x \in V$ is regular, so are all the points in the trajectories $\varphi_f(t,x)$ and $\varphi_{f_V}(t,x)$, by the equivariance of the vector fields f and f_V.

Let $\omega_{f_V}(x)$ and $\omega_f(x)$ denote the ω-limit sets of a point x under the flows of f_V and f, respectively. The next result shows how the two ω-limit sets are related.

Theorem 5.3 *Let G be a compact Lie group with a polar action on the finite-dimensional vector space X with a Cartan subspace $V \subset X$ and let f be a smooth G-equivariant vector field in X. Suppose there is a regular point $x \in V$ such that $\omega_{f_V}(x) = \{A\}$ with $A \in V$, $f_V(A) = 0$. Then $G(A)$ is a relative equilibrium for the flow of f, with $\omega_f(x)$ not empty and contained in $G(A)$.*

Proof First note that if $x \in V$ is regular and $\varphi_{f_V}(t,x)$ is defined for all $t > 0$, then the curve $\gamma(t)$ of Theorem 5.2 is also defined for all $t > 0$, since the half-trajectory $\{\varphi_{f_V}(t,x) \quad t \geq 0\}$ may be covered by a countable and locally finite family of neighbourhoods where the theorem holds.

If $\omega_{f_V}(x) = \{A\}$ then there exists a sequence $t_n \to +\infty$ such that $\varphi_{f_V}(t_n,x) \to A$ and by Theorem 5.2 there is a sequence $\gamma_n \in G$ such that $\varphi_f(t_n,x) = \gamma_n \cdot \varphi_{f_V}(t_n,x)$. Since G is compact, there is a converging subsequence $\gamma_{n_j} \to \gamma_\infty \in G$. Thus $\varphi_f(t_{n_j},x) = \gamma_{n_j} \cdot \varphi_{f_V}(t_{n_j},x)$ converges to $\gamma_\infty \cdot A$ and $\omega_f(x)$ is not empty.

A point y lies in $\omega_f(x)$ if and only if there is a sequence $t_n \to +\infty$ such that $\varphi_f(t_n, x) \to y$. If $\widetilde{\gamma}$ is any accumulation point of the sequence $(\gamma_n)^{-1} \in G$ such that $(\gamma_n)^{-1} \cdot \varphi_f(t_n, x) = \varphi_{f_V}(t_n, x)$ then taking the limit it follows that $\widetilde{\gamma} \cdot y = A$ and $y \in G(A)$.

For any $y \in \omega_f(x)$ we know that $\gamma \cdot y = A$ for some $\gamma \in G$ and thus $G(A) = G(y)$. Since ω-limit sets are flow invariant, then for any time s we have $\varphi_f(s, y) \in \omega_f(x) \subset G(A) = G(y)$. By the equivariance of f, this also holds for any $y \in G(A)$, showing that $G(A)$ is a f-relative equilibrium. ∎

Remark 5.4 Suppose that $\omega_{f_V}(x) = \{A_\omega\}$ and $\alpha_{f_V}(x) = \{A_\alpha\}$ are, respectively, the ω- and α-limit sets of a regular point x for the flow of f_V. Then the ω- and α-limit sets of x for the flow of f are not empty and contained in $G(A_\omega)$ and $G(A_\alpha)$, respectively, following the proof of the previous theorem. If A_α is a saddle and A_ω is a sink for the flow of f_V then $G(A_\omega)$ attracts all neighbouring f-trajectories and the connections from $G(A_\alpha)$ to $G(A_\omega)$ persist under small symmetry-preserving perturbations.

6 Examples

Example 1 The first example, studied by Melbourne *et al* [14], connects equilibria to periodic orbits. Let $\mathbf{O}(2) \times \mathbf{SO}(2)$ act on $\mathbf{R}^6 \equiv \mathbf{C}^3$ by:

$$(\phi, \theta).(z_0, z_1, z_2) = (e^{i\phi} z_0, e^{i(\theta+\phi)} z_1, e^{i(\theta-\phi)} z_2)$$

$$k.(z_0, z_1, z_2) = (\overline{z}_0, z_2, z_1).$$

Using the notation: $\mathbf{Z}_2(k) = \{1, k\}$, ${\mathbf{Z}_2}^c = \{1, (\pi, \pi)\}$, $\mathbf{Z}_2(k.(\pi, \pi)) = \{1, k.(\pi, \pi)\}$, we find in the isotropy lattice the heteroclinic cycle of isotropy subgroups:

$$\begin{tikzcd}
T_1 & & T_2 & & T_3 \sim T_1 \\
\mathbf{Z}_2(k) \times \mathbf{SO}(2) & & \mathbf{Z}_2(k) \oplus {\mathbf{Z}_2}^c & & \mathbf{Z}_2\left((\frac{\pi}{2}, 0) \cdot k \cdot (-\frac{\pi}{2}, 0)\right) \times \mathbf{SO}(2) \\
& \Sigma_1 \arrow[ul] \arrow[ur] & & \Sigma_2 \arrow[ul] \arrow[ur] & \\
& \mathbf{Z}_2(k) & & \mathbf{Z}_2(k \cdot (\pi, \pi)) &
\end{tikzcd}$$

The data for the construction of the heteroclinic cycle is given in the tables below, where $z_0, z_1, z_2 \in \mathbf{C}$ and $x, r \in \mathbf{R}$. For the subgroups Σ_j we give the type of regular G_j-orbits inside $\mathrm{Fix}(\Sigma_j)$:

Σ_j	$\mathrm{Fix}(\Sigma_j)$	$G_j = N_\Gamma(\Sigma_j)/\Sigma_j$	regular G_j orbit type	V_j
$\mathbf{Z}_2(k)$	(x, z_1, z_1)	$\mathbf{Z}_2(\pi, 0) \times \mathbf{SO}(2)$	circle	(x, r, r)
$\mathbf{Z}_2(k.(\pi, \pi))$	(ix, z_1, z_1)	$\mathbf{Z}_2(\pi, 0) \times \mathbf{SO}(2)$	circle	(ix, r, r)

To construct the table with the data for T_j we use T_1 to compute $\mathrm{Fix}(T_1) \cap V_1$ and its conjugate T_3 to compute $\mathrm{Fix}(T_1) \cap V_2$.

T_j	$\mathrm{Fix}(T_j)$	$\mathrm{Fix}(T_j) \cap V_1$	$\mathrm{Fix}(T_j) \cap V_2$
$T_1 = \mathbf{Z}_2(k) \times \mathbf{SO}(2)$	$(x, 0, 0)$	$(x, 0, 0)$	$(ix, 0, 0)$
$T_2 = \mathbf{Z}_2(k) \oplus \mathbf{Z}_2^c$	$(0, z_1, z_1)$	$(0, r, r)$	$(0, r, r)$

The action of G_j on $\mathrm{Fix}(\Sigma_j)$ is polar and G_j-regular orbits have codimension two in $\mathrm{Fix}(\Sigma_j)$. The effective action on V_j is that of $\mathbf{Z}_2 \times \mathbf{Z}_2$. Theorem 3.2 then guarantees the existence of a heteroclinic cycle connecting the orbit of a point in $\mathrm{Fix}(T_1)$ to the orbit of a circle contained in $\mathrm{Fix}(T_2)$.

Example 2 We construct a new example with a cycle connecting points and closed curves. Let $\Gamma = \mathbf{O}(2) \times \mathbf{O}(2)$ act on $\mathbf{R}^6 = \mathbf{R}^3 \times \mathbf{R}^3$ with the diagonal action (each $\mathbf{O}(2)$ acts on one $\mathbf{R}^3$ and fixes the other). The group $\mathbf{O}(2)$ acts on $\mathbf{R}^3$ by reflection on the xy plane and rotation on the z axis. Thus the group $\mathbf{O}(2)$ is generated by θ and k, where

$$\begin{aligned}
\theta \cdot (x, y, z) &= (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, z), \theta \in [0, 2\pi[\\
k \cdot (x, y, z) &= (x, y, -z).
\end{aligned}$$

The isotropy lattice has four maximal and six submaximal isotropy subgroups. It contains the following heteroclinic cycle of isotropy subgroups:

$$\begin{tikzcd}[column sep=tiny]
T_1 & & T_2 & & T_3 & & T_4 & & T_5 = T_1 \\
\mathbf{O}(2) \times \mathbf{Z}_2 & & \mathbf{O}(2) \times \mathbf{SO}(2) & & \mathbf{SO}(2) \times \mathbf{O}(2) & & \mathbf{Z}_2 \times \mathbf{O}(2) & & \mathbf{O}(2) \times \mathbf{Z}_2 \\
& \Sigma_1 \arrow[ul] \arrow[ur] & & \Sigma_2 \arrow[ul] \arrow[ur] & & \Sigma_3 \arrow[ul] \arrow[ur] & & \Sigma_4 \arrow[ul] \arrow[ur] & \\
& \mathbf{O}(2) \times \mathbf{1} & & \mathbf{SO}(2) \times \mathbf{SO}(2) & & \mathbf{1} \times \mathbf{O}(2) & & \mathbf{Z}_2 \times \mathbf{Z}_2 &
\end{tikzcd}$$

Using coordinates $(x_1, y_1, z_1, x_2, y_2, z_2)$ in $\mathbf{R}^3 \times \mathbf{R}^3$, the data for Σ_j is:

Σ_j	$\mathrm{Fix}(\Sigma_j)$	$G_j = N_\Gamma(\Sigma_j)/\Sigma_j$	regular G_j orbit type	V_j
$\mathbf{O}(2) \times \mathbf{1}$	$\{0\} \times \mathbf{R}^3$	$\mathbf{1} \times \mathbf{O}(2)$	circle	$\{0\} \times \{(0, y_2, z_2)\}$
$\mathbf{SO}(2) \times \mathbf{SO}(2)$	$\{(0, 0, z_1, 0, 0, z_2)\}$	$\mathbf{Z}_2 \times \mathbf{Z}_2$	four points	$\{(0, 0, z_1, 0, 0, z_2)\}$
$\mathbf{1} \times \mathbf{O}(2)$	$\mathbf{R}^3 \times \{0\}$	$\mathbf{O}(2) \times \mathbf{1}$	circle	$\{(0, y_1, z_1)\} \times \{0\}$
$\mathbf{Z}_2 \times \mathbf{Z}_2$	$\{(x_1, y_1, 0, x_2, y_2, 0)\}$	$\mathbf{SO}(2) \times \mathbf{SO}(2)$	torus	$\{(0, y_1, 0, 0, y_2, 0)\}$

The data for T_j is:

T_j	$\mathrm{Fix}(T_j)$	$\mathrm{Fix}(T_j) \cap V_{j-1}$	$\mathrm{Fix}(T_j) \cap V_j$
$\mathbf{O}(2) \times \mathbf{Z}_2$	$\{0\} \times \{(x_2, y_2, 0)\}$	$\{0\} \times \{(0, y_2, 0)\}$	$\{0\} \times \{(0, y_2, 0)\}$
$\mathbf{O}(2) \times \mathbf{SO}(2)$	$\{0\} \times \{(0, 0, z_2)\}$	$\{0\} \times \{(0, 0, z_2)\}$	$\{0\} \times \{(0, 0, z_2)\}$
$\mathbf{SO}(2) \times \mathbf{O}(2)$	$\{(0, 0, z_1)\} \times \{0\}$	$\{(0, 0, z_1)\} \times \{0\}$	$\{(0, 0, z_1)\} \times \{0\}$
$\mathbf{Z}_2 \times \mathbf{O}(2)$	$\{(x_1, y_1, 0)\} \times \{0\}$	$\{(0, y_1, 0)\} \times \{0\}$	$\{(0, y_1, 0)\} \times \{0\}$

Note that the action of G_j on $\mathrm{Fix}(\Sigma_j)$ is polar, regular G_j-orbits in $\mathrm{Fix}(\Sigma_j)$ have codimension two and the effective action on V_j is that of $\mathbf{Z}_2 \times \mathbf{Z}_2$.

From Theorem 3.2 it follows that there is a persistent heteroclinic cycle connecting a closed curve in $\mathrm{Fix}(T_1)$ to a point in $\mathrm{Fix}(T_2)$, then to a point in $\mathrm{Fix}(T_3)$, then to a closed curve in $\mathrm{Fix}(T_4)$ and finally to the initial closed curve.

7 Discussion

Theorem 3.2 provides conditions for the existence of a persistent heteroclinic cycle under polar actions on fixed-point spaces. An important particular case is the following:

Corollary 7.1 *Let Γ be a compact Lie group acting linearly on $\mathbf{R}^n$ with a heteroclinic cycle of isotropy subgroups. Suppose that subgroups of the cycle satisfy:*

$$\dim \mathrm{Fix}(\Sigma_j) = 2 \qquad \dim \mathrm{Fix}(T_j) = 1 \text{ and}$$

$$N(T_j)/T_j \cong \mathbf{Z}_2 \qquad N(\Sigma_j)/\Sigma_j \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2.$$

Then there are smooth Γ-equivariant vector fields on $\mathbf{R}^n$ with a persistent heteroclinic cycle.

The cycle found by Guckenheimer and Holmes in [8] is an example of application of this corollary. In the corollary, the hypothesis concerning the action on fixed-point spaces being polar is unnecessary as $\mathrm{Fix}(\Sigma)$ is a 2-dimensional space itself.

Polar actions allow us to deal with the problem of the existence of heteroclinic cycles in higher dimensions. The polar action hypothesis is not very restrictive:

many interesting examples occur for polar actions and there is a vast collection of such actions.

Although we do not address the issue of existence of homoclinic cycles (cf. Sottocornola [15, 16, 17] for recent results), homoclinic connections may be obtained using the proof of theorem 3.2. These appear when the conjugacy class of the isotropy subgroups T_j is nontrivial and the cycle of isotropy subgroups contains only two isotropy classes.

The next logical step after having proved the existence of an object is to study its stability. This may be achieved using the results of Krupa and Melbourne [12, 13]. There is no reason for the heteroclinic cycles, whose existence is guaranteed by theorem 3.2, to have an *a priori* well-defined stability. The stability depends on characteristics of the vector field more particular than those needed to establish existence.

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