The variety of strict pseudosemilattices

Luís Oliveira

October 23, 2009

Abstract

The goal of this paper is to present a basis of identities B for the variety of all strict pseudosemilattices. We will then use B to obtain some properties for the variety of all strict pseudosemilattices.

1 Introduction

A regular semigroup is a semigroup S for which every $x \in S$ has an $x' \in S$ such that $xx'x = x$. We shall denote the set of idempotents of a regular semigroup S by $E(S)$. We define the following two binary relations on $E(S)$:

$$e\,\omega^r f \;\Leftrightarrow\; e = fe \quad \text{ and } \quad e\,\omega^l f \;\Leftrightarrow\; e = ef.$$

We consider also the binary relation $\omega = \omega^r \cap \omega^l$ on $E(S)$. Then ω^r and ω^l are quasi-orders on $E(S)$, while ω is a partial order on $E(S)$. We shall denote by $\omega^r(f)$ the set of idempotents e such that $e\,\omega^r f$. Similarly, we define $\omega^l(f)$ and $\omega(f)$.

Locally inverse semigroups are regular semigroups for which any two idempotents e and f have another idempotent g such that $\omega^r(e) \cap \omega^l(f) = \omega(g)$. This idempotent g is unique for each pair of idempotents e and f, and shall be denoted by $e \wedge f$. Thus, any locally inverse semigroup S originates a new binary algebra $(E(S), \wedge)$ called the pseudosemilattice of idempotents of S. These new binary algebras form a variety **PS** defined by the following three identities together with the left-right duals of the last two (Nambooripad, [8]):

(i) $x \wedge x \approx x$;

(ii) $(x \wedge y) \wedge (x \wedge z) \approx (x \wedge y) \wedge z$;

(iii) $((x \wedge y) \wedge (x \wedge z)) \wedge (x \wedge w) \approx (x \wedge y) \wedge ((x \wedge z) \wedge (x \wedge w))$.

Abstractly, a pseudosemilattice is a binary algebra satisfying these five identities, and so, any pseudosemilattice is the pseudosemilattice of idempotents of some locally inverse semigroup.

An e-variety of regular semigroups is a class of these semigroup closed under formation of homomorphic images, regular subsemigroups and direct products (see [3, 4]). The class **LI** of all locally inverse semigroup is an example of an e-variety of regular semigroups. Auinger [2] showed that the mapping

$$\varphi : \mathcal{L}_e(\mathbf{LI}) \longrightarrow \mathcal{L}(\mathbf{PS}), \quad \mathbf{V} \longmapsto \{(E(S), \wedge) \,|\, S \in \mathbf{V}\}$$

is a well-defined complete homomorphism from the lattice $\mathcal{L}_e(\mathbf{LI})$ of e-varieties of locally inverse semigroups onto the lattice $\mathcal{L}(\mathbf{PS})$ of varieties of pseudosemilattices.

Pseudosemilattices are not semigroups in general. The largest variety of pseudosemilattices whose algebras are also semigroups is the variety **NB** of all normal bands and the smallest variety of pseudosemilattices with algebras that are not semigroups is the variety **SPS** of all strict pseudosemilattices. It is well known that $\mathbf{NB} \subseteq \mathbf{SPS}$. A strict pseudosemilattice is the pseudosemilattice of idempotents of some [combinatorial] strict regular semigroup and a strict regular semigroup is a subdirect product of completely simple and/or 0-simple semigroups.

Free pseudosemilattices have been studied in [5, 7] and one solution for the word problem for free pseudosemilattices has been presented in [9]. The free strict pseudosemilattices were studied by Auinger [1] who gave also a solution for the word problem for them. In this paper we shall present a basis of identities for the variety **SPS** of all strict pseudosemilattices and draw some consequences from it.

In the next section we present a model for the free pseudosemilattice on X involving labeled connected cycle free bipartite graphs. Several other models have been presented in [10]. However, the model presented here seems to be more effective for our intentions. In Section 3 we reduce our scope of identities for defining varieties of pseudosemilattices. We define the notion of elementary identity and prove that, for varieties of pseudosemilattices, any identity satisfied by all strict pseudosemilattices is equivalent to a finite set of elementary identities.

In Section 4 we define a set B of elementary identities and prove that B is a basis of identities for **SPS**. Then, we use this basis to obtain some properties for **SPS**. For instance, we shall prove that **SPS** has infinite axiomatic rank and no independent basis of identities. We shall prove also that **SPS** is $\wedge$-irreducible in the lattice of varieties of pseudosemilattices.

2 Free pseudosemilattices

Throughout this paper we shall denote by X an arbitrary alphabet and by $(F_2(X), \wedge)$ the absolutely free binary algebra on X. A bipartite graph can be described as a triple (L, D, R) where $L \cup R$ is the set of vertices with $L \cap R = \emptyset$ and $D \subseteq L \times R$ is the set of edges. Let $\mathcal{B}$ be the set of all 6-tuples (l, L, D, R, r, φ) such that

(*a*) (L, D, R) is a connected cycle free bipartite graph with $(l, r) \in D$;

(*b*) $\varphi : L \cup R \to X$ is a labeling for the vertices of (L, D, R).

Let $D\varphi = \{(a\varphi, b\varphi) \,:\, (a, b) \in D\} \cup \{(a\varphi, a\varphi) \,:\, a \in L \cup R\}$.

We shall now inductively associate a 6-tuple $\alpha_u = (l_u, L_u, D_u, R_u, r_u, \varphi_u)$ from $\mathcal{B}$ to each word $u \in F_2(X)$:

(*i*) $|L_x| = |D_x| = |R_x| = 1$ and $l_x\varphi_x = r_x\varphi_x = x$ for any $x \in X$;

(*ii*) if $u = x \wedge y$ with $x, y \in X$, then $|L_u| = |D_u| = |R_u| = 1$, $l_u\varphi_u = x$ and $r_u\varphi_u = y$;

(*iii*) if $u = v \wedge w$, then $\alpha_u = (l_v, L_v \cup L_w, D_v \cup D_w \cup \{(l_v, r_w)\}, R_v \cup R_w, r_w, \varphi_u)$ where φ_u is the natural labeling induced by φ_v and φ_w (note that we are assuming that L_v, L_w, R_v and R_w are pairwise disjoint sets).

Clearly each α_u belongs to $\mathcal{B}$. The reverse is not so evident but it is still true, that is, we can construct, for each $\alpha \in \mathcal{B}$, a word $u \in F_2(X)$ such that $\alpha = \alpha_u$. In fact, we can usually construct several words for each α. In Section 3 of [9] we introduced an equivalence relation σ on $F_2(X)$. We can check easily that $\alpha_u = \alpha_v$ if and only if $(u, v) \in \sigma$ for each $u \in F_2(X)$ such that $|c(u)| \geq 2$, where $c(u)$ is the content of u, that is, the set of letters from X that appear in u.

Next, we introduce the concepts of labeled subgraph and strong labeled subgraph for members of $\mathcal{B}$. These concepts will be useful later. Let $\alpha_i =$

$(l_i, L_i, D_i, R_i, r_i, \varphi_i) \in \mathcal{B}$ for $i = 1, 2$. We say that α_1 is a labeled subgraph of α_2 if there exists an injective map $\psi : L_1 \cup R_1 \to L_2 \cup R_2$ such that

(*i*) $D_1\psi = \{(a\psi, b\psi) : (a, b) \in D_1\} \subseteq D_2$;

(*ii*) $a\psi\varphi_2 = a\varphi_1$ for all $a \in L_1 \cup R_1$.

Observe that we must have $L_1\psi \subseteq L_2$ and $R_1\psi \subseteq R_2$ since $D_1\psi \subseteq D_2$. If we have also $l_1\psi = l_2$ and $r_1\psi = r_2$, then we say that α_1 is a strong labeled subgraph of α_2.

Let $\mathcal{A}$ be the set of 6-tuples (l, L, D, R, r, φ) from $\mathcal{B}$:

(*c*) with no degree 1 vertex $a \notin \{l, r\}$ such that $a\varphi = b\varphi$ for $(a, b) \in D$ or $(b, a) \in D$; and

(*d*) with neither $(a, c), (b, c) \in D$ nor $(c, a), (c, b) \in D$ such that $a\varphi = b\varphi$ and $a \neq b$.

The 6-tuples from $\mathcal{A}$ correspond to the $\mathcal{R}$-reduced σ-equivalence classes introduced in [9].

We introduce an operation $\wedge$ on $\mathcal{A}$ as follows. Given

$$\alpha_1 = (l_1, L_1, D_1, R_1, r_1, \varphi_1) \in \mathcal{A} \ \text{ and } \ \alpha_2 = (l_2, L_2, D_2, R_2, r_2, \varphi_2) \in \mathcal{A},$$

consider the 6-tuple

$$\alpha' = (l_1, L_1 \cup L_2, D_1 \cup D_2 \cup \{(l_1, r_2)\}, R_1 \cup R_2, r_2, \varphi') \in \mathcal{B}$$

where φ' is the natural labeling induced by φ_1 and φ_2. Then, we apply the following rules to α' while possible:

(*i*) delete degree 1 vertices $a \notin \{l_1, r_2\}$ if $a\varphi' = b\varphi'$ and (a, b) or (b, a) is the only edge with vertex a;

(*ii*) if $a\varphi' = b\varphi'$ for $(a, c), (b, c) \in D_1 \cup D_2 \cup \{(l_1, r_2)\}$ or for $(c, a), (c, b) \in D_1 \cup D_2 \cup \{(l_1, r_2)\}$, then identify the vertices a and b (and the correspondent edges).

From [9, Proposition 4.8] this process always ends up with the same 6-tuple $\alpha = (l_1, L, D, R, r_2, \varphi) \in \mathcal{A}$, where φ is the expected labeling induced by φ' (there is no ambiguity in defining φ since we are either eliminating vertices or identifying vertices with the same label). We define $\alpha_1 \wedge \alpha_2 = \alpha$. From [9, Theorem 5.7], $(\mathcal{A}, \wedge)$ is a model for the free pseudosemilattice on X.

The maximal subsemilattices of $\mathcal{A}$ are the disjoint sets

$$\mathcal{A}_{x,y} = \{(l, L, D, R, r, \varphi) \in \mathcal{A} \, : \, l\varphi = x \text{ and } r\varphi = y\}$$

for $x, y \in X$; and for $\alpha_i = (l_i, L_i, D_i, R_i, r_i, \varphi_i) \in \mathcal{A}$, $i = 1, 2$, we have $\alpha_1 \, \omega \, \alpha_2$ if and only if α_2 is a strong labeled subgraph of α_1. Thus, if $\alpha_1 \, \omega \, \alpha_2$, then $[\alpha_1, \alpha_2]$ is a finite semilattice with maximal subchains with at most $|L_1 \cup R_1| - |L_2 \cup R_2| + 1$ elements. Further, α_2 is an ω-cover of α_1 if and only if

- $L_1 \cup R_1 = L_2 \cup R_2 \cup \{a_1, \cdots, a_n\}$ and $\varphi_2 = \varphi_1|_{L_2 \cup R_2}$;
- $D_1 = D_2 \cup \{(a_{i-1}, a_i) \, : \, 1 \leq i \leq n\}$;
- $a_0\varphi_1 = a_1\varphi_1 = \cdots = a_{n-1}\varphi_1$ and $a_n\varphi_1 \neq a_0\varphi_1$.

Lemma 2.1 *With the notations introduced above, if $\alpha_1 = \alpha_u$ for some $u \in F_2(X)$ and α_2 is an ω-cover of α_1 with $L_1\varphi_1 \cap R_1\varphi_1 = \emptyset$, then $\alpha_2 = \alpha_v$ for some $v \in F_2(X)$ obtained from u by replacing either a subword $a_0\varphi_1 \wedge a_1\varphi_1$ or a subword $a_1\varphi_1 \wedge a_0\varphi_1$ with $a_0\varphi_1$.*

Proof: Observe that $n = 1$ if $L_1\varphi_1 \cap R_1\varphi_1 = \emptyset$. It is now obvious how to obtain a word $v \in F_2(X)$ from u satisfying the conditions in the statement of this lemma. ■

3 Elementary identities

Let $u, v \in F_2(X)$ with $\alpha_u, \alpha_v \in \mathcal{A}$ and assume that

$$l_u\varphi_u = l_v\varphi_v \, , \; r_u\varphi_u = r_v\varphi_v \, , \; D_u\varphi_u = D_v\varphi_v \text{ and } L_u\varphi_u \cap R_u\varphi_u = \emptyset .$$

Let $\alpha = (l, L, D, R, r, \varphi) = \alpha_1 \wedge \alpha_2$. Then $D\varphi = D_u\varphi_u$ and $L\varphi \cap R\varphi = \emptyset$. Let

(*i*) $\alpha = \alpha_n \, \omega \, \alpha_{n-1} \, \omega \, \cdots \, \omega \, \alpha_1 = \alpha_u$ be a maximal subchain of $[\alpha, \alpha_u]$; and

(*ii*) $\alpha = \alpha_n \, \omega \, \alpha_{n+1} \, \omega \, \cdots \, \omega \, \alpha_m = \alpha_v$ be a maximal subchain of $[\alpha, \alpha_v]$.

Observe that if $\alpha_i = (l_i, L_i, D_i, R_i, r_i, \varphi_i)$ for $1 \leq i \leq m$, then $l_i\varphi_i = l\varphi$, $r_i\varphi_i = r\varphi$ and $D_i\varphi_i = D\varphi$. Thus $L_i\varphi_i \cap R_i\varphi_i = \emptyset$.

Since $\alpha_n \,\omega\, \alpha_{n-1} \,\omega\, \cdots \,\omega\, \alpha_1$ is a maximal subchain of $[\alpha, \alpha_u]$ and $L_i\varphi_i \cap R_i\varphi_i = \emptyset$ for $1 \leq i \leq n$, each α_i has one more vertex than α_{i-1}. Further, if $\alpha_n = \alpha_{w_n}$ for some $w_n \in F_2(X)$, then from Lemma 2.1 we can construct a sequence of words w_n, w_{n-1}, $\cdots$, w_1 such that

- $\alpha_i = \alpha_{w_i}$ for $1 \leq i \leq n$;
- each w_i is obtained from w_{i-1} by replacing either $x_i \wedge y_i$ or $y_i \wedge x_i$ with x_i for some $x_i, y_i \in X$.

Similarly, we can construct a sequence of words w_n, w_{n+1}, $\cdots$, w_m for the maximal subchain $\alpha_n \,\omega\, \alpha_{n+1} \,\omega\, \cdots \,\omega\, \alpha_m$ of $[\alpha, \alpha_v]$.

Lemma 3.1 *With the notations introduced above, a variety* $\mathbf{V}$ *of pseudosemilattices satisfies the identity* $u \approx v$ *if and only if it satisfies all identities* $w_i \approx w_{i+1}$ *for* $1 \leq i < m$.

Proof: Let δ be the congruence on $\mathcal{A}$ such that $\mathcal{A}/\delta$ is the free algebra on $\mathbf{V}$. Then $\mathbf{V}$ satisfies $u \approx v$ if and only if $(\alpha_u, \alpha_v) \in \delta$. Let $x = l\varphi$ and $y = r\varphi$. Since $\mathcal{A}_{x,y}$ is a semilattice and $\alpha = \alpha_u \wedge \alpha_v$, then $(\alpha_u, \alpha_v) \in \delta$ if and only if $(\alpha_i, \alpha_{i+1}) \in \delta$ for all $1 \leq i < m$, and thus if and only if $\mathbf{V}$ satisfies all identities $w_i \approx w_{i+1}$ for $1 \leq i < m$. ■

Let $\beta = (l, L', D', R', r, \varphi') \in \mathcal{A}$ be an ω-cover for $\alpha = (l, L, D, R, r, \varphi) \in \mathcal{A}$. Hence, we can assume that $L \cup R = L' \cup R' \cup \{a\}$, D has one more edge than D' and $b\varphi = b\varphi'$ for each $b \in L' \cup R'$. Assume $L\varphi \cap R\varphi = \emptyset$ and let $(l, s) \in D'$ with $s \neq r$. Then $\alpha_1 = (l, L, D, R, s, \varphi) \in \mathcal{A}$ and $\beta_1 = (l, L', D', R', s, \varphi') \in \mathcal{A}$. Further, β_1 is an ω-cover for α_1.

Lemma 3.2 *Let* ρ *be a congruence on* $\mathcal{A}$. *With the notations introduced above,* $(\alpha, \beta) \in \rho$ *if and only if* $(\alpha_1, \beta_1) \in \rho$.

Proof: Let

$$\alpha_r = (l, \{l\}, \{(l,r)\}, \{r\}, r, \varphi_r) \quad \text{and} \quad \alpha_s = (l, \{l\}, \{(l,s)\}, \{s\}, s, \varphi_s)$$

where $\varphi_r = \varphi|_{\{l,r\}}$ and $\varphi_s = \varphi|_{\{l,s\}}$ and observe that

$$\alpha \wedge \alpha_s = \alpha_1 \quad \text{and} \quad \alpha_1 \wedge \alpha_r = \alpha\,.$$

Similarly, $\beta \wedge \alpha_s = \beta_1$ and $\beta_1 \wedge \alpha_r = \beta$, and thus $(\alpha, \beta) \in \rho$ if and only if $(\alpha_1, \beta_1) \in \rho$. ■

If $a \in R$, then there exist $b \in L$ and $c \in R'$ such that $(b, a), (b, c) \in D$; let $\alpha' = (b, L, D, R, c, \varphi)$ and $\beta' = (b, L', D', R', c, \varphi')$. If $a \in L$, then there exist $b \in R$ and $c \in L'$ such that $(a, b), (c, b) \in D$; let $\alpha' = (c, L, D, R, b, \varphi)$ and $\beta' = (c, L', D', R', b, \varphi')$. Clearly α' and β' belong to $\mathcal{A}$, and β' is an ω-cover for α'. Applying Lemma 3.2 and its dual several times if necessary, we obtain the following corollary:

Corollary 3.3 *Let ρ be a congruence on $\mathcal{A}$. Then $(\alpha, \beta) \in \rho$ if and only if $(\alpha', \beta') \in \rho$.*

Let $u, v \in F_2(X)$. We shall say that $u \approx v$ is an elementary identity if:

(i) $\alpha_u, \alpha_v \in \mathcal{A}$;

(ii) $(l_u\varphi_u, D_u\varphi_u, r_u\varphi_u) = (l_v\varphi_v, D_v\varphi_v, r_v\varphi_v)$ and $L_u\varphi_u \cap R_u\varphi_u = \emptyset$;

(iii) there exists $(x, y) \in D_u\varphi_u$ such that either $l_u\varphi_u = x$ and v is obtained from u by replacing the first x in u with $(x \wedge y)$, or $r_u\varphi_u = y$ and v is obtained from u by replacing the last y in u with $(x \wedge y)$.

We shall say that a pair (α, β) of 6-tuples from $\mathcal{A}$ is elementary if there exist words $u, v \in F_2(X)$ such that $\alpha = \alpha_u$, $\beta = \alpha_v$ and $u \approx v$ is an elementary identity.

The following result follows now from Lemma 3.1 and Corollary 3.3:

Corollary 3.4 *For varieties of pseudosemilattices, an identity $u \approx v$ satisfying the first two conditions of the definition of elementary identity is equivalent to a finite set of elementary identities.*

Let $u_1, v_1 \in F_2(X)$. Auinger [1, Corollary 4.4] proved that $u_1 \approx v_1$ is satisfied by all strict pseudosemilattices if and only if

$$(l_{u_1}\varphi_{u_1}, D_{u_1}\varphi_{u_1}, r_{u_1}\varphi_{u_1}) = (l_{v_1}\varphi_{v_1}, D_{v_1}\varphi_{v_1}, r_{v_1}\varphi_{v_1}).$$

In particular $c(u_1) = c(v_1) = \{x_1, \cdots, x_n\} \subseteq X$. Consider a set

$$Y = \{x_{1,l}\,,\ x_{1,r}\,,\ \cdots\,,\ x_{n,l}\,,\ x_{n,r}\,\} \subseteq X$$

such that no two letters from Y are the same. Let u' and v' be the word obtained from u_1 and v_1 by replacing each letter x_i with $x_{i,l} \wedge x_{i,r}$. The two identities $u_1 \approx v_1$ and $u' \approx v'$ are clearly equivalent.

Recall the two rules introduced earlier to transform some $\alpha \in \mathcal{B}$ into $\alpha' \in \mathcal{A}$. If we apply these two rules to $\alpha_{u'}$, we obtain $\alpha_u \in \mathcal{A}$ for some $u \in F_2(X)$. The identity $u' \approx u$ is satisfied by all pseudosemilattices due to [9, Proposition 4.10]. We can do the same for v' and obtain a word v. Thus, for varieties of pseudosemilattices, the identity $u_1 \approx v_1$ is equivalent to the identity $u \approx v$. Clearly $u \approx v$ satisfies the two first conditions in the definition of elementary identity. If we consider now the Corollary 3.4, we obtain the following result:

Proposition 3.5 *Let $u_1 \approx v_1$ be an identity satisfied by all strict pseudosemilattices. Then, for varieties of pseudosemilattices, the identity $u_1 \approx v_1$ is equivalent to a finite set of elementary identities.*

A byproduct of the previous proposition is that every variety $\mathbf{V}$ of pseudosemilattices containing $\mathbf{SPS}$ has a basis of identities constituted by elementary identities only. Further, to prove that a set B of elementary identities satisfied by all pseudosemilattices from $\mathbf{V}$ is a basis of identities for $\mathbf{V}$ it is enough to show that every elementary identity satisfied by all pseudosemilattices from $\mathbf{V}$ is a consequence of the identities in B.

Let $\mathbf{V}$ be a variety of pseudosemilattices containing $\mathbf{SPS}$. There exists a congruence ρ on $\mathcal{A}$ such that $\mathcal{A}/\rho$ is the free algebra in $\mathbf{V}$ on the set X. This congruence ρ is fully invariant in the sense that $(\alpha\psi, \beta\psi) \in \rho$ if $(\alpha, \beta) \in \rho$ for all endomorphism ψ of $\mathcal{A}$. Assuming X is countably infinite, a set I of elementary identities is a basis of identities for $\mathbf{V}$ if and only if the congruence ρ is the fully invariant congruence on $\mathcal{A}$ generated by the set of elementary pairs

$$\Sigma_I = \{(\alpha_u, \alpha_v) \,:\, u \approx v \in I\}.$$

Let $\alpha_i = (l_i, L_i, D_i, R_i, r_i, \varphi_i) \in \mathcal{A}$, $i = 1, 2, 3$, such that α_3 is a labeled subgraph of α_1 induced by $\theta_1 : L_3 \cup R_3 \to L_1 \cup R_1$ and α_3 is a strong labeled subgraph of α_2 induced by $\theta_2 : L_3 \cup R_3 \to L_2 \cup R_2$. Consider the disconnected labeled bipartite graph

$$(l_1, L_1 \cup L_2, D_1 \cup D_2, R_1 \cup R_2, r_1, \varphi_1 \cup \varphi_2),$$

and identify all vertices $a \in L_1 \cup R_1$ and $b \in L_2 \cup R_2$ (and the corresponding edges) such that $c\theta_1 = a$ and $c\theta_2 = b$ for some $c \in L_3 \cup R_3$. We obtain a 6-tuple

$$\alpha = (l_1, L, D, R, r_1, \varphi) \in \mathcal{B}.$$

Now, we apply the two rules introduced in Section 2 to transform α into an element of $\mathcal{A}$. We shall denote this element of $\mathcal{A}$ by $\alpha_1 \wedge_{\alpha_3} \alpha_2$. Observe that α_1 is always a strong labeled subgraph of $\alpha_1 \wedge_{\alpha_3} \alpha_2$.

Let Σ be a set of elementary pairs of $\mathcal{A}$ and observe that if ψ is an endomorphism of $\mathcal{A}$ and $(\alpha_1, \alpha_2) \in \Sigma$, then $\alpha_1\psi$ is naturally a strong labeled subgraph of $\alpha_2\psi$. A pair (α, β) of elements of $\mathcal{A}$ is said to be Σ-derivable if there exist:

- a sequence $\alpha = \gamma_0,\ \gamma_1,\ \cdots,\ \gamma_n = \beta$ of words from $\mathcal{A}$;
- elementary pairs $(\alpha_i, \beta_i) \in \Sigma$, for $1 \le i \le n$; and
- endomorphisms ψ_i of $\mathcal{A}$, for $1 \le i \le n$;

such that either $\alpha_i\psi_i$ is a labeled subgraph of γ_{i-1} and $\gamma_i = \gamma_{i-1} \wedge_{\alpha_i\psi_i} \beta_i\psi_i$, or $\alpha_i\psi_i$ is a labeled subgraph of γ_i and $\gamma_{i-1} = \gamma_i \wedge_{\alpha_i\psi_i} \beta_i\psi_i$, for $1 \le i \le n$. The sequence $\gamma_0,\ \gamma_1,\ \cdots,\ \gamma_n$ is called a Σ-derivation. Observe that if $\gamma_i = (l_i, L_i, D_i, R_i, r_i, \varphi_i)$ for $0 \le i \le n$, then $D_{i-1}\varphi_{i-1} = D_i\varphi_i$ since the pairs in Σ are elementary.

We end this section with a standard result in this field of research. We omit the proof since the argumentation is the usual one.

Lemma 3.6 *Let Σ be a set of elementary pairs of $\mathcal{A}$ and let ρ be the fully invariant congruence generated by Σ. Then $(\alpha, \beta) \in \rho$ if and only if (α, β) is Σ-derivable.*

4 The variety SPS

Let $n \ge 2$ and let $\{x_1, x_2, \cdots, x_{2n}\}$ be a set of $2n$ distinct letters from X. Let

- $L_n = \{2i+1\ :\ 0 \le i \le n\}$ and $R_n = \{2i\ :\ 1 \le i \le n\}$;
- $D_n = \{(i,j)\ :\ i \in L_n,\ j \in R_n$ and $|i-j| = 1\}$;
- $\varphi_n : L_n \cup R_n \to X$ with $i\varphi_n = x_i$ for $1 \le i \le 2n$ and $(2n+1)\varphi_n = x_1$;
- $R'_n = R_n \cup \{0\}$, $D'_n = D_n \cup \{(1,0)\}$ and $\varphi'_n : L_n \cup R'_n \to X$ such that $0\varphi'_n = x_{2n}$ and $i\varphi'_n = i\varphi_n$ for $1 \le i \le 2n+1$.

Let $\alpha_n = (1, L_n, D_n, R_n, 2, \varphi_n) \in \mathcal{A}$ and $\alpha'_n = (1, L_n, D'_n, R'_n, 2, \varphi'_n) \in \mathcal{A}$.

Observe there is only one word $u_n \in F_2(X)$ such that $\alpha_{u_n} = \alpha_n$ and only one word $v_n \in F_2(X)$ such that $\alpha_{v_n} = \alpha'_n$. The identity $u_n \approx v_n$ is clearly an elementary identity, and thus they are satisfied by all strict pseudosemilattices. Further, $u_n \approx v_n$ implies $u_m \approx v_m$ for $m < n$. Let

$$B = \{u_n \approx v_n \ : \ n \geq 2\}.$$

We shall prove that B is a basis of elementary identities for **SPS**.

Let $L^*_n = R_n$, $R^*_n = L_n$ and $D^*_n = \{(a, b) \ : \ (b, a) \in D_n\}$, and define $u^*_n \in F_2(X)$ such that $\alpha_{u^*_n} = (2, L^*_n, D^*_n, R^*_n, 1, \varphi_n)$. Similarly we define $v^*_n \in F_2(X)$ and obtain elementary identities $u^*_n \approx v^*_n$.

Lemma 4.1 *For varieties of pseudosemilattices, the identity* $u_{n+1} \approx v_{n+1}$ *implies the identity* $u^*_n \approx v^*_n$.

Proof: Consider a homomorphism $\theta : F_2(X) \to F_2(X)$ such that $x_1\theta = x_1$, $x_i\theta = x_{i-1}$ for $1 < i < 2n+2$, and $x_{2n+2}\theta = x_{2n} \wedge x_1$. Thus, if $u = x_2 \wedge (u_{n+1}\theta)$ and $v = x_2 \wedge (v_{n+1}\theta)$, then $u_{n+1} \approx v_{n+1}$ implies $u \approx v$. Observe now that $\alpha_{u^*_n} \in \mathcal{A}$ is the 6-tuple obtained from $\alpha_u \in \mathcal{B}$ applying the two rules introduced earlier; whence $u \approx u^*_n$ is satisfied by all pseudosemilattices. In the same way we can verify that $v \approx v^*_n$ is satisfied by all pseudosemilattices. Therefore, for varieties of pseudosemilattices, the identity $u_{n+1} \approx v_{n+1}$ implies the identity $u^*_n \approx v^*_n$. ∎

Theorem 4.2 *The set* B *is a basis of identities for* **SPS**.

Proof: Obviously, every strict pseudosemilattice satisfies the identities from B. Thus, to prove this proposition, it is enough to show that B implies every elementary identity. Let $u \approx v$ be an elementary identity. Then α_v has one more edge than α_u: (l_u, a) for $a \neq r_u$ or (a, r_u) for $a \neq l_u$. By duality and Lemma 4.1 we can assume that (l_u, a) is the edge in α_v not in α_u. Then

$$\alpha_v = (l_u, L_u, D_u \cup \{(l_u, a)\}, R_u \cup \{a\}, r_u, \varphi_v),$$

where $b\varphi_v = b\varphi_u$ for $b \neq a$ and $a\varphi_v$ is such that $(l_u\varphi_u, a\varphi_v) \in D_u$.

Let $(b, c) \in D_u$ such that $(b\varphi_u, c\varphi_u) = (l_u\varphi_u, a\varphi_v)$ and let

$$\alpha = (l, L, D, R, r, \varphi)$$

be the smallest (connected) labeled subgraph of α_v containing the edges (l_u, a) and (b, c). Let β be the 6-tuple obtained form α by deleting the vertex a and the edge (l_u, a); then β is a labeled subgraph of α_u. Let $u', v' \in F_2(X)$ such that $\alpha_{u'} = \beta$ and $\alpha_{v'} = \alpha$. Obviously, for varieties of pseudosemilattices, the identity $u' \approx v'$ implies $u \approx v$ by construction of u' and v'.

Let i be the number of vertices in β and let n be the smallest integer such that $i \leq 2n+1$. It is not hard to check that $u' \approx v'$ is a consequence of $u_n \approx v_n$ (we just have to relabel the vertices of α_{u_n} and α_{v_n}, and choose a different pair for (l, r)). Thus, for varieties of pseudosemilattices, the identities from B imply $u \approx v$. We can now conclude that B is a basis of identities for **SPS**. ■

Let $w_n \in F_2(X)$ such that

$$\alpha_{w_n} = (1, L_n, D_n \cup \{(2n+1, 0)\}, R'_n, 2, \varphi_{n,1})$$

with $0\varphi_{n,1} = x_2$ and $i\varphi_{n,1} = i\varphi_n$ for $1 \leq i \leq 2n+1$. Then $\alpha_{w_n} \in \mathcal{A}$. Observe that α_{v_n} and α_{w_n} are the only elements of $\mathcal{A}$ with $D_{v_n}\varphi'_n = D_{w_n}\varphi_{n,1} = D_{u_n}\varphi_n$ for which α_{u_n} is an ω-cover.

Let $u'_n, w'_n \in F_2(X)$ such that $\alpha_{u'_n} = (2n+1, L_n, D_n, R_n, 2n, \varphi_n)$ and $\alpha_{w'_n} = (2n+1, L_n, D_n \cup \{(2n+1, 0)\}, R'_n, 2n, \varphi_{n,1})$. By Corollary 3.3, the identities $u_n \approx w_n$ and $u'_n \approx w'_n$ are equivalent for varieties of pseudosemilattices. Further, $u'_n \approx w'_n$ and $u_n \approx v_n$ are also equivalent identities since is just a question of relabeling the vertices. Hence $u_n \approx v_n$ and $u_n \approx w_n$ are equivalent identities for varieties of pseudosemilattices.

Lemma 4.3 *Let I be a set of elementary identities. Then $u_n \approx v_n$ is a consequence of I if and only if it is a consequence of some $u \approx v \in I$.*

Proof: Obviously, we just have to prove the direct implication. Assume $u_n \approx v_n$ is a consequence of I. Then $(\alpha_{u_n}, \alpha_{v_n})$ is Σ_I-derivable. Let

$$\alpha_{u_n} = \gamma_0,\ \gamma_1, \cdots,\ \gamma_k = \alpha_{v_n}$$

be a Σ_I-derivation. We can assume that $\gamma_1 \neq \gamma_0$, and thus one of these bipartite graphs is a proper strong labeled subgraph of the other.

Observe there is no proper strong labeled subgraph (l, L, D, R, r, φ) of α_{u_n} such that $D\varphi = D_n\varphi_n$. Thus γ_0 must be a proper strong labeled subgraph of γ_1, and there exist $(\alpha, \beta) \in \Sigma_I$ and an endomorphism ψ of $\mathcal{A}$ such that

$\alpha\psi$ is a labeled subgraph of γ_0 and $\gamma_1 = \gamma_0 \wedge_{\alpha\psi} \beta\psi$. Let $u \approx v \in I$ such that $\alpha = \alpha_u$ and $\beta = \alpha_v$.

Since α_{v_n} and α_{w_n} are the only elements from $\mathcal{A}$ ω-covered by α_{u_n} such that $D_{u_n}\varphi_{u_n} = D_{v_n}\varphi_{v_n} = D_{w_n}\varphi_{w_n}$, we must have $\alpha_{v_n} \in [\gamma_1, \gamma_0]$ or $\alpha_{w_n} \in [\gamma_1, \gamma_0]$. Thus $(\alpha_{u_n}, \alpha_{v_n})$ or $(\alpha_{u_n}, \alpha_{w_n})$ belongs to the fully invariant congruence on $\mathcal{A}$ generated by $\{(\alpha, \beta)\}$; whence $u_n \approx v_n$ or $u_n \approx w_n$ is a consequence of $u \approx v$. By the comments made prior to this lemma, $u_n \approx v_n$ and $u_n \approx w_n$ are equivalent identities, and therefore $u_n \approx v_n$ is a consequence of $u \approx v \in I$. ■

Lemma 4.4 *Let $u \approx v$ be an elementary identity. If $u \approx v$ implies $u_n \approx v_n$, then $|c(u)| \geq 2n - 2$.*

Proof: We begin assuming that $R_v = R_u \cup \{a\}$ for some vertex a. By the proof of the previous result there exists an endomorphism ψ of $\mathcal{A}$ such that $\alpha_u\psi$ is a labeled subgraph of α_{u_n} and α_{v_n} is a strong labeled subgraph of $\alpha_{u_n} \wedge_{\alpha_u\psi} \alpha_v\psi$. In particular, $\alpha_u\psi$ is a proper strong labeled subgraph of $\alpha_v\psi$.

Let $x = l_u\varphi_u$ and $\alpha = \alpha_x\psi$. Since $D_v = D_u$ and $\alpha_u\psi \neq \alpha_v\psi$, $\alpha_u\psi$ must contain two distinct labeled subgraphs α. Further, there must exist $b \in L_u$ with $b\varphi_u = l_u\varphi_u$ such that b and l_u induce two different labeled subgraphs α in $\alpha_u\psi$. Let $\{(a_{i-1}, a_i) \,:\, 1 \leq i \leq m\}$ be the set of edges in the path from l_u to b.

Clearly α_{x_1} is the only labeled subgraph of α_{u_n} that appears in two different places. Since $\alpha_u\psi$ is a labeled subgraph of α_{u_n}, we conclude that $\alpha = \alpha_{x_1}$, and so

$$\alpha_u\psi = (i, L_n, D_n, R_n, j, \varphi_n) \ \text{ with } \ (i, j) \in D_n\,.$$

Observe now that $\{a_i\varphi_u \,:\, 0 \leq i \leq m\}$ must contain at least $2n$ distinct letters from X for b to exist since $L_u \cap R_u = \emptyset$. We have shown that if $R_v = R_u \cup \{a\}$, then $|c(u)| \geq 2n$.

Assume now the other case, that is, $L_v = L_u \cup \{a\}$ for some vertex a. From Lemma 4.1 the identity $u \approx v$ implies $u^*_{n-1} \approx v^*_{n-1}$. By duality, we conclude that $|c(u)| \geq 2n - 2$ for this case. ■

A variety $\mathbf{V}$ of pseudosemilattices has finite axiomatic rank if there exist $k \in \mathbb{N}$ and a basis of identities I for $\mathbf{V}$ such that $|c(u)| \leq k$ and $|c(v)| \leq k$ for every $u \approx v \in I$. Otherwise, we say that $\mathbf{V}$ has infinite axiomatic rank. Clearly, any infinite axiomatic rank variety has no finite basis of identities.

Proposition 4.5 *The variety* **SPS** *of all strict pseudosemilattices has infinite axiomatic rank.*

Proof: If I is a basis of elementary identities for **SPS**, then there is no $k \in \mathbb{N}$ such that $|c(u)| = |c(v)| \leq k$ for all $u \approx v \in I$ by Theorem 4.2 and Lemmas 4.3 and 4.4. Further, by Proposition 3.5, if $u' \approx v'$ is an identity satisfied by all strict pseudosemilattices with $|c(u')| = |c(v')| = k$, then $u' \approx v'$ is equivalent to a finite set I_1 of elementary identities such that $|c(u)| = |c(v)| \leq 2k$ for all $u \approx v \in I_1$.

Let I' be a basis of identities for **SPS**. Then, we can replace each identity from I' by a finite set of elementary identities and obtain another basis of identities I for **SPS** constituted by elementary identities only. Since there is no $k \in \mathbb{N}$ such that $|c(u)| = |c(v)| \leq 2k$ for all $u \approx v \in I$, then there is no $k \in \mathbb{N}$ such that $|c(u')| = |c(v')| \leq k$ for all $u' \approx v' \in I'$. Therefore, the variety **SPS** has infinite axiomatic rank. ■

A basis of identities I for a variety **V** is said to be independent if no proper subset of I is a basis of identities for **V**.

Proposition 4.6 *The variety* **SPS** *of all strict pseudosemilattices has no independent basis of identities. Further, if I is a basis of identities for* **SPS***, then every co-finite subset of I is also a basis of identities for* **SPS**.

Proof: The first part of this proposition follows immediately from the second part. Thus, we shall prove only that if I is a basis of identities for **SPS**, then every co-finite subset I' of I is also a basis of identities for **SPS**.

By Proposition 3.5 and Lemma 4.3, each identity $u_n \approx v_n \in B$ is a consequence of some $u \approx v \in I$. Since each identity $u_n \approx v_n$ implies all identities $u_k \approx v_k$ with $k < n$, either I' or $I_1 = I \setminus I'$ implies all identities from B. Since I_1 is finite, it cannot be I_1 by Proposition 3.5 and Lemmas 4.3 and 4.4 . Therefore I' implies all identities from B and it is a basis of identities for **SPS**. ■

Let $\mathcal{L}$ be a lattice. An element $a \in \mathcal{L}$ is $\wedge$-irreducible if whenever $a = b \wedge c$ for some $b, c \in \mathcal{L}$, then $a = b$ or $a = c$. The element a is $\wedge$-prime if whenever $b \wedge c \leq a$, then $b \leq a$ or $c \leq a$. It is well known that a $\wedge$-prime element is also $\wedge$-irreducible.

Proposition 4.7 *The variety* **SPS** *of all strict pseudosemilattices is* $\wedge$*-prime and* $\wedge$*-irreducible in the lattice* $\mathcal{L}(\mathbf{PS})$*. Further,* **SPS** *has no covers in* $\mathcal{L}(\mathbf{PS})$*.*

Proof: It is well known that the lattice $\mathcal{L}(\mathbf{PS})$ is the disjoint union of the intervals $[\mathbf{T}, \mathbf{NB}]$ and $[\mathbf{SPS}, \mathbf{PS}]$ with $\mathbf{NB} \subset \mathbf{SPS}$, where $\mathbf{T}$ is the variety of all trivial binary algebras. Thus, for $\mathbf{U}$, $\mathbf{V} \in \mathcal{L}(\mathbf{PS})$,

$$\mathbf{U} \wedge \mathbf{V} \subset \mathbf{SPS} \qquad \Longrightarrow \qquad \mathbf{U} \subseteq \mathbf{NB} \quad \vee \quad \mathbf{V} \subseteq \mathbf{NB}$$

Assume that $\mathbf{SPS} = \mathbf{U} \wedge \mathbf{V}$ and let I and I_1 be basis of identities for $\mathbf{U}$ and $\mathbf{V}$, respectively. Then $I \cup I_1$ implies all identities from B. As in the proof of the previous result, we can conclude that I or I_1 implies all identities from B. Thus $\mathbf{U} = \mathbf{SPS}$ or $\mathbf{V} = \mathbf{SPS}$. We have shown that $\mathbf{SPS}$ is $\wedge$-prime and $\wedge$-irreducible in $\mathcal{L}(\mathbf{PS})$.

Assume now that $\mathbf{U}$ is a cover for $\mathbf{SPS}$. Then, there exists $u_n \approx v_n \in B$ not satisfied by all pseudosemilattices from $\mathbf{U}$. Let $\mathbf{V}$ be the variety of pseudosemilattices generated by the identity $u_n \approx v_n$. Then $\mathbf{U}$ is not contained in $\mathbf{V}$, and so $\mathbf{SPS} = \mathbf{U} \wedge \mathbf{V}$. Since $\mathbf{V} \neq \mathbf{SPS}$, we must have $\mathbf{U} = \mathbf{SPS}$, which means that $\mathbf{U}$ is not a cover for $\mathbf{SPS}$. Therefore, $\mathbf{SPS}$ has no cover in the lattice $\mathcal{L}(\mathbf{PS})$. ■

Acknowledgments: This work was partly supported by Fundação para a Ciência e a Tecnologia (FCT) through the Centro de Matemática da Universidade do Porto (CMUP) and by the Project PTDC/MAT/65481/2006, which is partly funded by the European Community Fund FEDER.

References

[1] K. Auinger, The word problem for the bifree combinatorial strict regular semigroup, *Math. Proc. Cambridge Philos. Soc.* **113** (1993), 519–533.

[2] K. Auinger, On the lattice of existence varieties of locally inverse semigroups, *Canad. Math. Bull.* **37** (1994), 13–20.

[3] T. Hall, Identities for existence varieties of regular semigroups, *Bull. Austral. Math. Soc.* **40** (1989), 59–77.

[4] J. Kaďourek and M. B. Szendrei, A new approach in the theory of orthodox semigroups, *Semigroup Forum* **40** (1990), 257–296.

[5] J. Meakin, The free local semilattice on a set, *J. Pure and Applied Algebra* **27** (1983), 263–275.

[6] J. Meakin and F. Pastijn, The structure of pseudo-semilattices, *Algebra Universalis* **13** (1981), 355–372.

[7] J. Meakin and F. Pastijn, The free pseudo-semilattice on two generators, *Algebra Universalis* **14** (1982), 297–309.

[8] K. S. S. Nambooripad, Pseudo-semilattices and biordered sets I, *Simon Stevin* **55** (1981), 103–110.

[9] L. Oliveira, A solution to the word problem for free pseudosemilattices, *Semigroup Forum* **68** (2004), 246–267.

[10] L. Oliveira, Models for free pseudosemilattices, *Algebra Universalis* **56** (2007), 315–336.