CLASSIFICATION OF LINEAR DIFFERENTIAL EQUATIONS NEAR INFINITY

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Abstract. Differential equations linear at infinity are shown to be analytically conjugate to a member of a 3-parameter family of differential equations; the parameters are identified as the constant and linear coefficients, which are also formal invariants, and an analytic invariant that can be computed as a fractional derivative.

In what follows $\mathbf{C}[[1/x]]$ denotes the set of formal power series in $\frac{1}{x}$ and $\mathbf{C}\{1/x\}$ the set of functions analytic in $\frac{1}{x}$.

We shall consider differential equations of the form:

$$y'(x) + g(x)y(x) + f(x) = 0 \tag{1}$$

where we demand that the coefficients belong to $\mathbf{C}\{1/x\}$, meaning they can be written as series in $\frac{1}{x}$ with a nonzero radius of convergence:

$$f(x) = \sum_{n=0}^{\infty} f_n \left(\frac{1}{x}\right)^n, \quad g(x) = \sum_{n=0}^{\infty} g_n \left(\frac{1}{x}\right)^n$$

We further demand that $f(\infty) = f_0 = 0$ (infinity is a critical point) and that $g(\infty) = g_0 = \lambda \neq 0$ (there is a linear part at infinity).

For an equation of the above form we will call the numbers $\lambda \equiv g_0$ and $\beta \equiv g_1$ the *constant coefficient* and the *linear coefficient* respectively.

These equations can be obtained from a holomorphic saddle-node vector field defined in a neighbourhood of the origin:

$$\dot{X} = X^2, \quad \dot{Y} = g\left(\frac{1}{X}\right) Y + f\left(\frac{1}{X}\right)$$

by a change of coordinates:

$$x = \frac{1}{X}, \quad y = Y$$

Consider linear changes of variables:

$$y(x) = A(x) + B(x)z(x) \tag{2}$$

such that $A(\infty) = 0$, $B(\infty) = 1$ (which we can interpret as meaning that $y \approx z$ for large x). If we use (2) to replace y by z in (1) we will get a new linear differential equation in z.

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If we can transform an equation into another by any change of variables with $A(x), B(x) \in \mathbf{C}[[1/x]]$ we say that both equations are *formally conjugated*. If the change of variables is such that $A(x), B(x) \in \mathbf{C}\{1/x\}$ we say that the equations are *analytically conjugated*.

It is well known [1] that λ and β are formal invariants when classifying differential equations of the form:

$$y'(x) = f(x, y), \qquad f \in \mathbf{C}\{1/x\} \text{ with } f(\infty, 0) = 0$$

i.e. there exists a formal change of variables $y = \Phi(x, z)$ that conjugates the above equation to:

$$z'(x) + \left(\lambda + \frac{\beta}{x}\right) z = 0$$

and that there exist a series of analytic invariants associated to the behaviour of a Borel transform at the points $\lambda\mathbf{N}$

We shall see that changes of variables of the type (2) are enough to conjugate any equation of the type (1) to a member of a three parameter family of simple equations: there is only an extra analytic invariant in our case, associated to the behaviour at just the point λ of the Borel transform of a function easily computed from f and g.

1. Pre-normal Form

For any given equation of the form $y'(x) + g(x)y(x) + f(x) = 0$, writing $A(x)$ and $B(x)$ as series in $\frac{1}{x}$ and substituting (2) into (1), it is easy to verify that λ and β are invariant under formal linear changes of variables. By a suitable change of variables we can set all other terms in $g(x)$ equal to zero:

Theorem 1 (Pre-normal Form)**.** *Any equation of the form:*

$$y'(x) + g(x)y(x) + f(x) = 0, \quad f(x), g(x) \in \mathbf{C}\{1/x\}$$

with:

$$g(x) = \lambda + \beta/x + \sum_{n=2}^{\infty} g_n/x^n$$

is analytically conjugated to:

$$y'(x) + h(x) + (\lambda + \beta/x)y(x) = 0$$

where:

$$h(x) = f(x) \exp\left(\sum_{n=1}^{\infty} \frac{g_{n+1}}{nx^n}\right) \in \mathbf{C}\{1/x\}$$

Proof. If we set

$$k(x) \equiv \sum_{n=2}^{\infty} g_n/x^n = g(x) - (\lambda + \beta/x), \qquad K(x) \equiv \sum_{n=1}^{\infty} \frac{g_{n+1}}{nx^n}$$

then $k(x) \in \mathbf{C}\{1/x\}$ by hypothesis and it is clear that $K(x)$ is also a $\mathbf{C}\{1/x\}$ function because $|K(x)| \leq |xk(x)|$. By termwise differentiation we have $K'(x) = k(x)$.

It is clear that $\exp(-K(x))$ will also be in $\mathbf{C}\{1/x\}$. The change of variables

$$y(x) = \exp(-K(x))z(x)$$

is thus analytical and $\exp(-K(\infty)) = \exp(0) = 1$, as required. Moreover:

$$y'(x) = \exp(-K(x))\left(z'(x) - k(x)z(x)\right)$$

Thus the differential equation $y'(x) + (\lambda + \beta/x + k(x))y(x) + f(x) = 0$ becomes:

$$\exp(-(K(x))\left[(z'(x) - k(x)z(x))) + (\lambda + \beta/x + k(x))z(x)\right] + f(x) = 0$$

and finally:

$$z'(x) + (\lambda + \beta/x)z(x) + h(x) = 0$$

□

The reason the argument does not apply to the elimation of λ and β is simply that λ and β/x do not have a primitive in $\mathbf{C}\{1/x\}$.

2. Borel transform

We have reduced the problem to the classification of equations of the type

$$y'(x) + \left(\lambda + \frac{\beta}{x}\right) y(x) + h(x) = 0 \tag{3}$$

where $\lambda \neq 0$ and $h(\infty) = 0$.

Provided we can find a particular solution y_0, we can use a change of variables $y(x) = y_0(x) + z(x)$ to reduce (3) to

$$z'(x) + \left(\lambda + \frac{\beta}{x}\right) z(x) = 0$$

and this is the best we can hope to do with linear changes of variables, due to the invariant nature of λ and β. The change of variables will be valid according to our definition if $y_0 \in \mathbf{C}\{1/x\}$ and $y_0(\infty) = 0$.

To determine when it is possible to find a particular solution of (3) in this conditions it is convenient to work not with (3) directly but with its Borel transform.

We shall require some properties of the Borel transform. Let $f(x)$ be a formal power series in $1/x$ without constant term:

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{x^{n+1}}$$

Then its Borel transform, which we shall denote by either $\mathcal{B}(f)(\xi)$ or $\breve{f}(\xi)$, is a formal power series in ξ defined as

$$\breve{f}(\xi) = \sum_{n=0}^{\infty} \frac{a_n}{n!}\xi^n$$

If the Borel transform is convergent for some ξ we can, under certain conditions, define the *Borel sum* of the original series. The Borel sum, $\mathcal{L}_\theta f(x)$, of a formal power series $f(x)$ with Borel transform $\breve{f}(\xi)$ along the direction θ is defined as

$$\mathcal{L}_\theta f(x) \equiv \int_0^{e^{i\theta}\infty} e^{-x\xi}\breve{f}(\xi)\mathrm{d}\xi$$

This is defined whenever the $\breve{f}(\xi)$ has an analytical continuation to the path of integration and the integral is convergent. For different θ we will have different domains of convergence and the functions thus obtained will be analytic continuations of one another.

Let $g(x)$ be also a formal power series. We have that

- $\mathcal{B}(af(x) + bg(x)) = a\breve{f}(\xi) + b\breve{g}(\xi)$ (Linearity)
- $\mathcal{B}((f \cdot g)(x)) = (\breve{f} * \breve{g})(\xi) \equiv \int_0^\xi \breve{f}(\zeta)\breve{g}(\xi - \zeta)\mathrm{d}\zeta$ (Product goes to convolution)
- $\mathcal{B}(f'(x)) = -\xi\breve{f}(\xi)$
- If $f(x) \in \mathbf{C}\{1/x\}$ then $\breve{f}(\xi)$ is entire and of exponential type
- If $\breve{f}(\xi)$ is entire and of exponential type then $f(x)$ has a positive radius of convergence and its Borel sum equals its sum.

By of exponential type we mean of exponential type in all directions, that is $|\breve{f}(\xi)| \leq Ce^{A|\xi|}$ for all $\xi \in \mathbf{C}$ for some $C, A \in \mathbf{R}^+$.

The last property is particularly useful for the case at hand. Given a formal solution to a differential equation we can check if it is a analytical solution by taking the Borel transform and checking if it is entire and of exponential type.

We can now apply the Borel transform to equation (3), assuming $y(\infty) = 0$, that is $y(x)$ has no constant term. Using the properties above we obtain an integral equation for $\breve{y}(\xi)$:

$$(\xi - \lambda)\,\breve{y}(\xi) - \int_0^\xi \beta\breve{y}(\zeta)\mathrm{d}\zeta - \breve{h}(\xi) = 0 \tag{4}$$

Using the properties of the Borel transform we know $\breve{h}$ is entire and of exponential type and that (3) will have the required $\mathbf{C}\{1/x\}$ solution if (4) has a solution that is entire and of exponential type. Supposing we have a solution we expand it and $\breve{h}$ in a power series around $\xi = \lambda$

$$\breve{y}(\xi) = \sum_{n=0}^{\infty} a_n (\xi - \lambda)^n \tag{5}$$

$$\breve{h}(\xi) = \sum_{n=0}^{\infty} h_n (\xi - \lambda)^n \tag{6}$$

Substituting these expansions into (4) we obtain

$$\sum_{n=0}^{\infty} \left(a_n (\xi - \lambda)^{n+1} - \frac{\beta}{n+1} a_n (\xi - \lambda)^{n+1} + \frac{\beta}{n+1} a_n (-\lambda)^{n+1} - h_n (\xi - \lambda)^n \right) = 0$$

Setting equal the coefficients of the same power of $\xi - \lambda$ we obtain:

$$\sum_{n=0}^{\infty} \frac{\beta}{n+1} a_n (-\lambda)^{n+1} = h_0 \tag{7}$$

$$(n + 1 - \beta) a_n = (n + 1) h_{n+1} \tag{8}$$

Now suppose that $\beta \notin \mathbf{N}$. This means that we can divide by $n+1-\beta$ for every n and we obtain from (8)

$$a_n = \frac{n+1}{n+1-\beta} h_{n+1} \tag{9}$$

Thus the a_n are completely determined. Substituting into (7) we obtain a relation among the h_n:

$$h_0 - \sum_{n=0}^{\infty} \frac{\beta}{n+1-\beta} h_{n+1} (-\lambda)^{n+1} = 0$$

which simplifies to:

$$\sum_{n=0}^{\infty} \frac{\beta}{\beta - n} h_n (-\lambda)^n = 0 \tag{10}$$

Notice that $\beta/(\beta - n) \to 0$ when $n \to \infty$ and therefore $|\beta/(\beta - n)|$ is bounded. This means that the left-hand side of (10) converges if and only if

$$\sum_{n=0}^{\infty} h_n (-\lambda)^n$$

converges. But this is simply $\breve{h}(0)$, and we know that $\breve{h}(\xi)$ has infinite radius of convergence at $\xi = 0$. Thus the left-hand side of (10) is a well defined number that depends on λ, β and $h(x)$.

For (4) (and thus (3)) to have a solution this number must be zero. We shall see that this condition is also sufficient.

In fact, under this condition (5) becomes:

$$\breve{y}(\xi)=\sum_{n=0}^{\infty}\frac{n+1}{n+1-\beta}h_{n+1}(\xi-\lambda)^n \tag{11}$$

But $(n+1)/(n+1-\beta)\to 1$ when $n\to\infty$ and thus $|(n+1)/(n+1-\beta)|$ is bounded; let M be a majorant of this sequence. Then, and as $\breve{h}(\xi)$ defined by (6) is entire, $\lim_{n\to\infty}\sup\sqrt[n]{|h_n|}=0$ and:

$$\lim_{n\to\infty}\sup\sqrt[n]{\left|\frac{n+1}{n+1-\beta}h_{n+1}\right|}\leq\lim_{n\to\infty}\sup\sqrt[n]{M\,|h_{n+1}|}=0 \tag{12}$$

Therefore $\breve{y}(\xi)$ defined by (11) is also entire.

We know that $\breve{h}(\xi)$ is of exponential type, that is:

$$|\breve{h}(\xi)|\leq Ce^{A|\xi|}\quad\text{when}\quad|\xi|>R,\qquad\text{for some } R,C,A\in\mathbf{R}^+$$

Then for $|\xi-\lambda|\geq 1$:

$$|\breve{y}(\xi)|\leq M\left|\sum_{n=0}^{\infty}h_{n+1}(\xi-\lambda)^n\right|\leq M\left|\sum_{n=0}^{\infty}h_{n+1}(\xi-\lambda)^{n+1}\right|=M|\breve{h}(\xi)-h_0|$$

and therefore, for $|\xi|>|\lambda|+1$:

$$|\breve{y}(\xi)|\leq M|\breve{h}(\xi)|+M|h_0|\leq MCe^{A|\xi|}+M|h_0|\leq M(C+|h_0|)e^{A|\xi|}$$

It follows that $\breve{y}(\xi)$ is of exponential type.

Now suppose that $\beta\in\mathbf{N}$. For $n+1=\beta$ equation (8) becomes simply $h_\beta=0$. Under this condition we can still use (8) to solve for a_n for $n+1\neq\beta$, $a_{\beta-1}$ being left undetermined. Substituting the a_n with $n\neq\beta-1$ in (7) we obtain:

$$\frac{\beta}{\beta}a_{\beta-1}(-\lambda)^\beta+\sum_{\substack{n=0\\ n\neq\beta-1}}^{\infty}\frac{\beta}{n+1-\beta}h_{n+1}(-\lambda)^{n+1}=h_0$$

and thus:

$$a_{\beta-1}=\sum_{\substack{n=0\\ n\neq\beta-1}}^{\infty}\frac{\beta}{\beta-n}h_n(-\lambda)^{n-\beta}$$

Arguing as above, the sum on the righthand side converges and $a_{\beta-1}$ is thus determined. We have:

$$\breve{y}(\xi)=a_{\beta-1}(\xi-\lambda)^{\beta-1}+\sum_{\substack{n=0\\ n\neq\beta-1}}^{\infty}\frac{n+1}{n+1-\beta}h_{n+1}(\xi-\lambda)^n \tag{13}$$

Since $a_{\beta-1}(\xi-\lambda)^{\beta-1}$ is always of exponential type, we conclude as before that $\breve{y}(\xi)$ as given by (13) is entire and of exponential type, and a solution of (4) provided that:

$$h_\beta = 0 \quad \text{or equivalently} \quad \check{h}^{(\beta)}(\lambda) = 0 \tag{14}$$

We are thus led to make the following

Definition 1. To each differential equation of the kind

$$y'(x) + \left(\lambda + \frac{\beta}{x}\right) y(x) + h(x) = 0$$

we associate the characteristic number α defined by:

$$\alpha(\lambda, \beta, h) \equiv \begin{cases} \displaystyle\sum_{n=0}^{\infty} \frac{\beta}{\beta - n} \frac{\check{h}^{(n)}(\lambda)}{n!}(-\lambda)^n & \text{if } \beta \notin \mathbf{N} \\ \\ \check{h}^{(\beta)}(\lambda) & \text{if } \beta \in \mathbf{N} \end{cases}$$

where $\check{h}(\xi) \equiv \mathcal{B}(h(x))$.

The arguments above establishes the following result:

Theorem 2. *The differential equation*

$$y_1'(x) + \left(\lambda + \frac{\beta}{x}\right) y_1(x) + h(x) = 0$$

is analytically conjugated to

$$y_2'(x) + \left(\lambda + \frac{\beta}{x}\right) y_2(x) = 0$$

if and only if $\alpha(\lambda, \beta, h) = 0$.

Notice that α is linear in h since the Borel transform is linear and it is easily seen from the definition that, for fixed λ and β, α is a linear function of $\mathcal{B}(h(x))$. This enables us to conclude that:

Corollary 3. *The differential equation*

$$y'(x) + (\lambda + \beta/x)y(x) + h_1(x) = 0$$

is analytically conjugated to

$$y'(x) + (\lambda + \beta/x)y(x) + h_2(x) = 0$$

if and only if $\alpha(\lambda, \beta, h_1) = \alpha(\lambda, \beta, h_2)$.

Proof. Consider the differential equation

$$y'(x) + \left(\lambda + \frac{\beta}{x}\right) y(x) + h_3(x) = 0$$

where $h_3(x) \equiv h_1(x) - h_2(x)$. By the previous theorem it will have an analytical solution $y_0(x)$ with $y_0(\infty) = 0$ if and only if:

$$\alpha(\lambda, \beta, h_3) = 0 \Longleftrightarrow \alpha(\lambda, \beta, h_1) = \alpha(\lambda, \beta, h_2)$$

Thus, under this hypothesis, the change of variables $y_1(x) = y_0(x) + y_2(x)$ is analytic. Substituting into the first equation we obtain the second. $\square$

For linear differential equations of arbitrary type, and in view of theorem 1, we define α to be:

Definition 2. To each equation of the type

$$y'(x) + g(x)y(x) + h(x) = 0$$

where

$$g(x) = \lambda + \beta/x + k(x), \qquad k(x) \in 1/x^2\mathbf{C}\{1/x\}$$

we associate the characteristic number α defined by:

$$\alpha(f, g) \equiv \alpha(\lambda, \beta, h) \tag{15}$$

where $h(x) = \exp\left(\int k(x)dx\right) f(x)$ and $\int k(x)dx$ is a primitive of $k(x)$ that is zero at infinity.

We can now state the

Theorem 4. *The linear differential equation*

$$y_1'(x) + g_1(x)y_1(x) + f_1(x) = 0 \tag{16}$$

is analytically conjugated to

$$y_2'(x) + g_2(x)y_2(x) + f_2(x) = 0 \tag{17}$$

if and only if the numbers λ, β and α are equal for both equations.

Proof. Let

$$h_i(x) \equiv \exp\left(\int g_i(x) - (\lambda + \beta/x)\right) f_i(x)$$

where $i = 1, 2$. Clearly

$$\alpha(f_1, g_1) = \alpha(f_2, g_2) \iff \alpha(\lambda, \beta, h_1) = \alpha(\lambda, \beta, h_2)$$

By Theorem 1, equation (16) is conjugated to:

$$y'(x) + (\lambda + \beta/x)y(x) + h_1(x) = 0$$

which, by Corolary 3, is conjugated to:

$$y'(x) + (\lambda + \beta/x)y(x) + h_2(x) = 0$$

which, again by Theorem 1, is conjugated to (17) $\square$

Remark 1. This shows that α is an analytic invariant for the type of equations we are considering.

3. Normal form

We can use the previous arguments to obtain a very simple normal form for this kind of equations.

Theorem 5 (Normal Form)**.** *Any equation of the form:*

$$y'(x) + g(x)y(x) + f(x) = 0$$

can be analytically conjugated to

$$y'(x) + (\lambda + \beta/x)y(x) + A/x^k = 0$$

where λ and β are the constant and linear coeficients of the original equation, $A = \alpha(f, g)$ and k is 1 if $\beta \notin \mathbf{N}$ and $1 + \beta$ otherwise.

Proof. We only have to prove that A *can* be chosen so that α is the same for both equations. Let $\bar{\alpha} = \alpha(\lambda + \beta/x, A/x^k)$ be the value of α for the normal form equation.

Consider the case $\beta \in \mathbf{N}$ first. Then $k = 1+\beta$. The Borel transform of A/x^k is $A\xi^{k-1}/(k-1)!$ and its $(k-1)$th derivative is A. According to the definition, $\bar{\alpha} = A$ and by choosing $A = \alpha(f, g)$ the equations are analytically conjugated.

Now consider that $\beta \notin N$ and $k = 1$. We have A as a Borel transform: $\mathcal{B}(A/x) = A$. Thus $\bar{\alpha} = A$, and setting $A = \alpha(f, g)$ we obtain the desired result. □

Remark 2. The normal form is not unique, when $\beta \notin \mathbf{N}$ we choose $k = 1$ for simplicity. In most cases any integer would do, but problems would arise if for a given differential equation a particular choice of k leads to a zero $\bar{\alpha}$ for any finite A: notice that for fixed A, β and k, $\bar{\alpha}$ will be a polynomial of degree $k-1$ in λ; if λ for the given equation is one of the zeros of this polynomial we cannot have analytical conjugation with this k unless α_1 is also zero.

4. Fractional derivative

The expression given for α is not very satisfactory in that distinct definitions are required depending on whether $\beta \in \mathbf{N}$. The fact that for $\beta \in \mathbf{N}$, α is simply a derivative of order β suggests that with an appropriate generalization of derivatives to complex orders we might obtain a uniform definition for α. We shall see that this is so.

There are several generalizations of derivatives to non-integer order, termed *fractional derivatives*. For a survey of this topic see [3]. The difference between definitions is mainly one of domain of applicability, when these intersect they will mostly agree.

In what follows we shall use the Riemann-Liouville fractional derivative [2].

Definition 3. Given a function $f(z)$ and a complex number η with $\operatorname{Re}\eta > 0$, we define the fractional integral of order η, or the fractional derivative of order $-\eta$, of the function $f(z)$ at the point z, denoted by $\mathsf{D}^{\eta} f(z)$, to be:

$$\mathsf{D}^{-\eta} f(z) \equiv \frac{1}{\Gamma(\eta)} \int_0^z (z-v)^{\eta-1} f(v) \mathrm{d}v \tag{18}$$

where Γ is the Euler Gamma function, and the integration is along the straight line joining 0 to η.

If $\mathrm{Re}\,\eta \leq 0$, let $k \in \mathbf{N}$ be the smallest positive integer such that $k + \mathrm{Re}\,\eta > 0$; then the fractional derivative of order $-\eta$ of the function $f(z)$ at the point z is:

$$\mathsf{D}^{-\eta} f(z) \equiv \frac{d^k}{dz^k} \mathsf{D}^{-k-\eta} f(z) \tag{19}$$

We shall now see that we can write α as a fractional derivative, restating Corollary 3 as:

Theorem 6. *The differential equation*

$$y'(x) = (\lambda + \beta/x)\, y(x) + h_1(x)$$

is analytically conjugated to

$$y'(x) = (\lambda + \beta/x)\, y(x) + h_2(x)$$

if and only if

$$\mathsf{D}^{\beta} \breve{h}_1(\lambda) = \mathsf{D}^{\beta} \breve{h}_2(\lambda)$$

Proof. Assume $\mathrm{Re}\,\beta < 0$ and begin with the series expansion of $\breve{h}(\xi)$ as in (6):

$$\breve{h}(\xi) = \sum_{n=0}^{\infty} h_n (\xi - \lambda)^n$$

Multiplying by the kernel of the fractional derivative of order β at λ we obtain:

$$(\lambda - \xi)^{-\beta-1} \breve{h}(\xi) = \sum_{n=0}^{\infty} h_n (-1)^n (\lambda - \xi)^{n-\beta-1}$$

We can integrate from 0 to λ. Interchanging the order of integration and sum we have:

$$\int_0^{\lambda} (\lambda - \xi)^{-\beta-1} \breve{h}(\xi) \mathrm{d}\xi = \sum_{n=0}^{\infty} h_n (-1)^n \int_0^{\lambda} (\lambda - \xi)^{n-\beta-1} \mathrm{d}\xi$$

and therefore:

$$\begin{aligned}
\mathsf{D}^\beta \check{h}(\lambda) =& \frac{1}{\Gamma(-\beta)} \sum_{n=0}^{\infty} h_n(-1)^n \left(- \frac{(\lambda-\xi)^{n-\beta}}{n-\beta}\Big|_0^\lambda \right) \\
=& \frac{1}{\Gamma(-\beta)} \sum_{n=0}^{\infty} h_n(-1)^n \frac{\lambda^{n-\beta}}{n-\beta} = \frac{\lambda^{-\beta}}{-\beta\Gamma(-\beta)} \sum_{n=0}^{\infty} \frac{\beta}{\beta-n} h_n(-\lambda)^n \\
=& \frac{\lambda^{-\beta}}{\Gamma(1-\beta)} \alpha(\lambda,\beta,h)
\end{aligned}$$

Assume Re $\beta \geq 0$ and let $k \in \mathbf{N}$ be the smallest positive integer such that $k - \text{Re}\ \beta > 0$; then:

$$\mathsf{D}^\beta \check{h}(\lambda) = \sum_{n=0}^{\infty} \frac{h_n}{\Gamma(k-\beta)} \frac{d^k}{dz^k}\Big|_{z=\lambda} \int_0^z (z-\xi)^{k-\beta-1}(\xi-\lambda)^n \mathrm{d}\xi$$

Let us denote the last integral by $I^\beta_{k,n}(z)$.

We begin by taking $k = 1$; integrating by parts just once, we obtain:

$$\begin{aligned}
I^\beta_{1,n}(z) =& \frac{z^{1-\beta}(-\lambda)^n}{1-\beta} + \frac{n}{1-\beta} I^{\beta-1}_{1,n-1}(z) \\
\frac{d}{dz}\Big|_{z=\lambda} I^\beta_{1,n}(z) =& (-1)^n \lambda^{n-\beta} + \frac{n}{1-\beta} \frac{d}{dz}\Big|_{z=\lambda} I^{\beta-1}_{1,n-1}(z)
\end{aligned} \tag{20}$$

Using the fact that:

$$I^\beta_{1,0}(z) = \frac{z^{1-\beta}}{1-\beta}, \quad \frac{d}{dz}\Big|_{z=\lambda} I^\beta_{1,0}(z) = \lambda^{-\beta}$$

and the above recurrence relation (20), it is easy to prove by induction that:

$$\frac{d}{dz}\Big|_{z=\lambda} I^\beta_{1,n}(z) = (-1)^{n+1} \frac{\beta}{n-\beta} \lambda^{n-\beta}$$

For arbitrary k, we obtain:

$$\begin{aligned}
I^\beta_{k,n}(z) =& \frac{z^{k-\beta}(-\lambda)^n}{k-\beta} + \frac{n}{k-\beta} I^{\beta-1}_{k,n-1}(z) \\
\frac{d^k}{dz^k}\Big|_{z=\lambda} I^\beta_{k,n}(z) =& (-1)^n \prod_{i=1}^{k-1}(i-\beta) \lambda^{n-\beta} + \frac{n}{k-\beta} \frac{d^k}{dz^k}\Big|_{z=\lambda} I^{\beta-1}_{k,n-1}(z)
\end{aligned} \tag{21}$$

Using now the fact that:

$$I^\beta_{k,0}(z) = \frac{z^{k-\beta}}{k-\beta}, \quad \frac{d^k}{dz^k}\Big|_{z=\lambda} I^\beta_{k,0}(z) = (k-1-\beta)\dots(1-\beta)\lambda^{-\beta}$$

and the above recurrence relation (21), it is easy to prove by induction that:

$$\left.\frac{d^k}{dz^k}\right|_{z=\lambda} I^{\beta}_{k,n}(z) = \left(\prod_{i=1}^{k-1}(i-\beta)\right) \left.\frac{d}{dz}\right|_{z=\lambda} I^{\beta}_{1,n}(z)$$
$$=(-1)^{n+1}\frac{\Gamma(k-\beta)}{\Gamma(1-\beta)}\frac{\beta}{n-\beta}\lambda^{n-\beta}$$

Therefore:

$$\mathsf{D}^{\beta}\check{h}(\lambda) = \sum_{n=0}^{\infty}\frac{h_n}{\Gamma(k-\beta)}(-1)^{n+1}\frac{\Gamma(k-\beta)}{\Gamma(1-\beta)}\frac{\beta}{n-\beta}\lambda^{n-\beta}$$
$$=\sum_{n=0}^{\infty}\frac{h_n}{\Gamma(1-\beta)}(-\lambda)^n\frac{-\beta}{n-\beta}\lambda^{-\beta}$$
$$=\frac{\lambda^{-\beta}}{\Gamma(1-\beta)}\alpha(\lambda,\beta,h)$$

□

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