On the lattice of varieties of pseudosemilattices

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Abstract

Inspired by a basis of identities for the variety of all strict pseudosemilattices obtained in [12], we define a class of identities and study the varieties defined by them. This study will give us some incite into the structure of the lattice of varieties of pseudosemilattices. Some interesting conclusions about this lattice will be drawn. In particular, we shall prove this lattice is uncountable.

1 Introduction

We shall denote the set of idempotents of a regular semigroup S by $E(S)$. Define the binary relation ω^r on $E(S)$ as follows:

$$e\,\omega^r f \text{ if and only if } e = fe.$$

Let ω^l be the dual relation of ω^r and let ω be the relation $\omega^r \cap \omega^l$. We shall denote by $\omega^r(f)$ the set of idempotents e such that $e\,\omega^r f$. Similarly, we define $\omega^l(f)$ and $\omega(f)$.

Locally inverse semigroup can be characterized as regular semigroups S such that, for any $e, f \in E(S)$, there exists (a unique) $g \in E(S)$ satisfying the equality $\omega^r(e) \cap \omega^l(f) = \omega(g)$. Thus, if S is a locally inverse semigroup, then we can consider the algebra $(E(S), \wedge)$ where $e \wedge f$ is the unique element $g \in E(S)$ such that $\omega^r(e) \cap \omega^l(f) = \omega(g)$. The algebras $(E(S), \wedge)$ are called pseudosemilattices.

Nambooripad [9] showed that the class of all pseudosemilattices constitutes a variety of algebras. This result was generalized by Auinger [3] who proved that the mapping

$$\varphi : \mathcal{L}_e(\mathbf{LI}) \longrightarrow \mathcal{L}(\mathbf{PS}), \quad \mathbf{V} \longmapsto \{(E(S), \wedge) \mid S \in \mathbf{V}\}$$

is a well-defined complete homomorphism from the lattice $\mathcal{L}_e(\mathbf{LI})$ of e-varieties of locally inverse semigroups (see [4, 5]) onto the lattice $\mathcal{L}(\mathbf{PS})$ of varieties of pseudosemilattices. Thus, any information about $\mathcal{L}(\mathbf{PS})$ is useful to understand the structure of $\mathcal{L}_e(\mathbf{LI})$.

A regular semigroup is called strict if it is a subdirect product of completely simple and/or 0-simple semigroups. A strict pseudosemilattice is the pseudosemilattice of idempotents of some [combinatorial] strict regular semigroup. The class **SPS** of all strict pseudosemilattices is a variety. In fact, the lattice $\mathcal{L}(\mathbf{PS})$ is divided into two disjoint intervals $[\mathbf{T},\mathbf{NB}]$ and $[\mathbf{SPS},\mathbf{PS}]$, where the former is the 8-element lattice of varieties of normal bands. Further, $\mathbf{NB} \subseteq \mathbf{SPS}$ and **SPS** is the smallest variety of pseudosemilattices with algebras that are not semigroups.

A basis of identities for the variety **SPS** was introduced in [12]. In this paper we shall generalize those identities and study the varieties define by these generalized identities. In the next section we recall some results and terminology used in [12] and introduce the identities $u_{n,k,i} \approx v_{n,k,i}$. In Section 3 we define the varieties $\mathbf{G}_{n,k,i}$ and study the inclusion relation between these varieties.

We can define the duals of the varieties $\mathbf{G}_{n,k,i}$. In Section 4 we study the connections between the varieties $\mathbf{G}_{n,k,i}$ and their duals. In this section we study also the varieties defined by the join or meet of infinite chains of varieties $\mathbf{G}_{n,k,i}$. Finally, in last section, we shall use the results obtained in the previous sections to show some properties of the lattice $\mathcal{L}(\mathbf{PS})$.

2 A class of identities

In this paper we shall denote by X a countably infinite alphabet, by $(F_2(X),\wedge)$ the absolutely free binary algebra on X and by $c(u)$ the content of $u \in F_2(X)$, that is, the set of letters from X that appear in u. The variety **PS** of all pseudosemilattices is defined by the identities [9]:

(i) $x \wedge x \approx x$;

(ii) $(x \wedge y) \wedge (x \wedge z) \approx (x \wedge y) \wedge z$;

(iii) $((x \wedge y) \wedge (x \wedge z)) \wedge (x \wedge w) \approx (x \wedge y) \wedge ((x \wedge z) \wedge (x \wedge w))$;

together with the right-left duals of the last two.

Free pseudosemilattices have been studied in [6, 8] and one solution to the word problem for free pseudosemilattices has been presented in [10]. Several models for the free pseudosemilattice on X are described in [11]. In [12] we gave another model for the free pseudosemilattice on X using bipartite graphs that we shall briefly describe next. The omitted details can be found in [12].

A bipartite graph can be defined as a triple (L, D, R) with $L \cap R = \emptyset$ and $D \subseteq L \times R$. The elements of $L \cup R$ are called vertices and the elements of D are called edges. Let $\mathcal{B}$ be the set of all 6-tuples (l, L, D, R, r, φ) such that

(*a*) (L, D, R) is a connected cycle free bipartite graph with $(l, r) \in D$;

(*b*) $\varphi : L \cup R \to X$ is a labeling for the vertices of (L, D, R).

Let $D\varphi = \{(a\varphi, b\varphi) \,:\, (a, b) \in D\} \cup \{(c\varphi, c\varphi) \,:\, c \in L \cup R\}$.

In [12, Section 2] we associated a natural 6-tuple

$$\alpha_u = (l_u, L_u, D_u, R_u, r_u, \varphi_u) \in \mathcal{B}$$

recursively for each $u \in F_2(X)$. We observed that, for every $\alpha \in \mathcal{B}$, there exists $u \in F_2(X)$ such that $\alpha = \alpha_u$, although we may have several possibilities for u. Let $\mathcal{A}$ be the 6-tuples $\alpha = (l, L, D, R, r, \varphi) \in \mathcal{B}$ verifying also the following two conditions:

(*c*) If $a \notin \{l, r\}$ is a vertex of degree 1 and $(a, b) \in D$ or $(b, a) \in D$, then $a\varphi \neq b\varphi$.

(*d*) If $(a, c), (b, c) \in D$ or $(c, a), (c, b) \in D$ with $a \neq b$, then $a\varphi \neq b\varphi$.

An operation $\wedge$ on $\mathcal{A}$ was introduced in [12, Section 2]. With this operation, the algebra $(\mathcal{A}, \wedge)$ becomes a model for the free pseudosemilattice on X.

Let $\alpha = (l, L, D, R, r, \varphi) \in \mathcal{B}$. A labeled subgraph of α is a 6-tuple $\alpha_1 = (l_1, L_1, D_1, R_1, r_1, \varphi_1) \in \mathcal{B}$ such that

$$D_1 \subseteq D \quad \text{and} \quad \varphi_1 = \varphi|_{L_1 \cup R_1}.$$

Observe that $L_1 \subseteq L$ and $R_1 \subseteq R$ since $D_1 \subseteq D$. If we have also $l_1 = l$ and $r_1 = r$, then we say that α_1 is a strong labeled subgraph of α.

Two elements $\alpha_i = (l_i, L_i, D_i, R_i, r_i, \varphi_i) \in \mathcal{B}$, $i = 1, 2$, are isomorphic if there exists a bijection $\psi : L_1 \cup R_1 \to L_2 \cup R_2$ such that (*i*) $D_1\psi = \{(a\psi, b\psi) \,:\, (a,b) \in D_1\} = D_2$;

(*ii*) $a\psi\varphi_2 = a\varphi_1$ for all $a \in L_1 \cup R_1$.

If β is isomorphic to a [strong] labeled subgraph of α, then we shall say also that β is a [strong] labeled subgraph of α. We observed in [12] that $\alpha\,\omega\,\beta$ if and only if β is a strong labeled subgraph of α.

Let $u, v \in F_2(X)$. If $D_u\varphi_u = D_v\varphi_v$, then $L_u\varphi_u = L_v\varphi_v$ and $R_u\varphi_u = R_v\varphi_v$. The identity $u \approx v$ is called an elementary identity if

(*i*) $\alpha_u, \alpha_v \in \mathcal{A}$;

(*ii*) $(l_u\varphi_u, D_u\varphi_u, r_u\varphi_u) = (l_v\varphi_v, D_v\varphi_v, r_v\varphi_v)$ and $L_u\varphi_u \cap R_u\varphi_u = \emptyset$;

(*iii*) there exists $(x, y) \in D_u\varphi_u$ such that either $l_u\varphi_u = x$ and v is obtained from u by replacing the first x in u with $(x \wedge y)$, or $r_u\varphi_u = y$ and v is obtained from u by replacing the last y in u with $(x \wedge y)$.

In particular, if $u \approx v$ is an elementary identity, then D_v has one more edge than D_u, either (l_u, a) or (a, r_u) for some vertex $a \notin L_u \cup R_u$.

Auinger [1] gave a solution to the word problem for the free strict pseudosemilattice on X. He proved that an identity $u \approx v$ is satisfied by all strict pseudosemilattices if and only if $(l_u\varphi_u, D_u\varphi_u, r_u\varphi_u) = (l_v\varphi_v, D_v\varphi_v, r_v\varphi_v)$. Thus, every elementary identity is satisfied by all strict pseudosemilattices. In [12, Proposition 3.5] we proved the following result:

Result 2.1 *Let $u \approx v$ be an identity satisfied by all strict pseudosemilattices with $|c(u)| = n$. Then, for varieties of pseudosemilattices, the identity $u \approx v$ is equivalent to a finite set I of elementary identities such that $|c(u')| \leq 2n$ for every $u' \approx v' \in I$.*

Let $n \geq 1$, $k \geq 0$ and $1 \leq i \leq 2n$, and consider a set $\{x_1, x_2, \cdots, x_{2n}\}$ of $2n$ distinct letters from X. Let

(*i*) $L_m = \{j \text{ odd} \,:\, 0 < j \leq m\}$ and $R_m = \{j \text{ even} \,:\, 0 < j \leq m\}$;

(*ii*) $D_m = \{(j, h) \,:\, j \in L_m,\, h \in R_m \text{ and } |j - h| = 1\}$;

(*iii*) $\varphi_{n,k,i} : L_{2nk+i} \cup R_{2nk+i} \to X$ with $j\varphi_{n,k,i} = x_h$ for $1 \leq h \leq 2n$ such that $j \equiv h \mod 2n$.

Define $\gamma_{n,0,1,0} = \alpha_{x_1}$ and

$$\gamma_{n,k,i,j} = \begin{cases} (j, L_{2nk+i}, D_{2nk+i}, R_{2nk+i}, j+1, \varphi_{n,k,i}) & \text{for } 1 \leq j < 2nk+i \text{ odd;} \\ (j+1, L_{2nk+i}, D_{2nk+i}, R_{2nk+i}, j, \varphi_{n,k,i}) & \text{for } 1 \leq j < 2nk+i \text{ even.} \end{cases}$$

Then each $\gamma_{n,k,i,j} \in \mathcal{A}$. In fact, for the Green relations $\mathcal{R} = \omega^r \cap (\omega^r)^{-1}$ and $\mathcal{L} = \omega^l \cap (\omega^l)^{-1}$ on $\mathcal{A}$,

$$\gamma_{n,k,i,j-1} \, \mathcal{R} \, \gamma_{n,k,i,j} \, \mathcal{L} \, \gamma_{n,k,i,j+1}$$

if j odd. Thus, for each $n \geq 1$, $k \geq 0$ and $1 \leq i \leq 2n$, the elements $\gamma_{n,k,i,j}$ with $1 \leq j < 2nk+i$ constitute an E-chain of idempotents from $\mathcal{A}$, which imply they all belong to the same $\mathcal{D}$-class of $\mathcal{A}$.

Let $\alpha_{n,k,i} = \gamma_{n,k,i,1}$. Define $R'_m = R_m \cup \{0\}$, $D'_m = D_m \cup \{(1,0)\}$ and

$$\varphi'_{n,k,i} : L_{2nk+i} \cup R'_{2nk+i} \to X$$

such that $0\varphi'_{n,k,i} = x_{2n}$ and $j\varphi'_{n,k,i} = j\varphi_{n,k,i}$ for $0 < j \leq 2nk+i$. Let

$$\beta_{n,k,i} = (\,1\,,\, L_{2nk+i}\,,\, D'_{2nk+i}\,,\, R'_{2nk+i}\,,\, 2\,,\, \varphi'_{n,k,i}\,).$$

Clearly $\alpha_{n,k,i}, \beta_{n,k,i} \in \mathcal{A}$ if $n \geq 2$ and there exist unique words $u_{n,k,i}, v_{n,k,i} \in F_2(X)$ such that $\alpha_{n,k,i} = \alpha_{u_{n,k,i}}$ and $\beta_{n,k,i} = \alpha_{v_{n,k,i}}$. Further, $u_{n,k,i} \approx v_{n,k,i}$ are elementary identities if $n \geq 2$ and $k \geq 1$. Note that if $n = 1$ then $\alpha_{n,k,i}, \beta_{n,k,i} \notin \mathcal{A}$, and if $k = 0$ then $u_{n,k,i} \approx v_{n,k,i}$ is not elementary.

Observe that, for $n \geq 2$, $u_{n,1,1}$ and $v_{n,1,1}$ were designated by u_n and v_n in [12], respectively. Thus, by [12, Theorem 4.2], we have the following result:

Result 2.2 $B = \{u_{n,1,1} \approx v_{n,1,1} \,:\, n \geq 2\}$ *is a basis of identities for* **SPS**.

By Lemmas 4.3 of [12], if $u_{n,1,1} \approx v_{n,1,1}$ is a consequence of a set I of elementary identities, then there exists $u \approx v \in I$ such that $u_{n,1,1} \approx v_{n,1,1}$ is a consequence of $u \approx v$. Further, the proof of Lemma 4.4 of [12] also tells us that $|c(u)| \geq 2n$ if $D_v = D_u \cup \{(l_u, a)\}$ for some vertex a, and that $|c(u)| \geq 2n-2$ if $D_v = D_u \cup \{(a, r_u)\}$ for some vertex a. If we look carefully into the proofs of these lemmas, we can check easily that they can be adapted for the general case of the identities $u_{n,k,i} \approx v_{n,k,i}$. Thus, we have the following lemma.

Lemma 2.3 *Let $n \geq 2$, $k \geq 1$ and $1 \leq i \leq 2n$. If $u_{n,k,i} \approx v_{n,k,i}$ is a consequence of a set I of elementary identities, then $u_{n,k,i} \approx v_{n,k,i}$ is a consequence of some $u \approx v \in I$. Further, $|c(u)| \geq 2n$ if $D_v = D_u \cup \{(l_u, a)\}$ or $|c(u)| \geq 2n-2$ if $D_v = D_u \cup \{(a, r_u)\}$, for some vertex a.*

From the previous lemma we conclude that if $u_{m,l,j} \approx v_{m,l,j}$ implies $u_{n,k,i} \approx v_{n,k,i}$, then $m \geq n$. Clearly, $u_{n,l,j} \approx v_{n,l,j}$ implies $u_{n,k,i} \approx v_{n,k,i}$ if $l < k$ or if $l = k$ and $j \leq i$ since, in these cases, $u_{n,l,j}$ and $v_{n,l,j}$ are strong labeled subgraphs of $u_{n,k,i}$ and $v_{n,k,i}$, respectively. The next proposition gives more information about these identities. However, we need to define a partial order $\preccurlyeq$ on $\mathbb{N} \times \mathbb{N}$ first. Let (l,j), $(k,i) \in \mathbb{N} \times \mathbb{N}$. Then

$$(l,j) \preccurlyeq (k,i) \text{ if } l < k \text{ or if } l = k \text{ and } j \geq i.$$

Note that $\preccurlyeq$ is not the lexicographic order on $\mathbb{N} \times \mathbb{N}$. We are considering the reverse order on $\mathbb{N}$ for the second component.

Proposition 2.4 *Let $n, m \geq 2$, $k, l \geq 1$, $1 \leq i \leq 2n$ and $1 \leq j \leq 2m$. If $m \geq n$ and $(l, 2m-j) \preccurlyeq (k, 2n-i)$, then $u_{m,l,j} \approx v_{m,l,j}$ implies $u_{n,k,i} \approx v_{n,k,i}$.*

Proof: Let ψ be an endomorphism of $\mathcal{A}$ such that

$$\alpha_{x_s}\psi = \begin{cases} \alpha_{x_1} & \text{if } s \leq 2m-2n\,; \\ \alpha_{x_{s-2m+2n}} & \text{if } 2m-2n < s \leq 2m\,, \end{cases}$$

and observe that $\alpha_{u_{m,l,j}}\psi \wedge \alpha_{x_2} = \alpha_{u_{n,l,j'}}$ and $\alpha_{v_{m,l,j}}\psi \wedge \alpha_{x_2} = \alpha_{v_{n,l,j'}}$, for $j' = \max\{1, j-2m+2n\}$. Thus $u_{m,l,j} \approx v_{m,l,j}$ implies $u_{n,l,j'} \approx v_{n,l,j'}$.

If $l < k$ then $(l, 2n-j') \preccurlyeq (k, 2n-i)$. If $l = k$, then $2m - j \geq 2n - i$, and so $2n - j' \geq 2n - i$. Thus, we have always $(l, 2n-j') \preccurlyeq (k, 2n-i)$. Consequently $u_{n,l,j'} \approx v_{n,l,j'}$ implies $u_{n,k,i} \approx v_{n,k,i}$, and we conclude that $u_{m,l,j} \approx v_{m,l,j}$ implies $u_{n,k,i} \approx v_{n,k,i}$. ■

3 The varieties $\mathbf{G}_{n,k,i}$

Let $\mathbf{G}_{n,k,i}$ be the variety of pseudosemilattices defined by the identity $u_{n,k,i} \approx v_{n,k,i}$, for $n \geq 2$, $k \geq 1$ and $1 \leq i \leq 2n$. Then $\mathbf{G}_{n,k,i}$ contains **SPS**. The following result is an obvious corollary of Proposition 2.4.

Corollary 3.1 *Let $n, m \geq 2$, $k, l \geq 1$, $1 \leq i \leq 2n$ and $1 \leq j \leq 2m$. If $m \geq n$ and $(l, 2m-j) \preccurlyeq (k, 2n-i)$, then $\mathbf{G}_{m,l,j} \subseteq \mathbf{G}_{n,k,i}$.*

The remainder of this section is devoted to the proof of the converse of this corollary. For vertices a and b of $\alpha \in \mathcal{A}$, let $d_\alpha(a,b)$ be the number of edges in the path from a to b. Let $\alpha_i = (l_i, L_i, D_i, R_i, r_i, \varphi_i)$, for $i = 1, 2$, be two isomorphic labeled subgraphs of $\alpha \in \mathcal{A}$ and let $\pi : L_1 \cup R_1 \to L_2 \cup R_2$ be the isomorphism from α_1 onto α_2. Note that π is unique since $\alpha_1 \in \mathcal{A}$. Define

$$d_\alpha(\alpha_1, \alpha_2) = \min\{d_\alpha(a, a\pi) \;:\; a \in L_1 \cup R_1\}\,.$$

If ψ is an endomorphism of $\mathcal{A}$, then $\alpha_1\psi$ and $\alpha_2\psi$ are two isomorphic labeled subgraphs of $\alpha\psi$. Further $d_\alpha(\alpha_1, \alpha_2) \geq d_{\alpha\psi}(\alpha_1\psi, \alpha_2\psi)$.

Fix $n \geq 2$, $k \geq 1$ and $1 \leq i \leq 2n$ for the remainder of this section. Let

$$A = \begin{cases} \{\alpha_{x_1}\} & \text{if } i = 1\ ; \\ \{\gamma_{n,0,i,j} \;:\; j < i\} & \text{if } i \neq 1\ , \end{cases}$$

and

$$B = \{\gamma_{n,1,i,j} \;:\; i < j \leq 2n\}\ .$$

Let $\mathcal{C}$ be the subpseudosemilattice of $\mathcal{A}$ generated by $C = A \cup B$.

The set

$$\mathcal{C}' = \{\alpha \in \mathcal{C} \;:\; D\varphi \not\subseteq D_{2n+1}\varphi_{2n+1} \text{ for } \alpha = (l, L, D, R, r, \varphi)\}$$

is an ideal of $\mathcal{C}$. Let $\mathcal{A}_{n,i}$ be the quotient algebra $\mathcal{C}/\mathcal{C}'$, that is, the algebra

$$\mathcal{A}_{n,i} = \{\alpha \;:\; \alpha \in \mathcal{C} \setminus \mathcal{C}'\} \cup \{0\}$$

where $\alpha_1 \wedge \alpha_2$ is defined to be 0 if $\alpha_1 \wedge \alpha_2 \in \mathcal{C}'$. In fact, if $i \neq 1$, then

$$\mathcal{A}_{n,i} = \{\gamma_{n,l,i,j} \;:\; l \geq 0 \text{ and } 1 \leq j < 2nl + i\} \cup \{0\}\ .$$

The case $i = 1$ is more complex. Beside the elements indicated above with $i = 1$, $\mathcal{A}_{n,1}$ contains also the elements

$$\{\alpha_{x_1}\} \cup \{\gamma_{n,l,1,2nj+1} \wedge \alpha_{x_1} \;:\; 0 \leq j < l\}\,.$$

Let $\mathcal{I}_k$ be the set of all 6-tuples from $\mathcal{A}_{n,i}$ with more than $2n(k+1)$ vertices, together with the element 0. Then $\mathcal{I}_k$ is an ideal of $\mathcal{A}_{n,i}$. We define the quotient algebra

$$\mathcal{A}_{n,k,i} = \mathcal{A}_{n,i}/\mathcal{I}_k\ .$$

Observe that if $\alpha \in \mathcal{A}_{n,k,i} \setminus \{0\}$ and α_1 and α_2 are two isomorphic labeled subgraphs of α, then $d_\alpha(\alpha_1, \alpha_2)$ is multiple of $2n$.

Lemma 3.2 *Let* $n, m \geq 2$, $k, l \geq 1$, $1 \leq i \leq 2n$ *and* $1 \leq j \leq 2m$. *Then* $\mathcal{A}_{n,k,i} \notin \mathbf{G}_{n,k,i}$. *Further,* $\mathcal{A}_{n,k,i} \in \mathbf{G}_{m,l,j}$ *if* $m < n$ *or* $(k, 2n-i) \prec (l, 2m-j)$.

Proof: Consider a homomorphism $\psi : \mathcal{A} \to \mathcal{A}_{n,i}$ such that

$$\alpha_{x_j}\psi = \begin{cases} \gamma_{n,0,i,j} & \text{if } \ j < i\,; \\ \gamma_{n,0,i,i-1} & \text{if } \ j = i\,; \\ \gamma_{n,1,i,j} & \text{if } \ i < j \leq 2n\,. \end{cases}$$

Then $\alpha_{n,k,i}\psi = \alpha_{n,k,i} = \gamma_{n,k,i,1}$ and $\beta_{n,k,i}\psi = \gamma_{n,k+1,i,2n+1}$. If π denotes the projection of $\mathcal{A}_{n,i}$ onto $\mathcal{A}_{n,k,i}$, then

$$\alpha_{n,k,i}\psi\pi = \alpha_{n,k,i} \quad \text{and} \quad \beta_{n,k,i}\psi\pi = 0\,.$$

Thus $\mathcal{A}_{n,k,i}$ fails to satisfy the identity $u_{n,k,i} \approx v_{n,k,i}$, and $\mathcal{A}_{n,k,i} \notin \mathbf{G}_{n,k,i}$.

Let us prove that $\mathcal{A}_{n,k,i} \in \mathbf{G}_{m,l,j}$ if $m < n$ or $(k, 2n-i) \prec (l, 2m-j)$. Let $\psi : \mathcal{A} \to \mathcal{A}_{n,k,i}$ be a homomorphism. If $\alpha_{m,l,j}\psi = 0$, then $\alpha_{m,l,j}\psi = \beta_{m,l,j}\psi$. Hence, assume $\alpha_{m,l,j}\psi \neq 0$.

The vertices of $\alpha_{m,l,j}$ labeled with x_1 are the vertices from

$$A = \{2ms + 1 \;:\; 0 \leq s \leq l\}\,.$$

Consider the labeled subgraphs of $\alpha_{m,l,j}\psi$ that correspond to the images of these vertices. These labeled subgraphs are isomorphic obviously. Taking into account the structure of $\alpha_{m,l,j}$, these labeled subgraphs are either all the same or pairwise distinct. Further, if the former case occurs, then

$$\alpha_{m,l,j}\psi = \alpha_{m,1,1}\psi = \beta_{m,l,j}\psi\,.$$

We shall prove that the latter case does not occur, thus concluding that $\alpha_{m,l,j}\psi = \beta_{m,l,j}\psi$ for any homomorphism $\psi : \mathcal{A} \to \mathcal{A}_{n,k,i}$. Hence $u_{m,l,j} \approx v_{m,l,j}$ is satisfied by $\mathcal{A}_{n,k,i}$, and $\mathcal{A}_{n,k,i} \in \mathbf{G}_{m,l,j}$.

Let α and β be the labeled subgraphs of $\alpha_{m,l,j}\psi$ that correspond to the images of the vertices 1 and $2m+1$. Since α and β are isomorphic distinct labeled subgraphs of $\alpha_{m,l,j}\psi \in \mathcal{A}_{n,k,i} \setminus \{0\}$, then

$$2n \leq d_{\alpha_{m,l,j}\psi}(\alpha, \beta) \leq d_{\alpha_{m,l,j}}(1, 2m+1) = 2m.$$

Thus $m \geq n$. Since A has $l+1$ vertices, $\alpha_{m,l,j}\psi$ has at least $l+1$ copies of $\alpha_{x_1}\psi$ (one for each vertex from A). However, every element of $\mathcal{A}_{n,k,i}$ has at most $k+1$ copies of some $\alpha \in \mathcal{A}$; whence $l \le k$. Then $(k, 2n-i) \prec (l, 2m-j)$ if and only if $l = k$ and $2n - i > 2m - j$.

Let $n \le m$, $k = l$ and $2n - i > 2m - j$. Let $\alpha = \alpha_{m,0,j}\psi \in \mathcal{A}_{n,k,i} \setminus \{0\}$. Observe that if α has more than $2n$ vertices, then $\alpha_{m,l,j}\psi$ has more than $2n(l+1)$ vertices. However, since $k = l$, no element of $\mathcal{A}_{n,k,i}$ has more than $2n(l+1)$ vertices. Thus, α has at most $2n$ vertices, and so $\alpha = \alpha_{x_1}$ if $i = 1$ or $\alpha = \gamma_{n,0,i,h}$ for some $1 \le h < i$ if $i \neq 1$.

Let $\alpha_s = (l'_s, L'_s, D'_s, R'_s, r'_s.\varphi'_s) = \alpha_{x_s}\psi$ for $1 \le s \le 2m$ and

$$y_s = \begin{cases} l'_s\varphi'_s & \text{if } s \text{ odd}; \\ r'_s\varphi'_s & \text{if } s \text{ even}. \end{cases}$$

Let $M = \{(y_s, y_t) \,:\, 1 \le s, t \le 2m \text{ and } |s-t| = 1\} \cup \{(y_1, y_{2m})\}$ and

$$N = \{(x_s, x_t) : 1 \le s, t \le 2n \text{ and } |s-t| = 1\} \cup \{(x_1, x_{2n})\} \subseteq D_{2n+1}\varphi_{2n+1}\,.$$

Observe that $N \subseteq M$ since otherwise the labeled subgraphs of $\alpha_{m,l,j}\psi$ corresponding to the images of the vertices 1 and $2m+1$ of $\alpha_{m,l,j}$ could not be distinct. Let

$$M_1 = \{(y_s, y_t) \in M \,:\, s, t \le j\} \text{ and } M_2 = \{(y_s, y_t) \in M \,:\, j \le s, t\}\,.$$

Then $|M| \le |M_1| + |M_2|$ and $|M_2| \le 2m - j < 2n - i$. Further, $|M_1|$ is less than the number of vertices of α since $\alpha = \alpha_{m,0,j}\psi$. Thus $|M_1| < i$ and $|M| < 2n-1$. Then N is not contained in M since $|N| = 2n-1$. We proved we cannot have $n \le m$, $k = l$ and $2n - i > 2m - j$. Then the latter case does not occur. ■

Proposition 3.3 *Let $n, m \ge 2$, $k, l \ge 1$, $1 \le i \le 2n$ and $1 \le j \le 2m$. Then $\mathbf{G}_{m,l,j} \subseteq \mathbf{G}_{n,k,i}$ if and only if $m \ge n$ and $(l, 2m-j) \preccurlyeq (k, 2n-i)$.*

Proof: The direct implication follows from Lemma 3.2 since if $m < n$ or $(k, 2n-i) \prec (l, 2m-j)$, then $\mathcal{A}_{n,k,i} \in \mathbf{G}_{m,l,j} \setminus \mathbf{G}_{n,k,i}$. The reverse implication is Corollary 3.1. ■

An obvious corollary is the following result.

Corollary 3.4 *Let $n, m \ge 2$, $k, l \ge 1$, $1 \le i \le 2n$ and $1 \le j \le 2m$. Then $\mathbf{G}_{n,k,i} = \mathbf{G}_{m,l,j}$ if and only if $(n, k, i) = (m, l, j)$.*

4 The varieties $\mathbf{G}^*_{n,k,i}$, $\mathbf{G}_{k,i}$ and $\mathbf{G}_k$

Let $L^*_m = R_m$, $R^*_m = L_m$ and $D^*_m = \{(h,j) : (j,h) \in D_m\}$. Let

$$\alpha^*_{n,k,i} = (2,\ L^*_{2nk+i}\,,\ D^*_{2nk+i}\,,\ R^*_{2nk+i}\,, 1,\ \varphi_{n,k,i}\,)\ ,$$

and let $u^*_{n,k,i}$ be the unique word of $F_2(X)$ such that $\alpha^*_{n,k,i} = \alpha_{u^*_{n,k,i}}$. Then $\alpha^*_{n,k,i}$ and $u^*_{n,k,i}$ are the duals of $\alpha_{u_{n,k,i}}$ and $u_{n,k,i}$, respectively. Similarly, we define $\beta^*_{n,k,i}$ and $v^*_{n,k,i}$, the duals of $\beta_{n,k,i}$ and $v_{n,k,i}$, respectively.

The results from the previous two section have their duals with respect to the words $u^*_{n,k,i}$ and $v^*_{n,k,i}$. Then

$$u^*_{n,k,i} \approx v^*_{n,k,i}$$

are elementary identities if and only if $n \geq 2$ and $k \geq 1$. Let $\mathbf{G}^*_{n,k,i}$ be the variety defined by the identity $u^*_{n,k,i} \approx v^*_{n,k,i}$, for $n \geq 2$, $k \geq 1$ and $1 \leq i \leq 2n$.

Proposition 4.1 *Let $n, m \geq 2$, $k, l \geq 1$, $1 \leq i \leq 2n$ and $1 \leq j \leq 2m$. Then $\mathbf{G}^*_{m,l,j} \subseteq \mathbf{G}^*_{n,k,i}$ if and only if $m \geq n$ and $(l, 2m-j) \preccurlyeq (k, 2n-i)$.*

The next three results compare the varieties $\mathbf{G}_{n,k,i}$ and $\mathbf{G}^*_{n,k,i}$.

Proposition 4.2 *Let $n, m \geq 2$, $k, l \geq 1$, $1 \leq i \leq 2n$ and $1 \leq j \leq 2m$ with i even. Then $\mathbf{G}^*_{n,k,i} = \mathbf{G}_{n,k,i}$. Further, $\mathbf{G}^*_{m,l,j} \subseteq \mathbf{G}_{n,k,i}$ if and only if $m \geq n$ and $(l, 2m-j) \preccurlyeq (k, 2n-i)$.*

Proof: We just need to prove that $\mathbf{G}_{n,k,i} \subseteq \mathbf{G}^*_{n,k,i}$ if i even since the equality $\mathbf{G}_{n,k,i} = \mathbf{G}^*_{n,k,i}$ follows then by duality and the second part of this proposition follows from Proposition 4.1. Recall that $\alpha_{n,k,i} = \gamma_{n,k,i,1}$ and

$$\beta_{n,k,i} = (\,1\,,\ L_{2nk+i}\,,\ D'_{2nk+i}\,,\ R'_{2nk+i}\,,\ 2\,,\ \varphi'_{n,k,i}\,)\,,$$

and define $\alpha = \gamma_{n,k,i,2nk+i-1}$ and

$$\beta = (\,2nk+i-1\,,\ L_{2nk+i}\,,\ D'_{2nk+i}\,,\ R'_{2nk+i}\,,\ 2nk+i\,,\ \varphi'_{n,k,i}\,)$$

(note that β is a well defined 6-tuple of $\mathcal{A}$ since i is even). Let $u, v \in F_2(X)$ such that $\alpha_u = \alpha$ and $\alpha_v = \beta$. Applying Lemma 3.2 of [12] and its dual several times if necessary, we conclude that $u_{n,k,i} \approx v_{n,k,i}$ and $u \approx v$ are equivalent identities.

Relabel the vertices of α_u and α_v using the mapping θ defined by

$$x_j\theta = \begin{cases} x_{i+1-j} & \text{if } j \leq i \,; \\ x_{2n+i+1-j} & \text{if } i+1 \leq j \leq 2n \,, \end{cases}$$

and observe that we obtain $\alpha_{u^*_{n,k,i}}$ from α_u, and $\alpha_{u^*_{n,k,i+1}}$ if $i \neq 2n$ or $\alpha_{u^*_{n,k+1,1}}$ if $i = 2n$ from α_v. Thus $u_{n,k,i} \approx v_{n,k,i}$ implies $u^*_{n,k,i} \approx u^*_{n,k,i+1}$ if $i \neq 2n$ or implies $u^*_{n,k,i} \approx u^*_{n,k+1,1}$ if $i = 2n$. We shall assume that $i \neq 2n$ and prove this case only. The argumentation works as well for $i = 2n$ but it needs some minor adaptations.

Let ψ be an endomorphism of $\mathcal{A}$ such that $\alpha_{x_j}\psi = \alpha_{x_{j+1}}$ for $j < 2n$ and $\alpha_{x_{2n}}\psi = \alpha_{x_{2n}\wedge x_1}$. Then

$$\alpha_{u_{n,k,i}}\psi \wedge \alpha_{x_1} = \alpha_{u^*_{n,k,i+1}} \quad \text{and} \quad \alpha_{v_{n,k,i}}\psi \wedge \alpha_{x_1} = \alpha_{v^*_{n,k,i+1}} \,.$$

Thus $u_{n,k,i} \approx v_{n,k,i}$ implies $u^*_{n,k,i+1} \approx v^*_{n,k,i+1}$, and so it implies the identity $u^*_{n,k,i} \approx v^*_{n,k,i+1}$. Finally, since

$$\alpha_{v^*_{n,k,i+1}} \;\omega\; \alpha_{v^*_{n,k,i}} \;\omega\; \alpha_{u^*_{n,k,i}} \,,$$

we conclude that $u_{n,k,i} \approx v_{n,k,i}$ implies $u^*_{n,k,i} \approx v^*_{n,k,i}$. Thus $\mathbf{G}_{n,k,i} \subseteq \mathbf{G}^*_{n,k,i}$. ∎

Proposition 4.3 *Let $n, m \geq 2$, $k \geq 1$, $1 \leq i \leq 2n$ and $1 \leq j \leq 2m$. If i odd and $2m - j = 2n - i$, then $\mathbf{G}^*_{m,k,j}$ and $\mathbf{G}_{n,k,i}$ are incomparable varieties in the lattice $\mathcal{L}(\mathbf{PS})$. In particular, $\mathbf{G}^*_{n,k,i}$ and $\mathbf{G}_{n,k,i}$ are incomparable.*

Proof: By Lemma 3.2 we just need to prove that $\mathcal{A}_{n,k,i} \in \mathbf{G}^*_{m,k,j}$ to conclude that $\mathbf{G}^*_{m,k,j} \nsubseteq \mathbf{G}_{n,k,i}$. The result follows by duality. Let $\psi : \mathcal{A} \to \mathcal{A}_{n,k,i}$ be a homomorphism. Mimicking the proof of Lemma 3.2, we can assume that

$$\alpha^*_{m,k,j}\psi = (l, L, D, R, r, \varphi) \in \mathcal{A}_{n,k,i} \setminus \{0\} \,.$$

Let $A = \{2ms + 1 \,:\, 0 \leq s \leq k\}$ and consider the labeled subgraphs of $\alpha^*_{m,k,j}\psi$ that correspond to the images under ψ of the vertices of A. Mimicking again the proof of Lemma 3.2, we can conclude that it is enough to show that these labeled subgraphs cannot be pairwise distinct. However, if they were pairwise distinct, we could prove that $|R| \geq (2nk + i + 1)/2$, but no non-zero element of $\mathcal{A}_{n,k,i}$ has such property. Therefore, these labeled subgraphs cannot be distinct and $\mathcal{A}_{n,k,i}$ satisfies the identity $u^*_{m,k,j} \approx v^*_{m,k,j}$. ∎

Proposition 4.4 *Let $n, m \geq 2$, $k, l \geq 1$, $1 \leq i \leq 2n$ and $1 \leq j \leq 2m$ with i odd. Then $\mathbf{G}^*_{m,l,j} \subseteq \mathbf{G}_{n,k,i}$ if and only if $m \geq n$ and $(l, 2m-j) \preccurlyeq (k, 2n-i+1)$.*

Proof: Assume $m \geq n$ and $(l, 2m-j) \preccurlyeq (k, 2n-i+1)$. If $i \neq 1$, then

$$\mathbf{G}^*_{m,l,j} \subseteq \mathbf{G}^*_{n,k,i-1} = \mathbf{G}_{n,k,i-1} \subseteq \mathbf{G}_{n,k,i}$$

by Propositions 4.1, 4.2 and 3.3. If $i = 1$ and $m = n$, then $2m - j = 2n - j$ and $2n - i + 1 = 2n$. Thus $l < k$ since otherwise $(k, 2n-i+1) \prec (l, 2m-j)$. Again by the results indicated above,

$$\mathbf{G}^*_{m,l,j} \subseteq \mathbf{G}^*_{n,k-1,2n} = \mathbf{G}_{n,k-1,2n} \subseteq \mathbf{G}_{n,k,1}\,.$$

If $i = 1$ and $m > n$, then $m \geq n+1$. Further, if $l = k$ then $2m - j \geq 2n > 2n - 2$. Thus $(l, 2m-j) \preccurlyeq (k, 2n-2)$ if $i = 1$ and $m > n$, and once more by the results above

$$\mathbf{G}^*_{m,l,j} \subseteq \mathbf{G}^*_{n+1,k,2} = \mathbf{G}_{n+1,k,2} \subseteq \mathbf{G}_{n,k,1}\,.$$

Assume $\mathbf{G}^*_{m,l,j} \subseteq \mathbf{G}_{n,k,i}$. If j even, then $\mathbf{G}_{m,l,j} \subseteq \mathbf{G}_{n,k,i}$ by Proposition 4.2; whence $m \geq n$ and $(l, 2m-j) \preccurlyeq (k, 2n-i)$ by Proposition 3.3. Since i is odd, we must have $(l, 2m-j) \preccurlyeq (k, 2n-i+1)$ as wanted. It remains to show the case j odd. By Propositions 3.3 and 4.2, and since $i < 2n$ (i is odd), we have

$$\mathbf{G}^*_{m,l,j} \subseteq \mathbf{G}_{n,k,i} \subseteq \mathbf{G}_{n,k,i+1} = \mathbf{G}^*_{n,k,i+1}\,.$$

Then $m \geq n$ and $(l, 2m-j) \preccurlyeq (k, 2n-i-1)$ by Proposition 4.1. If $l < k$, then $(l, 2m-j) \preccurlyeq (k, 2n-i+1)$ as wanted. If $l = k$, then $(l, 2m-j) \preccurlyeq (k, 2n-i)$ since both i and j are odd numbers; and by Proposition 4.3, we have in fact $(l, 2m-j) \preccurlyeq (k, 2n-i+1)$.

We have shown that $\mathbf{G}^*_{m,l,j} \subseteq \mathbf{G}_{n,k,i}$ if and only if $m \geq n$ and $(l, 2m-j) \preccurlyeq (k, 2n-i+1)$. ■

In this section we have study the dual varieties of $\mathbf{G}_{n,k,i}$ until now. From now on we are going to study another class of varieties. For $k \geq 1$ and $i \geq 0$, we define the following varieties of pseudosemilattices:

$$\mathbf{G}_{k,i} = \cap\{\mathbf{G}_{n,k,2n-i} \,|\, n \geq 2\} \quad \text{and} \quad \mathbf{G}_k = \cap\{\mathbf{G}_{n,k,1} \,|\, n \geq 2\}.$$

Then $G_{k,i} = \{u_{n,k,2n-i} \approx v_{n,k,2n-i} \,|\, n \geq 2\}$ is a basis of identities for $\mathbf{G}_{k,i}$ and $G_k = \{u_{n,k,1} \approx v_{n,k,1} \,|\, n \geq 2\}$ is a basis of identities for $\mathbf{G}_k$. Then $\mathbf{G}_1 = \mathbf{SPS}$ by Result 2.2.

Next, we present a list of some trivial consequences of Lemma 2.3 and Proposition 3.3:

(i) The varieties $\mathbf{G}_{k,i}$ and $\mathbf{G}_k$ are pairwise distinct varieties and they all contain the variety **SPS**.

(ii) No $\mathbf{G}_{m,l,j}$ is contained in $\mathbf{G}_{k,i}$ or in $\mathbf{G}_k$.

(iii) $\mathbf{G}_k \subseteq \mathbf{G}_{m,l,j}$ if and only if $k \leq l$; and $\mathbf{G}_{k,i} \subseteq \mathbf{G}_{m,l,j}$ if and only if $k < l$ or $k = l$ and $2m - j \leq i$.

(iv) The varieties $\mathbf{G}_{k,i}$ and $\mathbf{G}_k$ form a subchain of $\mathcal{L}(\mathbf{PS})$:

$$\mathbf{G}_k = \cap_{i \geq 0} \mathbf{G}_{k,i} \subset \cdots \subset \mathbf{G}_{k,i+1} \subset \mathbf{G}_{k,i} \subset \cdots \subset \mathbf{G}_{k,0} \subset \mathbf{G}_{k+1} \ .$$

(v) If $(\mathbf{U}_s)_{s \geq 1}$ is a sequence of varieties $\mathbf{G}_{m,l,j}$ such that $\mathbf{U}_{s+1} \subset \mathbf{U}_s$, then $\cap_{s \geq 1} \mathbf{U}_s$ is one of the varieties $\mathbf{G}_{k,i}$ or $\mathbf{G}_k$.

We shall discuss the dual varieties $\mathbf{G}^*_{k,i}$ and $\mathbf{G}^*_k$ briefly now. Clearly,

$$G^*_{k,i} = \{u^*_{n,k,2n-i} \approx v^*_{n,k,2n-i} \,|\, n \geq 2\}$$

is a basis of identities for $\mathbf{G}^*_{k,i}$ and

$$G^*_k = \{u^*_{n,k,1} \approx v^*_{n,k,1} \,|\, n \geq 2\}$$

is a basis of identities for $\mathbf{G}^*_k$. If i even, then $\mathbf{G}_{k,i} = \mathbf{G}^*_{k,i}$ by Proposition 4.2. If i odd, no identity from $G_{k,i}$ is a consequence of an identity from $G^*_{k,i}$ by Proposition 4.3; whence $\mathbf{G}_{k,i} \neq \mathbf{G}^*_{k,i}$ by Lemma 2.3. By Propositions 4.4, $\mathbf{G}^*_{n+1,k,1} \subseteq \mathbf{G}_{n,k,1}$; whence $\mathbf{G}^*_k \subseteq \mathbf{G}_k$. By duality we conclude that $\mathbf{G}^*_k = \mathbf{G}_k$ for all $k \geq 1$.

5 The lattice $\mathcal{L}(\mathbf{PS})$

We begin this section showing that any finite pseudosemilattice is contained in some $\mathbf{G}_k$.

Lemma 5.1 *Let E be a pseudosemilattice with t elements. Then $E \in \mathbf{G}_t$.*

Proof: In [12] we proved that $u_{n,1,1} \approx v_{n,1,1}$ and $u_{n,1,1} \approx u_{n,1,2}$ are equivalent identities (see the comments made before Lemma 4.3 in [12]). In the same way we can show that $u_{n,k,1} \approx v_{n,k,1}$ and $u_{n,k,1} \approx u_{n,k,2}$ are equivalent identities too, for each $k \geq 1$. We shall prove that E satisfies $u_{n,t,1} \approx u_{n,t,2}$ for all $n \geq 2$.

Fix $n \geq 2$ and let $\psi : \mathcal{A} \to E$ be a homomorphism. Let $e_l = \alpha_{n,l,1}\psi$ for $l \geq 1$. Then $e_{l+1} \, \omega \, e_l$. Since E has t elements, there must exist $s, r \geq 1$ such that $e_s = e_{s+r}$ and $s + r \leq t + 1$. However, due to the structure of $\alpha_{n,k,1}$, we must have in fact $e_s = e_l$ for all $l \geq s$. In particular, $e_t = e_{t+1}$. Thus $\alpha_{n,t,2}\psi = \alpha_{n,t,1}\psi$ since

$$\alpha_{n,t+1,1} \; \omega \; \alpha_{n,t,2} \; \omega \; \alpha_{n,t,1} \, .$$

We conclude that E satisfies the identities $u_{n,t,1} \approx u_{n,t,2}$ for all $n \geq 2$, and so $E \in \mathbf{G}_t$. ■

Corollary 5.2 $\mathbf{PS} = \vee_{k \geq 1} \mathbf{G}_k$.

Proof: The e-variety **LI** of all locally inverse semigroups is generated by its finite combinatorial members by [2, Corollary 5.14]. Hence, **PS** is generated by the finite pseudosemilattices. The previous lemma tells us that $\vee_{k \geq 1} \mathbf{G}_k$ contains all finite pseudosemilattices. Consequently $\mathbf{PS} = \vee_{k \geq 1} \mathbf{G}_k$. ■

In Figure 1 we depict the inclusion relation (not the actual sublattice) between the varieties $\mathbf{G}_{n,k,i}$, $\mathbf{G}_{k,i}$ and $\mathbf{G}_k$. The dashed and dotted lines represent the meet and join of infinite chains of these varieties.

A variety $\mathbf{V}$ has finite axiomatic rank if there exist $k \geq 1$ and a basis of identities V for $\mathbf{V}$ such that $|c(u)|$, $|c(v)| \leq k$ for all $u \approx v \in V$. Otherwise, we say that $\mathbf{V}$ has infinite axiomatic rank. An infinite axiomatic rank variety has no finite basis of identities obviously. A basis of identities V for a variety $\mathbf{V}$ is independent if no proper subset of V is a basis of identities for $\mathbf{V}$. An element a of a lattice $\mathcal{L}$ is $\wedge$-prime if whenever $b \wedge c \leq a$, then $b \leq a$ or $c \leq a$; and it is $\wedge$-irreducible if whenever $b \wedge c = a$, then $b = a$ or $c = a$. Clearly, a $\wedge$-prime element is $\wedge$-irreducible.

In [12] we proved that the variety **SPS** has infinite axiomatic rank and no independent basis of identities. In fact, we proved that every cofinite subset of a basis of identities for **SPS** still is a basis of identities for **SPS**. We proved also that **SPS** is a $\wedge$-prime element and a $\wedge$-irreducible element of $\mathcal{L}(\mathbf{PS})$, and has no covers. These results follow from the fact that if a set I of identities imply $u_{n,1,1} \approx v_{n,1,1}$, then there exists $u \approx v \in I$ with $|c(u)| \geq 2n - 2$ such that $u \approx v$ implies $u_{n,1,1} \approx v_{n,1,1}$. Using Lemma 2.3 and its dual, we can replicate, for $\mathbf{G}_k$, $\mathbf{G}_{k,i}$ and $\mathbf{G}^*_{k,i}$, the proofs presented in [12] for the previous results. Therefore, we just state bellow those results for the varieties $\mathbf{G}_k$, $\mathbf{G}_{k,i}$ and $\mathbf{G}^*_{k,i}$.

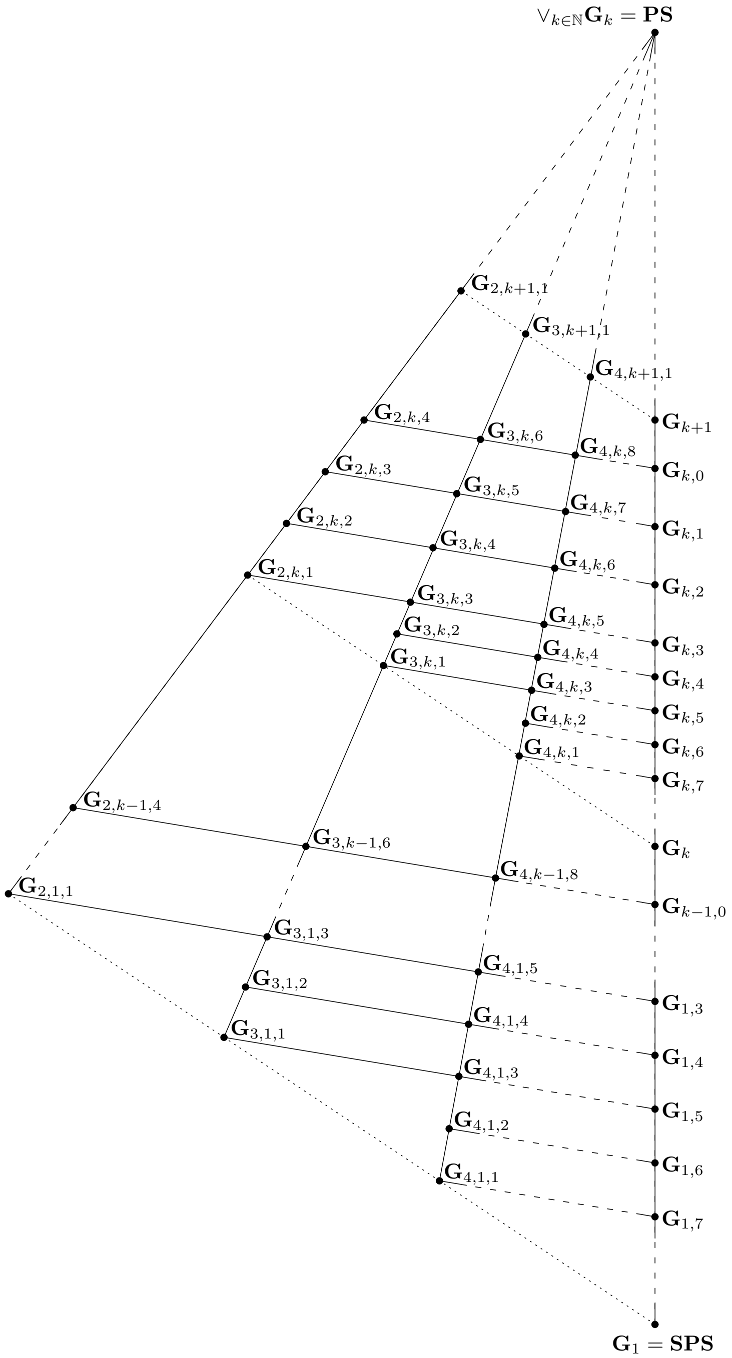

Figure 1: Inclusion relation in the lattice $\mathcal{L}(\mathbf{PS})$.

Proposition 5.3 *Let $k \geq 1$ and $i \geq 0$.*

(i) $\mathbf{G}_k$, $\mathbf{G}_{k,i}$ *and* $\mathbf{G}^*_{k,i}$ *have infinite axiomatic rank.*

(ii) A cofinite subset of a basis of identities for $\mathbf{G}_k$ $[\mathbf{G}_{k,i}, \mathbf{G}^*_{k,i}$, *respectively*$]$ *stills a basis of identities for* $\mathbf{G}_k$ $[\mathbf{G}_{k,i}, \mathbf{G}^*_{k,i}$, *respectively*$]$.

(iii) $\mathbf{G}_k$, $\mathbf{G}_{k,i}$ *and* $\mathbf{G}^*_{k,i}$ *have no independent basis of identities.*

(iv) $\mathbf{G}_k$, $\mathbf{G}_{k,i}$ *and* $\mathbf{G}^*_{k,i}$ *are* $\wedge$*-prime elements and* $\wedge$*-irreducible elements of* $\mathcal{L}(\mathbf{PS})$.

(v) $\mathbf{G}_k$, $\mathbf{G}_{k,i}$ *and* $\mathbf{G}^*_{k,i}$ *have no covers in the lattice* $\mathcal{L}(\mathbf{PS})$.

The proof for the $\wedge$-prime and $\wedge$-irreducible properties work also for the varieties $\mathbf{G}_{n,k,i}$ and $\mathbf{G}^*_{n,k,i}$.

Proposition 5.4 *Let $n \geq 2$, $k \geq 1$ and $1 \leq i \leq 2n$. Then, the varieties* $\mathbf{G}_{n,k,i}$ *and* $\mathbf{G}^*_{n,k,i}$ *are* $\wedge$*-prime and* $\wedge$*-irreducible elements of* $\mathcal{L}(\mathbf{PS})$.

We end this paper showing that $\mathcal{L}(\mathbf{PS})$ is uncountable.

Theorem 5.5 *The lattice* $\mathcal{L}(\mathbf{PS})$ *of varieties of pseudosemilattices is uncountable.*

Proof: Let $\mathbf{U}_k = \mathbf{G}_{k+1,k,1}$ for $k \geq 1$. Then $\{\mathbf{U}_k \,|\, k \geq 1\}$ is a set of pairwise incomparable varieties of pseudosemilattices by Proposition 3.3. Let A and B be two subsets of $\mathbb{Z}^+$, and let

$$\mathbf{U} = \cap_{k\in A}\mathbf{U}_k \quad \text{and} \quad \mathbf{V} = \cap_{k\in B}\mathbf{U}_k.$$

Then, by Lemma 2.3 and Proposition 3.3, $\mathbf{U} = \mathbf{V}$ if and only if $A = B$. Therefore, for any subset of $\mathbb{Z}^+$, we have a new variety of pseudosemilattices, and so $\mathcal{L}(\mathbf{PS})$ is uncountable. ∎

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