THREE-DIMENSIONAL CONSERVATIVE STAR FLOWS ARE ANOSOV

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Abstract. A divergence-free vector field satisfy the star property if any divergence-free vector field in some C^1-neighborhood has all the singularities and all closed orbits hyperbolic. In this article we prove that any divergence-free star vector field defined in a closed three-dimensional manifold is Anosov. Moreover, we prove that a C^1-structurally stable three-dimensional conservative flow is Anosov.

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1. Introduction, basic definitions and statement of the results

Let M be a three-dimensional closed and connected C^∞ Riemannian manifold endowed with a volume-form and let μ denote the Lebesgue measure associated to it. We say that a vector field $X\colon M \to TM$ is *divergence-free* if its divergence is equal to zero or equivalently if the measure μ is invariant for the associated flow, X^t, $t \in \mathbb{R}$. In this case we say that the flow is *conservative* or *volume-preserving*. We denote by $\mathfrak{X}^r_\mu(M)$ $(r \geq 1)$ the space of C^r divergence-free vector fields on M and we endow this set with the usual C^1 Whitney topology. Let also denote by $\mathfrak{X}^r(M) \supset \mathfrak{X}^r_\mu(M)$ $(r \geq 1)$ the space of C^r (dissipative) vector fields on M.

Given $X \in \mathfrak{X}^1(M)$ let $Sing(X)$ denote the set of *singularities* of X and $R := M \setminus Sing(X)$ the set of *regular* points.

Given $x \in R$ we consider its normal bundle $N_x = X(x)^\perp \subset T_xM$ and define the *linear Poincaré flow* by $P^t_X(x) := \Pi_{X^t(x)} \circ DX^t_x$ where $\Pi_{X^t(x)} : T_{X^t(x)}M \to N_{X^t(x)}$ is the projection along the direction of $X(X^t(x))$. Let $\Lambda \subset R$ be an X^t-invariant set and $N = N^1 \oplus N^2$ be a P^t_X-invariant splitting over Λ; as X is conservative these bundles are one-dimensional. We say that this splitting is an ℓ*-dominated splitting* for the linear Poincaré flow if there exists an $\ell \in \mathbb{N}$ such that for all

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$x \in \Lambda$ we have:

$$\|P_X^\ell(x)|_{N_x^2}\|.\|P_X^{-\ell}(X^\ell(x))|_{N^1_{X^\ell(x)}}\| \le \frac{1}{2}.$$

This definition is weaker than hyperbolicity where it is required that

$$\|P_X^\ell(x)|_{N_x^2}\| \le \frac{1}{2} \text{ and also that } \|P_X^{-\ell}(X^\ell(x))|_{N^1_{X^\ell(x)}}\| \le \frac{1}{2}.$$

When Λ is compact this definition is equivalent to the usual definition of hyperbolic flow ([8, Proposition 1.1]).

The simplest examples of hyperbolic sets are singularities and closed orbits and it is well-know that these sets are stable by C^1-perturbations, that is, any other sufficiently C^1-close system has equivalent behavior or, in other words, it is possible to find a change of coordinates conjugating locally the two dynamics (for more details see [10]). Other classical examples are the Anosov ones where M is hyperbolic, and they form an open set of $\mathfrak{X}^1_\mu(M)$ (see e.g. [12]).

We say that a vector field is *Axiom A* if the closure of the union of the closed orbits and the singularities is the non-wandering set, denoted by $\Omega(X)$, and this set is hyperbolic. Since, by Poincaré recurrence theorem, for conservative vector fields the non-wandering set is equal to M, a conservative vector field that is Axiom A is actually an Anosov system. In the dissipative case, in order to obtain stability we must check if there exists no cycles. Recall that, by the spectral decomposition of an Axiom A flow, we have that $\Omega(X) = \cup_{i=1}^k \Lambda_i$ where each Λ_i is a basic piece. We define an order relation by $\Lambda_i \prec \Lambda_j$ if there exists $x \in M \setminus (\Lambda_i \cup \Lambda_j)$ such that $\alpha(x) \subset \Lambda_i$ and $\omega(x) \subset \Lambda_j$. We say that X has a cycle if there exists a cycle with respect to $\prec$ (see [12] for details).

We say that $X \in \mathfrak{X}^1(M)$ is a *star flow* if there exists a C^1-neighborhood $\mathcal{V} \ni X$ such that if $Y \in \mathcal{V}$, then the all the closed orbits and all the singularities of Y are hyperbolic. Denote the set of star flows in M by $\mathcal{G}^1(M)$.

Recently, Gan and Wen ([9]) proved a remarkable result about dissipative star flows defined in a d-dimensional manifold, where $d \ge 3$:

Theorem 1.1. *If $X \in \mathcal{G}^1(M^d)$ and $Sing(X) = \emptyset$ then X is Axiom A without cycles.*

In this paper we deal with these issues in the setting of three-dimensional divergence-free vector fields and our approach is of a completely different nature. We consider flows that are star flows *restricted to the conservative setting*, which we denote by $\mathcal{G}^1_\mu(M)$. That is, $X \in \mathcal{G}^1_\mu(M)$ if there exists a neighbourhood $\mathcal{V}$ of X in $\mathfrak{X}^1_\mu(M)$ such that any $Y \in \mathcal{V}$, has all the closed orbits and all the singularities hyperbolic. Our main result states that such a flow has no singularities and is hyperbolic (Anosov). We note that Gan and Wen must consider non-singular flows due, in particular, to the fact that the Lorenz strange attractor is in $\mathcal{G}^1(M)$. However, Arbieto and Matheus ([1, Corollary 4.1]) proved that, in the conservative setting, there are no geometrical Lorenz sets, which could indicate that it should be possible to remove the hypothesis of the non-existence of singularities.

Let us now state our main result.

Theorem 1. *If $X \in \mathcal{G}^1_\mu(M)$ then $Sing(X) = \emptyset$ and actually X is Anosov.*

We point out that the proof of this result is a consequence of several recent results on conservative three-dimensional flows. We believe that the previous result is also true in any dimension and its proof should be obtained by generalizing these recent results to any dimension[1] and eventually following the strategy of the cited work of Gan and Wen, namely by using the fact that vector fields in $\mathcal{G}^1_\mu(M)$ cannot have heterodimensional cycles.

Let $\mathcal{A}^3_\mu$ denote the open set of divergence-free Anosov vector fields on a three-dimensional manifold M.

It is clear that $\mathcal{G}^1(M) \cap \mathfrak{X}^1_\mu(M) \subset \mathcal{G}^1_\mu(M)$; Theorem 1 implies that

$$\mathcal{G}^1(M) \cap \mathfrak{X}^1_\mu(M) = \mathcal{G}^1_\mu(M) = \mathcal{A}^3_\mu.$$

As a consequence of Theorem 1 we also obtain the following result.

Corollary 1.2. *The boundary of $\mathcal{A}^3_\mu$ has no isolated points.*

A vector field $X \in \mathfrak{X}^1_\mu(M)$ is said to be C^1-structurally stable in the conservative setting if there exists a C^1 neighbourhood, $\mathcal{V}$, of X in $\mathfrak{X}^1_\mu(M)$ such that every $Y \in \mathcal{V}$ is topological equivalent to X (see, for example [10]).

Combining Theorem 1 with previous results of the first author with P. Duarte ([4]) and with V. Araújo ([2]) we are able to prove the stability conjecture for C^1 conservative 3-flows.

Theorem 2. *If $X \in \mathfrak{X}^1_\mu(M)$ is a C^1-structurally stable three-dimensional flow then X is Anosov.*

2. Some main tools

If p is a regular point of $X \in \mathfrak{X}^1_\mu(M)$, define the segment of orbit $\Gamma(p,\tau) = \{X^t(p);\ t \in [0,\tau]\}$. Now consider V, V' sub-spaces of N_p, with $dim(V) = j$, for some $2 \le j \le n-1$, and $N_p = V \oplus V'$. A *one-parameter linear family* $\{A_t\}_{t\in\mathbb{R}}$ associated to $\Gamma(p,\tau)$ and V is defined as follows:

- $A_t \colon N_p \to N_p$ is a linear map, for all $t \in \mathbb{R}$,
- $A_t = Id$, for all $t \le 0$, and $A_t = A_\tau$, for all $t \ge \tau$,

[1] In [6] the authors obtain a generalization of one of these results.

- $A_t|_V \in SL(j,\mathbb{R})$, and $A_t|_{V'} \equiv Id$, $\forall t \in [0,\tau]$, in particular we have $\det(A_t) = 1$, for all $t \in \mathbb{R}$, and
- the family A_t is C^∞ on the parameter t.

In this paper we will consider $n = 3$ and so $V = N_p$ and $\dim(V) = 2$.

The following result is a kind of Franks' Lemma for volume-preserving flows and was proved in [5].

Theorem 2.1. *Given $\epsilon > 0$ and a vector field $X \in \mathfrak{X}^4_\mu(M)$ there exists $\xi_0 = \xi_0(\epsilon, X)$ such that $\forall \tau \in [1,2]$, for any periodic point p of period greater than 2, for any sufficient small flowbox $\mathcal{T}$ of $\Gamma(p,\tau)$ and for any one-parameter linear family $\{A_t\}_{t\in[0,\tau]}$ such that $\|A'_t A_t^{-1}\| < \xi_0$, $\forall t \in [0,\tau]$, there exists $Y \in \mathfrak{X}^1_\mu(M)$ satisfying the following properties*

1. *Y is ϵ-C^1-close to X;*
2. *$Y^t(p) = X^t(p)$, for all $t \in \mathbb{R}$;*
3. *$P^\tau_Y(p) = P^\tau_X(p) \circ A_\tau$, and*
4. *$Y|_{\mathcal{T}^c} \equiv X|_{\mathcal{T}^c}$.*

Let us state a useful and direct application of Theorem 2.1 that, under certain conditions, allows us to create elliptic periodic points.

Corollary 2.2. *Let $X \in \mathfrak{X}^4_\mu(M)$ and $\epsilon > 0$. There exists $\delta > 0$ such that if $p \in M$ is a X^t-closed orbit of period $\pi \geq 2$ and $\|P^\pi_X(p) - Id\| < \delta$ then there exists $Y \in \mathfrak{X}^1_\mu(M)$, ϵ-C^1-close to X, such that p is an elliptic point of period π of Y.*

Remark 2.1. *Actually, using Theorem 2.1, we conclude that if π is large enough then the condition $\|P^\pi_X(p) - Id\| \approx 0$ can be replaced by the condition $\|P^\pi_X(p)\| \approx 1$.*

We also recall the C^1-Closing Lemma adapted to the setting of volume-preserving flows by Pugh and Robinson [11, Section 8(c)]. The X^t-orbit of a recurrent point x can be approximated for a very long time $T > 0$ by a closed orbit of a flow Y which is C^1-close to X. In fact, given $r, T > 0$ we can find a ϵ-C^1-neighborhood $\mathfrak{U} \subset \mathfrak{X}^1_\mu(M)$ of X, a closed orbit p of Y with period π and a map $\tau\colon [0,T] \to [0,\pi]$ close to the identity such that

- $dist\big(X^t(x), Y^{\tau(t)}(p)\big) < r$ for all $0 \leq t \leq T$;
- $Y = X$ over $M \setminus \bigcup_{0\leq t\leq\pi} \big(B(p,r) \cap B(Y^t(p), r)\big)$.

Another ingredient of the proofs of our theorems is a generalization of Bochi's dichotomy (see [7, Theorem A]) for the continuous-time class. This result was obtained recently by combining a Theorem of [3, Theorem 1], corresponding to the case when X has no singularities, and a Theorem of [2, Theorem A], that corresponds to case when X can have singularities. More precisely the following result was obtained.

Theorem 2.3. *There exists a C^1-residual set $\mathcal{R} \subset \mathfrak{X}^1_\mu(M)$ such that if $X \in \mathcal{R}$ then X is Anosov or else almost every point in M has zero Lyapunov exponents.*

3. Proofs of the results

Let us recall that a singularity σ of a vector field X is *linear hyperbolic* if σ is a hyperbolic singularity and there exists a smooth local change of coordinates around σ that conjugates X and DX_σ (cf. [13, Definition 4.1]).

The proof of Theorem 1 is made in two steps. First we prove that if $X \in \mathcal{G}^1_\mu(M)$ then X has no singularities and P^t_X admits a dominated splitting over M (Lemma 3.1) and then we prove that if $X \in \mathcal{G}^1_\mu(M)$ is such that P^t_X admits a dominated splitting over M then X is Anosov (Lemma 3.2).

Lemma 3.1. *If $X \in \mathcal{G}^1_\mu(M)$ then X has no singularities and P^t_X admits a dominated splitting over M.*

Proof. Let us first observe that $\mathcal{G}^1_\mu(M)$ is C^1 open in $\mathfrak{X}^1_\mu(M)$.

To prove the lemma let us fix $X \in \mathcal{G}^1_\mu(M)$ and a C^1-neighbourhood $\mathcal{V}$ of X in $\mathcal{G}^1_\mu(M)$. Let us choose $Y \in \mathcal{V}$ such that all the singularities of Y are linear hyperbolic. If $M \setminus Sing(Y)$ admits a dominated splitting for the linear Poincaré flow of Y then [13, Proposition 4.1] implies that $Sing(Y) = \emptyset$. It follows that there exists $\mathcal{U} \subset \mathcal{V}$, $Y \in \mathcal{U}$, whose elements do not have singularities and admit a dominated splitting for the associated linear Poincaré flow. So let us now assume that $M \setminus Sing(Y)$ does not admit a dominated splitting for the linear Poincaré flow of Y.

We claim that for all $m \in \mathbb{N}$, there exists a Y^t-invariant set $\Gamma_m \subset M \setminus Sing(Y)$ such that $\mu(\Gamma_m) > 0$ and Γ_m do not have dominated splitting for P^t_Y. In fact if this claim was false, then there would exist m such that $M \setminus Sing(Y)$ has an m-dominated splitting which contradicts our assumption.

Using the techniques involved in the proof of Theorem 2.3 (see [3]) it is straightforward to conclude that for m sufficiently large and any $t > T_0$ there exists $Y_1 \in \mathcal{V}$, C^1-close to Y, and such that $\|P^t_{Y_1}(x)\| \approx 1$, for a.e. $x \in \Gamma_m$. Actually, let $\hat{U} \subset \Gamma_m$ be a measurable set with positive measure. Let $R \subset \hat{U}$ be the set given by the Poincaré recurrence theorem with respect to Y_1. Then every $x \in R$ returns to $\hat{U}$ infinitely many times under the flow Y^t_1 and is not a periodic point. Let $\mathcal{Z}$ denote the subset of Γ_m having zero Lyapunov exponents for Y_1.

Given $x \in \mathcal{Z} \cap R$ and $\delta > 0$, there exists $T_x \in \mathbb{R}$ such that

$$e^{-\delta t} < \|P^t_{Y_1}(x)\| < e^{\delta t} \text{ for every } t \geq T_x.$$

By the C^1-Closing Lemma the Y^t_1-orbit of x can be approximated, for a very long recurrent time $T > T_x$, by a periodic orbit of a C^1-close flow Y_2: given $r, T > 0$ we can find a small C^1-neighborhood $\mathcal{U}$ of Y_1 in $\mathfrak{X}^1_\mu(M)$, a vector field $Y_2 \in \mathcal{U}$, a periodic orbit p of Y_2 with period $\pi > T$ and a map $g : [0, T] \to [0, \pi]$ close to the identity such that

- $dist\big(Y_1^t(x), Y_2^{g(t)}(p)\big) < r$ for all $0 \leq t \leq T$;
- $Y_2 = Y_1$ over $M \setminus \bigcup_{0\leq t\leq \pi}\big(B(p,r) \cap B(Y_2^t(p), r)\big)$.

Letting $r > 0$ be small enough we obtain also that

$$e^{-\delta\pi} < \|P^{\pi}_{Y_2}(p)\| < e^{\delta\pi}.$$

Now, using Zuppa's theorem ([14]), as p is a hyperbolic periodic point, we can approximate Y_2 by $Y_3 \in \mathfrak{X}^4_\mu(M) \cap \mathcal{V}$ such that the analytic continuation of p, p_0, is a periodic point of period close to π for Y_3, and such that

$$e^{-\delta\pi} < \|P^{\pi}_{Y_3}(p_0)\| < e^{\delta\pi}.$$

Finally, using Remark 2.1, we are able to obtain $Y_4 \in \mathcal{G}^1_\mu(M) \cap \mathcal{V}$, close to Y_3 and such that $\|P^{\pi}_{Y_4}(p_0)\| = 1$ and moreover p_0 is an elliptic point, which is a contradiction.

Therefore $Sing(Y) = \emptyset$ and P^t_Y admits a dominated splitting over M.

Let us now prove that X has no singularities and P^t_X admits a dominated splitting over M. In fact if X has a singularity then there exists $Y_0 \in \mathcal{V}$ such that Y_0 has at least one linear hyperbolic singularity. Now we proceed as before to $Y \in \mathcal{V}$, arbitrarily close to Y_0, having all the singularities linear hyperbolic and with $Sing(Y) \neq \emptyset$. Repeating the arguments above we get that $Sing(Y) = \emptyset$, which is a contradiction. Therefore $Sing(X) = \emptyset$ and, in previous arguments, we can take $Y = X$ and then conclude that X has a dominated splitting for the linear Poincaré flow.

□

Lemma 3.2. *If $X \in \mathcal{G}^1_\mu(M)$ is such that P^t_X admits a dominated splitting over M then X is Anosov.*

Proof. Since P^t_X admits a dominated splitting over M one gets that there exists $\ell \in \mathbb{N}$ such that

$$\Delta(x,\ell) = \|P^{\ell}_X(x)|_{N^1_x}\|\|P^{-\ell}_X(X^{\ell}(x))|_{N^2_{X^{\ell}(x)}}\| \leq \frac{1}{2}, \ \forall x \in M,$$

where $N = N^1 \oplus N^2$, and these subbundles are P^t_X-invariant and are one dimensional.

For any $i \in \mathbb{N}$ we have $\Delta(x, i\ell) \leq 1/2^i$. For every $t \in \mathbb{R}$ we may write $t = i\ell + r$ and since $\|P^r_X\|$ is bounded, say by L, take $C = 2^{\frac{r}{\ell}}L^2$ and $\sigma = 2^{-\frac{1}{\ell}}$ to get $\Delta(x,t) \leq C\sigma^t$ for every $x \in M$ and $t \in \mathbb{R}$. Denote by α_t the angle $\measuredangle(N^1_{X^t(x)}, N^2_{X^t(x)})$. We already know by domination that this angle is bounded bellow from zero, say by β. Since we do not have singularities there exists $K > 1$ such that for all $x \in M$, $K^{-1} \leq \|X(x)\| \leq K$. Since the flow is conservative and the subbundles are both one dimensional we have that

$$\sin(\alpha_0) = \|P^t_X(x)|_{N^1_x}\|\|P^t_X(x)|_{N^2_x}\| \sin(\alpha_t)\frac{\|X(X^t(x))\|}{\|X(x)\|}.$$

So,

$$\begin{aligned}\|P_X^t(x)|_{N_x^2}\|^2 &= \frac{\sin(\alpha_0)}{\sin(\alpha_t)}\frac{\|X(x)\|}{\|X(X^t(x))\|}\Delta(x,t)\\ &\leq \Delta(x, i\ell + r)\sin(\beta)^{-1}K^2\\ &\leq \sigma^t C\sin(\beta)^{-1}K^2.\end{aligned}$$

Analogously we get

$$\begin{aligned}\|P_X^{-t}(x)|_{N_x^1}\|^2 &= \frac{\sin(\alpha_t)}{\sin(\alpha_0)}\frac{\|X(X^t(x))\|}{\|X(x))\|}\Delta(x,t)\\ &\leq \Delta(x, i\ell + r)\sin(\beta)^{-1}K^2\\ &\leq \sigma^t C\sin(\beta)^{-1}K^2.\end{aligned}$$

These two inequalities show that M is hyperbolic for the linear Poincaré flow. Then by [8, Proposition 1.1] we obtain that M is a hyperbolic set, thus X is Anosov. This end the proof of the lemma. □

Proof. (of Corollary 1.2) We claim that an isolated point X of the boundary of $\mathcal{A}^3_\mu$ do not have singularities. In fact if $Sing(X) \neq \emptyset$ then, since Anosov vector fields do not have singularities, the singularities of X must be all nonhyperbolic. A nonhyperbolic singularity can be made hyperbolic by a small perturbation, thus there are vector fields arbitrarily close to X having (stably) hyperbolic singularities which is a contradiction because X is an isolated point of the boundary of $\mathcal{A}^3_\mu$.

Now we just have to follow the proof of Theorem 1, taking $Y = X$ (where we don't need to assume anymore that $X \in \mathcal{G}^1_\mu(M)$), concluding that linear Poincaré flow of X admits a dominated splitting over M. Now, as in the proof of the previous corollary, it follows that X is Anosov. □

Proof. (of Theorem 2) Let us fix a C^1-structurally stable vector field in $\mathfrak{X}^1_\mu(M)$ and choose a neighbourhood $\mathcal{V}$ of X whose elements are topologically equivalent to X. If $X \notin \mathcal{A}^3_\mu = \mathcal{G}^1_\mu(M)$ then it follows that $\mathcal{V} \cap \mathcal{A}^3_\mu = \emptyset$. Using [4, 2] one gets that there exists a residual subset $\mathcal{R} \subset \mathcal{V}$ such that for every $Y \in \mathcal{R}$ the set of elliptic closed orbits is dense in M. Let us fix $Y \in \mathcal{R}$ and choose a small neighbourhood of Y, $\mathcal{W} \subset \mathcal{V}$.

Let x be an elliptic point of large period, say π. Using Zuppa's theorem ([14]) and the stability of elliptic points, we can approximated Y, in the C^1 topology, by a C^4-vector field $Z \in \mathcal{W}$ such that the analytic continuation of x is also an elliptic point with period close to π. Now, if π is large enough, we apply Theorem 2.1 several times, by concatenating small rotations (the maps A_t), in order to obtain a new vector field $W \in \mathcal{W}$ exhibiting a parabolic closed orbit. Since the existence of a parabolic point prevents structural stability and $W \in \mathcal{W}$ we get a contradiction. Therefore $X \in \mathcal{A}^3_\mu$, which ends the proof.

$\square$

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References

[1] A. Arbieto and C. Matheus, *A pasting lemma and some applications for conservative systems*, Ergod. Th. & Dynam. Sys., 27 (2007), 1399–1417.
[2] V. Araújo and M. Bessa, *Dominated splitting and zero volume for incompressible three flows*, Nonlinearity 21, 7 (2008), 1637–1653.
[3] M. Bessa, *The Lyapunov exponents of generic zero divergence three-dimensional vector fields*, Ergod. Th. & Dynam. Sys., 27 (2007), 1445–1472.
[4] M. Bessa and P. Duarte, *Abundance of elliptical dynamics on conservative 3-flows*, Dynamical Systems: An international Journal, 23, 4, (2008).
[5] M. Bessa and J. Rocha, *On C^1-robust transitivity of volume-preserving flows*, Journal Diff. Eq., 245, 11 (2008), 3127–3143.
[6] M. Bessa and J. Rocha, *Contributions to the geometric and ergodic theory of conservative flows.*, Preprint arXiv:0810.3855, 2008.
[7] J. Bochi, *Genericity of zero Lyapunov exponents*, Ergod. Th. & Dynam. Sys., 22 (2002), 1667–1696.
[8] C. Doering, *Persistently transitive vector fields on three-dimensional manifolds*, Proceedings on Dynamical Systems and Bifurcation Theory, Vol. 160 (1987), 59–89, Pitman.
[9] S. Gan and L. Wen, *Nonsingular star flows satisfy Axiom A and the no-cycle condition.*, Invent. Math., **164** (2006), 279–315.
[10] J. Palis and W. de Melo, "Geometric Theory Of Dynamical Systems", Springer-Verlag, 1982.
[11] C. Pugh and C. Robinson, *The C^1 Closing lemma, including hamiltonians*, Ergod. Th. & Dynam. Sys., 3, 2 (1983), 261–313.
[12] M. Shub, "Global stability of dynamical systems", Springer-Verlag, New York, 1987.
[13] T. Vivier, *Projective hyperbolicity and fixed points*, Ergod. Th. & Dynam. Sys., 26 (2006), 923–936.
[14] C. Zuppa, *Regularisation C^∞ des champs vectoriels qui préservent l'elément de volume*, Bol. Soc. Bras. Mat., 10, 2 (1979), 51–56.

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