On the question of embedding a semigroup into an idempotent generated one

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Abstract

In this paper we present a new embedding of a semigroup into a semiband (idempotent-generated semigroup) of depth 4 (every element is the product of 4 idempotents) using a semidirect product construction. Our embedding does not assume that S is a monoid (although it assumes a weaker condition), and works also for (non-monoid) regular semigroups. In fact, this semidirect product is particularly useful for regular semigroups since we can defined another embedding for these semigroups into a smaller semiband of depth 2. In this paper we shall compare our construction with other known embeddings, and we shall see that some properties of S are preserved by it.

1 Introduction

The reader should consult [9] for undefined terms used in this paper. Let S be a semigroup. As usual, we shall denote by S^1 the monoid induced by S, that is, $S^1 = S$ if S is a monoid, or otherwise S^1 is the semigroup S with an extra identity element adjoined. We shall denote by 1 the identity element of S^1. Also, we shall denote by $E(S)$ the set of idempotents of S (we shall write only E if no ambiguity occurs) and by $\langle E(S)\rangle$ the subsemigroup of S generated by its idempotents. A semigroup S is a semiband if $S = \langle E\rangle$. If E^n denotes the set of elements of S that are the product of n idempotents, then

$$\langle E\rangle = \cup_{n=1}^{\infty} E^n \,.$$

Clearly $E^n \subseteq E^{n+1}$ since we are talking about products of idempotents, and if $E^n = E^{n+1}$ then $\langle E\rangle = E^n$. The depth of an element $a \in \langle E\rangle$ is the positive integer k such that $a \in E^k \setminus E^{k-1}$ ($E^0 = \emptyset$). We shall say that a semiband S has depth k if $E^{k-1} \neq E^k = E^{k+1}$, and has infinite depth if $E^n \neq E^{n+1}$ for all n. Note that finite semibands have always finite depth.

There have been described in the literature several ways of embedding a semigroup into a semiband. The first such known embedding is due to Howie [7] and it embeds a semigroup S into the (regular) subsemigroup $\langle E(\mathcal{T}_X)\rangle$ of some full transformation semigroup $\mathcal{T}_X$. Howie's embedding allows to embed a semigroup of order n (that is, with n elements) into a regular semiband of order at most $(n+3)^{n+3} - (n+3)!$. Another embedding also involving an idempotent generated subsemigroup of some full transformation semigroup appeared in [2] where it was attributed to Perrot. This later embedding was investigated by Pastijn [15] who proved that the mentioned semiband has depth 4, and if S has order n, then the semiband has order at most $2n^2 + 4n + 1$. However, unlike Howie's embedding, this semiband is regular only when S is regular.

To answer to a question posed by Howie in [8], Laffey [12] and Hall presented each one a way to embed a finite semigroup into a finite semiband of depth 2 (the proof of the latter appears in [3]). Laffey's embedding is based on linear algebra techniques, while Hall's embedding makes use of the Rees matrix semigroup construction. Higgins [6] presented another proof that every semigroup can be embedded into a regular semiband of depth 2 in a way that preserves finiteness. Higgins' proof relies on embedding a full transformation semigroup $\mathcal{T}_X$ into a regular subsemiband of depth 2 of $\mathcal{T}_{X\cup X'}$ where X' is another set disjoint from X but with the same cardinality.

In [16], Pastijn and Yan introduced another way to embed a semigroup into a semiband of depth 4. This new semiband is given by a presentation and it follows closed to the results obtained in [15]. The semiband used in [15] can be seen as a quotient of the semiband presented in [16]. Petrich [17] presented another idempotent generated Rees matrix semigroup construction in which we can embed a given semigroup. This idempotent generated Rees matrix semigroup is isomorphic to the semigroup presented in [16].

Recently, Almeida and Moura [1] studied the question of embedding a semigroup into a semiband inside some pseudovarieties of semigroups using some known embeddings, but also some profinite techniques. As an example, they were able to show that every finite semigroup S whose regular $\mathcal{D}$-classes are aperiodic semigroups is embeddable into a semiband with the same characteristic using profinite techniques that do not give us an explicit semiband construction. All the known embeddings of a semigroup into a semiband do not preserve this property of S, and we must say that the new embeddings presented in this paper continue to fail to preserve this property of S. So, the question of finding an explicit semiband construction that allows to embed a semigroup whose regular $\mathcal{D}$-classes are aperiodic semigroups into a semiband with the same characteristic is still open.

In the following section we shall recall some of the embedding described above. In Section 3 we shall present a new embedding. We shall embed a semigroup S with a certain property (which includes both monoids and regular semigroups) into a semiband $T(S)$ of depth 4 that is a subsemigroup of a semidirect product of $S \times S$ by R_2, where R_2 denotes the two element right-zero semigroup. Note that this embedding will work for any semigroup S even if it does not have the referred property since we can always consider the monoid S^1 instead. The properties of S preserved by $T(S)$ is the focus of study of Section 4.

The semidirect product introduced in Section 3 is particularly interesting if we consider only regular semigroups. If S is a regular semigroup, we will embed S into another subsemiband of the referred semidirect product that has only depth 2. This will be the topic of study of Section 5.

2 Recalling some embeddings

In this section we shall briefly describe some known embeddings of semigroups into semibands, but we will not do it chronologically. We begin with the Pastijn and Yan's embedding [16]. Let $\overline{X} = \{\overline{h}\} \cup \{\overline{s} \;:\; s \in S^1\}$ and consider the semigroup $F(S)$ given by the following presentation

$$F(S) = \langle\, \overline{X} \mid \overline{x}^2 = \overline{x}\,,\ \overline{h}\overline{1} = \overline{1}\,,\ \overline{1}\overline{h} = \overline{h}\,,\ \overline{s}\overline{t} = \overline{s}\,,\ \overline{h}\overline{s}\overline{h}\overline{t} = \overline{h}\overline{st}\,\rangle\,,$$

where $\overline{x} \in \overline{X}$ and $s, t \in S^1$. Then $F(S)$ is generated by the set of idempotents $\overline{X}$ and by [16, Lemma 16], the list

$$\overline{h}\,,\ \overline{1}\,,\ \overline{s}\,,\ \overline{h}\overline{s}\,,\ \overline{s}\overline{h}\,,\ \overline{h}\overline{s}\overline{h}\,,\ \overline{s}\overline{h}\overline{t}\,,\ \overline{s}\overline{h}\overline{t}\overline{h}\,, \qquad s, t \in S^1 \setminus \{1\}\,,$$

presents all elements of $F(S)$ (and the elements of this list are all pairwise distinct). Further, since $\overline{h} = \overline{1}\overline{h}\overline{1}\overline{h}$, $\overline{1} = \overline{1}\overline{h}\overline{1}$, $\overline{s} = \overline{s}\overline{h}\overline{1}$, $\overline{h}\overline{s} = \overline{1}\overline{h}\overline{s}$, $\overline{s}\overline{h} = \overline{s}\overline{h}\overline{1}\overline{h}$ and $\overline{h}\overline{s}\overline{h} = \overline{1}\overline{h}\overline{s}\overline{h}$, we conclude that

$$F(S) = \{\overline{s}\overline{h}\overline{t}\,,\ \overline{s}\overline{h}\overline{t}\overline{h} \;:\; s, t \in S^1\}$$

with any two elements of this set distinct from each other.

The mapping $S \to F(S)$, $s \to \overline{h}\overline{s} = \overline{1}\overline{h}\overline{s}$ clearly embeds S into the semiband $F(S)$ of depth 4. Note that if S has order n and it is a monoid, then $F(S)$ has order $2n^2$; but if is not a monoid, then S is embeddable into the semiband $F(S) \setminus \{\overline{1}\}$ that has order $2n^2 + 4n + 1$.

The semiband used in [15] to embed a semigroup into a semiband can be seen as a quotient of the semigroup $F(S)$. We can easily check that

$$\rho = \{(\overline{s}\overline{h}\overline{t}, \overline{r}\overline{h}\overline{t}),\, (\overline{s}\overline{h}\overline{t}\overline{h}, \overline{r}\overline{h}\overline{t}\overline{h}) : \; st = rt \;\text{ for }\; r, s, t \in S^1\}$$

is a congruence on $F(S)$. Let $A(S) = F(S)/\rho$. This quotient semigroup is isomorphic to the semigroup of [15] denoted by $\widetilde{\mathcal{A}(S)}$. Clearly the mapping $S \to A(S)$, $s \to \overline{h}\overline{s}\rho$ embeds S into $A(S)$ since $\overline{h}\overline{s}\rho = \overline{h}\overline{r}\rho$ if and only if $s = r$. However, in [15], it was used instead the following embedding: $S \to A(S)$, $s \to \overline{h}\overline{s}\overline{h}\rho$.

Let $\Sigma = \{\sigma, \tau\}$ and consider the Rees matrix semigroup

$$\Phi S = \mathcal{M}(S^1, S^1, \Sigma; Q)$$

where $Q = (q_{\alpha s})$ is the $\Sigma \times S^1$-matrix defined by $q_{\sigma s} = 1$ and $q_{\tau s} = s$ for $s \in S^1$. By [17, Theorem 3.1] there is an isomorphism χ from $F(S)$ onto ΦS such that $\overline{h}\chi = (1, 1, \tau)$ and $\overline{s}\chi = (1, s, \sigma)$ for $s \in S^1$. Thus ΦS is a semiband of depth 4. Clearly the mapping $S \to \Phi S$, $s \to (1, s, \sigma)$ embeds S into ΦS.

The last embedding we shall recall is the one introduced by Higgins [6] where the full transformation semigroup $\mathcal{T}_X$ is embedded into a subsemiband of the full transformation semigroup $\mathcal{T}_Y$ for $Y = X \cup X'$ and X' a set disjoint from X but with the same cardinality. Let $x \to x'$ be a bijection from X onto X' and, for each $\alpha \in \mathcal{T}_X$, define $\alpha' \in \mathcal{T}_Y$ by $x\alpha' = x\alpha = x'\alpha'$ for $x \in X$. Let also

$$T = \{\alpha \in \mathcal{T}_Y \;:\; X\alpha = X'\alpha \subseteq X \text{ or } X\alpha = X'\alpha \subseteq X'\}\,.$$

It was shown in [6] that T is a regular subsemiband of $\mathcal{T}_Y$ of depth 2 and that the mapping $\mathcal{T}_X \to T$, $\alpha \to \alpha'$ embeds $\mathcal{T}_X$ into T.

3 A new embedding

Let R and T be two semigroups. A (left) action of R on T is a mapping $R \times T \to T$, $(a, t) \to {}^a t$ such that ${}^a(ts) = {}^a t \cdot {}^a s$ and ${}^{ab}t = {}^a({}^b t)$ for all $a, b \in R$

and all $t, u \in T$. The semidirect product $T * R$ of T by R with respect to this action is the semigroup obtained by defining on $T \times R$ the product

$$(t, a) \cdot (u, b) = (t \cdot {}^a u, ab) \qquad (a, b \in R,\ t, u \in T)\,.$$

Let $R_2 = \{\sigma, \tau\}$ be the two element right-zero semigroup and let $T = S \times S$. Consider the mapping $R_2 \times T \to T$ defined by

$${}^\sigma(t, s) = (s, s) \quad \text{and} \quad {}^\tau(t, s) = (t, t)\,.$$

Clearly this mapping is an action of R_2 on $T = S \times S$. Consider the semidirect product $T * R_2$ induced by this action. We shall look at the elements of $T * R_2$ as triples from $S \times S \times R_2$. Then

$$(t_1, s_1, \alpha_1)(t_2, s_2, \alpha_2) = \begin{cases} (t_1 s_2\,,\ s_1 s_2\,,\ \alpha_2) & \text{if } \alpha_1 = \sigma \\ (t_1 t_2\,,\ s_1 t_2\,,\ \alpha_2) & \text{if } \alpha_1 = \tau \end{cases}$$

for $t_1, t_2, s_1, s_2 \in S$ and $\alpha_1, \alpha_2 \in R_2$.

Lemma 3.1 $E(T * R_2) = \{(s, e, \sigma)\,,\ (e, s, \tau)\ :\ e \in E(S),\ se = s\}\,.$

Proof: Clearly, since $(t, u, \sigma)^2 = (tu, u^2, \sigma)$, $(t, u, \sigma) \in E(T * R_2)$ if and only if $tu = t$ and $u \in E(S)$. Similarly, $(t, u, \tau) \in E(T * R_2)$ if and only if $t \in E(S)$ and $ut = u$ because $(t, u, \tau)^2 = (t^2, ut, \tau)$. ∎

For the purpose of this paper, we shall say that an element $s \in S$ is idempotent covered in S if $se = s = fs$ for some $e, f \in E(S)$, that is, if $s \in Se \cap fS$ for some $e, f \in E(S)$. We shall say that a semigroup S is idempotent covered if all elements of S are idempotent covered in S. Note that all monoids and all regular semigroups are idempotent covered semigroups and that S is an idempotent covered semigroup if and only if

$$S = \cup_{e \in E(S)} Se \quad \text{and} \quad S = \cup_{e \in E(S)} eS\,,$$

that is, if and only if S is both the union of its idempotent generated principal left ideals and the union of its idempotent generated principal right ideals. We shall denote by $\mathcal{IC}$ the class of all idempotent covered semigroups.

Note that if $S \in \mathcal{IC}$ then the set $E(T * R_2)$ is not empty. If $s \in Se$ for some $e \in E(S)$, then (e, s, τ) and (s, e, σ) are idempotents of $T * R_2$; and every idempotent of $T * R_2$ is obtained in this manner. Consider now the subset of idempotents

$$E = \{(s, t, \alpha) \in E(T * R_2) \,:\, s \in S^1 t\}\,,$$

and let $T(S) = \langle E \rangle$.

Proposition 3.2 *Let $S \in \mathcal{IC}$. Then:*

(i) $E = \{(s, e, \sigma) : e \in E(S),\, se = s\} \cup \{(e, s, \tau) : e \in E(S),\, e\mathcal{L}s\}$.

(ii) $T(S) = \{(s, t, \alpha) \in T * R_2 \,:\, s \in S^1 t\}$ *is a semiband of depth 4 with* $E(T(S)) = E$.

(iii) The mapping $\varphi : S \to T(S)$, $s \to (s, s, \sigma)$ *embeds* S *into* $T(S)$.

Proof: (i). By Lemma 3.1 the elements of $\{(s, e, \sigma) : e \in E(S),\, se = s\} \cup \{(e, s, \tau) : e \in E(S),\, e\mathcal{L}s\}$ are idempotents of $T * R_2$ that clearly belong to E. Further, if $(a, b, \sigma) \in E \subseteq E(T * R_2)$, then $b \in E(S)$ and $ab = a$. Finally, let $(a, b, \tau) \in E$. Then $a \in S^1 b$ and once more by Lemma 3.1, $a \in E(S)$ and $ba = b$, whence $a\mathcal{L}b$. We have proved (i).

(ii). It is trivial to verify that $A = \{(s, t, \alpha) \in T * R_2 \,:\, s \in S^1 t\}$ is a subsemigroup of $T * R_2$ containing E, and so it contains $T(S)$ also. Let $s, t \in S$ such that $s \in S^1 t$. Then let $e, f \in E(S)$ and $a \in S^1$ such that $t \in Se \cap fS$ and $s = at$. Note that (af, f, σ), (f, f, τ), (t, e, σ) and (e, e, τ) are idempotents from E such that

$$(s, t, \sigma) = (af, f, \sigma)(f, f, \tau)(t, e, \sigma) \quad \text{and} \quad (s, t, \tau) = (s, t, \sigma)(e, e, \tau)$$

Thus $(s, t, \sigma), (s, t, \tau) \in T(S)$ and $T(S) = A$ is a semiband of depth 4. Finally, by the definition of E and since $T(S) = A$, it is obvious that $E(T(S)) = E$.

(iii). The mapping φ is well defined by (ii) and it is now obvious that it embeds S into $T(S)$. ∎

Note that if S is not idempotent covered, then we can consider S^1 instead of S and embed S into $T(S^1)$. Thus we have another proof that every semigroup can be embedded into a semiband.

The following result compares the two semibands $T(S)$ and $A(S)$ of depth 4 in which we can embed $S \in \mathcal{IC}$.

Proposition 3.3 *$T(S)$ and $A(S)$ are isomorphic for every monoid S. If $S \in \mathcal{IC}$ is not a monoid, then $T(S)$ is isomorphic to the subsemigroup $A_1 = \{(\overline{h}\overline{s})\rho,\ (\overline{h}\overline{s}\overline{h})\rho,\ (\overline{r}\overline{h}\overline{s})\rho,\ (\overline{r}\overline{h}\overline{s}\overline{h})\rho \,:\, r, s \in S\}$ of $A(S)$.*

Proof: Define $\psi : T(S^1) \to A(S)$ as follows:

$$(s,t,\sigma)\psi = (\overline{a}\overline{h}\overline{t})\rho \quad \text{and} \quad (s,t,\tau)\psi = (\overline{a}\overline{h}\overline{t}\overline{h})\rho\,,$$

where $a \in S^1$ is such that $s = at$. Note that if $s = bt$ for some $b \in S^1$, then $(\overline{a}\overline{h}\overline{t})\rho = (\overline{b}\overline{h}\overline{t})\rho$ and $(\overline{a}\overline{h}\overline{t}\overline{h})\rho = (\overline{b}\overline{h}\overline{t}\overline{h})\rho$ by definition of ρ, and so ψ is well defined. Further, the description of $A(S)$ also allows us to conclude immediately that ψ is a bijection. Finally, to prove that ψ is an isomorphism we must check that

$$((s_1,t_1,\alpha_1)(s_2,t_2,\alpha_2))\psi = (s_1,t_1,\alpha_1)\psi \cdot (s_2,t_2,\alpha_2)\psi$$

for any (s_1,t_1,α_1), $(s_2,t_2,\alpha_2) \in T(S^1)$. We have four straightforward cases depending on whether α_1 and α_2 are σ and/or τ that we leave to the reader to verify.

We can now conclude that $T(S)$ and $A(S)$ are isomorphic if S is a monoid. If $S \in \mathcal{IC}$ is not a monoid, then $T(S) = \{(s,t,\alpha) \in T(S^1) \,:\, t \in S\}$. Observe now that for $t \in S$,

$$(s,t,\sigma)\psi = \begin{cases} (\overline{h}\overline{t})\rho & \text{if } s = t \\ (\overline{r}\overline{h}\overline{t})\rho & \text{if } s \neq t \text{ and } s = rt\,, \end{cases}$$

and

$$(s,t,\tau)\psi = \begin{cases} (\overline{h}\overline{t}\overline{h})\rho & \text{if } s = t \\ (\overline{r}\overline{h}\overline{t}\overline{h})\rho & \text{if } s \neq t \text{ and } s = rt\,. \end{cases}$$

So $(T(S))\psi = A_1$ and A_1 is a subsemigroup of $A(S)$ isomorphic to $T(S)$. ∎

Note that the inverse isomorphism $\psi^{-1} : A(S) \longrightarrow T(S^1)$ is defined by

$$((\overline{a}\overline{h}\overline{t})\rho)\psi^{-1} = (at,t,\sigma) \quad \text{and} \quad ((\overline{a}\overline{h}\overline{t}\overline{h})\rho)\psi^{-1} = (at,t,\tau)\,.$$

We will mention ψ^{-1} again in the last section of this paper.

4 Properties preserved by $T(S)$

In this section we shall see that some properties of $S \in \mathcal{IC}$ are preserved by $T(S)$.

Proposition 4.1 *A semigroup $S \in \mathcal{IC}$ is finite, periodic or regular if and only if the semiband $T(S)$ is respectively finite, periodic or regular.*

Proof: The finiteness case is obvious. Let $s, t \in S$ such that $s \in S^1 t$. Since $\langle (s,t,\sigma) \rangle = \{(st^{n-1}, t^n, \sigma) \,:\, n \geq 1\}$, we know that $\langle (s,t,\sigma) \rangle$ is finite if and only if $\langle t \rangle$ is finite. Similarly, since $\langle (s,t,\tau) \rangle = \{(s^n, ts^{n-1}, \tau) \,:\, n \geq 1\}$, we know that $\langle (s,t,\tau) \rangle$ is finite if and only if $\langle s \rangle$ is finite. Thus S is periodic if and only if $T(S)$ is periodic. For the regularity case, observe that $tt't = t$ for some $t' \in S$ if and only if $(s,t,\alpha)(t',t',\sigma)(s,t,\alpha) = (s,t,\alpha)$ for $\alpha \in R_2$; and so S is regular if and only if $T(S)$ is regular. ∎

The next lemma will be useful to characterize the Green's relations on $T(S)$.

Lemma 4.2 *Let $S \in \mathcal{IC}$ and let (s,t,α), $(u,v,\beta) \in T(S)$ such that $u = av$ for some $a \in S^1$.*

(i) $(s,t,\tau)\,\mathcal{R}\,(s,t,\sigma)$.

(ii) *If* $(s,t,\alpha)\,\mathcal{L}\,(u,v,\beta)$, *then* $\alpha = \beta$.

(iii) $(s,t,\tau)\,\mathcal{L}\,(u,v,\tau)$ *if and only if* $(s,t,\sigma)\,\mathcal{L}\,(u,v,\sigma)$.

(iv) $(s,t,\sigma)\,\mathcal{L}\,(u,v,\sigma)$ *if and only if there exist* (a,b,σ), $(c,d,\sigma) \in T(S)$ *such that* $(u,v,\sigma) = (a,b,\sigma)(s,t,\sigma)$ *and* $(s,t,\sigma) = (c,d,\sigma)(u,v,\sigma)$.

Proof: The proof of (i) is trivial since if $e \in E(S)$ is such that $te = t$, then $(s,t,\tau)(e,e,\sigma) = (s,t,\sigma)$ and $(s,t,\sigma)(e,e,\tau) = (s,t,\tau)$. Statement (ii) is also obvious since $(a,b,\gamma)(s,t,\alpha) = (u,v,\beta)$ only occurs if $\alpha = \beta$. Statement (iii) follows from the fact that $(a,b,\alpha)(a_1,b_1,\tau) = (a_2,b_2,\tau)$ if and only if $(a,b,\alpha)(a_1,b_1,\sigma) = (a_2,b_2,\sigma)$. Finally, the statement (iv) holds true since $(a,b,\tau)(c,d,\alpha) = (ac_1, bc_1, \sigma)(c,d,\alpha)$ for $c = c_1 d$ in S^1. ∎

The natural partial order $\leq$ on a semigroup S is defined by

$$s \leq t \quad \text{if and only if} \quad s = at = tb = sb \text{ for some } a, b \in S\,.$$

This natural partial order was introduced by Mitsch [13] for any semigroup and it generalizes the more usual natural partial order for regular semigroups introduced independently by Hartwig [5] and Nambooripad [14]. The next result describes the natural partial order and the Green's relations on $T(S)$. We shall use the notation $a(s,t)$ to represent the pair (as, at) for $a \in S^1$ and $(s,t) \in S \times S$.

Proposition 4.3 *Let $S \in \mathcal{IC}$ and (s,t,α), $(u,v,\beta) \in T(S)$.*

(i) $(s,t,\alpha)\,\mathcal{R}\,(u,v,\beta)$ *if and only if* $t\mathcal{R}v$ *and* $(s,u) = a(t,v)$ *for some* $a \in S^1$.

(ii) $(s,t,\alpha)\,\mathcal{L}\,(u,v,\beta)$ *if and only if* $\alpha = \beta$ *and* $t\mathcal{L}v$.

(iii) $(s,t,\alpha)\,\mathcal{H}\,(u,v,\beta)$ *if and only if* $\alpha = \beta$, $t\mathcal{H}v$ *and* $(s,u) = a(t,v)$ *for some* $a \in S^1$.

(iv) $(s,t,\alpha)\,\mathcal{D}\,(u,v,\beta)$ *if and only if* $t\mathcal{D}v$.

(v) $(s,t,\alpha)\,\mathcal{J}\,(u,v,\beta)$ *if and only if* $t\mathcal{J}v$.

(vi) $(s,t,\alpha) \leq (u,v,\beta)$ *if and only if* $\alpha = \beta$, $t \leq v$ *and* $(s,u) = a(t,v)$ *for some* $a \in S^1$.

Proof: (i). Assume $(s,t,\alpha)\,\mathcal{R}\,(u,v,\beta)$. Then $(s,t,\sigma)\,\mathcal{R}\,(u,v,\sigma)$ by Lemma 4.2.(i) and there exist $(u_1,v_1,\sigma), (s_1,t_1,\sigma) \in T(S)$ such that

$$(u,v,\sigma) = (s,t,\sigma)(u_1,v_1,\sigma) = (sv_1, tv_1, \sigma)$$

and

$$(s,t,\sigma) = (u,v,\sigma)(s_1,t_1,\sigma) = (ut_1, vt_1, \sigma)\,.$$

Thus $v\mathcal{R}t$. If $a \in S^1$ is such that $s = at$, then $u = sv_1 = atv_1 = av$.

Assume now that $t\mathcal{L}v$ and $(s,u) = a(t,v)$ for some $a \in S^1$. Let $v_1, t_1 \in S^1$ such that $t = vt_1$ and $v = tv_1$. Then

$$(s,t,\sigma)(v_1,v_1,\sigma) = (sv_1, tv_1, \sigma) = (atv_1, tv_1, \sigma) = (av, v, \sigma) = (u,v,\sigma)$$

and

$$(u,v,\sigma)(t_1,t_1,\sigma) = (ut_1, vt_1, \sigma) = (avt_1, vt_1, \sigma) = (at, t, \sigma) = (s,t,\sigma)\,.$$

Again by Lemma 4.2.(i) we conclude that $(s,t,\alpha)\,\mathcal{R}\,(u,v,\beta)$.

(ii). Assume $(s,t,\alpha)\,\mathcal{L}\,(u,v,\beta)$. Then $\alpha=\beta$ and $(s,t,\sigma)\,\mathcal{L}\,(u,v,\sigma)$ by (ii) and (iii) of Lemma 4.2, and by (iv) of the same lemma there exist $(u_1,v_1,\sigma),(s_1,t_1,\sigma)\in T(S)$ such that

$$(u,v,\sigma) = (u_1,v_1,\sigma)(s,t,\sigma) = (u_1t, v_1t, \sigma)$$

and

$$(s,t,\sigma) = (s_1,t_1,\sigma)(u,v,\sigma) = (s_1v, t_1v, \sigma)\,.$$

Thus $t\mathcal{L}v$.

Assume now that $\alpha=\beta$ and $t\mathcal{L}v$. Let $a,b,s_1,u_1\in S^1$ such that $v=at$, $t=bv$, $s=s_1t$ and $u=u_1v$. Then

$$(u_1a, a, \sigma)(s,t,\sigma) = (u_1at, at, \sigma) = (u,v,\sigma)$$

and

$$(s_1b, b, \sigma)(u,v,\sigma) = (s_1bv, bv, \sigma) = (s,t,\sigma)\,,$$

and by Lemma 4.2.(ii) and (iii) we conclude that $(s,t,\alpha)\,\mathcal{L}\,(u,v,\beta)$.

(iii). The statement (iii) follows now from (i) and (ii).

(iv). Assume $(s,t,\alpha)\,\mathcal{D}\,(u,v,\beta)$ and let $(s_1,t_1,\alpha_1)\in T(S)$ such that

$$(s,t,\alpha)\,\mathcal{R}\,(s_1,t_1,\alpha_1)\,\mathcal{L}\,(u,v,\beta)\,.$$

By (i) and (ii) we must have $t\mathcal{R}t_1\mathcal{L}v$, and so $t\mathcal{D}v$.

Conversely, assume $t\mathcal{D}v$ and let $t_1\in S$ such that $t\mathcal{R}t_1\mathcal{L}v$. Let $s=at$ for some $a\in S^1$. Observe now that $(s,t,\alpha)\,\mathcal{R}\,(at_1,t_1,\beta)$ by (i), and that $(at_1,t_1,\beta)\,\mathcal{L}\,(u,v,\beta)$ by (ii). Hence $(s,t,\alpha)\,\mathcal{D}\,(u,v,\beta)$.

(v). By (iv) it is enough to show that $(t,t,\sigma)\,\mathcal{J}\,(v,v,\sigma)$ if and only if $t\mathcal{J}v$. But this is obvious since $a=bc$ for $a,b,c\in S$ if and only if $(a,a,\sigma)=(b,b,\sigma)(c,c,\sigma)$.

(vi). Assume $(s,t,\alpha)\le(u,v,\beta)$ and let (a,b,γ), $(c,d,\delta)\in T(S)$ such that

$$(s,t,\alpha) = (a,b,\gamma)(u,v,\beta) = (u,v,\beta)(c,d,\delta) = (s,t,\alpha)(c,d,\delta)\,.$$

Then $\alpha = \beta = \delta$. We shall assume that $\alpha = \sigma$ and prove only this case since the case $\alpha = \tau$ is shown similarly. Then $t = vd = td$ and $s = ud = sd$. Let $u_1 \in S^1$ such that $u = u_1 v$. Then $s = ud = u_1 vd = u_1 t$ and $(s, u) = u_1(t, v)$. If $\gamma = \sigma$, then $t = bv$ and $t \leq v$. If $\gamma = \tau$, then $t = bu = bu_1 v$ and $t \leq v$ again. We have shown the direct implication.

Assume now that $\alpha = \beta$, $t \leq v$ and $(s, u) = a(t, v)$, and let $b, c \in S^1$ such that $t = bv = vc = tc$. Once again we have two cases to consider, $\alpha = \sigma$ and $\alpha = \tau$, but since they are similar we shall prove only one. Thus assume that $\alpha = \sigma$. Then

$$(ab, b, \sigma)(u, v, \sigma) = (abv, bv, \sigma) = (at, t, \sigma) = (s, t, \sigma)\,,$$

$$(u, v, \sigma)(c, c, \sigma) = (uc, vc, \sigma) = (avc, vc, \sigma) = (at, t, \sigma) = (s, t, \sigma)\,,$$

$$(s, t, \sigma)(c, c, \sigma) = (sc, tc, \sigma) = (atc, tc, \sigma) = (at, t, \sigma) = (s, t, \sigma)\,,$$

and $(s, t, \sigma) \leq (u, v, \sigma)$. ■

The description of the Green's relations given by the previous result allows us to immediately conclude the following:

Corollary 4.4 *Let $S \in \mathcal{IC}$ and let φ be the embedding of S into $T(S)$ considered earlier. The restriction to $S\varphi$ of a Green's relation on $T(S)$ gives us precisely the corresponding Green's relation on $S\varphi$; in other words, if $\mathcal{K} \in \{\mathcal{H}, \mathcal{R}, \mathcal{L}, \mathcal{D}, \mathcal{J}\}$ and $a \in S\varphi$, then*

$$K_a^{T(S)} \cap S\varphi = K_a^{S\varphi}\,.$$

Further, $H_a^{T(S)} = H_a^{S\varphi}$ and each $\mathcal{J}$-class [$\mathcal{D}$-class] of $T(S)$ contains exactly one $\mathcal{J}$-class [$\mathcal{D}$-class] of $S\varphi$.

The local submonoids of a semigroup S are the subsemigroups eSe of S with $e \in E(S)$. In the next result we show that each maximal subgroup and each local submonoid of $S \in \mathcal{IC}$ is respectively isomorphic to some maximal subgroup and some local submonoid of $T(S)$, and vice-versa.

Proposition 4.5 *Let $S \in \mathcal{IC}$ and $e = (s, t, \alpha) \in E(T(S))$.*

(*i*) *If $\alpha = \sigma$, then $\psi : H_e \longrightarrow H_t$, $(s_1, t_1, \sigma) \longmapsto t_1$ is an isomorphism from the (maximal) subgroup H_e of $T(S)$ onto the (maximal) subgroup H_t of S; if $\alpha = \tau$, then $\psi : H_e \longrightarrow H_s$, $(s_1, t_1, \tau) \longmapsto s_1$ is an isomorphism from the (maximal) subgroup H_e of $T(S)$ onto the (maximal) subgroup H_s of S.*

(*ii*) *If* $\alpha = \sigma$*, then* $\chi : e\,T(S)\,e \longrightarrow tSt,\ (s_1, t_1, \sigma) \longmapsto t_1$ *is an isomorphism from the local submonoid* $e\,T(S)\,e$ *of* $T(S)$ *onto the local submonoid* tSt *of* S*; if* $\alpha = \tau$*, then* $\chi : e\,T(S)\,e \longrightarrow sSs,\ (s_1, t_1, \tau) \longmapsto s_1$ *is an isomorphism from the local submonoid* $e\,T(S)\,e$ *of* $T(S)$ *onto the local submonoid* sSs *of* S*.*

Proof: (*i*). If $\alpha = \sigma$, then $t \in E(S)$ and by Proposition 4.3.(*iii*),

$$H_e = \{(s_1, t_1, \sigma) \;:\; t_1\,\mathcal{H}\,t \;\text{ and }\; (s, s_1) = a(t, t_1) \;\text{ for some }\; a \in S^1\}\,.$$

It is now evident that ψ is a well-defined surjective homomorphism. If $at = bt = s$ for some $a, b \in S^1$, then $at_1 = bt_1$ and so ψ is also injective.

If $\alpha = \tau$, then $s \in E(S)$ and $t\mathcal{L}s$. Let $g = (s_1, t_1, \beta) \in T(S)$. Then $g\mathcal{H}e$ if and only if $\beta = \tau$, $t_1\mathcal{H}t$ and $(s, s_1) = a(t, t_1)$ for some $a \in S$. Since $s\mathcal{L}t$, the conditions $t_1\mathcal{H}t$ and $(s, s_1) = a(t, t_1)$ are equivalent to the conditions $s_1\mathcal{H}s$ and $(t, t_1) = b(s, s_1)$ for some $b \in S^1$. Thus

$$H_e = \{(s_1, t_1, \sigma) \;:\; s_1\,\mathcal{H}\,s \;\text{ and }\; (t, t_1) = b(s, s_1) \;\text{ for some }\; b \in S^1\}\,.$$

The proof now follows similarly to the case $\alpha = \sigma$.

(*ii*). If $\alpha = \sigma$, then $t \in E(S)$ and

$$eT(S)e = \{(sut, tut, \sigma),\, (sus, tus, \sigma) \;:\; u \in S\} = \{(av, v, \sigma) \;:\; v \in tSt\}$$

for $a \in S^1$ such that $s = at$. If $\alpha = \tau$, then $s \in E(S)$ and $s\mathcal{L}t$. Let $b \in S^1$ such that $t = bs$. Then

$$eT(S)e = \{(sut, tut, \tau),\, (sus, tus, \tau) \;:\; u \in S\} = \{(v, bv, \tau) \;:\; v \in sSs\}\,.$$

It is now trivial to check that for either $\alpha = \sigma$ or $\alpha = \tau$, the mapping χ is an isomorphism. ■

Since the subgroups and the local submonoids of $S \in \mathcal{IC}$ and $T(S)$ are isomorphic by the previous result, we now have:

Corollary 4.6 *Let* $S \in \mathcal{IC}$ *and let* $\mathcal{P}$ *be any group property and* $\mathcal{Q}$ *be any semigroup property.*

(*i*) *The (maximal) subgroups of* S *have property* $\mathcal{P}$ *if and only if the (maximal) subgroups of* $T(S)$ *have property* $\mathcal{P}$*.*

(*ii*) *The local submonoids of S have property $\mathcal{Q}$ if and only if the local submonoids of $T(S)$ have property $\mathcal{Q}$.*

An e-variety of regular semigroups [4, 11] is a class of these semigroups closed for homomorphic images, regular subsemigroups and direct products. We know that the local submonoids of a regular semigroup are regular too. Thus, if $\mathbf{V}$ is an e-variety of regular semigroup, we can define the class $\mathbf{LV}$ of all regular semigroups whose local submonoids belong to $\mathbf{V}$. It is well known that $\mathbf{LV}$ is again an e-variety of regular semigroups, and we call locally $\mathbf{V}$ the semigroups from $\mathbf{LV}$. Corollary 4.6.(*ii*) now implies the next result.

Corollary 4.7 *Let $\mathbf{V}$ be an e-variety of regular semigroups. Then $S \in \mathbf{LV}$ if and only if $T(S) \in \mathbf{LV}$.*

In particular, we can conclude that S is a completely simple semigroup (locally a group) if and only if $T(S)$ is a completely simple semigroup; S is a combinatorial strict semigroup (locally a semilattice) if and only if $T(S)$ is a combinatorial strict semigroup; S is a strict semigroup (locally a semilattice of groups) if and only if $T(S)$ is a strict semigroup; and S is a locally inverse semigroup (locally an inverse semigroup) if and only if $T(S)$ is a locally inverse semigroup.

The Corollary 4.7 cannot be obtained directly from the previously known constructions of an embedding of a semigroup S into a semiband $B(S)$ since those constructions usually assume that S is a monoid. For example, the usual conclusions were that $B(S)$ is completely simple if and only if S is a group, or that $B(S)$ is locally inverse if and only if S is inverse. However, for some of those constructions, we can obtain Corollary 4.7 if we consider instead a subsemiband $B^*(S)$ of $B(S^1)$ and embed the semigroup S into $B^*(S)$.

A semigroup S is simple [bisimple] if it has only one $\mathcal{J}$-class [$\mathcal{D}$-class]. A semigroup S with element 0 is 0-simple [0-bisimple] if $\{0\}$ and $S \setminus \{0\}$ are the only $\mathcal{J}$-classes [$\mathcal{D}$-classes] of S and S is not a null semigroup, that is, $S^2 \neq \{0\}$. A non-zero idempotent $e \in E(S)$ is called primitive if for all $f \in E(S)$,

$$ef = fe = f \neq 0 \implies e = f\,.$$

It is well known that a 0-simple [simple] semigroup has a primitive idempotent if and only if all non-zero idempotents are primitive. A completely 0-simple [completely simple] semigroup is a 0-simple [simple] semigroup with a primitive idempotent. Note that we have used above that a completely simple semigroup is a regular semigroup whose local submonoids are groups. It is well known that the two definitions are equivalent.

Let S be a semigroup and $a \in S$. Let $J(a)$ be the principal ideal generated by a and J_a the $\mathcal{J}$-class of a. Then $I(a) = J(a) \setminus J_a$ is an ideal of $J(a)$ and $J(a)/I(a) \cong J_a \cup \{0\}$ is either a 0-simple semigroup or a null semigroup. Note further that $J(a)/I(a)$ is 0-simple if and only if there exist $b, c \in J_a$ such that $bc \in J_a$. The semigroups $J(a)/I(a)$ are called the principal factors of S, and S is called semisimple if all its principal factors are 0-simple semigroups. Further, if all principal factors of S are completely 0-simple semigroups, then we say that S is completely semisimple.

We need one more definition for the next result. A semigroup S is [left, right] cryptic if the Green's $\mathcal{H}$-relation is a [left, right] congruence on S.

Proposition 4.8 *Let $\mathcal{P}$ be one of the following properties: simple, bisimple, completely simple, semisimple, completely semisimple and [left, right] cryptic. Then $S \in \mathcal{IC}$ has property $\mathcal{P}$ if and only if $T(S)$ has property $\mathcal{P}$.*

Proof: The bisimple and simple cases follow respectively from Proposition 4.3.(iv) and (v), and the completely simple case follows from Corollary 4.7. Since each $\mathcal{J}$-class of $T(S)$ contains exactly one $\mathcal{J}$-class of $S\varphi$ (Corollary 4.4), if S is semisimple (and so $S\varphi$ is semisimple), then $T(S)$ is semisimple. Let J be a $\mathcal{J}$-class of S and let J_1 be the $\mathcal{J}$-class of $T(S)$ containing $J\varphi$. If $T(S)$ is semisimple, then there exist $(s_i, t_i, \alpha_i) \in J_1$ for $i = 1, 2$ such that $(s_1, t_1, \alpha_1)(s_2, t_2, \alpha_2) \in J_1$. Thus $t_1, t_2 \in J$. If $\alpha_1 = \sigma$, then $t_1 t_2 \in J$. If $\alpha_1 = \tau$, then $t_1 s_2 \in J$; but since $s_2 \in S^1 t_2$, we must have also $s_2 \in J$. We conclude that if $T(S)$ is semisimple, then S is semisimple too. We have shown the semisimple case.

For the completely semisimple case, we already know that $S \in \mathcal{IC}$ is semisimple if and only if $T(S)$ is semisimple. Let $e \in E(S)$. Observe that $(s, t, \alpha)(e, e, \sigma) = (s, t, \alpha)$ implies $\alpha = \sigma$, and that $(e, e, \sigma)(s, t, \alpha) = (s, t, \alpha)$ implies $s = t$. It is now trivial to check that e is a primitive idempotent of a principal factor of S if and only if (e, e, σ) is a primitive idempotent of the corresponding principal factor of $T(S)$. Hence, S is completely semisimple if and only if $T(S)$ is completely semisimple.

If $T(S)$ is [left, right] cryptic, then S is [left, right] cryptic since

$$\mathcal{H}^{T(S)} \cap S\varphi = \mathcal{H}^{S\varphi}$$

and S is isomorphic to $S\varphi$. Consider now $(s,t,\alpha),(u,v,\beta),(p,q,\gamma) \in T(S)$ such that $(s,t,\alpha)\,\mathcal{H}\,(u,v,\beta)$. Then $\alpha = \beta$, $t\mathcal{H}v$ and $(s,u) = a(t,v)$. If S is left cryptic, then

$$(p,q,\gamma)(s,t,\alpha)\,\mathcal{H}\,(p,q,\gamma)(u,v,\beta)$$

since $(p,q,\gamma)(s,t,\alpha) = (p_1t, q_1t, \alpha)$ and $(p,q,\gamma)(u,v,\beta) = (p_1v, q_1v, \alpha)$ for $(p_1,q_1) = (p,q)$ (if $\gamma = \sigma$) or $(p_1,q_1) = (pa, qa)$ (if $\gamma = \tau$). Hence, $T(S)$ is left cryptic if S is left cryptic. If S is right cryptic, then $tr\mathcal{H}vr$ for any $r \in S$, and so

$$(s,t,\alpha)(p,q,\gamma)\,\mathcal{H}\,(u,v,\beta)(p,q,\gamma)$$

since $(s,t,\alpha)(p,q,\gamma) = (sr, tr, \gamma)$ and $(u,v,\beta)(p,q,\gamma) = (ur, vr, \gamma)$ for $r = p$ (if $\alpha = \tau$) or $r = q$ (if $\alpha = \sigma$). Hence, $T(S)$ is right cryptic if S is right cryptic. We have shown that $S \in \mathcal{IC}$ is [left, right] cryptic if and only if $T(S)$ is [left, right] cryptic. ■

Let $S \in \mathcal{IC}$ be a semigroup with 0. Then $\overline{0} = \{(0,0,\sigma),(0,0,\tau)\}$ is the kernel of $T(S)$, that is, the minimal ideal of $T(S)$. Let $T^*(S) = T(S)/\overline{0}$. Clearly $T^*(S)$ is a semiband with 0 and the mapping $\varphi^* : S \longrightarrow T^*(S)$ defined by

$$s\varphi^* = \begin{cases} s\varphi & \text{if } s \neq 0 \\ \overline{0} & \text{if } s = 0 \end{cases}$$

embeds S into $T^*(S)$.

Proposition 4.9 *Let $S \in \mathcal{IC}$ be a semigroup with 0 element. Then S is 0-simple, 0-bisimple or completely 0-simple if and only if $T^*(S)$ is respectively 0-simple, 0-bisimple or completely 0-simple.*

Proof: If S is 0-simple, then $\overline{0}$ and $T(S) \setminus \overline{0}$ are the only $\mathcal{J}$-classes of $T(S)$ by Proposition 4.3.(v), and so $T^*(S)$ is a principal factor of $T(S)$. By Proposition 4.8 the semiband $T(S)$ is semisimple (S is semisimple), whence $T^*(S)$ is a 0-simple semigroup. If $T^*(S)$ is 0-simple, then S has only two $\mathcal{J}$-classes, namely $\{0\}$ and $S \setminus \{0\}$, again by Proposition 4.3.(v). Note that if S is a null semigroup, then $T^*(S)$ is a null semigroup too. Hence S is 0-simple. We have shown that S is 0-simple if and only if $T^*(S)$ is 0-simple.

A 0-bisimple semigroup is a 0-simple semigroup with only two $\mathcal{D}$-classes. Thus, the 0-bisimple case follows from Proposition 4.3.(iv) and the 0-simple case. As in the proof of the completely simple case of Proposition 4.8, to show the completely 0-simple case we need only to show that e is a primitive idempotent of S if and only if (e, e, σ) is a primitive idempotent of $T^*(S)$. This is trivial to check and therefore we leave the details to the reader. ■

5 Regular semigroups

In this section we shall assume that S is always a regular semigroup. Then $T(S)$ is also regular by Proposition 4.1. Let R be a regular subsemigroup of $T(S)$ containing $S\varphi$. Then

$$\mathcal{K}^{T(S)} \cap R = \mathcal{K}^R$$

for the Green relation $\mathcal{K} \in \{\mathcal{H}, \mathcal{L}, \mathcal{R}\}$ since R is a regular subsemigroup. In the next result we show that the same equality holds true for $\mathcal{K} = \mathcal{D}$ and for $\mathcal{K} = \mathcal{J}$. We shall prove also that $(s, t, \alpha) \leq^{T(S)} (u, v, \beta)$ if and only if $(s, t, \alpha) \leq^R (u, v, \beta)$ for any $(s, t, \alpha), (u, v, \beta) \in R$.

Proposition 5.1 *Let S be a regular semigroup and let R be a regular subsemigroup of $T(S)$ containing $S\varphi$. Then*

$$\mathcal{K}^{T(S)} \cap R = \mathcal{K}^R \quad \textit{and} \quad \leq^{T(S)} \cap R = \leq^R$$

for $\mathcal{K} \in \{\mathcal{H}, \mathcal{L}, \mathcal{R}, \mathcal{D}, \mathcal{J}\}$.

Proof: As observed above, we only need to show this proposition for $\mathcal{K} \in \{\mathcal{D}, \mathcal{J}\}$. It is also clear that

$$\mathcal{K}^R \subseteq \mathcal{K}^{T(S)} \cap R \quad \text{and} \quad \leq^R \subseteq \leq^{T(S)} \cap R\,.$$

Let $(s, t, \alpha), (u, v, \beta) \in R$ such that $(s, t, \alpha)\mathcal{J}^{T(S)}(u, v, \beta)$. Then

$$(s, t, \alpha)\mathcal{L}^R(t, t, \alpha), \quad (u, v, \beta)\mathcal{L}^R(v, v, \beta) \quad \text{and} \quad (t, t, \alpha)\mathcal{J}^{S\varphi}(v, v, \beta)\,.$$

Hence $(s, t, \alpha)\mathcal{J}^R(u, v, \beta)$, and $\mathcal{J}^{T(S)} \cap R = \mathcal{J}^R$. The proof for $\mathcal{K} = \mathcal{D}$ is similar.

Assume now that $(s, t, \alpha) \leq^{T(S)} (u, v, \beta)$ with $(s, t, \alpha), (u, v, \beta) \in R$. By Proposition 4.3.$(vi)$ we know that $\alpha = \beta$, $t \leq v$ and $(s, u) = a(t, v)$ for some $a \in S^1$. Since S is regular, there are idempotents $e, f \in S$ such that $t = ev = vf$ and $e\mathcal{R}t\mathcal{L}f$. Let t' be the inverse of t such that $tt' = e$ and $t't = f$. Then $(ae, e, \sigma) = (s, t, \alpha)(t', t', \sigma) \in R$ and $(f, f, \alpha) = (t', t', \sigma)(s, t, \alpha) \in R$. Observe now that (ae, e, σ) and (f, f, α) are idempotents of R such that

$$(ae, e, \sigma)(u, v, \beta) = (s, t, \alpha) \quad \text{and} \quad (u, v, \beta)(f, f, \alpha) = (s, t, \alpha)\,.$$

Thus $(s, t, \alpha) \leq^R (u, v, \beta)$, and $\leq^{T(S)} \cap R = \leq^R$. ■

Let $R(S)=\{(s,t,\alpha)\in T(S)\,:\, s\mathcal{L}t\,\}$. Since $\mathcal{L}$ is a right congruence on S, $R(S)$ is a subsemigroup of $T(S)$. The idempotent of $R(S)$ are the elements $(s,t,\alpha)\in R(S)$ such that either $\alpha=\sigma$ and $t\in E(S)$, or $\alpha=\tau$ and $s\in E(S)$.

Proposition 5.2 *Let S be a regular semigroup. Then $R(S)$ is a regular semiband of depth 2 and $\varphi : S \longrightarrow R(S)$, $s \longmapsto (s,s,\sigma)$ embeds S into $R(S)$. Further,*

$$\mathcal{K}^{T(S)}\cap R(S)=\mathcal{K}^{R(S)} \qquad \textit{and} \qquad \leq^{T(S)}\cap R(S)=\leq^{R(S)}$$

for $\mathcal{K}\in\{\mathcal{H},\mathcal{L},\mathcal{R},\mathcal{D},\mathcal{J}\}$.

Proof: Let $s,t\in S$ such that $s\mathcal{L}t$. Let s' and t' be inverses of s and t respectively. Then (st',tt',σ), $(t't,t,\tau)\in E(R(S))$ and

$$(s,t,\tau)=(st',tt',\sigma)(t't,t,\tau)\,.$$

Similarly (ss',ts',τ), $(s,s's,\sigma)\in E(R(S))$ and

$$(s,t,\sigma)=(ss',ts',\tau)(s,s's,\sigma).$$

Thus $R(S)$ is a semiband of depth 2. Observe now that (t',t',σ) is an inverse of (s,t,σ) and that (s',s',τ) is an inverse of (s,t,τ). Hence $R(S)$ is a regular semigroup. Clearly φ embeds S into $R(S)$ and the second statement of this proposition is a particular case of Proposition 5.1. ∎

In the previous section we showed that $T(S)$ preserves many properties of S. We can now show that $R(S)$ preserves the same properties. Some of those properties follow immediately from the fact that $S\varphi\subseteq R(S)\subseteq T(S)$ and Proposition 5.2, but for others we have to mimic the proof presented in the previous section for $T(S)$. In the next three result we register those properties preserved by $R(S)$, but we shall not include their proofs since is just a question of rephrasing the proofs presented in the previous section.

Proposition 5.3 *Let $\mathcal{O}$ be any group property and let $\mathcal{Q}$ be any semigroup property. Let $\mathcal{P}$ be one of the following properties: finite, periodic, simple, bisimple, completely simple, semisimple, completely semisimple, [left, right] cryptic, the (maximal) subgroups have property $\mathcal{O}$ or the local submonoids have property $\mathcal{Q}$. Then, a regular semigroup S has property $\mathcal{P}$ if and only if the regular semigroup $R(S)$ has property $\mathcal{P}$.*

Proposition 5.4 *Let* $\mathbf{V}$ *be an e-variety of regular semigroup. Then* $S \in \mathbf{LV}$ *if and only if* $R(S) \in \mathbf{LV}$.

If S is a semigroup with a 0-element, then we can consider also the semigroup $R^*(S) = R(S)/\overline{0}$. We can easily see that S embeds naturally into $R^*(S)$ and we can obtain a version for $R^*(S)$ of Proposition 4.9.

Proposition 5.5 *Let* S *be a regular semigroup with* 0*-element. Then* S *is 0-simple, 0-bisimple or completely 0-simple if and only if* $R^*(S)$ *is respectively 0-simple, 0-bisimple or completely 0-simple.*

There are however properties of S preserved by $R(S)$ that are not preserved by $T(S)$. One of those properties is the complete regularity. For example, if $S = \{0, 1\}$ with the usual product, then $(0, 1, \tau) \in T(S)$ and $(0, 1, \tau)^2 = (0, 0, \tau)$ is not $\mathcal{H}$-related to $(0, 1, \tau)$. Hence $T(S)$ is not completely regular. We shall prove next that if S is completely regular, then $R(S)$ is also completely regular.

Proposition 5.6 *A regular semigroup* S *is completely regular if and only if* $R(S)$ *is completely regular.*

Proof: We just have to prove the direct implication since a regular subsemigroup of a completely regular semigroup is again completely regular. Let $s, t \in S$ such that $s\mathcal{L}t$, that is, $s = at$ and $t = bs$ for some $a, b \in S$. Let $e \in H_s \cap E(S)$ and $f \in H_t \cap E(S)$. Then $be \in H_t$ and $abe = abss^{-1} = ats^{-1} = ss^{-1} = e$ where s^{-1} is the inverse of s in the group H_s. It is now clear that (af, f, σ) and (e, be, τ) are idempotents of $R(S)$. Further $(af, f, \sigma)\mathcal{H}(s, t, \sigma)$ and $(e, be, \tau)\mathcal{H}(s, t, \tau)$. Thus $R(S)$ is completely regular. ∎

The previous result allows to conclude that every (finite) complete regular semigroup is embeddable into a (finite) complete regular semiband. This result was first proved by Pastijn [15] using a subsemiband of $A(S)$ (see also [1] for another proof using Petrich's embeding [17] instead). If S is a completely regular monoid, then S is a semilattice Y of completely simple semigroups D_α, $\alpha \in Y$. Pastijn [15, Theorem 3.5] showed that the subset

$$A_1 = \{(\overline{s}\overline{h}\overline{t})\rho,\ (\overline{s}\overline{h}\overline{t}\overline{h})\rho\ :\ s \in D_\nu,\ t \in D_\mu,\ \nu, \mu \in Y,\ \nu \geq \mu\}.$$

of $A(S)$ is a complete regular subsemiband. The mapping $\psi_1 : s \longrightarrow (\overline{1}\overline{h}\overline{s})\rho$ embeds S into A_1 (in [15] was used the embedding $s \longrightarrow (\overline{1}\overline{h}\overline{s}\overline{h})\rho$ instead).

Let $s \in D_\nu$ and $t \in D_\mu$ with $\nu \geq \mu$ and let $s_1 = stt^{-1} = st^{-1}t$. Then $s_1 \mathcal{L} t$, and $(\overline{s}\overline{h}\overline{t})\rho = (\overline{s_1}\overline{h}\overline{t})\rho$ and $(\overline{s}\overline{h}\overline{t}\overline{h})\rho = (\overline{s_1}\overline{h}\overline{t}\overline{h})\rho$. Hence

$$A_1 = \{(\overline{s}\overline{h}\overline{t})\rho,\, (\overline{s}\overline{h}\overline{t}\overline{h})\rho \;:\; s\mathcal{L}t\}\,.$$

It is now trivial to check that $\psi^{-1}_{|A_1} : A_1 \longrightarrow R(S)$ is an isomorphism for $\psi^{-1} : A(S) \longrightarrow T(S)$ the isomorphism defined at the end of Section 3. Furthermore, $\psi_1\psi^{-1} = \varphi$ for $\varphi : S \longrightarrow R(S)$ the embedding defined in Proposition 5.2. In particular, A_1 is a semiband of depth 2 (in [15] it was shown that every element of A_1 is the product of at most 4 idempotents).

Let $\mathcal{T}_X$ be the full transformation semigroup on a set X and let $X' = \{x' : x \in X\}$ be a set disjoint from X but with the same cardinality. Recall the Higgins' embedding [6] of $\mathcal{T}_X$ into a subsemiband T of $\mathcal{T}_Y$ for $Y = X \cup X'$ (see Section 2). Next, we shall compare the two semibands $R(\mathcal{T}_X)$ and T of depth 2. It is well known that for $\lambda, \mu \in \mathcal{T}_X$, $\lambda\mathcal{L}\mu$ if and only if $X\lambda = X\mu$. Thus

$$R(\mathcal{T}_X) = \{(\lambda,\mu,\alpha) \in \mathcal{T}_X \times \mathcal{T}_X \times R_2 \;:\; X\lambda = X\mu\}\,.$$

For $\Delta = (\lambda,\mu,\alpha) \in R(\mathcal{T}_X)$ define $\overline{\delta} \in \mathcal{T}_Y$ as follows:

$$\begin{cases} x\overline{\delta} = x\mu \\ x'\overline{\delta} = x\lambda \end{cases} \text{if } \alpha = \sigma \qquad \text{or} \qquad \begin{cases} x\overline{\delta} = (x\mu)' \\ x'\overline{\delta} = (x\lambda)' \end{cases} \text{if } \alpha = \tau$$

Then $\overline{\delta} \in T$ since $X\lambda = X\mu$. Further, $X\overline{\delta} = X'\overline{\delta} \subseteq X$ if $\alpha = \sigma$, and $X\overline{\delta} = X'\overline{\delta} \subseteq X'$ if $\alpha = \tau$. Consider now the mapping $\psi : R(\mathcal{T}_X) \longrightarrow T$ defined by $\Delta\psi = \overline{\delta}$.

Proposition 5.7 *The semibands $R(\mathcal{T}_X)$ and T are isomorphic and the mapping ψ defined above is an isomorphism from $R(\mathcal{T}_X)$ onto T.*

Proof: The mapping ψ is clearly a bijection. So, we need to show only that ψ is a homomorphism. Let $\Delta_1 = (\lambda_1,\mu_1,\alpha_1)$ and $\Delta_2 = (\lambda_2,\mu_2,\alpha_2)$ be two elements of $R(\mathcal{T}_X)$, and let $\Delta = \Delta_1\Delta_2$. Let $\overline{\delta_1} = \Delta_1\psi$, $\overline{\delta_2} = \Delta_2\psi$ and $\overline{\delta} = \Delta\psi$. We want to prove that $\overline{\delta} = \overline{\delta_1}\,\overline{\delta_2}$. If $\alpha_1 = \sigma$ and $\alpha_2 = \sigma$, then $\Delta = (\lambda_1\mu_2, \mu_1\mu_2, \sigma)$. Further, for $x \in X$,

$$x\,\overline{\delta_1}\,\overline{\delta_2} = x\mu_1\overline{\delta_2} = x\mu_1\mu_2 = x\overline{\delta} \quad \text{and} \quad x'\,\overline{\delta_1}\,\overline{\delta_2} = x'\lambda_1\overline{\delta_2} = x'\lambda_1\mu_2 = x'\overline{\delta}\,.$$

Thus $\overline{\delta} = \overline{\delta_1\delta_2}$ if $\alpha_1 = \alpha_2 = \sigma$. The other three cases are shown similarly. ■

Let C be a class of semigroups. For $n \geq 2$ let $\sigma_C^{(2)}(n)$ denote the smallest integer $k \geq n$ such that every semigroup of C of order not greater than n can be embeddable into a semiband of depth 2 and order not greater than k. Let G be the class of all groups and let Reg be the class of all regular semigroups. Giraldes and Howie [3] showed that

$$\sigma_G^{(2)}(n) \leq 2n^2 \qquad \text{and} \qquad \sigma_{Reg}^{(2)}(n) \leq (n+1)^3 \,.$$

We can now improve the upper bound for $\sigma_{Reg}^{(2)}(n)$.

Proposition 5.8 $\sigma_{Reg}^{(2)}(n) \leq 2n^2$ *for all* $n \geq 2$.

Proof: We just need to observe that $R(S)$ has order less than or equal to $2n^2$ if S has order n. ∎

Let S be regular semigroup of order n. Note that $R(S)$ has order $2n^2$ only when S is a left group; but if S is not a left group, then the order of $R(S)$ can decrease significantly. Note further that we can do a left-right dual construction of $R(S)$ and obtain another semiband $L(S)$ of depth 2 in which S can be embedded. We can check also that $L(S)$ has order $2n^2$ only when S is a right group. Therefore, if S is a non-group regular semigroup of order n, then S can be embedded in a regular semiband of depth 2 of order less than $2n^2$. It seems reasonable to define now the integer

$$\sigma_C^{(2)}(n,m)$$

as follows. For a class C of regular semigroups and an integer $n \geq 2$, let $\sigma_C^{(2)}(n,m)$ denote the smallest integer $k \geq n$ such that every semigroup of C of order not greater than n and with subgroups of order not greater that m, can be embedded into a semiband of depth 2 of order not greater than k. Of course that $\sigma_C^{(2)}(n,m)$ makes sense only for classes C of regular semigroups containing other semigroups besides groups.

Proposition 5.9 *Let C be a class of regular semigroups. Then*

$$\sigma_C^{(2)}(n,m) \leq 2n\sqrt{n}\sqrt{m} = 2\sqrt{n^3 m}\,.$$

Proof: Let $S \in C$ be a regular semigroup of order not greater than n and with subgroups of order not greater than m. If l and r are respectively the sizes of the largest $\mathcal{L}$-class and the largest $\mathcal{R}$-class of S, then $R(S)$ has order less than $2nl$ and $L(S)$ has order less than $2nr$. Note that $lr \leq mn$, and for $h = \min\{l, r\}$, $h \leq \sqrt{mn}$. Hence S is embeddable into a semiband of depth 2 and order not greater than $2nh \leq 2n\sqrt{n}\sqrt{m}$. ■

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References

[1] J. Almeida and A. Moura, Idempotent-generated semigroups and pseudovarieties, *Proc. Edinburgh Math. Soc.* **54** (2011), 545–568.

[2] C. Benzaken and H. Mayr, Notion de Demi-bande: Demi-bande de type deux, *Semigroup Forum* **10** (1975), 115–128.

[3] E. Giraldes and J. Howie, Embedding finite semigroups in finite semibands of minimal depth, *Semigroup Forum* **28** (1984), 135–142.

[4] T. Hall, Identities for existence varieties of regular semigroups, *Bull. Austral. Math. Soc.* **40** (1989), 59–77.

[5] R. Hartwig, How to partially order regular elements, *Math. Japon.* **25** (1980), 1–13.

[6] P. Higgins, Embedding semigroups in semibands of depth two, *Semigroup Forum* **51** (1995), 395–396.

[7] J. Howie, The subsemigroup generated by the idempotents of a full transformation semigroup, *J. London Math. Soc.* **41** (1966), 707–716.

[8] J. Howie, Embedding semigroups in semibands: some arithmetical results, *Quart. J. Math. Oxford (2)* **32** (1981), 323–337.

[9] J. Howie, Fundamentals of Semigroups Theory, *Clarendon Press, Oxford*, 1995.

[10] J. Howie and J. Selfridge, A semigroup embedding problem and an arithmetical function, *Math. Proc. Camb. Phil. Soc.* **109** (1991), 277–286.

[11] J. Kad'ourek and M. B. Szendrei, A new approach in the theory of orthodox semigroups, *Semigroup Forum* **40** (1990), 257–296.

[12] T. J. Laffey, A note on embedding finite semigroups in finite semibands, *Quart. J. Math. Oxford (2)* **34** (1983), 453–454.

[13] H. Mitsch, A natural partial order for semigroups, *Proc. Amer. Math. Soc.* **97** (1986), 384–388.

[14] K. Nambooripad, The natural partial order on a regular semigroup, *Proc. Edinburgh Math. Soc.* **23** (1980), 249–260.

[15] F. Pastijn, Embedding semigroups in semibands, *Semigroup Forum* **14** (1977), 247–263.

[16] F. Pastijn and X. Yan, Varieties of semigroups and varieties of completely regular semigroups closed for certain extensions, *J. Algebra* **163** (1994), 777–794.

[17] M. Petrich, Embedding semigroups into idempotent generated ones, *Monatsh. Math.* **141** (2004), 315–322.

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