PHYSICAL MEASURES FOR INFINITE-MODAL MAPS

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ABSTRACT. We analyse certain parametrized families of one-dimensional maps with infinitely many critical points from the measure-theoretical point of view. We prove that such families have absolutely continuous invariant probability measures for a positive Lebesgue measure subset of parameters. Moreover we show that both the densities of these measures and their entropy vary continuously with the parameter. In addition we obtain sub-exponential rate of mixing for these measures and also that they satisfy the Central Limit Theorem.

CONTENTS

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1. INTRODUCTION

One of the main goals of Dynamical Systems is to describe the global asymptotic behavior of the iterates of most points under a transformation of a compact manifold, either from a topological or from a probabilistic (or ergodic) point of view. The notion of *uniform hyperbolicity*, introduced by Smale in [Sm], and of *non-uniform hyperbolicity*, introduced by Pesin [P], have been the main tools to rigorously establish general results in the field.

While uniform hyperbolicity is defined using only a finite number of iterates of a given transformation, non-uniform hyperbolicity is a asymptotic notion to begin with, demanding the existence of non-zero Lyapunov exponents almost everywhere with respect to some invariant probability measure.

On the one hand, the study of consequences of both notions in a general setting has a long history, see [M, S, KH, B, BP, Y, BDV] for details and thourough references.

On the other hand, it is rather hard in general to verify non-uniform hyperbolicity, since we must take into account the behavior of the iterates of the given map when time goes to infinity. This was first achieved in the groundbreaking work of Jakobson [J] on the quadratic family, which was extended for more general one-dimensional families with a unique critical point by many other mathematicians, see e.g. [BC1, R, MS, T, TTY]. One-dimensional families with two critical points were first considered in [Ro] and multimodal maps and maps with critical points and singularities with unbounded derivative were treated in [LT, LV, BLS]. To the best of our knowledge, maps with *infinitely many critical points* were first dealt with in [PRV].

The aim of this paper is prove that the dynamics of the family considered in [PRV], for a positive Lebesgue measure subset of parameters, is non-uniformly hyperbolic and to deduce some consequences from the ergodic point of view. These families naturally appear as one-dimensional models for the dynamical behavior near the unfolding of a double saddle-focus homoclinic connection of a flow in a three-dimensional manifold, see Figure 1 and [Sh]. The main novelty is that we prove *global stochastic behavior* for a family of maps with *infinitely many regions of contraction*.

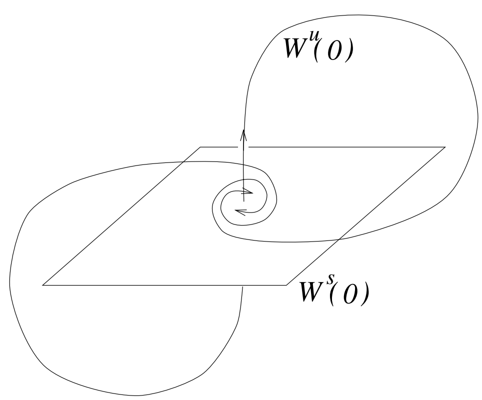

FIGURE 1. Double saddle-focus homoclinic connections

Roughly speaking, the family f_μ of one-dimensional circle maps which we consider here is obtained from first-return maps of the three-dimensional flow in Figure 1 to appropriate cross-sections and disregarding one of the variables. This reduction to a one-dimensional model greatly simplifies the study of this kind of unfolding and provides important insight to its behavior. However as we shall see the dynamics of the reduced model is still highly complex.

This family of maps is obtained translating the left-hand side and right-hand side, vertically in opposite directions, of the graph of the map $f = f_0$ described in Figure 2. This family approximates the behavior of any generic unfolding of f_0. Such unfolding was first studied in [PRV], where it was shown that for a positive Lebesgue measure subset S of parameters the map f_μ, for $\mu \in S$, exhibits a chaotic attractor. This was achieved by proving that the orbits of the critical values of f_μ have positive Lyapunov exponent and that f_μ has a dense orbit.

Here we complement the topological description of the dynamics of f_μ provided by [PRV] for $\mu \in S$ with a probabilistic description constructing for the same parameters a *physical* probability measure ν_μ. We say that an invariant probability measure ν is *physical* or *Sinai-Ruelle-Bowen* (SRB) if there is a positive Lebesgue measure set of points $x \in \mathbf{S}^1$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi\left(f_\mu^k(x)\right) = \int \varphi \, d\nu,$$

for any observable (continuous function) $\varphi : \mathbf{S}^1 \to \mathbf{R}$. The set of points $x \in \mathbf{S}^1$ with this property is called the *basin* of ν. SRB measures provide a statistical description of the asymptotic behavior of a large subset of orbits. Combining this with the results from [PRV] we have that f_μ has non-zero Lyapunov exponent almost everywhere with respect to ν_μ, i.e. f_μ is non-uniformly hyperbolic for $\mu \in S$.

The main feature needed for the construction of such measures is to obtain positive Lyapunov exponent for Lebesgue almost every point under the action of f_μ, $\mu \in S$. The presence of critical points is a serious obstruction to achieve an asymptotic expansion rate on the derivative of most points. Therefore the control of derivatives along orbits of the critical values is a central subject in the ergodic theory of one-dimensional maps.

The crucial role of the orbits of the critical values on the statistical description of the global dynamics of one-dimensional maps was already present in the pioneer work of Jakobson [J], who considered quadratic maps and obtained SRB measures for a positive Lebesgue measure subset of parameters.

This was later followed by the celebrated papers of Benedicks and Carleson [BC1, BC2], where the parameter exclusion technique was used to show that, for a positive Lebesgue measure subset of parameters, the derivative along the orbit of the unique critical value has exponential growth and satisfies what is nowadays called a *slow recurrence* condition to the critical point. This is enough to construct SRB measures for those parameters.

Recently, in the unimodal setting it was established that indeed the existence of SRB measures, and the exponential growth of the derivative along the orbit of the critical value, are equivalent conditions for Lebesgue almost every parameter for which there are no sinks, see [ALM, AM1, AM2]. See also [BLS] for multimodal maps.

In [PRV] the technique of exclusion of parameters was extended to deal with infinitely many critical orbits. Here we refine this technique to obtain exponential growth of the derivatives and slow recurrence to *the whole critical set for Lebesgue almost every orbit*. By [ABV] this ensures the existence of SRB measures for every parameter $\mu \in S$, see Subsection 1.2 and Theorem A.

Moreover we are able to control the measure of the set of points whose orbits are too close to the critical set during the first n iterates, showing that its Lebesgue measure is sub-exponential in n, see Theorem B. In addition, the Lebesgue measure of the set of points whose derivative does not grow exponentially fast in the first n iterates decreases exponentially fast with n, see Theorem C. By recent general results on the ergodic theory of non-uniformly hyperbolic systems [ALP, G], both estimates above taken together imply sub-exponential decay of correlations for Hölder continuous observables for ν_μ and also that ν_μ satisfies the Central Limit Theorem, for all $\mu \in S$, see Subsection 1.3 and Corollary D. We remark that these properties are likewise satisfied by uniformly expanding maps of $\mathbf{S}^1$, which are the touchstone of chaotic dynamics (except that their correlation decay rate is exponential, see [B, V]), in spite of the presence of infinitely many points with unbounded contraction (critical points).

Furthermore analyzing our arguments we observe that all the estimates obtained do not depend on the choice of the parameter $\mu \in S$. This shows after [A, AOT] that the density $d\nu_\mu/d\lambda$ of the SRB measure ν_μ with respect to Lebesgue measure and its entropy $h_{\nu_\mu}(f_\mu)$ vary continuously with $\mu \in S$, see Subsection 1.4 and Corollary E. This type of result was recently obtained in [F] for quadratic maps on the set of parameters constructed in [BC1, BC2] using a similar strategy.

Hence statistical properties of the maps f_μ for $\mu \in S$ are stable under small variations of the parameter, i.e. this family is *statistically stable* over S.

The paper is organized as follows. We first state precisely our results in Subsections 1.2 to 1.4. We sketch the proof in Section 2. In Section 3 we explain how a sequence $(\mathcal{P}_n)_{n\geq 0}$ of partitions of $\mathbf{S}^1$ whose atoms have bounded distortion under action of f_μ^n is constructed. Basic lemmas are stated and proved in Section 4. These are used to obtain the main estimates in Section 5. In Sections 6 and 7 we use the main estimates to deduce slow recurrence to the critical set and fast expansion for most points. Finally in Section 8 we keep track of the estimates obtained during our constructs and show that they do not depend on the parameter $\mu \in S$.

1.1. **Statement of the results.** Let $\hat f$ be the interval map $\hat f : [-\varepsilon_1, \varepsilon_1] \to [-1,1]$ given by

$$\hat f(z) = \begin{cases} az^\alpha \sin(\beta \log(1/z))) & \text{if } z > 0 \\ -a|z|^\alpha \sin(\beta \log(1/|z|))) & \text{if } z < 0, \end{cases} \tag{1.1}$$

where $0 < \alpha < 1$, $\beta > 0$ and $\varepsilon_1 > 0$, see Figure 2.

Maps $\hat f$ as above have infinitely many critical points, of the form

$$x_k = \hat x \exp(-k\pi/\beta) \text{ and } x_{-k} = -x_k \text{ for each large } k > 0 \tag{1.2}$$

where $\hat x > 0$ is independent of k. Let $k_0 \geq 1$ be the smallest integer such that x_k is defined for all $|k| \geq k_0$, and x_{k_0} is a local minimum.

We extend this expression to the whole circle $S^1 = I/\{-1 \sim 1\}$, where $I = [-1,1]$, in the following way. Let $\tilde f$ be an orientation-preserving expanding map of S^1 such that $\tilde f(0) = 0$ and $\tilde f' > \tilde\sigma$ for some constant $\tilde\sigma >> 1$. We define $\varepsilon = 2 \cdot x_{k_0}/(1 + e^{-\pi/\beta})$, so that x_{k_0} is the middle

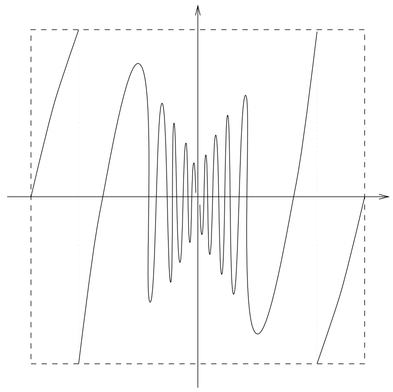

FIGURE 2. Graph of the circle map f.

point of the interval $(e^{-\pi/\beta}\varepsilon, \varepsilon)$ and fix two points $x_{k_0} < \hat{y} < \tilde{y} < \varepsilon$, with $|\hat{f}'(\hat{y})| >> 1$. Then we take f to be any smooth map on S^1 coinciding with $\hat{f}$ on $[-\hat{y}, \hat{y}]$, coinciding with $\tilde{f}$ on $S^1 \setminus [-\tilde{y}, \tilde{y}]$, and monotone on each interval $\pm[\hat{y}, \tilde{y}]$.

Finally let f_μ be the following one-parameter family of circle maps unfolding the dynamics of $f = f_0$

$$f_\mu(z) = \begin{cases} f(z) + \mu & \text{for } z \in (0, \varepsilon] \\ f(z) - \mu & \text{for } z \in [-\varepsilon, 0) \end{cases} \tag{1.3}$$

for $\mu \in (-\varepsilon, \varepsilon)$. For $z \in \mathbf{S}^1 \setminus [-\varepsilon, \varepsilon]$ we assume only that $\left|\frac{\partial}{\partial\mu} f_\mu(z)\right| \geq 2$. In what follows we write $z_k^{\pm}(\mu) = f_\mu(x_k)$ for $|k| \geq k_0$.

Theorem 1.1. [PRV, Theorem A] *For a given $\sigma \in (1, \sqrt{\tilde{\sigma}})$ there exists an integer N such that taking $k_0 > N$ in the construction of f, we can find a small positive constant $\tilde{\rho}$ such that for $0 < \rho < \tilde{\rho}$ there exists a positive Lebesgue measure subset $S \subset (-\varepsilon, \varepsilon)$ satisfying for every $\mu \in S$*

(1) *for all $n \geq 1$ and all $k_0 \leq |k| \leq \infty$*
  (a) $\left|\left(f_\mu^n\right)'\left(z_k^{\pm}(\mu)\right)\right| \geq \sigma^n$*;*
  (b) *either $|f_\mu^n(f_\mu(x_l))| > \varepsilon$ or $|f_\mu^n(f_\mu(x_l)) - x_{m(n)}| \geq e^{-\rho n}$;*
  where $x_{m(n)}$ is the critical point nearest $f_\mu^n(f_\mu(x_l))$.
(2) $\liminf_{n\to+\infty} n^{-1} \log|(f_\mu^n)'(z)| \geq \log\sigma/3$ *for Lebesgue almost every point $z \in S^1$;*
(3) *there exists $z \in S^1$ whose orbit $\{f_\mu^n(z) : n \geq 0\}$ is dense in S^1.*

The statement of Theorem 1.1 is slightly different from the main statement of [PRV] but the proof is contained therein.

1.2. **Existence of absolutely continuous invariant probability measures.** The purpose of this work is to prove that for parameters $\mu \in S$ the map f_μ admits a unique absolutely continuous invariant probability measure ν_μ, whose basin covers Lebesgue almost every point of $\mathbf{S}^1$, and to study some of the main statistical and ergodic properties of these measures.

In what follows we write λ for the normalized Lebesgue measure on $\mathbf{S}^1$. Our first result shows the existence of the *SRB* measure.

Theorem A. *Let $\mu \in S$ be given. Then there exists a f_μ-invariant probability measure ν_μ which is absolutely continuous with respect to λ and such that for λ-almost every $x \in \mathbf{S}^1$ and every continuous $\varphi : \mathbf{S}^1 \to \mathbf{R}$*

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f_\mu^j(x)) = \int \varphi \, d\nu_\mu. \tag{1.4}$$

The proof is based on the technique of parameter exclusion developed in [PRV] to prove Theorem 1.1 and on recent results on hyperbolic times for non-uniformly expanding maps with singularities and criticalities, from [ABV].

In our setting *non-uniform expansion* means the same as item (2) of Theorem 1.1. However due to the presence of (infinitely many) criticalities and the singularity at 0, an extra condition is needed to construct the *SRB* measure: we need to control the average distance to the critical set along most orbits.

We say that f_μ has *slow recurrence to the critical set* $\mathcal{C} = \{x_k : |k| \geq k_0\} \cup \{0\}$ if, for every $\delta > 0$, there exists $\gamma > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} -\log \operatorname{dist}_\gamma\left(f_\mu^k(x), \mathcal{C}\right) < \delta \quad \text{for Lebesgue almost every} \quad x \in \mathbf{S}^1, \tag{1.5}$$

where γ is a small positive value, and $\operatorname{dist}_\gamma(x,y) = |x-y|$ if $|x-y| \leq \gamma$ and 1 otherwise.

Let $f_0 : I \setminus \mathcal{C} \to I$ be a C^2 map. We say that $\mathcal{C}$ is a *non-flat critical set* if there exist constants $B > 1$ and $\beta > 0$ such that

S1: $\displaystyle \frac{1}{B}\operatorname{dist}(x,\mathcal{C})^\beta \leq |f_0'(x)| \leq B \operatorname{dist}(x,\mathcal{C})^{-\beta}$;

S2: $\displaystyle \big|\log|f_0'(x)| - \log|f_0'(y)|\big| \leq B\frac{|x-y|}{\operatorname{dist}(x,\mathcal{C})^\beta}$;

for every $x, y \in I \setminus \mathcal{C}$ with $|x-y| < \operatorname{dist}(x,\mathcal{C})/2$.

The following result ensuring the existence of finitely many physical probability measures is proved in [ABV].

Theorem 1.2. *If f_0 satisfies (S1), (S2), is non-uniformly expanding and has slow recurrence to the critical set $\mathcal{C}$, then there are finitely many $\mu_1, \dots, \mu_l$ ergodic absolutely continuous f_0-invariant probability measures such that Lebesgue almost every point in I belongs in the basin of μ_i for some $i \in \{1, \dots, l\}$.*

The maps f_μ satisfy conditions (S1)-(S2) above. Indeed we define $y_k = 2 \cdot x_k/(1+e^{-\pi/\beta})$, for each $k \geq k_0$, so that x_k is the middle point of the interval (y_{k+1}, y_k). We note that x_k is the *closest* critical point to any $y \in (y_{k+1}, y_k)$. We also use a similar notation for $k \leq -k_0$. We will argue using the following lemmas, which correspond to Lemmas 3.2 and 3.3 proved in [PRV].

Lemma 1.3. *There exists $C > 0$ depending on $\hat{f}$ only (not depending on ε or μ) such that, for every $x \in (y_{l+1}, y_l)$ and $l \geq k_0$, respectively, $x \in (y_l, y_{l-1})$ and $l \leq -k_0$, we have*

(1) $C^{-1}|x_l|^{\alpha-2}|x-x_l|^2 \le |f(x)-f(x_l)| \le C|x_l|^{\alpha-2}|x-x_l|^2$;
(2) $C^{-1}|x_l|^{\alpha-2}|x-x_l| \le |f'_\mu(x)| \le C|x_l|^{\alpha-2}|x-x_l|$.

Lemma 1.4. *Let $s,t\in[y_{l+1},y_l]$ with $l\ge k_0$, respectively, $s,t\in[y_l,y_{l-1}]$ with $l\le -k_0$. Then*

$$\left|\frac{f'_\mu(s)-f'_\mu(t)}{f'_\mu(t)}\right| \le K_1\frac{|s-t|}{|t-x_l|}$$

where $K_1>0$ is independent of l,s,t,ε and μ.

On the one hand since $0<\alpha<1$, $x\in(y_{l+1},y_l)$ and $|x_l|<1$, then from item 2 of Lemma 1.3

$$C|x_l|^{\alpha-2}|x-x_l| = \left(C|x_l|^{\alpha-2}|x-x_l|^2\right)|x-x_l|^{-1} \le \left(C|x_l|^{\alpha-2}|x_l|^2\right)|x-x_l|^{-1} \le C|x-x_l|^{-1}.$$

On the other hand since $\alpha-2<0$ and $|x_l|<1$ we get $C^{-1}|x_l|^{\alpha-2}|x-x_l| \ge C^{-1}|x-x_l|$, showing that (S1) holds for f_μ with $B=C$ and $\beta=1$.

To check that (S2) also holds we write

$$\frac{|f'_\mu(x)|}{|f'_\mu(y)|} = \frac{|f'_\mu(x)-f'_\mu(y)+f'_\mu(y)|}{|f'_\mu(y)|} \le 1+\frac{|f'_\mu(x)-f'_\mu(y)|}{|f'_\mu(y)|}$$

and then because $\log(1+z)\le z$ for $z>-1$ we get

$$|\log|f'_0(x)|-\log|f'_0(y)|| \le \frac{|f'_\mu(x)-f'_\mu(y)|}{|f'_\mu(y)|} \le K_1\frac{|x-y|}{|x-x_l|}.$$

Thus according to Theorem 1.2 and after Theorem 1.1, we only need to show that f_μ has slow recurrence to the critical set for $\mu\in S$ to achieve the result stated in Theorem A. This is done in Sections 4 to 6, where a much stronger result is obtained, since we do not use the truncated distance dist_γ in our arguments.

1.3. **Stretched exponential decay of correlations and Central Limit Theorem.** Using some recent developments on the statistical behavior of non-uniformly expanding maps [ALP, G] we are able to obtain sub-exponential bounds on the decay of correlations between Hölder continuous observables for ν_μ with $\mu\in S$. In addition it follows from standard techniques that ν_μ also satisfies the Central Limit Theorem. In order to achieve this we refined the arguments in [PRV] extending the estimates obtained therein for critical orbits to Lebesgue almost every orbit, yielding a sub-exponential bound on the Lebesgue measure of the set of points whose average distance to the critical set during the first n iterates is small, as follows.

We first define the average distance to the critical set without truncation

$$\mathcal{C}_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} -\log\mathrm{dist}\left(f^k_\mu(x),\mathcal{C}\right). \tag{1.6}$$

We note that $-\log\mathrm{dist}_\gamma(x,\mathcal{C}) \le -\log\mathrm{dist}(x,\mathcal{C})$ for every $\gamma>0$ and $x\in I$. Then we are able to prove the following.

Theorem B. *Let $\mu \in S$ and $\delta > 0$ be given. Then there are constants $C_1, \xi_1 > 0$ dependent on $\hat{f}$, σ, k_0 and δ only such that $\mathcal{R}(x) = \min\{N \geq 1 : \mathcal{C}_n(x) < \delta, \forall n \geq N\}$ satisfies*

$$\lambda\Big(\{x \in \mathbf{S}^1 : \mathcal{R}(x) > n\}\Big) \leq C_1 \cdot e^{-\xi_1 \sqrt{n}}.$$

We note that in particular this shows that f_μ has slow recurrence to the critical set, since we have

$$\lim_{n\to\infty} \mathcal{C}_n(x) = 0 \quad \text{for Lebesgue almost every } x \in \mathbf{S}^1,$$

and thus this ensures the existence of the *SRB* measure ν_μ for $\mu \in S$ by Theorem 1.2. Moreover this also yields that $\log \operatorname{dist}(x, \mathcal{C})$ *is non-integrable with respect to* ν_μ, for otherwise we would have $\int \log \operatorname{dist}(x, \mathcal{C})\, d\nu_\mu = 0$ by the Ergodic Theorem, since ν_μ is absolutely continuous with respect to Lebesgue, leading to a contradiction with the fact that $-\log \operatorname{dist}(x, \mathcal{C}) \geq -\log(1 - x_{k_0}) > 0$ for all $x \in I$.

We are also able to obtain, using the same techniques, an exponential bound on the set of points whose expansion rate up to time n is less than the one prescribed by item (2) of Theorem 1.1. This is detailed in Section 7.

Theorem C. *Let $\mu \in S$ be given. Then there exist constants $C_2, \xi_2 > 0$ dependent on $\hat{f}$, ρ and k_0 only such that $\mathcal{E}(x) = \min\{N \geq 1 : \left|(f_\mu^n)'(x)\right| > \sigma^{n/3}, \forall n \geq N\}$ satisfies*

$$\lambda\Big(\{x \in \mathbf{S}^1 : \mathcal{E}(x) > n\}\Big) \leq C_2 \cdot e^{-\xi_2 \cdot n}.$$

In particular we obtain a new proof of item (2) of Theorem 1.1, which *does not follow directly from Theorem A plus the Ergodic Theorem* since it is not obvious whether $\log|f'|$ is ν_μ integrable.

Theorems B and C together ensure that for $\mu \in S$ there are constants $C_3 > 0$ and $\xi_3 \in (0,1)$ such that $\Gamma_n = \{x \in \mathbf{S}^1 : \mathcal{E}(x) > n \quad \text{or} \quad \mathcal{R}(x) > n\}$ satisfies

$$\lambda(\Gamma_n) \leq C_3 \cdot e^{-\xi_3 \sqrt{n}} \tag{1.7}$$

for all $n \geq 1$. This fits nicely into the following statements.

Theorem 1.5. *Let $g : \mathbf{S}^1 \to \mathbf{S}^1$ be a transitive C^2 local diffeomorphism outside a non-flat critical set $\mathcal{C}$ such that* (1.7) *holds. Then*

(1) [ALP, Theorem 1] *there exists an absolutely continuous invariant probability measure* ν *and some finite power of g is mixing with respect to* ν*;*
(2) [G, Theorem 1.1] *there exist constants $C, c > 0$ such that the correlation function* $\operatorname{Corr}_n(\varphi, \psi) = \left|\int (\varphi \circ g^n) \cdot \psi\, d\nu - \int \varphi\, d\nu \int \psi\, d\nu\right|$*, for Hölder continuous observables $\varphi, \psi : \mathbf{S}^1 \to \mathbf{R}$, satisfies for all $n \geq 1$*

$$\operatorname{Corr}_n(\varphi, \psi) \leq C \cdot e^{-c\sqrt{n}}.$$

(3) [ALP, Theorem 4] ν *satisfies the Central Limit Theorem: given a Hölder continuous function $\phi : \mathbf{S}^1 \to \mathbf{R}$ which is not a coboundary ($\phi \neq \psi \circ g - \psi$ for any $\psi : \mathbf{S}^1 \to \mathbf{R}$) there exists $\theta > 0$ such that for every interval $J \subset \mathbf{R}$*

$$\lim_{n\to\infty} \nu\left(\left\{x \in \mathbf{S}^1 : \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \left(\phi(g^j(x)) - \int \phi\, d\nu\right) \in J\right\}\right) = \frac{1}{\theta\sqrt{2\pi}} \int_J e^{-t^2/2\theta^2}\, dt.$$

It is then straightforward to deduce the following conclusion.

Corollary D. *For every $\mu \in S$ the map f_μ has sub-exponential decay of correlations for Hölder continuous observables and satisfies the Central Limit Theorem with respect to the SRB measure ν_μ.*

1.4. **Continuous variation of densities and of entropy.** We note that during the arguments in Sections 2 to 7 the constants used in every estimation depend uniformly on the values of ρ, σ and ε which can be set right from the start of the construction that proves Theorems B and C. This enables us to use recent results of *statistical stability* and *continuity of the SRB entropy* from [A, AOT], showing that *both the densities of the SRB measures ν_μ and the entropy vary continuously with $\mu \in S$.*

Let $\mathcal{F}$ be a family of C^2 maps of $\mathbf{S}^1$ such that for any given $f \in \mathcal{F}$ and $\varepsilon > 0$ there exists $\delta > 0$ satisfying for every measurable subset $E \subset \mathbf{S}^1$

$$\lambda(E) < \delta \implies \lambda(f^{-1}(E)) < \varepsilon,$$

that is $f_*(\lambda) \ll \lambda$. We say that a family $\mathcal{F}$ as above is a *non-degenerate family of maps*.

Theorem 1.6. *Let a non-degenerate family $\mathcal{F}$ of C^2 maps of $\mathbf{S}^1$ be given such that every $f \in \mathcal{F}$ has a non-flat critical set $\mathcal{C}_f$ and the corresponding functions $\mathcal{E}, \mathcal{R} : \mathbf{S}^1 \to \mathbf{N}$ define a family $(\Gamma_n)_{n\geq 1}$ satisfying* (1.7) *with constants C_3, ξ_3 not depending on $f \in \mathcal{F}$. Then*

1. [A, Theorem A] *the map $(\mathcal{F}, d_{C^2}) \to (L^1(\lambda), \|\cdot\|_1), f \mapsto \frac{d\nu_f}{d\lambda} \in L^1(\lambda)$ is continuous, where d_{C^2} is the C^2 distance and $\|\cdot\|_1$ the L^1-norm;*
2. [AOT, Corollary C] *the map $(\mathcal{F}, d_{C_2}) \to \mathbf{R}, f \mapsto h_{\nu_f}(f)$ is continuous.*

We observe that $\mathcal{F} = \{f_\mu : \mu \in S\}$ satisfies all the above conditions since

- $\hat{f}$ is a C^∞ map whose non-zero singularities, albeit infinitely many, are of quadratic type, and near zero $\hat{f}$ is asymptotic to $|z|^\alpha$;
- f_μ is obtained from $\hat{f}$ through a local diffeomorphism extension plus two translations (or rigid rotations when viewed on $\mathbf{S}^1$);
- the values of $\beta, \varepsilon, \sigma, \rho$ can be chosen so that
  - S is given by Theorem 1.1 with positive Lebesgue measure;
  - f_μ for $\mu \in S$ satisfies (1.7) with $C_3, \xi_3 > 0$ depending only on $\varepsilon, \sigma, \rho$ — this is detailed in Section 8.

Thus we deduce the following corollary which shows that statistical properties of f_μ are stable under small variations of the parameter μ within the set S.

Corollary E. *The following maps are both continuous:*

$$\begin{array}{rcl} S & \to & (L^1(\lambda), \|\cdot\|_1) \\ \mu & \mapsto & \frac{d\nu_\mu}{d\lambda} \end{array} \quad \textit{and} \quad \begin{array}{rcl} S & \to & \mathbf{R} \\ \mu & \mapsto & h_{\nu_\mu}(f) \end{array}.$$

2. Idea of the proof

From now on we fix a parameter $\mu \in S$ and write $\mathcal{C}_\infty = \cup_{n=0}^{\infty}(f^n)^{-1}(\mathcal{C})$ for the set of pre-orbits of the critical set. Following [PRV] we consider a convenient partition $\{I(l,s,j)\}$ of the phase space into subintervals, with a bounded distortion property: trajectories with the same itinerary with respect to this partition have derivatives which are comparable, up to a multiplicative constant. This is done as follows. Let $l \geq k_0$ and $y_l \in (x_l, x_{l-1})$ be as defined in Subsections 1.1 and 1.2: x_l is the middle point of (y_{l+1}, y_l). We partition (x_l, y_l) into subintervals

$$I(l,s) = (x_l + e^{-(\pi/\beta)s}\cdot(y_l - x_l), x_l + e^{-(\pi/\beta)(s-1)}\cdot(y_l - x_l)), \qquad s \geq 1.$$

We denote by $I(l,-s)$ the subinterval of (y_{l+1}, x_l) symmetrical to $I(l,s)$ with respect to x_l.

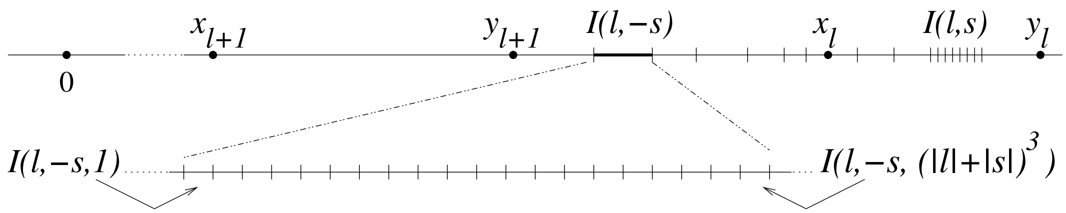

Figure 3. The initial partition $\mathcal{P}_0$.

We subdivide $I(l,\pm s)$ into $(l+|s|)^3$ intervals $I(l,\pm s,j)$, $1 \leq j \leq (l+|s|)^3$ with equal length and j increasing as $I(l,\pm s,j)$ is closer to x_l, see Figure 3. We also perform entirely symmetric constructions for $l \leq -k_0$. Let $I(\pm k_0,1,1)$ be the intervals having $\pm\varepsilon$ in their boundaries. Clearly we may suppose that $I(\pm k_0,1,1)$ are contained in the region $\mathbf{S}^1 \setminus [-\tilde{y},\tilde{y}]$ where f coincides with $\tilde{f}$, and so $|f'| > \sigma_0 > 1$. Finally, for completeness, we set $I(0,0,0) = I(0,0) = \mathbf{S}^1 \setminus [-\varepsilon,\varepsilon]$.

Remark 2.1. *By the definition of* $I(l,s,j)$

$$|I(l,s,j)| = a_1\frac{e^{-(\pi/\beta)(|l|+|s|)}}{(|l|+|s|)^3} \quad \textit{and} \quad a_2 e^{-(\pi/\beta)(|l|+|s|)} \leq \operatorname{dist}(I(l,s,j),x_l) \leq a_2 e^{-(\pi/\beta)(|l|+|s|-1)}$$

where $|I|$ *denotes the length of the interval* I,

$$a_1 = \hat{x}\frac{(e^{(\pi/\beta)}-1)^2}{e^{(\pi/\beta)}+1} \quad \textit{and} \quad a_2 = \hat{x}\frac{e^{(\pi/\beta)}-1}{e^{(\pi/\beta)}+1} < 1.$$

Moreover for any $m \geq 1$ *we have* $|x_m - x_{m+1}| = \hat{x}\cdot(1-e^{-\pi/\beta})\cdot e^{-\frac{\pi}{\beta}m}$.

We will separate the orbit of a point $x_0 \in I \setminus \mathcal{C}_\infty$ into sequences of consecutive iterates according to whether the point is near $\mathcal{C}$ or is in the expanding region $I(0,0,0)$. When $x_n = f_\mu^n(x_0)$ is near $\mathcal{C}$, we say that n is a *return time* and the expansion may be lost. But since we know that for $\mu \in S$ the derivatives along the critical orbits grow exponentially fast, we shadow the orbit of x_n during a *binding period* by the orbit of the nearest critical point and borrow its expansion. At the end of this binding period, the expansion is completely recovered, which will be explained precisely in Section 4.

This picture is complicated by the infinite number of critical points and by the possible returns near another critical point during a binding period. Iterates outside binding periods and return times are *free iterates*, where the derivative is uniformly expanded.

Our main objective is to obtain slow recurrence to $\mathcal{C}$, which means that the returns of generic orbits are not too close to $\mathcal{C}$ on the average. However *even at a free iterate the orbits may be very close to the critical set*, by the geometry of the graph of f_0, which demands a deeper analysis to achieve slow recurrence to the critical set.

Using the slow recurrence we show that the derivative along the orbit of Lebesgue almost every point grows exponentially fast. Using the estimates from Sections 3 to 5 we are able to obtain more: we deduce the exponential estimates on Theorems B and C in Sections 6 and 7.

Finally the dependence of the constants on the choices made during the entire construction is taken into account in Section 8, where we conclude that the estimates are uniform on $\mu \in S$.

3. Refining the partition

We are going to build inductively a sequence of partitions $\mathcal{P}_0, \mathcal{P}_1, \dots$ of I (modulus a zero Lebesgue measure set) into intervals. We will define inductively the sets $R_n(\omega) = \left\{r_1, \dots, r_{\gamma(n)}\right\}$ which is the set of the return times of $\omega \in \mathcal{P}_n$ up to n and a set $Q_n(\omega) = \left\{(l_1, s_1, j_1), \dots, (l_{\gamma(n)}, s_{\gamma(n)}, j_{\gamma(n)})\right\}$, which records the indexes of the intervals such that $f_\mu^{r_i}(\omega) \subset I(l_i, s_i, j_i)$, $i = 1, \dots, r_{\gamma(n)}$.

In the process we will show inductively that for all $n \in \mathbf{N}_0$

$$\forall \omega \in \mathcal{P}_n \quad f_\mu^{n+1}|_\omega \text{ is a diffeomorphism,} \tag{3.1}$$

which is essential for the construction itself. For $n = 0$ we define

$$\mathcal{P}_0 = \{I(0,0,0)\} \cup \left\{I(l,s,j) : |l| \geq k_0,\, s \geq 1,\, 1 \leq j \leq (l+s)^3\right\}.$$

It is obvious that $\mathcal{P}_0$ satisfies (3.1) for $n = 0$. We set $R_0(I(0,0,0)) = \emptyset$ and $R_0(I(l,s,j)) = \{0\}, Q_0(I(l,s,j)) = \{(l,s,j)\}$ for all possible $(l,s,j) \neq (0,0,0)$.

Remark 3.1. *This means that* every $I(l,s,j)$ *with* $|l| \geq k_0$, $|s| \geq 1$ *and* $j = 1, \dots, (|l| + |s|)^3$ *has a return at time* 0*, by definition. This will be important in Section 6.*

For each (l,s) with $|l| \geq k_0$ and $|s| \geq 1$ such that

$$e^{-(\pi/\beta)|s|} \cdot \frac{1 - e^{-(\pi/\beta)}}{1 + e^{-(\pi/\beta)}} < \tau, \quad \text{i.e.} \quad |s| > s(\tau) = -\frac{\beta}{\pi} \log\left(\tau \cdot \frac{1 + e^{-(\pi/\beta)}}{1 - e^{-(\pi/\beta)}}\right), \tag{3.2}$$

we define the *binding period* $p(x)$ of $x \in I(l,s)$ to be the largest integer $p \geq 0$ such that

$$\begin{array}{c} |f_\mu^h(x_l)| \leq \varepsilon \quad \text{and} \quad |f_\mu^{h+n}(x) - f_\mu^h(x_l)| \leq |f_\mu^h(x_l) - x_{m(h-1)}| e^{-\tau h} \\ \text{or} \\ |f_\mu^h(x_l)| > \varepsilon \quad \text{and} \quad |f_\mu^{h+n}(x) - f_\mu^h(x_l)| \leq \varepsilon^{1+\tau} e^{-\tau h} \end{array} \tag{3.3}$$

for all $1 \leq h \leq p$, where $x_{m(h)}$ is the critical point nearest $f_\mu^h(f_\mu(x_l))$ and $\tau > 0$ is a small constant to be specified during the construction.

Failing condition (3.2) means that $I(l,s)$ is not close enough to $\mathcal{C}$ since

$$|f_\mu^n(x) - x_l| \geq e^{-(\pi/\beta)|s|} \cdot (y_l - x_l) \geq e^{-(\pi/\beta)|s|} \cdot \frac{1-e^{-(\pi/\beta)}}{1+e^{-(\pi/\beta)}} \cdot |x_l| \geq \tau |x_l|$$

for all $x \in I(l,s)$, and in this case there is no expansion loss at time n. Indeed by Lemma 1.3 and using the definition of x_l from (1.2) we get

$$|f_\mu'(f_\mu^n(x))| \geq C^{-1} \cdot |x_l|^{\alpha-2} \cdot |f_\mu^n(x) - x_l| \geq C^{-1} \cdot |x_l|^{\alpha-2} \cdot \tau |x_l| = \frac{\tau \hat{x}^{\alpha-2}}{C} \cdot e^{(1-\alpha)\frac{\pi}{\beta}|l|} \tag{3.4}$$

Since $1-\alpha > 0$ and $|l| \geq k_0$, this ensures that $|f_\mu'(x)| > 1$ if we take k_0 big enough.

Remark 3.2. *As we will explain along the proof, the values of k_0 and τ^{-1} will both need to be taken sufficiently big. We note that $k_0 \to \infty$ when $\tau \to 0^+$. For more on these dependencies see Section 8.*

We define *the binding period $p(l,s)$ of the interval $I(l,s)$ to be the smallest binding period of all points of this interval*, that is $p(l,s) = \inf\{p(x) : x \in I(l,s)\}$.

For (l,s,j) with $|l| \geq k_0$, $|s| > s(\tau)$ and $1 \leq j \leq (|l|+|s|)^3$, we write

$$I(l,s,j)^+ = I(l,s_1,j_1) \cup I(l,s,j) \cup I(l,s_2,j_2),$$

where $I(l,s_1,j_1)$ and $I(l,s_2,j_2)$ are the intervals adjacent to $I(l,s,j)$ in $\mathcal{P}_0$. We also define the non-return set

$$\mathcal{N}_\tau = \bigcup \{I(l,s) : |l| \geq k_0, |s| \leq s(\tau)\} \cup I(0,0,0).$$

Now we assume that $\mathcal{P}_{n-1}$ is defined, satisfies (3.1) and R_{n-1}, Q_{n-1} are also defined on each element of $\mathcal{P}_{n-1}$. Fixing an interval $\omega \in \mathcal{P}_{n-1}$ there are three possible situations.

(1) If $R_{n-1}(\omega) \neq \emptyset$ and $n < r_{\gamma(n-1)} + p(l_{\gamma(n-1)}, s_{\gamma(n-1)})$ then we say that n is a *bound time* for ω, put $\omega \in \mathcal{P}_n$ and set $R_n(\omega) = R_{n-1}(\omega)$, $Q_n(\omega) = Q_{n-1}(\omega)$.

(2) If either $R_{n-1}(\omega) = \emptyset$, or $n \geq r_{\gamma(n-1)} + p(l_{\gamma(n-1)}, s_{\gamma(n-1)})$ and $f_\mu^n(\omega) \subset \mathcal{N}_\tau$, then we say that n is a *free time* for ω, put $\omega \in \mathcal{P}_n$ and set $R_n(\omega) = R_{n-1}(\omega)$, $Q_n(\omega) = Q_{n-1}(\omega)$.

(3) If the two conditions above fail then n is a *return time* for ω. We consider two cases:

   (a) $f_\mu^n(\omega)$ does not cover completely any $I(l,s,j)$, with $|l| \geq k_0, |s| > s(\tau)$ and $l = 1,\dots,(|l|+|s|)^3$. Because f_μ^n is continuous and ω is an interval, $f_\mu^n(\omega)$ is also an interval and thus is contained in some $I(l,s,j)^+$, for a certain $|l| \geq k_0, |s| > s(\tau)$ and $l = 1,\dots,(|l|+|s|)^3$, which is called the *host interval* of the return. This n is an *inessential return time* for ω and we set $R_n(\omega) = R_{n-1}(\omega) \cup \{n\}$, $Q_n(\omega) = Q_{n-1}(\omega) \cup \{(l,s,j)\}$.

   (b) $f_\mu^n(\omega)$ contains at least an interval $I(l,s,j)$, with $|l| \geq k_0, |s| > s(\tau)$ and $j = 1,\dots,(|l|+|s|)^3$, in which case we say that n is an *essential return time* for ω. Then we consider the sets

$$\begin{aligned} \omega'_{l,s,j} &= f_\mu^{-n}(I(l,s,j)) \cap \omega \text{ for } |l| \geq k_0,\ |s| > s(\tau), 1 \leq j \leq (|l|+|s|)^3; \\ \omega'_{0,0,0} &= f_\mu^{-n}(\mathcal{N}_\tau) \cap \omega. \end{aligned}$$

Denoting by I the set of indexes (l,s,j) such that $\omega'_{l,s,j} \neq \emptyset$ we have

$$\omega \setminus f_\mu^{-n}(\mathcal{C}) = \bigcup_{(l,s,j)\in I} \omega'_{l,s,j}. \tag{3.5}$$

By the induction hypothesis $f_\mu^n|_\omega$ is a diffeomorphism and then each $\omega'_{l,s,j}$ is an interval. Moreover $f_\mu^n(\omega'_{l,s,j})$ covers the whole $I(l,s,j)$ for $|l| \geq k_0$, $|s| > s(\tau), 1 \leq j \leq (|l|+|s|)^3$, except eventually for one or two end intervals. When $f_\mu^n(\omega'_{l,s,j})$ does not cover entirely $I(l,s,j)$ we enlarge $\omega'_{l,s,j}$ gluing it with its adjacent intervals in (3.5), getting a new decomposition of $\omega \setminus f_\mu^{-n}(\mathcal{C} \cup \mathcal{N}_\tau)$ into intervals $\omega_{l,s,j}$ such that

$$I(l,s,j) \subset f_\mu^n(\omega_{l,s,j}) \subset I(l,s,j)^+ \text{ for } |l| \geq k_0,\ |s| > s(\tau),\ 1 \leq j \leq (|l|+|s|)^3.$$

We put $\omega_{l,s,j} \in \mathcal{P}_n$ for all (l,s,j) such that $\omega_{l,s,j} \neq \emptyset$, with $|l| \geq k_0$. This results in a refinement of $\mathcal{P}_{n-1}$ at ω.
We set $R_n(\omega_{l,s,j}) = R_{n-1}(\omega) \cup \{n\}$ and n is an *essential return time* for $\omega_{l,s,j}$. The interval $I(l,s,j)^+$ is the *host interval* of $\omega_{l,s,j}$ and $Q_n(\omega_{l,s,j}) = Q_{n-1}(\omega) \cup \{(l,s,j)\}$.

At last, if $\omega'_{0,0,0} \neq \emptyset$, then $\omega'_{0,0,0}$ either contains one of $I(l, \pm s(\tau), (|l|+|s|)^3)$ for some $|l| \geq k_0$, or not.
In the latter case, we join $\omega'_{0,0,0}$ with the adjacent return interval $\omega_{l,\pm(s(\tau)+1),1}$, replace $\omega_{l,\pm(s(\tau)+1),1}$ by the new interval $\tilde{\omega}$ in $\mathcal{P}_n$ and set n as an essential return time for $\tilde{\omega}$, with $R_n(\tilde{\omega}) = R_{n-1}(\omega) \cup \{n\}$ and $Q_n(\tilde{\omega}) = Q_{n-1}(\omega) \cup \{(l, \pm(s(\tau)+1), 1)\}$.
In the former case, we say that n is a *free time* for $\omega'_{0,0,0}$, put $\omega'_{0,0,0} \in \mathcal{P}_n$ and set $R_n(\omega'_{0,0,0}) = R_{n-1}(\omega)$ and $Q_n(\omega'_{0,0,0}) = Q_{n-1}(\omega)$.

To complete the induction step all we need is to check that (3.1) holds for $\mathcal{P}_n$. Since for any interval $J \subset \mathbf{S}^1$

$$\left.\begin{array}{l} f_\mu^n|_J \text{ is a diffeomorphism} \\ \mathcal{C} \cap f_\mu^n(J) = \emptyset \end{array}\right\} \Rightarrow f_\mu^{n+1}|_J \text{ is a diffeomorphism,}$$

all we are left to prove is that $\mathcal{C} \cap f_\mu^n(\omega) = \emptyset$ for all $\omega \in \mathcal{P}_n$.

Remark 3.3. *We note that if n is a free time for z, then $x = f_\mu^n(z)$ either is in the region $\mathbf{S}^1 \setminus [-\varepsilon, \varepsilon]$ and thus $|f'_\mu(x)| \gg 1$, or satisfies the inequality* (3.4)*. Hence on free times we always have expansion of derivatives bounded from below by some uniform constant $\sigma_0 > 1$. We stress that we may and will assume that $\sigma_0 > \sqrt{\tilde{\sigma}}$ in what follows.*

Let $\omega \in \mathcal{P}_n$. If n is a free time for ω then we are done. If n is a return time for ω, essential or inessential, by construction we have that $f_\mu^n(\omega) \subset I(l,s,j)^+$ for some $|l| \geq k_0$, $|s| \geq s(\tau)$, $j = 1, \ldots, (|l|+|s|)^3$ and thus $\mathcal{C} \cap f_\mu^n(\omega) = \emptyset$. For the binding case we use the following important estimate.

Proposition 3.4. *Let $n \geq 1$ and $\omega \in \mathcal{P}_n$ be such that n is a binding time for ω. Then*

$$\textit{either} \quad |f_\mu^n(x)| > \hat{y} \quad \textit{or} \quad \operatorname{dist}(f_\mu^n(x), \mathcal{C}) \geq \rho_0 \cdot e^{-\rho(n-r)}$$

for all $x \in \omega$, where $r = r_{\gamma(n-1)}$ is the last return time for ω with $n < r + p(f_\mu^r(\omega))$ and $\rho_0 = \min\{1-\varepsilon, 1-e^{-\rho}\}$.

This result is enough to conclude that $\mathcal{C} \cap f_\mu^n(\omega) = \emptyset$, completing the induction step.

Proof. If n is a binding time for ω, then because $\mu \in S$, we know from 1.1(1b) that for every $h \geq 1$ and for all $|l| \geq k_0$ either

$$|f_\mu^h(f_\mu(x_l))| > \varepsilon \quad \text{or} \quad |f_\mu^h(f_\mu(x_l)) - x_{m(h)}| \geq e^{-\rho h}, \tag{3.6}$$

where $x_{m(h)}$ is the critical point closest to $f_\mu^h(f_\mu(x_l))$ as before. In the former case by the the definition of binding period we get for all $x \in \omega$ that

$$\begin{aligned} |f_\mu^n(x)| &\geq |f_\mu^{n-r_{\gamma(n-1)}}(x_{l_{\gamma(n-1)}})| - |f_\mu^n(x) - f_\mu^{n-r_{\gamma(n-1)}}(x_{l_{\gamma(n-1)}})| \\ &\geq \varepsilon - \varepsilon^{1+\tau} e^{-\tau(n-r_{\gamma(n-1)})} \geq \frac{\varepsilon}{x_{k_0}}(1-\varepsilon^\tau)x_{k_0} = 2\frac{1-\varepsilon^\tau}{1+e^{-\pi/\beta}}x_{k_0} > \hat{y}, \end{aligned}$$

as long as ε is taken small enough, which can be achieved choosing a bigger k_0 if needed.

In the latter case in (3.6) we get that $|f_\mu^n(x) - x_{m(n-r_{\gamma(n-1)})}|$ is bounded by

$$\begin{aligned} &|f_\mu^{n-r_{\gamma(n-1)}}(x_{l_{\gamma(n-1)}}) - x_{m(n-r_{\gamma(n-1)})}| - |f_\mu^n(x) - f_\mu^{n-r_{\gamma(n-1)}}(x_{l_{\gamma(n-1)}})| \\ &\geq e^{-\rho(n-r_{\gamma(n-1)})} - |f_\mu^{(n-r_{\gamma(n-1)})}(x_{l_{\gamma(n-1)}}) - x_{m(n-r_{\gamma(n-1)})}| e^{-\tau(n-r_{\gamma(n-1)})} \\ &\geq e^{-\rho(n-r_{\gamma(n-1)})} \cdot (1-\varepsilon) > 0, \end{aligned}$$

by definition of binding (3.3) and because we assume that $\rho < \tau$.

To complete the proof we consider the case when $x_{l_{\gamma(n-1)}}$ is not the closest critical point to $f_\mu^n(x)$. We first argue that no $x' \in \mathcal{C}$ is between $f_\mu^n(x)$ and $f_\mu^{n-r_{\gamma(n-1)}}(x_{l_{\gamma(n-1)}})$. For otherwise using (3.3) and the definition of $x_{l_{\gamma(n-1)}}$ we would have

$$\begin{aligned} \frac{1}{2} \cdot |x' - x_{m(n-r_{\gamma(n-1)})}| &< |f_\mu^n(x) - f_\mu^{n-r_{\gamma(n-1)}}(x_{l_{\gamma(n-1)}})| \\ &\leq |f_\mu^{n-r_{\gamma(n-1)}}(x_{l_{\gamma(n-1)}}) - x_{m(n-r_{\gamma(n-1)})}| \cdot e^{-\tau(n-r_{\gamma(n-1)})} \\ &\leq \frac{e^{-\tau(n-r_{\gamma(n-1)})}}{2} \cdot |x' - x_{m(n-r_{\gamma(n-1)})}|, \end{aligned}$$

a contradiction because $e^{-\tau(n-r_{\gamma(n-1)})} < 1$. Hence there exists $x' \in \mathcal{C}$ such that x' and $x_{m(n-r_{\gamma(n-1)})}$ are consecutive critical points in $\mathcal{C}$ and both $f_\mu^n(x)$ and $f_\mu^{n-r_{\gamma(n-1)}}(x_{l_{\gamma(n-1)}})$ are between x' and

$x_{m(n-r_{\gamma(n-1)})}$. But then

$$\begin{aligned}
|x' - f_\mu^n(x)| &\geq |x' - f_\mu^{n-r_{\gamma(n-1)}}(x_{l_{\gamma(n-1)}})| - |f_\mu^n(x) - f_\mu^{n-r_{\gamma(n-1)}}(x_{l_{\gamma(n-1)}})| \\
&\geq \frac{1}{2}|x' - x_{m(n-r_{\gamma(n-1)})}| - |f_\mu^{n-r_{\gamma(n-1)}}(x_{l_{\gamma(n-1)}}) - x_{m(n-r_{\gamma(n-1)})}| e^{-\tau(n-r_{\gamma(n-1)})} \\
&\geq \frac{1}{2}|x' - x_{m(n-r_{\gamma(n-1)})}| - \frac{1}{2}\cdot|x' - x_{m(n-r_{\gamma(n-1)})}|\cdot e^{-\tau(n-r_{\gamma(n-1)})} \\
&\geq \frac{1}{2}|x' - x_{m(n-r_{\gamma(n-1)})}|\cdot\left(1 - e^{-\tau(n-r_{\gamma(n-1)})}\right).
\end{aligned}$$

Setting $m = m(n - r_{\gamma(n-1)})$ for simplicity, we observe that since x' and x_m are consecutive critical points we have that x' is either x_{m+1} or x_{m-1}, thus

$$|x' - x_m| \geq 2|f_\mu^{n-r_{\gamma(n-1)}}(x_{l_{\gamma(n-1)}}) - x_m| \geq 2e^{-\rho(n-r_{\gamma(n-1)})}.$$

Combining the two last inequalities and taking into account that $\rho < \tau$ gives

$$|x' - f_\mu^n(x)| \geq e^{-\rho(n-r_{\gamma(n-1)})}\cdot\left(1 - e^{-\tau(n-r_{\gamma(n-1)})}\right) \geq e^{-\rho(n-r_{\gamma(n-1)})}\cdot\left(1 - e^{-\rho(n-r_{\gamma(n-1)})}\right),$$

and this finishes the proof since $1 - e^{-\rho(n-r_{\gamma(n-1)})} \geq 1 - e^{-\rho}$. $\square$

4. Auxiliary lemmas

Here we collect some intermediate results needed for the proofs of the main estimates. In all that follows we write C for a constant depending only on the initial map $\hat{f}$ or f_0.

Lemma 4.1 (Bounded distortion on binding periods)**.** *There exists $A = A(C,\tau) > 1$ such that for all $x \in I(l,s)$ we have*

$$\frac{1}{A} \leq \left|\frac{(f_\mu^j)'(\xi)}{(f_\mu^j)'(f_\mu(x_l))}\right| \leq A$$

for every $1 \leq j \leq p(l,s)$ and every $\xi \in [f_\mu(x_l), f_\mu(x)]$.

Proof. We let $\eta = f_\mu(x_l)$ and consider $0 \leq i < j$. There are two cases to treat, corresponding to the two possibilities in (3.3). If $|f_\mu^i(\eta)| \leq \varepsilon$ then, by Lemma 1.4,

$$\left|\frac{f'(f_\mu^i(\xi)) - f'(f_\mu^i(\eta))}{f'(f_\mu^i(\eta))}\right| \leq C\frac{|f_\mu^i(\xi) - f_\mu^i(\eta)|}{|f_\mu^i(\eta) - x_{m(i-1)}|} \leq Ce^{-\tau i}.$$

If $|f_\mu^i(\eta)| > \varepsilon$, then $|f_\mu^i(\xi) - f_\mu^i(\eta)| \leq \varepsilon^{1+\tau}e^{-\tau i} << \varepsilon$ and so the interval bounded by $f_\mu^i(\xi)$ and $f_\mu^i(\eta)$ is contained in the region $S^1 \setminus [-\tilde{y}, \tilde{y}]$, where $f = \tilde{f}$. Thus,

$$\left|\frac{f'(f_\mu^i(\xi)) - f'(f_\mu^i(\eta))}{f'(f_\mu^i(\eta))}\right| \leq C|f_\mu^i(\xi) - f_\mu^i(\eta)| \leq C\varepsilon^{1+\tau}e^{-\tau i} \leq Ce^{-\tau i}.$$

Putting together all the above we get $\sum_{i=0}^{j-1}\left|\frac{f'(f_\mu^i(\xi))-f'(f_\mu^i(\eta))}{f'(f_\mu^i(\eta))}\right| \le C\sum_{i=0}^{j} e^{-\tau i} \le C$. Thus

$$\log\left|\frac{(f_\mu^j)'(\xi)}{(f_\mu^j)'(\eta)}\right| \le \sum_{i=0}^{j-1}\log\left(1+\left|\frac{f_\mu'(f_\mu^i(\xi))}{f_\mu'(f_\mu^i(\eta))}-1\right|\right) \le \sum_{i=0}^{j-1}\left|\frac{f_\mu'(f_\mu^i(\xi))}{f_\mu'(f_\mu^i(\eta))}-1\right| \le C,$$

and the statement of the lemma follows. □

The proof of the following elementary result can be found in [PRV, Lemma 3.1].

Lemma 4.2. *Given $\alpha_1,\alpha_2,\beta_1,\beta_2$ with $\frac{\alpha_1}{\alpha_2}\ne\frac{\beta_1}{\beta_2}$, there exists $\delta>0$ such that, for every x, at least one of the following assertions hold:*

$$|\alpha_1\sin x+\beta_1\cos x|\ge\delta \quad \text{or} \quad |\alpha_2\sin x+\beta_2\cos x|\ge\delta.$$

Using this we obtain the following property of bounded distortion for the second derivative near critical points.

Lemma 4.3. *There exists a constant $C>0$ depending only on $\hat f$ such that for every $k\ge k_0$ and $t\in[y_{k+1},y_k]$ we have*

$$\frac{1}{C}\le\frac{|f''(t)|}{|f''(x_k)|}\le C.$$

Proof. Indeed we have

$$f'(x)=a|x|^{\alpha-1}[\alpha\sin(\beta\log|x|^{-1})-\beta\cos(\beta\log|x|^{-1})]$$
$$f''(x)=a|x|^{\alpha-2}[A\sin(\beta\log|x|^{-1})+B\cos(\beta\log|x|^{-1})]$$

for some A and B depending only on α and β. Applying the previous lemma we get, since $f'(x_k)=0$, that

$$\left|\frac{x_{k-1}}{x_k}\right|^{\alpha-2}\cdot\frac{\min\{\big||A|\pm|B|\big|\}}{|A|+|B|}\le\frac{|f''(t)|}{|f''(x_k)|}\le\left|\frac{x_{k+1}}{x_k}\right|^{\alpha-2}\cdot\frac{|A|+|B|}{\delta}.$$

Thus by (1.2) we obtain

$$e^{\frac{\pi}{\beta}(\alpha-2)}\cdot\frac{\min\{\big||A|\pm|B|\big|\}}{|A|+|B|}\le\frac{|f''(t)|}{|f''(x_k)|}\le e^{-\frac{\pi}{\beta}(\alpha-2)}\cdot\frac{|A|+|B|}{\delta}$$

with $\min\{\big||A|\pm|B|\big|\}>0$. □

Lemma 4.4 (Expansion during binding periods)**.** *There are constants $A_0=A_0(\varepsilon,\rho,\tau)>1$ and $0<\zeta=\zeta(\rho+\tau)<\min\{10^{-2},\frac{\pi}{10\beta}\}$ such that for $n\ge1$ and $\omega\in\mathcal{P}_n$ with $R_n(\omega)\ne\emptyset$, if r is the last return time for ω and $f_\mu^r(\omega)\subset I(l,s,j)$, then setting $p=p(l,s)>0$ we have*

(a) $p\le\frac{2\pi}{\beta\log\sigma}(|l|+|s|)$;
(b) $|(f_\mu^{p+1})'(f_\mu^r(x))|\ge e^{(1-2\zeta)\frac{\pi}{\beta}(|l|+|s|)}$, *for every $x\in\omega$;*
(c) $|(f_\mu^{p+1})'(f_\mu^r(x))|\ge A_0\cdot\sigma^{(p+1)/3}>1$ *for every $x\in\omega$.*

Proof. To prove item (a), we use the definition of the partition and the construction of the refinement. As $p>0$, we have $|f_\mu^r(x)-x_l|\le\tau|x_l|<\varepsilon$ for all $x\in\omega$. In particular $(l,s,j)\ne(\pm k_0,1,1)$ and so $|f_\mu^r(x)-x_l|\ge a_2\cdot e^{-(\pi/\beta)(|l|+|s|)}$, where we used the estimates from Remark 2.1. Using second-order Taylor approximation and Lemma 4.3 we get

$$|f_\mu^{r+1}(x)-f_\mu(x_l)|\ge\frac{1}{C}|f''(x_l)|(a_2e^{-(\pi/\beta)(|l|+|s|)})^2\ge\frac{1}{C}\cdot e^{-\frac{\pi}{\beta}|l|(\alpha-2)}\cdot e^{-2(\pi/\beta)(|l|+|s|)},$$

where $|f''(x_l)|\ge C^{-1}|x_l|^{\alpha-2}=C^{-1}\cdot\hat{x}^{\alpha-2}\cdot e^{-\frac{\pi}{\beta}|l|(\alpha-2)}$ by Lemma 1.3(2). Then, for each $0\le j\le p$, there is some ξ between $f_\mu(x_l)$ and $f_\mu^{r+1}(x)$ such that

$$\begin{aligned}|f_\mu^{j+r+1}(x)-f_\mu^{j+1}(x_l)| &= |(f_\mu^j)'(\xi)||f_\mu^{r+1}(x)-f_\mu(x_l)|\\ &\ge C^{-1}\cdot e^{-\frac{\pi}{\beta}|l|(\alpha-2)}\cdot e^{-2(\pi/\beta)(|l|+|s|)}|(f_\mu^j)'(\xi)|.\end{aligned}\tag{4.1}$$

Now since we can take $|l|\ge k_0$ very big, as a consequence of Lemma 4.1 and of the exponential growth of the derivative at the critical orbits, we get the following bound

$$2\cdot e^{-2(\pi/\beta)(|l|+|s|)}\sigma^j\le|f_\mu^{j+r+1}(x)-f_\mu^{j+1}(x_l)|\le 2$$

Hence, $e^{-2(\pi/\beta)(|l|+|s|)}\sigma^j\le 1$ for all $1\le j\le p$. In particular,

$$-2(\pi/\beta)(|l|+|s|)+p\log(\sigma)\le 0,\quad\text{implying}\quad p\le\frac{2(\pi/\beta)(|l|+|s|)}{\log\sigma},$$

thus proving (a).

Now we prove (b). Since $p+1$ is not a binding time we must have by definition either

$$|f_\mu^{p+1}(x_l)-f_\mu^{p+r+1}(x)|>\varepsilon^{1+\tau}\cdot e^{-\tau(p+1)}$$

or

$$|f_\mu^{p+1}(x_l)-f_\mu^{r+p+1}(x)|>|f_\mu^{p+1}(x_l)-x_{m(p)}|e^{-\tau(p+1)}\ge e^{-\rho p}e^{-\tau(p+1)}>e^{-(\rho+\tau)(p+1)},$$

where we have used Theorem 1.1(1b). We set $\Delta_{p+1}=\min\{\varepsilon^{1+\tau}\cdot e^{-\tau(p+1)},e^{-(\rho+\tau)(p+1)}\}$ and note that for some $\xi\in[f_\mu(x_l),f_\mu^{r+1}(x)]$ by Lemma 4.1 and using second-order Taylor expansion of f_μ near x_l together with Lemma 4.3

$$\begin{aligned}\Delta_{p+1} &\le |f_\mu^{p+1}(x_l)-f_\mu^{p+r+1}(x)|=|(f_\mu^p)'(\xi)|\cdot|f_\mu(x_l)-f_\mu^{r+1}(x)|\\ &\le C\cdot|(f_\mu^p)'(f_\mu^{r+1}(x))|\cdot|f_\mu''(x_l)|\cdot|f_\mu^r(x)-x_l|^2\\ &\le C^2|(f_\mu^p)'(f_\mu^{r+1}(x))|\cdot\hat{x}^{\alpha-2}\cdot e^{-\frac{\pi}{\beta}|l|(\alpha-2)}\cdot a_2^2\cdot e^{-2\frac{\pi}{\beta}(|l|+|s|-1)}\\ &= C^2\cdot\hat{x}^{\alpha-2}\cdot a_2^2\cdot e^{\frac{\pi}{\beta}(2-\alpha|l|-2|s|)}\cdot|(f_\mu^p)'(f_\mu^{r+1}(x))|.\end{aligned}\tag{4.2}$$

Hence

$$|(f_\mu^p)'(f_\mu^{r+1}(x))|\ge\frac{1}{C^2\cdot\hat{x}^{\alpha-2}\cdot a_2^2}\cdot\Delta_{p+1}\cdot\exp\left[\frac{\pi}{\beta}(\alpha|l|+2|s|-2)\right].\tag{4.3}$$

Using again Lemma 4.1 we take $y \in \omega$ and write by Lemma 1.3

$$\begin{aligned}
|(f_\mu^{p+1})'(f_\mu^r(x))| &= |f_\mu'(f_\mu^r(x))| \cdot |(f_\mu^p)'(f_\mu^{r+1}(x))| \\
&\geq C^{-1}|x_l|^{\alpha-2}|f_\mu^r(x)-x_l| \cdot \frac{|(f_\mu^p)'(f_\mu^{r+1}(x))|}{|(f_\mu^p)'(f_\mu^{r+1}(y)|}|(f_\mu^p)'(f_\mu^{r+1}(y))| \\
&\geq C^{-1} \cdot \hat{x}^{\alpha-2}e^{-\frac{\pi}{\beta}|l|(\alpha-2)} \cdot a_2 e^{-\frac{\pi}{\beta}(|l|+|s|)} \cdot \frac{1}{A} \cdot |(f_\mu^p)'(f_\mu^{r+1}(y))|.
\end{aligned}$$

Since the previous bounds do not depend on the point $x \in \omega$, we can use (4.3) in the last expression obtaining after cancellation

$$|(f_\mu^{p+1})'(f_\mu^r(x))| \geq \frac{1}{A \cdot C^3 \cdot a_2} \cdot \Delta_{p+1} \cdot e^{\frac{\pi}{\beta}(|l|+|s|-2)}.$$

We observe that because $0 < \varepsilon < 1$ and by item (a) of the lemma we get

$$\Delta_{p+1} \geq \varepsilon^{1+\tau} \cdot e^{-(\rho+\tau)(p+1)} \geq \varepsilon^{1+\tau} \cdot \exp\big[-(\tau+\rho)\big(\frac{2\pi}{\log\sigma}(|l|+|s|)-1\big)\big].$$

Altogether we arrive at

$$\begin{aligned}
|(f_\mu^{p+1})'(f_\mu^r(x))| &\geq \frac{\varepsilon^{1+\tau} \cdot e^{\tau+\rho-\frac{2\pi}{\beta}}}{A \cdot C^3 \cdot a_2} \exp\left[\big(\frac{\pi}{\beta}-(\tau+\rho)\frac{2\pi}{\log\sigma}\big)(|l|+|s|)\right] \\
&\geq \frac{\varepsilon^{1+\tau} \cdot e^{\tau+\rho-\frac{2\pi}{\beta}}}{A \cdot C^3 \cdot a_2} \cdot e^{(1-\zeta)\frac{\pi}{\beta}(|l|+|s|)} \geq e^{(1-2\zeta)\frac{\pi}{\beta}(|l|+|s|)},
\end{aligned}$$

where $0 < \zeta < \{10^{-2}, \frac{\pi}{10\beta}\}$ as long as $s(\tau)$ is big enough and $\rho+\tau$ is small enough, concluding the proof of (b).

In order to prove (c) we use Lemma 4.1 once again, the inequality (4.2), Lemma 1.3(2) and Lemma 4.3 to get

$$\begin{aligned}
|(f_\mu^{p+1})'(f_\mu^r(x)|^2 &= |(f_\mu^p)'(f_\mu^{r+1}(x))|^2 \cdot |f_\mu'(f_\mu^r(x))|^2 \\
&\geq C^{-1}|(f_\mu^p)'(f_\mu(x_l))| \cdot |(f_\mu^p)'(f_\mu^{r+1}(x))| \cdot C^{-1}|f_\mu''(x_l)|^2 \cdot |f_\mu^r(x)-x_l|^2 \\
&= C^{-1}|(f_\mu^p)'(f_\mu(x_l))| \cdot |f_\mu''(x_l)| \cdot \big(|(f_\mu^p)'(f_\mu^{r+1}(x))| \cdot |f_\mu''(x_l)| \cdot |f_\mu^r(x)-x_l|^2\big) \\
&\geq C^{-1}|(f_\mu^p)'(f_\mu(x_l))| \cdot |f_\mu''(x_l)| \cdot \big(|(f_\mu^p)'(\xi)| \cdot |f_\mu(x_l)-f_\mu^{r+1}(x)|\big) \\
&= C^{-1}|(f_\mu^p)'(f_\mu(x_l))| \cdot |f_\mu''(x_l)| \cdot |f_\mu^{p+1}(x_l)-f_\mu^{r+p+1}(x)| \\
&\geq C^{-1}\sigma^p \cdot C^{-1}|x_l|^{\alpha-2} \cdot |f_\mu^{p+1}(x_l)-f_\mu^{r+p+1}(x)| \qquad (4.4) \\
&= C^{-2}\sigma^p \cdot \hat{x}^{\alpha-2} \cdot e^{-\frac{\pi}{\beta}|l|(\alpha-2)} \cdot |f_\mu^{p+1}(x_l)-f_\mu^{r+p+1}(x)|.
\end{aligned}$$

Now we must consider two cases. On the one hand, if $|f_\mu^{p+1}(x_l)| > \varepsilon$ then, by the definition of p in (3.3), we must have $|f_\mu^{p+1}(x_l)-f_\mu^{r+p+1}(x)| > \varepsilon^{1+\tau}e^{-\tau(p+1)}$. Because $\alpha-2<0$ and $|x_l| \leq \varepsilon$, equation (4.4) implies $|(f_\mu^{p+1})'(f_\mu^r(x)|^2 \geq C^{-2}\sigma^p \cdot \varepsilon^{\alpha-2} \cdot |f_\mu^{p+1}(x_l)-f_\mu^{r+p+1}(x)|$ and we may write $|(f_\mu^{p+1})'(f_\mu^r(x)|^2 \geq C^{-1}\sigma^{p+1}\varepsilon^{\alpha-1+\tau}e^{-\tau(p+1)} \geq A_0^2 \cdot \sigma^{2(p+1)/3}$, as long as we fix $\tau <$

$\min\{1-\alpha, \log\sigma/3\}$ and suppose ε small enough. On the other hand, if $|f_\mu^{p+1}(x_l)| \le \varepsilon$ then, by (3.3)

$$|f_\mu^{p+1}(x_l) - f_\mu^{r+p+1}(x)| > |f_\mu^{p+1}(x_l) - x_{m(p)}|e^{-\tau(p+1)}.$$

We note that now there is only one possibility according to item (1b) of Theorem 1.1: $|f_\mu^{p+1}(x_l) - x_{m(p)}| \ge e^{-\rho p}$ and thus $|f_\mu^{p+1}(x_l) - f_\mu^{r+p+1}(x)| > C^{-1}\varepsilon e^{-(\rho+\tau)(p+1)}$. Hence $|(f_\mu^{p+1})'(f_\mu^r(x)|^2 \ge C^{-1}\sigma^{p+1}\varepsilon^{\alpha-1}e^{-(\rho+\tau)(p+1)} \ge A_0^2 \cdot \sigma^{2(p+1)/3}$ as long as we take ε, ρ and τ small enough. This concludes the proof of the lemma. □

Now we will obtain estimates of the length of $|f_\mu^n(\omega)|$.

Lemma 4.5 (Lower bounds on the length at return times)**.** *Suppose that r is a return time for $\omega \in \mathcal{P}_{n-1}$, with host interval $I(l,s,j)^+$, and let $p = p(l,s)$ denote the length of its binding period. Then for k_0 and $s(\tau)$ sufficiently big (depending on ζ from Lemma 4.4) the following holds.*

(1) *Assuming that $r^* \le n-1$ is the next return time for ω (either essential or inessential) $\left|f_\mu^{r^*}(\omega)\right| \ge \sigma_0^q \cdot e^{(1-2\zeta)\frac{\pi}{\beta}(|l|+|s|)} \cdot |f_\mu^r(\omega)| \ge 2\cdot\left|f_\mu^r(\omega)\right|$.*
(2) *If r is the last return time for ω up to iterate $n-1$ and also an* essential *return, and n is a return time for ω, then setting $q = n-(r+p)$ we have*

$$\left|f_\mu^{r_*}(\omega)\right| \ge a_1 \cdot \sigma_0^q \cdot e^{-3\zeta\frac{\pi}{\beta}(|l|+|s|)}.$$

Proof. We start by assuming $r^* \le n-1$ as in item (1). By the mean value theorem we have $|f_\mu^{r^*}(\omega)| \ge |(f_\mu^{r^*-r})'(f_\mu^r(\zeta))| \cdot |f_\mu^r(\omega)|$ for some $\zeta \in \omega$.

Using Remark 3.3 and Lemma 4.4 we get defining $q = r^* - (r+p)$

$$\begin{aligned}
\left|f_\mu^{r^*}(\omega)\right| &\ge \left|\left(f_\mu^q\right)'\left(f_\mu^{r+p}(\zeta)\right)\right|\left|\left(f_\mu^p\right)'\left(f_\mu^r(\zeta)\right)\right|\left|f_\mu^r(\omega)\right| \\
&\ge \sigma_0^q \cdot e^{(1-2\zeta)\frac{\pi}{\beta}(|l|+|s|)} \cdot |f_\mu^r(\omega)| \qquad (4.5)\\
&\ge \sigma_0^q \cdot e^{(1-3\zeta)\frac{\pi}{\beta}(|l|+|s|)} \cdot e^{\zeta\frac{\pi}{\beta}(|l|+|s|)} \cdot |f_\mu^r(\omega)|. \qquad (4.6)
\end{aligned}$$

If r is an essential return time for ω, then $I(l,s,j) \subset f_\mu^r(\omega)$ and $|f_\mu^r(\omega)| \ge a_1 \frac{e^{-(\pi/\beta)(|l|+|s|)}}{(|l|+|s|)^3}$, hence

$$\begin{aligned}
\left|f_\mu^{r^*}(\omega)\right| &\ge \sigma_0^q \cdot e^{\zeta\frac{\pi}{\beta}(|l|+|s|)} \cdot e^{(1-3\zeta)\frac{\pi}{\beta}(|l|+|s|)} \cdot a_1 \frac{e^{-(\pi/\beta)(|l|+|s|)}}{(|l|+|s|)^3} \\
&\ge a_1 \cdot \sigma_0^q \cdot \frac{e^{\zeta\frac{\pi}{\beta}(|l|+|s|)}}{(|l|+|s|)^3} \cdot e^{-3\zeta\frac{\pi}{\beta}(|l|+|s|)} \ge a_1 \cdot \sigma_0^q \cdot e^{-3\zeta\frac{\pi}{\beta}(|l|+|s|)}
\end{aligned}$$

as long as k_0 and $s(\tau)$ are big enough in order that $e^{\zeta\frac{\pi}{\beta}(|l|+|s|)} \ge (|l|+|s|)^3$. This proves item (2) by taking $r^* = n$.

In this setting we must also have $e^{\zeta\frac{\pi}{\beta}(|l|+|s|)} \ge 2$ in (4.6), which together with (4.5) prove item (1) and concludes the proof of the lemma. □

Lemma 4.6 (Bounded Distortion). *There is a constant $D_0 = D_0(\rho,\tau,\sigma) > 0$ such that for $\omega \in \mathcal{P}_{n-1}$, $n \in \mathbf{N}$, and for every $x, y \in \omega$ we have*

$$\left|\frac{(f_\mu^n)'(x)}{(f_\mu^n)'(y)}\right| \le D_0.$$

Proof. Let $R_{n-1}(\omega) = \{r_1,\dots,r_\gamma\}$ and $Q_{n-1}(\omega) = \{(l_1,s_1,j_1),\dots,(l_\gamma,s_\gamma,j_\gamma)\}$, be the sets of return times and indexes of host intervals of ω, respectively, as defined during the construction of the partition. Let $\omega_i = f_\mu^{r_i}(\omega)$, $p_i = p(l_i,s_i)$ for $i = 1,\dots,\gamma$ and, for $y,z \in \omega$, let $y_k = f_\mu^k(y)$ and $z_k = f_\mu^k(z)$ for $k = 0,\dots,n-1$. Observe that $\omega_i \subset I(l_i,s_i,j_i)^+$ for all i and

$$\left|\frac{(f_\mu^n)'(z)}{(f_\mu^n)'(y)}\right| = \prod_{k=0}^{n-1}\left|\frac{f'(z_k)}{f'(y_k)}\right| \le \prod_{k=0}^{n-1}\left(1+\left|\frac{f'(z_k)-f'(y_k)}{f'(y_k)}\right|\right). \tag{4.7}$$

On free iterates, if $y_k \in [-\varepsilon,\varepsilon]$, then by Lemma 1.4

$$\left|\frac{f'(z_k)-f'(y_k)}{f'(y_k)}\right| \le K_1\cdot\left|\frac{z_k-y_k}{y_k-\tilde{x}_k}\right| \le K_1\cdot\frac{|f_\mu^k(\omega)|}{\Delta_k(\omega)}, \tag{4.8}$$

where we define $\Delta_k(\omega) = \operatorname{dist}(f_\mu^k(\omega),\mathcal{C}) = \inf_{x\in\omega}\operatorname{dist}(f_\mu^k(x),\mathcal{C})$ and $\tilde{x}_k$ is the critical point closest to y_k. We observe that in this case the interval $f_\mu^k(\omega)$ is between two consecutive critical points, x_l and x_{l+1}, and the greatest positive integer s satisfying

$$f_\mu^k(\omega)\cap I(l,-s)\neq\emptyset \quad\text{or}\quad f_\mu^k(\omega)\cap I(l+1,s))\neq\emptyset \quad\text{is such that}\quad s\le s(\tau).$$

We then set $(\hat{l}_k,\hat{s}_k)$ to be the index of the partition interval satisfying the above condition and note that by the exponential character of the initial partition, we have

$$|f_\mu^k(\omega)| \le C\cdot|I(\hat{l}_k,\hat{s}_k,1)^+| \quad\text{and}\quad \Delta_k(\omega)\ge C^{-1}\cdot|I(\hat{l}_k,\hat{s}_k)| \tag{4.9}$$

for some constant $C > 0$ depending only on $\hat{f}$ and τ.

Otherwise for free iterates $y_k \in S^1\setminus[-\varepsilon,\varepsilon]$ we get

$$\sum_{\substack{r_i+p_i<k<r_{i+1}\\|y_k|>\varepsilon}}\left|\frac{f'(z_k)-f'(y_k)}{f'(y_k)}\right| \le \frac{L}{\tilde{\sigma}}\sum_{\substack{r_i+p_i<k<r_{i+1}\\|y_k|>\varepsilon}}|z_k-y_k| \le \frac{L}{\tilde{\sigma}}\sum_{\substack{r_i+p_i<k<r_{i+1}\\|y_k|>\varepsilon}}|f_\mu^k(\omega)|$$

$$\le \frac{L}{\tilde{\sigma}}\sum_{r_i+p_i<k<r_{i+1}}\sigma_0^{k-r_{i+1}}\cdot|f^{r_{i+1}}(\omega)| \le K_2\cdot\frac{|\omega_{i+1}|}{\Delta_{r_{i+1}}(\omega)} \tag{4.10}$$

by definition of f_μ on $S^1\setminus[-\varepsilon,\varepsilon]$, since $|f'_\mu \mid S^1\setminus[-\varepsilon,\varepsilon]| > \tilde{\sigma}$ and $|f''_\mu \mid S^1\setminus[-\varepsilon,\varepsilon]| \le L$ for some constant L. We recall also that $\Delta_{r_{i+1}}(\omega) < 1$ by definition.

Next we find a bound for iterates during binding times. Let us fix $i = 1,\dots,\gamma$. Then for $k = r_i$ we have the same bound (4.8). For $r_i < k \le r_i + p_i$ we get for some $\xi\in\omega$

$$\begin{aligned}|z_k-y_k| &= |(f^{k-r_i})'(f_\mu^{r_i}(\xi))|\cdot|z_{r_i}-y_{r_i}| \le |(f^{k-r_i})'(f_\mu^{r_i}(\xi))|\cdot|f_\mu^{r_i}(\omega)|\\ &\le C\cdot|(f^{k-r_i-1})'(f_\mu^{r_i+1}(\xi))|\cdot|f''(x_{l_i})|\cdot|f_\mu^{r_i}(\xi)-x_{l_i}|\cdot|\omega_i|,\end{aligned}$$

where we have used the Taylor expansion of f' near the critical point x_{l_i} together with Lemma 4.3. By definition of p_i we have two possibilities. On the one hand, for the first case in (3.3) there exists $w \in [f_\mu(x_{l_i}), f_\mu^{r_i+1}(\xi)]$ such that, using second order Taylor expansion and Lemma 4.3 again

$$\begin{aligned} |f_\mu^{k-r_i}(x_{l_i}) - x_{m(k-r_i-1)}|e^{-\tau(k-r_i)} &\geq |f_\mu^k(\xi) - f_\mu^{k-r_i}(x_{l_i})| \\ &= |(f^{k-r_i-1})'(w)| \cdot |f_\mu^{r_i+1}(\xi) - f_\mu(x_{l_i})| \\ &\geq C^{-1}|(f^{k-r_i-1})'(w)| \cdot |f''(x_{l_i})| \cdot |f_\mu^{r_i}(\xi) - x_{l_i}|^2 \\ &\geq (AC)^{-1}|(f^{k-r_i-1})'(f_\mu^{r_i+1}(\xi))| \cdot |f''(x_{l_i})||f_\mu^{r_i}(\xi) - x_{l_i}|^2, \end{aligned} \tag{4.11}$$

where we have used Lemma 4.1 in the last inequality. The last two expression together show that

$$|z_k - y_k| \cdot |f_\mu^{r_i}(\xi) - x_{l_i}| \leq (AC^2) \cdot |f_\mu^{k-r_i}(x_{l_i}) - x_{m(k-r_i-1)}|e^{-\tau(k-r_i)} \cdot |\omega_i|.$$

This and Lemma 1.4 provide

$$\begin{aligned} \left|\frac{f'(z_k) - f'(y_k)}{f'(y_k)}\right| &\leq K_1 \left|\frac{z_k - y_k}{y_k - \tilde{x}_k}\right| \leq AC^2 K_1 e^{-\tau(k-r_i)} \cdot \frac{|\omega_i| \cdot |f_\mu^{k-r_i}(x_{l_i}) - x_{m(k-r_i-1)}|}{|f_\mu^{r_i}(\xi) - x_{l_i}| \cdot |y_k - \tilde{x}_k|} \\ &\leq D \cdot e^{-\tau(k-r_i)} \cdot \frac{|\omega_i|}{\Delta_i(\omega)} \cdot \frac{|f_\mu^{k-r_i}(x_{l_i}) - x_{m(k-r_i-1)}|}{|f_\mu^{k-r_i}(x_{l_i}) - x_{m(k-r_i-1)}| - |y_k - f_\mu^{k-r_i}(x_{l_i})|} \\ &\leq D \cdot \frac{e^{-\tau(k-r_i)}}{1 - e^{-\tau(k-r_i)}} \cdot \frac{|\omega_i|}{\Delta_i(\omega)} \leq D_1 \cdot e^{-\tau(k-r_i)} \cdot \frac{|\omega_i|}{\Delta_{r_i}(\omega)}. \end{aligned}$$

On the other hand, for the second case in (3.3) we get a similar inequality in (4.11) providing

$$|z_k - y_k| \cdot |f_\mu^{r_i}(\xi) - x_{l_i}| \leq (AC^2) \cdot \varepsilon^{1+\tau} e^{-\tau(k-r_i)} \cdot |\omega_i|$$

and thus by definition of $\tilde{f}$ we get

$$\begin{aligned} \left|\frac{f'(z_k) - f'(y_k)}{f'(y_k)}\right| &\leq \frac{L \cdot |z_k - y_k|}{\tilde{\sigma}} \leq \frac{ACL}{\tilde{\sigma}} e^{-\tau(k-r_i)} \frac{|\omega_i| \cdot \varepsilon^{1+\tau}}{|f_\mu^{r_i}(\xi) - x_{l_i}|} \\ &\leq D_2 \cdot e^{-\tau(k-r_i)} \cdot \frac{|\omega_i|}{\Delta_{r_i}(\omega)}. \end{aligned}$$

This shows that for every $i = 1, \ldots, \gamma$ we have

$$\sum_{k=r_i}^{r_i+p_i} \left|\frac{f'(z_k) - f'(y_k)}{f'(y_k)}\right| \leq D_3 \cdot \frac{|\omega_i|}{\Delta_{r_i}(\omega)} \leq \frac{1}{C} \cdot \frac{|I(l_i, s_i, j_i)^+|}{|I(l_i, s_i)|}, \tag{4.12}$$

where we have used the definition of ω_i and of host interval, together with the same estimate as in (4.9). Taking into account (4.8), (4.10) and (4.12) and summing over all iterates we arrive at

$$\sum_{k=0}^{n-1} \left|\frac{f'(z_k) - f'(y_k)}{f'(y_k)}\right| \leq D_4 \sum_{k \in F_1} \frac{|f_\mu^k(\omega)|}{\Delta_k(\omega)} + \frac{L}{\tilde{\sigma}} \sum_{k \in F_2} |f_\mu^k(\omega)|. \tag{4.13}$$

Here the left hand side sum is over the set F_1 of free iterates inside $[-\varepsilon, \varepsilon]$, free iterates in between returns together with return iterates from $k = 0$ to $k = n-1$. The right hand side sum is over the set F_2 of free iterates on $S^1 \setminus [-\varepsilon, \varepsilon]$ which are not followed by any return, from $r_\gamma + p_\gamma$ to n.

Moreover D_4 is a constant depending only on ε, τ and $\tilde{\sigma}$. So if we can bound (4.13) uniformly we then find a uniform bound to (4.7) also and complete the proof of the lemma.

The right hand side sum in (4.13) is easily bounded as follows

$$\sum_{k\in F_2} |f_\mu^k(\omega)| \le \sum_{k=0}^{n-1} \tilde{\sigma}^{k-n} |f_\mu^n(\omega)| \le C,$$

since $|f_\mu^n(\omega)|$ is always less than 1.

Now we bound the left hand side sum

$$\sum_{k\in F_1} \frac{|f_\mu^k(\omega)|}{\Delta_k(\omega)} \le \sum_{|l|\ge k_0} \sum_{|s|\ge 1} \sum_{\substack{k\in F_1\\ (\hat{l}_k,\hat{s}_k)=(l,s)}} \frac{|f_\mu^k(\omega)|}{\Delta_k(\omega)} \le \sum_{|l|\ge k_0} \sum_{|s|\ge 1} \frac{\sigma_1}{\sigma_1 - 1} \cdot \frac{1}{C} \cdot \frac{|f_\mu^{q(l,s)}(\omega)|}{|I(l,s)|},$$

by (4.9), where $q(l,s) = \max\{0 \le k \le n-1 : (\hat{l}_k, \hat{s}_k) = (l,s)\}$ and we convention that whenever $\{0 \le k \le n-1 : (\hat{l}_q, \hat{s}_q) = (l,s)\} = \emptyset$ we have $\frac{|f_\mu^{q(l,s)}(\omega)|}{|I(l,s)|} = 0$. We have used the following estimate for any given fixed value of (l,s)

$$\sum_{\{k:\hat{s}_k=s\}} |f_\mu^k(\omega)| \le |f_\mu^{q(l,s)}(\omega)| \sum_{\{k:(\hat{l}_k,\hat{s}_k)=(l,s)\}} \sigma_1^{k-q(l,s)} \le \frac{\sigma_1}{\sigma_1-1} \cdot |f_\mu^{q(l,s)}(\omega)| \le C \cdot |I(l,s,j)^+|,$$

because writing $\{k : \hat{s}_k = s\} = \{k_1 < k_2 < \cdots < k_h\}$ we have $|f_\mu^{k_i}(\omega)| \le \sigma_1^{-1} \cdot |f_\mu^{k_{i+1}}(\omega)|$ for $i = 1, \dots, h$, where $1 < \sigma_1 = \min\{\sigma_0, e^{(1-2\zeta)\cdot\frac{\pi}{\beta}(k_0+1)}\} \le \min\{\sigma_0, e^{(1-2\zeta)\cdot\frac{\pi}{\beta}(|l|+|s|)}\}$, after Lemma 4.4(b) together with Remark 3.3.

We observe that by construction we must have $|I(l,s,j)^+|/|I(l,s)| \le 9(|l|+|s|)^{-3}$ and so we arrive at

$$\sum_{k\in F_1} \frac{|f_\mu^k(\omega)|}{\Delta_k(\omega)} \le C \sum_{|l|\ge k_0} \sum_{|s|\ge 1} \frac{9}{(|l|+|s|)^3} < \infty,$$

finishing the proof of the lemma.

$\square$

5. Main estimates

Here we use the results from Section 4 to relate the indexes of host intervals at inessential returns, which we call the *depth* of the return, with the previous essential return depth.

We use this information to obtain a bound on the time it takes from one essential return to the next and also an estimation for the probability of points whose orbit has a given sequence of host intervals at essential return times.

5.1. **Returns between consecutive essential returns.**

Next we show that the depth of an inessential return is not greater than the depth of the essential return that precedes it.

Lemma 5.1. *Let t_i be an essential return for $\omega \in \mathcal{P}_n$ with $I(l_i, s_i, j_i) \subset f_\mu^{t_i}(\omega) \subset I(l_i, s_i, j_i)^+$. Then for each consecutive inessential return $t_i < t_i(1) < \cdots < t_i(v) < n$ before the next essential* *return, the host interval* $I(l_i(k),s_i(k),j_i(k)) \supset f_\mu^{t_i(k)}(\omega)$, $k=1,\dots,v$ *is such that* $|l_i(k)|+|s_i(k)|<|l_i|+|s_i|$.

Proof. By lemma 4.5(1) we have $|f_\mu^{t_i(k)}(\omega)|>|f_\mu^{t_i}(\omega)|>|I(l_i,s_i,j_i)|$. Thus because each $t_i(k)$ is an inessential return we get for $k=1,\dots,v$

$$a_1\frac{e^{-(\pi/\beta)(|l_i(k)|+|s_i(k)|)}}{(|l_i(k)|+|s_i(k)|)^3}>|f_\mu^{t_i(k)}(\omega)|>a_1\frac{e^{-(\pi/\beta)(|l_i|+|s_i|)}}{(|l_i|+|s_i|)^3}.$$

As $z^{-3}\cdot e^{-(\pi/\beta)z}$ is decreasing for $z>0$, we conclude that $|l_i(k)|+|s_i(k)|<|l_i|+|s_i|$. □

Now we prove a similar result for returns during binding periods.

Lemma 5.2. *Let r be a return time (either essential or inessential) for $\omega\in\mathcal{P}_n$, with $f_\mu^r(\omega)\subset I(l,s,j)^+$. Let $p=p(l,s)$ be the binding period associated to this return. Then for $|l|\geq k_0$ big enough and ρ small enough (depending only on σ and ε through ρ_0) and for every $x\in\omega$, if $f_\mu^{r+k}(x)\in I(l_k,s_k,j_k)$ for $0<k<p$ with $|l_k|\geq k_0$ and $|s_k|\geq 1$, then $|l_k|+|s_k|<|l|+|s|$.*

Proof. Let us fix a point $x\in\omega$ and $k\in\{1,\dots,p-1\}$ such that $f_\mu^{r+k}(x)\in I(l_k,s_k,j_k)$ as in the statement. We split the proof in two cases.

On the one hand, if $|f_\mu^k(x_l)|>\varepsilon$, then

$$\begin{aligned}|f_\mu^{r+k}(x)| &\geq |f_\mu^k(x_l)|-|f_\mu^{r+k}(x)-f_\mu^k(x_l)|\geq\varepsilon-\varepsilon^{1+\tau}\cdot e^{-\tau k}\\ &\geq \frac{\varepsilon}{x_{k_0}}(1-\varepsilon^\tau)x_{k_0}=2\frac{1-\varepsilon^\tau}{1+e^{-\pi/\beta}}x_{k_0}>\hat{y},\end{aligned}$$

which cannot happen if $f_\mu^{r+k}(x)$ is indeed a return.

On the other hand, if $|f_\mu^k(x_l)|\leq\varepsilon$, then by Remark 2.1, Proposition 3.4 and Lemma 4.4(a) we get

$$\begin{aligned}a_2\cdot e^{-\frac{\pi}{\beta}(|l_k|+|s_k|-1)} &\geq \operatorname{dist}(f_\mu^{r+k}(x),\mathcal{C})\geq\rho_0\cdot e^{-\rho k},\quad\text{that is}\\ e^{-\frac{\pi}{\beta}(|l_k|+|s_k|-1)} &\geq \frac{\rho_0}{a_2}\exp\Big(-\rho\cdot\frac{2\pi}{\log\sigma}(|l|+|s|)\Big)\\ &\geq \exp\Big[-\frac{\pi}{\beta}(|l|+|s|)\cdot\big(\frac{2\rho\beta}{\log\sigma}-\frac{\beta\log(\rho_0/a_2)}{\pi(|l|+|s|)}\big)\Big]\\ &\geq e^{-\frac{\pi}{\beta}(|l|+|s|)},\end{aligned}$$

because $2\rho\beta/\log\sigma-\beta\log(\rho_0/a_2)/(\pi(|l|+|s|))\leq 1$ if ρ is very small and $|l|+|s|\geq k_0+1$ is very big. This proves the statement of the lemma. □

5.2. **Probability of deep essential returns.** For each $x\in I\setminus\mathcal{C}_\infty$, let $u_n(x)$ denote the number of essential returns of the orbit of x between 1 and n, let $0\leq t_1\leq\dots\leq t_{u_n}\leq n$ be the instants of occurrence of the essential returns and let $(l_1,s_1,j_1),\dots,(l_{u_n},s_{u_n},j_{u_n})$ be the corresponding critical points and depths. Note that by construction $t_1(x)=0$ for all $x\in I\setminus(\mathcal{C}\cup I(0,0,0))$ and $t_1(x)=1$ for any $x\in I(0,0,0)$, see Remark 3.1.

Lemma 5.3 (No return probability)**.** *For every $n \geq 0$ there exists* no *non-degenerate interval $\omega \in \mathcal{P}_n$ such that $\omega \in \mathcal{P}_{n+k}$ for all $k \geq 1$. Moreover there exist constants $0 < \xi_0 < 1$ and $K_0 > 0$ (depending only on σ, σ_0 and on ζ from Lemma 4.4), and $n_0 \geq 1$ such that for every $n > n_0$*

$$\lambda\big(\bigcup\{\omega \in \mathcal{P}_n : u_n \mid \omega = 1\}\big) \leq K_0 \cdot e^{-\xi_0 n}.$$

Proof. If $\omega \in \mathcal{P}_{n+k}$ for all $k \geq 0$, then ω is not refined in all future iterates. This means that $f_\mu^{n+k}(\omega)$ has no essential returns for $k \geq 1$. Hence every iterate is either free or a binding time associated to a inessential return. Let $p_0, p_1, p_2, \dots$ be the length of the binding period associated to every (if any) inessential return time $t = r_0 < r_1 < r_2 < \dots$ for ω after t, where $0 \leq t \leq n$ is the last essential return time before n. Let $k \geq 0$ and $r_i + p_i \leq n + k < r_{i+1}$ for some $i \geq 0$, where we set $r_{i+1} = +\infty$ if r_i is the last inessential return time after t (it may happen that $r_1 = +\infty$ in which case $i = 0$ and there are no inessential returns after t). Lemma 4.5 ensures that

$$2 \geq |f_\mu^{n+k}(\omega)| \geq 2^i \cdot \sigma_0^{n+k-t-\sum_{k=0}^{i} p_i} |f_\mu^t(\omega)|, \tag{5.1}$$

for arbitrarily big values of $k \geq 0$. Note that if there are no inessential returns, then $i = 0$ and so

$$|f_\mu^t(\omega)| \leq 2 \cdot \sigma_0^{t+p_0-n-k}. \tag{5.2}$$

Otherwise there is at least a free iterate at the end of each binding period, hence

$$|f_\mu^t(\omega)| \leq 2^{1-i} \cdot \sigma_0^{-(n+k+i)}.$$

We conclude that $|f_\mu^t(\omega)| = 0$ which is not possible for a non-degenerate interval. This proves the first part of of the Lemma.

Now let $\omega \in \mathcal{P}_n$ be such that the restriction $u_n \mid \omega$ of u_n to ω is constant and equal to 1. Then either $\omega = I(0,0,0)$ with $t_1 = 1$ the unique essential return up to iterate n and $f_\mu(\omega) = I(l,s,j)$ with $(l,s,j) \neq (0,0,0)$; or $\omega = I(l,s,j)$ with $|l| \geq k_0, |s| \geq 1$ and $j = 1, \dots, (|l| + |s|)^3$, having a single essential return $t_1 = 0$ up to iterate n. We concentrate on the latter case and write $p_0, p_1, p_2, \dots$ and $0 = t_1 = r_0 < r_1 < r_2 < \dots$ the binding periods associated to their respective inessential return times of the orbit of ω as before. Then by (5.1) and (5.2) either $n \leq p_0$ or

$$|\omega| \leq 2 \cdot \sigma_0^{p_0 - n} \quad \text{if } p_0(\omega) < n \leq r_1(\omega) \leq +\infty, \quad \text{or}$$
$$|\omega| \leq 2^{1-i} \cdot \sigma_0^{-(n - \sum_{k=0}^{i-1} p_k(\omega))} \quad \text{if there is } i \geq 1 \text{ such that } r_i(\omega) \leq n < r_{i+1}(\omega).$$

We note that in the case of $f_\mu(\omega) = I(l,s,j)$, that is, when $t_1(\omega) = 1$, we can repeat the arguments for $\omega_1 = f_\mu(\omega)$, arriving at the same bounds for $|\omega|$ except for an extra factor of $\tilde\sigma$ since $|f_\mu(\omega)| \geq \tilde\sigma|\omega|$. Hence we may write according to three cases above

$$\begin{aligned}
\lambda\big(\cup\{\omega \in \mathcal{P}_n : u_n \mid \omega = 1\}\big) \quad &\leq \sum_{n \leq p_0(\omega)} |\omega| + \sum_{\substack{|\omega| \leq 2\sigma_0^{p_0(\omega)-n} \\ p_0(\omega) \leq n \leq r_1(\omega) \leq +\infty}} |\omega| + \sum_{\substack{|\omega| \leq 2^{1-i} \cdot \sigma_0^{-n+\sum_{k=0}^{i-1} p_k(\omega)} \\ r_i(\omega) \leq n < r_{i+1}(\omega)}} |\omega| + S_3 \\
&= S_0 + S_1 + S_2 + S_3,
\end{aligned}$$

where, by the above comment, we may assume that every sum ranges over $\omega \in \mathcal{P}_0 \setminus I(0,0,0)$ and S_3 corresponds to the sum over the partition elements in $I(0,0,0) \cap \mathcal{P}_1$, which is bounded by $(S_0 + S_1 + S_2)/\tilde\sigma$.

For S_0 we use Lemma 4.4(a) to deduce that the summands in S_0 are the elements of $\mathcal{P}_0$ such that

$$n \leq p_0(\omega) \leq \frac{2\pi}{\beta \log \sigma}(|l|+|s|), \quad \text{i.e.} \quad |l|+|s| \geq \frac{\beta \log \sigma}{2\pi} \cdot n,$$

thus by Remark 2.1, setting $C_0 = \frac{\beta \log \sigma}{2\pi}$

$$\begin{aligned} S_0 &\leq \sum_{|l|+|s| \geq C_0 \cdot n} a_1 \frac{e^{-(\pi/\beta)(|l|+|s|)}}{(|l|+|s|)^3} \leq a_1 \sum_{k \geq C_0 \cdot n} k \cdot \frac{e^{-(\pi/\beta)k}}{k^3} \\ &\leq \frac{a_1}{(C_0 \cdot n)^2} \cdot \frac{e^{-(\pi/\beta)C_0 \cdot n}}{(1-e^{-C_0 \cdot n})} \leq K_0' \cdot \sigma^{-n/2}. \end{aligned} \tag{5.3}$$

We write $S_1 = S_{11} + S_{12}$ where

$$S_{11} = \sum_{\substack{p_0(\omega) \leq n/2 \\ |\omega| \leq 2\sigma_0^{-n/2}}} |\omega| \quad \text{and} \quad S_{12} = \sum_{n/2 < p_0(\omega)} |\omega| \leq K_0' \cdot \sigma^{-n/4}$$

and we have used the bound (5.3). We also split $S_2 = S_{21} + S_{22}$ according to whether $n - \sum_{k=0}^{i-1} p_k(\omega) \geq n/2$ or not, obtaining

$$S_{21} = \sum_{|\omega| \leq 2^{1-i} \cdot \sigma_0^{-n/2}} |\omega| \quad \text{and} \quad S_{22} = \sum_{\substack{|\omega| \leq 2^{1-i} \cdot \sigma_0^{-n+\sum_{k=0}^{i-1} p_k(\omega)} \\ n - \sum_{k=0}^{i-1} p_k(\omega) < n/2}} |\omega|.$$

Since $2^{1-i} \leq 2$ we get $S_{11} + S_{21} \leq 2 \cdot S_{11}$ and the summands $\omega \in \mathcal{P}_0$ satisfy $|\omega| \leq 2 \cdot \sigma_0^{-n/2}$, thus by Remark 2.1 we get

$$|l|+|s| \geq \frac{n}{2} \cdot \frac{\log \sigma_0}{3+\pi/\beta} + \frac{3 - \log(2/a_1)}{3+\pi/\beta} \geq C_1 \cdot \frac{n}{4},$$

where $C_1 = \log \sigma_0 / (3+\pi/\beta)$ for every big enough n. Then $S_{11} + S_{21} \leq 2K_0'' \cdot \sigma_0^{-n/5}$ by the same calculations as in (5.3) with slightly different constants.

For S_{22} we note that $n - \sum_{k=0}^{i-1} p_k(\omega) < n/2$ implies $\sum_{k=0}^{i-1} p_k(\omega) > n/2$ and so, by Lemma 4.4(b) we get

$$2 \geq |f^{r_i}(\omega)| > e^{(1-2\zeta)\frac{\pi}{\beta}\sum_{k=0}^{i-1} p_k(\omega)} \cdot |\omega| > e^{n\pi(1-2\zeta)/(2\beta)} \cdot |\omega|$$

and hence again by Remark 2.1, for every big enough n, we must have $|l|+|s| > C_2 \cdot n/4$ where $C_2 = \pi(1-2\zeta)/(3\beta+\pi)$. We deduce that $S_{22} \leq K_0''' \cdot e^{-(\pi/\beta)C_2 \cdot n/4}$ following the same calculations in (5.3).

Putting all together we see that there are constants $0 < \xi_0 < 1$, $K_0 > 0$ and $n_0 \geq 1$ such that $S_0 + S_1 + S_2 + S_3 \leq K_0 \cdot e^{-\xi_0 n}$ for all $n \geq n_0$, with ξ_0 and K_0 dependent on σ, σ_0 and ζ, as stated. □

Given an integer $u \leq n$ and u pairs of positive integers $(\eta_1, \upsilon_1), \ldots, (\eta_u, \upsilon_u)$, where each υ_i is larger than $s(\tau)$ (recall the definition of $s(\tau)$ in (3.2)), we define the event:

$$A^u_{(\eta_1,\upsilon_1),\ldots,(\eta_u,\upsilon_u)}(n) = \{x \in I \,:\, u_n(x) = u \text{ and } \left|f_\mu^{t_i}(x)\right| \in I(\eta_i, \upsilon_i), \quad i \in \{1,\ldots,u\}\}.$$

Proposition 5.4 (Probability of essential returns with specified depths)**.** *If k_0 is large enough (depending on ζ from Lemma 4.4 and on D_0 from Lemma 4.6), then for every big $n \ge u \ge 1$*

$$\lambda\left(A^u_{(\eta_1,\upsilon_1),\dots,(\eta_u,\upsilon_u)}(n)\right) \le \exp\left[\left(5\zeta - \frac{\pi}{\beta}\right)\sum_{i=1}^{u}(\eta_i+\upsilon_i)\right].$$

Proof. We start by fixing $n\in\mathbf{N}$, $u\in\{1,\dots,n\}$ and taking $\omega_0\in\mathcal{P}_0$. For $m=1,\dots,u$ we write $\omega_m=\omega((l_1,s_1,j_1),\dots,(l_m,s_m,j_m))\in\mathcal{P}_{t_m}$ the subset of ω_0 satisfying

$$f_\mu^{t_i}(\omega_m)\subset I(l_i,s_i,j_i)^+,\ i\in\{1,\dots,m-1\}\ \text{ and }\ I(l_m,s_m,j_m)\subset f_\mu^{t_m}(\omega_m)\subset I(l_m,s_m,j_m)^+.$$

Since $\omega_{i-1}\in\mathcal{P}_{i-1+h}$ for all $h=0,\dots,t_i-t_{i-1}-1$ by construction of the sequence of partitions, we can estimate through Lemma 4.6

$$\begin{aligned}
\frac{|\omega_u|}{|\omega_0|} &= \prod_{i=1}^{u}\frac{|\omega_i|}{|\omega_{i-1}|}\le\prod_{i=1}^{u}D_0\cdot\frac{\left|f_\mu^{t_i}(\omega_i)\right|}{\left|f_\mu^{t_i}(\omega_{i-1})\right|}\\
&\le \prod_{i=1}^{u}\left[D_0\cdot 9\cdot a_1\cdot\frac{e^{-(\pi/\beta)(|l_i|+|s_i|)}}{(|l_i|+|s_i|)^3}\cdot\left(a_1\cdot e^{-3\zeta(\pi/\beta)(|l_{i-1}|+|s_{i-1}|)}\right)^{-1}\right]\\
&= \left(\prod_{i=1}^{u}\frac{9D_0}{(|l_i|+|s_i|)^3}\right)\cdot\exp\left[-\frac{\pi}{\beta}\sum_{i=1}^{u}(|l_i|+|s_i|)+3\zeta\sum_{i=1}^{u-1}(|l_i|+|s_i|)+3\zeta(|l_0|+|s_0|)\right]\\
&\le \left(\prod_{i=1}^{u}\frac{9D_0}{(|l_i|+|s_i|)^3}\right)\cdot e^{3\zeta(|l_0|+|s_0|)}\cdot e^{(3\zeta-\frac{\pi}{\beta})\sum_{i=1}^{u-1}(|l_i|+|s_i|)}.
\end{aligned}$$

Observing that the number of distinct elements of $\mathcal{P}_{t_m}$ having the prescribed itinerary at essential returns

$$\#\left\{\omega=\omega((l_1,s_1,j_1),\dots,(l_m,s_m,j_m))\in\mathcal{P}_{t_m}:(|l_i|,|s_i|)=(\eta_i,\upsilon_i),i=1,\dots,u\right\}$$

is bounded by $4^u\cdot(\eta_1+\upsilon_1)^3\dots(\eta_u+\upsilon_u)^3$, we bound $\lambda\left(A^u_{(\eta_1,\upsilon_1),\dots,(\eta_u,\upsilon_u)}(n)\right)$ by

$$\begin{aligned}
&\left(\prod_{i=1}^{u}4(\eta_i+\upsilon_i)^3\right)\cdot\left(\prod_{i=1}^{u}\frac{9D_0}{(\eta_i+\upsilon_i)^3}\right)\cdot e^{(3\zeta-\frac{\pi}{\beta})\sum_{i=1}^{u-1}(\eta_i+\upsilon_i)}\cdot\sum_{\omega_0\in\mathcal{P}_0}e^{3\zeta(|l_0|+|s_0|)}\cdot|\omega_0|\\
&\le (36D_0)^u\cdot e^{(3\zeta-\frac{\pi}{\beta})\sum_{i=1}^{u}(\eta_i+\upsilon_i)}\cdot\left(2(1-\varepsilon)+\sum_{|l|\ge k_0,|s|\ge 1}e^{3\zeta(|l|+|s|)}\cdot a_1\cdot e^{-\frac{\pi}{\beta}(|l|+|s|)}\right).
\end{aligned}$$

Since the double sum in l,s of the last expression is bounded by some constant, we can further bound by

$$\exp\left((3\zeta-\frac{\pi}{\beta})\sum_{i=1}^{u}(\eta_i+\upsilon_i)+K\right)=\exp\left((3\zeta-\frac{\pi}{\beta}+\frac{K}{\sum_{i=1}^{u}(\eta_i+\upsilon_i)})\sum_{i=1}^{u}(\eta_i+\upsilon_i)\right),$$

where $K=K(\zeta,D_0)>0$. Finally taking k_0 sufficiently big, since $\eta_i\ge k_0$ for $i=1,\dots,u$, we can bound the last expression by $\exp[(5\zeta-\frac{\pi}{\beta})\sum_{i=1}^{u}(\eta_i+\upsilon_i)]$ as stated. ∎

Now we set using the same notations as before

$$\begin{aligned}
A^{j,u}_{(\eta,\upsilon)}(n) &= \{x\in I : u_n(x)=u \quad \text{and} \quad |f_\mu^{t_j}(x)|\in I(\eta,\upsilon)\},\\
A^{u}_{(\eta,\upsilon)}(n) &= \{x\in I : u_n(x)=u \quad \text{and} \quad \exists j\in\{1,\dots,u\} \quad \text{such that} \quad |f_\mu^{t_j}(x)|\in I(\eta,\upsilon)\},\\
A_{(\eta,\upsilon)}(n) &= \{x\in I : \exists t\le n \quad \text{such that } t \text{ is an essential return and} \quad |f_\mu^{t}(x)|\in I(\eta,\upsilon)\},
\end{aligned}$$

and derive the following corollary which will be used during the final arguments.

Corollary 5.5. *We have for $1\le j\le u\le n$ that*

(1) $\lambda\big(A^{j,u}_{(\eta,\upsilon)}(n)\big)\le e^{(5\zeta-\pi/\beta)(\eta+\upsilon)}$ *for all $\eta>k_0$ and $\upsilon>s(\tau)$;*
(2) $\lambda\big(A^{u}_{(\eta,\upsilon)}(n)\big)\le u\cdot e^{(5\zeta-\pi/\beta)(\eta+\upsilon)}$ *, and*
(3) $\lambda\big(A_{(\eta,\upsilon)}(n)\big)\le \frac{n(n+1)}{2}\cdot e^{(5\zeta-\pi/\beta)(\eta+\upsilon)}$.

Proof. We note that since

$$A^{j,u}_{(\eta,\upsilon)}(n)\subset \bigcup_{\eta_i\ge k_0,\upsilon_i\ge s(\tau),i\ne j} A^{u}_{(\eta_1,\upsilon_1),\dots,(\eta_{i-1},\upsilon_{i-1}),(\eta,\upsilon),(\eta_{i+1},\upsilon_{i+1}),\dots,(\eta_u,\upsilon_u)}(n)$$

then $\lambda\big(A^{j,u}_{(\eta,\upsilon)}(n)\big)\le\Big(\sum_{l\ge k_0,s\ge s(\tau)}e^{(5\zeta-\pi/\beta)(l+s)}\Big)^{u-1}\cdot e^{(5\zeta-\pi/\beta)(\eta+\upsilon)}\le e^{(5\zeta-\pi/\beta)(\eta+\upsilon)}$, as long as k_0 is big enough in order to have $\sum\limits_{l\ge k_0,\,s\ge s(\tau)} e^{(5\zeta-\pi/\beta)(l+s)}\le 1$. From this we get items (2) and (3) since

$$A^{u}_{(\eta,\upsilon)}(n)\subset\bigcup_{j=1}^{u}A^{j,u}_{(\eta,\upsilon)}(n) \quad \text{and} \quad A_{(\eta,\upsilon)}(n)\subset\bigcup_{u=1}^{n}A^{u}_{(\eta,\upsilon)}(n).$$

□

5.3. **Time between consecutive essential returns.** The next lemma gives an upper bound for the time we have to wait between two essential return situations.

Lemma 5.6. *Let us take t_i an essential return for $\omega\in\mathcal{P}_n$, with $I(l_i,s_i,j_i)\subset f_\mu^{t_i}(\omega)\subset I(l_i,s_i,j_i)^+$. Then the next essential return situation $t_{i+1}<n$ satisfies*

$$t_{i+1}-t_i<\frac{10\pi}{\beta\log\sigma}(|l_i|+|s_i|)$$

as long as $k_0+s(\tau)$ is big enough (depending only on ζ from Lemma 4.4).

Proof. Let $t_i(1)<\dots<t_i(v)$ denote the inessential returns between t_i and t_{i+1}, with host intervals $I(l_i(1),s_i(1),j_i(1)),\dots,I(l_i(v),s_i(v),j_i(v))$, respectively. We also consider $t_i(0)=t_i$; $t_i(v+1)=t_{i+1}$; $\omega_k=f_\mu^{t_i(k)}(\omega)$ and $q_k=t_i(k+1)-(t_i(k)+p_k)$ for $k=0,\dots,v$, where p_k is the binding period associated to each return $t_i(k)$. We split the argument in two cases.

Case A: $v=0$, that is, there is no inessential return in between t_i and t_{i+1}.

In this situation $t_{i+1}-t_i=p_0+q_0$. Applying lemma 4.5(2) we get that

$$2\ge|\omega_1|\ge a_1\sigma_0^{q_0}e^{-3\zeta\frac{\pi}{\beta}(|l_i|+|s_i|)},\quad \text{so}\quad q_0\le\frac{\log(2/a_1)}{\log\sigma_0}+\frac{3\zeta\pi}{\beta\log\sigma_0}(|l_i|+|s_i|).$$

Taking $|l_i|\ge k_0$ sufficiently large we can write $q_0\le\frac{5\zeta\pi}{\beta\log\sigma_0}(|l_i|+|s_i|)$ and therefore using Lemma 4.4(a) we arrive at

$$t_{i+1}-t_i=p_0+q_0\le\frac{2\pi}{\beta\log\sigma}(|l_i|+|s_i|)+\frac{5\zeta\pi}{\beta\log\sigma_0}(|l_i|+|s_i|)\le\frac{(2+5\zeta)\pi}{\beta\log\sigma}(|l_i|+|s_i|),$$

since $\sigma_0>\sigma$, recall Remark 3.3.

Case B: $v>0$, i.e., there are v inessential returns between t_i and t_{i+1}.

In this case $t_{i+1}-t_i=\sum_{k=0}^{v}(p_k+q_k)=p_0+\left(\sum_{k=1}^{v-1}p_k+\sum_{k=0}^{v-1}q_k\right)+(p_v+q_v)$, and we control each of the three parts above separately.

(i) For p_0 we have by Lemma 4.4(a) that $p_0\le\frac{2\pi}{\beta\log\sigma}(|l_i|+|s_i|)$.

(ii) For $\sum_{k=1}^{v-1}p_k+\sum_{k=0}^{v-1}q_k$ we proceed as follows. By lemma 4.5 we get for $k=1,\dots,v-1$

$$|\omega_1|\ge a_1\cdot\sigma_0^{q_0}\cdot e^{-3\zeta\frac{\pi}{\beta}(|l_i|+|s_i|)}\quad\text{and}\quad\frac{|\omega_{k+1}|}{|\omega_k|}\ge\sigma_0^{q_k}\cdot e^{(1-2\zeta)\frac{\pi}{\beta}(|l_i(k)|+|s_i(k)|)}.$$

Since $p_k\le\frac{2\pi}{\beta\log\sigma}(|l_i(k)|+|s_i(k)|)$ writing $|\omega_v|=|\omega_1|\prod_{k=1}^{v-1}\frac{|\omega_{k+1}|}{|\omega_k|}$, and taking into account that $\omega_v\subset I(l_i(v),s_i(v),j_i(v))$, with $|l_i(v)|+|s_i(v)|\ge k_0+s(\tau)$, then $|\omega_v|\le a_1\cdot e^{-(\pi/\beta)(k_0+s(\tau))}$ and hence

$$a_1\cdot e^{-(\pi/\beta)(k_0+s(\tau))}\ge a_1\cdot\exp\Big(\log\sigma_0\sum_{k=0}^{v-1}q_k-3\zeta\frac{\pi}{\beta}(|l_i|+|s_i|)+\sum_{k=1}^{v-1}(1-2\zeta)\frac{\pi}{\beta}(|l_i(k)|+|s_i(k)|)\Big).$$

Consequently

$$3\zeta\frac{\pi}{\beta}(|l_i|+|s_i|)-\frac{\pi}{\beta}(k_0+s(\tau))\ge\log\sigma_0\sum_{k=0}^{v-1}q_k+\sum_{k=1}^{v-1}(1-2\zeta)\frac{\pi}{\beta}(|l_i(k)|+|s_i(k)|),$$

thus

$$\begin{aligned}\sum_{k=1}^{v-1}p_k+\sum_{k=0}^{v-1}q_k &\le \frac{1}{\log\sigma_0}\Big(\log\sigma_0\sum_{k=0}^{v-1}q_k\Big)+\frac{2}{(1-2\zeta)\log\sigma_0}\Big(\sum_{k=1}^{v-1}(1-2\zeta)\frac{\pi}{\beta}(|l_i(k)|+|s_i(k)|)\Big)\\ &\le \frac{2}{(1-2\zeta)\log\sigma_0}\Big(\log\sigma_0\sum_{k=0}^{v-1}q_k+\sum_{k=1}^{v-1}(1-2\zeta)\frac{\pi}{\beta}(|l_i(k)|+|s_i(k)|)\Big)\\ &\le \frac{2}{(1-2\zeta)\log\sigma_0}\cdot\Big(3\zeta\frac{\pi}{\beta}(|l_i|+|s_i|)-\frac{\pi}{\beta}(k_0+s(\tau))\Big).\end{aligned}$$

(iii) For the last term p_v+q_v we do as follows. By lemma 4.5(1) we have

$$\frac{|\omega_{v+1}|}{|\omega_v|}\ge\sigma_0^{q_v}\cdot e^{(1-2\zeta)\frac{\pi}{\beta}(|l_i(v)|+|s_i(v)|)},\quad\text{and also}\quad|\omega_v|\ge 2^{v-1}|\omega_1|\ge|\omega_1|.$$

Hence $2 \geq |\omega_{v+1}| \geq |\omega_1| \cdot \frac{|\omega_{v+1}|}{|\omega_v|}$ and so

$$2a_1^{-1}e^{3\zeta\frac{\pi}{\beta}(|l_i|+|s_i|)} \geq 2a_1^{-1}\sigma_0^{-q_0}e^{3\zeta\frac{\pi}{\beta}(|l_i|+|s_i|)} \geq \frac{2}{|\omega_1|} \geq \frac{|\omega_{v+1}|}{|\omega_v|} \geq \sigma_0^{q_v}e^{(1-2\zeta)\frac{\pi}{\beta}(|l_i(v)|+|s_i(v)|)}$$

implying

$$\begin{aligned} p_v+q_v &\leq \frac{2}{(1-2\zeta)\log\sigma_0}\Big((1-2\zeta)\frac{\pi}{\beta}(|l_i(v)|+|s_i(v)|)+q_v\log\sigma_0\Big) \\ &\leq \frac{2}{(1-2\zeta)\log\sigma_0}\Big(\log\frac{2}{a_1}+3\zeta\frac{\pi}{\beta}(|l_i|+|s_i|)\Big). \end{aligned}$$

Putting all together we get

$$\begin{aligned} t_{i+1}-t_i &= p_0+\left(\sum_{k=1}^{v-1}p_k+\sum_{k=0}^{v-1}q_k\right)+(p_v+q_v) \\ &\leq \frac{2\pi}{\beta\log\sigma}(|l_i|+|s_i|)+\frac{2}{(1-2\zeta)\log\sigma_0}\Big(6\zeta\frac{\pi}{\beta}(|l_i|+|s_i|)+\log\frac{2}{a_1}-\frac{\pi}{\beta}(k_0+s(\tau))\Big) \\ &\leq \frac{2\pi}{\beta\log\sigma}(|l_i|+|s_i|)\cdot\Big(1+\frac{6\zeta}{1-2\zeta}+\frac{(\beta/\pi)\log(2/a_1)}{(1-2\zeta)(|l_i|+|s_i|)}-\frac{k_0+s(\tau)}{|l_i|+|s_i|}\Big) \\ &\leq \frac{2\pi}{\beta\log\sigma}(|l_i|+|s_i|)\cdot\Big(\frac{1+4\zeta}{1-2\zeta}+\frac{(\beta/\pi)\log(2/a_1)}{k_0+s(\tau)}\Big) \leq \frac{10\pi}{\beta\log\sigma}(|l_i|+|s_i|), \end{aligned}$$

as long as $k_0+s(\tau)$ is sufficiently big. $\square$

6. Slow recurrence to the critical set

Now we make use of the lemmas from Section 5 to prove Theorem B and consequently also Theorem A. We start by recalling the definition of $\mathcal{C}_n(x)$ from (1.6) and that $u_n(x)$ is the number of essential returns of the f_μ-orbit of x between 0 and n. We let $0 \leq t_1 < \ldots < t_{u_n} \leq n$ be the essential returns times of the orbit of x and write $(l_1,s_1,j_1),\ldots,(l_{u_n},s_{u_n},j_{u_n})$ for the corresponding critical points and depths at each essential return, as in Section 5. Then we define

$$\mathcal{D}_n(x) = \sum_{k=1}^{u_n(x)}(|l_k|+|s_k|)^2$$

which is constant on the elements of $\mathcal{P}_n$ and get the following bound.

Proposition 6.1. *For every $\omega \in \mathcal{P}_n$ such that $u_n \mid \omega \geq 2$ we have $\mathcal{C}_n(x) \leq \frac{B_0}{n}\cdot\mathcal{D}_n(x)$ for all $x \in \omega$, where $B_0 = B_0(\sigma,\rho,\tau) > 0$.*

Proof. Let us fix $x \in \omega \in \mathcal{P}_n$ with $u_n \mid \omega \geq 2$ and $i \in \{1,\ldots,u_n(x)-1\}$. According to Remark 2.1 and the definition of essential return we have

$$\operatorname{dist}(f_\mu^{t_i}(x),\mathcal{C}) \geq a_2\cdot e^{-\frac{\pi}{\beta}(|l_i|+|s_i|)}. \tag{6.1}$$

On the one hand, letting $p_i = p(l_i, s_i)$ denote the binding period length of the essential return t_i, Proposition 3.4, Lemma 4.4(a) and (6.1) ensure that

$$\begin{aligned} \mathcal{C}(t_i, t_i + p_i) &= \sum_{k=t_i}^{t_i+p_i} -\log \operatorname{dist}\left(f_\mu^k(x), \mathcal{C}\right) \\ &\le -\log a_2 - p_i \log \rho_0 + \frac{\pi}{\beta}(|l_i| + |s_i|) + \rho \frac{p_i(p_i+1)}{2} \qquad (6.2) \\ &\le \left[\frac{\pi}{\beta}\big(1 + \frac{2\rho - 2\log\rho_0}{\log\sigma}\big) - \frac{\log a_2}{(p_i+1)(|l_i|+|s_i|)}\right] \cdot (|l_i| + |s_i|) \cdot (p_i + 1). \end{aligned}$$

On the other hand, we clearly get the same bound (6.1) for any inessential return $t_i + p_i < t_i(1) < \cdots < t_i(v) < t_{i+1}$ by Lemma 5.1. Hence denoting p_k the binding period associated to the inessential return $f_\mu^{t_i(k)}(x)$ for $k = 1, \ldots, v$ we also get

$$\mathcal{C}(t_i(k), t_i(k) + p_k) \le \left[\frac{\pi}{\beta}\big(1 + \frac{2\rho - 2\log\rho_0}{\log\sigma}\big) - \frac{\log a_2}{(p_k+1)(|l_i|+|s_i|)}\right] \cdot (|l_i| + |s_i|) \cdot (p_k + 1).$$

Moreover for the free times we have the following lemma.

Lemma 6.2. *Given $\omega \in \mathcal{P}_n$, let k be a free time for ω between two consecutive essential returns $t_i + p_i < k < t_{i+1}$. Then*

$$-\log \operatorname{dist}(f_\mu^k(\omega), \mathcal{C}) \le \left(3 + \frac{\pi}{\beta} - \frac{3 + \log(\tau a_2) + \log(\varepsilon - x_{k_0})}{|l_i| + |s_i|}\right) \cdot (|l_i| + |s_i|).$$

Proof of Lemma 6.2. Since by Lemma 4.4(b) expansion is recovered during binding periods, we have that

$$|f_\mu^k(\omega)| > |f_\mu^{t_i}(\omega)| \ge a_1 \frac{e^{-\frac{\pi}{\beta}(|l_i|+|s_i|)}}{(|l_i|+|s_i|)^3}.$$

But either $f_\mu^k(\omega)$ is in $S^1 \setminus (-\varepsilon, \varepsilon)$ or it is between two consecutive critical points x_l and x_{l-1}. In the former case we get $-\log \operatorname{dist}(f_\mu^k(\omega), \mathcal{C}) \le -\log(\varepsilon - x_{k_0})$. In the latter case we have $|f_\mu^k(\omega)| < |x_{l-1} - x_l| = (e^{\pi/\beta} - 1) \cdot |x_l|$, and because k is a free time and by definition of τ in (3.2) we get

$$\operatorname{dist}(f_\mu^k(\omega), \mathcal{C}) \ge \tau \cdot |x_l| > \frac{\tau \cdot |f_\mu^k(\omega)|}{e^{\pi/\beta} - 1} \ge \frac{\tau a_1}{e^{\pi/\beta} - 1} \cdot \frac{e^{-\frac{\pi}{\beta}(|l_i|+|s_i|)}}{(|l_i|+|s_i|)^3} = \tau a_2 \frac{e^{-\frac{\pi}{\beta}(|l_i|+|s_i|)}}{(|l_i|+|s_i|)^3}.$$

Collecting the two cases we arrive at

$$-\log \operatorname{dist}(f_\mu^k(\omega), \mathcal{C}) \le \left(3 + \frac{\pi}{\beta} - \frac{3 + \log(\tau a_2) + \log(\varepsilon - x_{k_0})}{|l_i| + |s_i|}\right) \cdot (|l_i| + |s_i|).$$

$\square$

Putting all together provides the bound

$$\sum_{k=t_i}^{t_{i+1}-1} -\log \operatorname{dist}\left(f_\mu^k(x),\mathcal{C}\right) \leq C(t_i,t_i+p_0)+\sum_{k=1}^{v} C(t_i(k),t_i(k)+p_k)+$$
$$\left(3+\frac{\pi}{\beta}-\frac{3+\log(\tau a_2(\varepsilon-x_{k_0}))}{|l_i|+|s_i|}\right)\cdot(|l_i|+|s_i|)\cdot(t_{i+1}-t_i-p_0-p_1-\cdots-p_v-v)$$
$$\leq B\cdot(|l_i|+|s_i|)\cdot(t_{i+1}-t_i)\leq B_0\cdot(|l_i|+|s_i|)^2$$

where we used Lemma 5.6 in the last inequality. The constants are as follows

$$B=3+\frac{\pi}{\beta}\big(2+\frac{2\rho-2\log\rho_0}{\log\sigma}\big)-\frac{1}{|l_i|+|s_i|}\left(\log a_2\sum_{k=0}^{v+1}\frac{1}{p_k+1}+3+\log(\tau a_2(\varepsilon-x_{k_0}))\right),$$

where we write $p_{v+1}=1$. Now since $\sum_{k=0}^{v+1}\frac{1}{p_k+1}<\sum_{k=0}^{v}p_k<t_{i+1}-t_i$, we use Lemma 5.6 to get

$$B\leq 4+\frac{\pi}{\beta}\big(2+\frac{2\rho-2\log\rho_0}{\log\sigma}\big)-\frac{10\pi\log a_2}{\beta\log\sigma}-\frac{\log(\tau a_2(\varepsilon-x_{k_0}))}{|l_i|+|s_i|}.$$

But by definition of $s(\tau)$ from (3.2)

$$-\frac{\log(\tau a_2(\varepsilon-x_{k_0}))}{|l_i|+|s_i|}\leq\frac{-\log(\tau a_2(\varepsilon-x_{k_0}))}{k_0+s(\tau)}\leq\frac{-\log(\tau a_2(\varepsilon-x_{k_0}))}{k_0-\frac{\beta}{\pi}\log(\tau\frac{\hat{x}}{a_2})}\leq\frac{10\pi}{\beta}$$

as long as τ is small enough, so B depend only on σ,ρ and τ. Moreover $B_0=\frac{10\pi}{\beta\log\sigma}\cdot B$.

Summing for all $i=1,\dots,u_n(x)-1$ and noting that during the times $t_{u_n(x)}<k<n$ the same bounds given by (6.1) and (6.2) still hold, whether n is a binding time for $t_{u_n(x)}$ or not, we get the statement of the result since $t_1(x)=0$ by Remark 3.1.

This finishes the proof of Proposition 6.1. □

6.1. **The expected value of the distance at return times.** The statement of Proposition 6.1 together with Lemma 5.3 and Proposition 5.4 ensure that, to obtain slow recurrence to the critical set, we need to bound $n^{-1}\mathcal{D}_n(x)$ for Lebesgue almost every $x\in I$. Indeed we have for every big enough n

$$\left\{x\in I:\mathcal{C}_n(x)>\delta\right\}\subset\bigcup\left\{\omega\in\mathcal{P}_n:u_n\mid\omega\equiv 1\right\}\cup\left\{x\in I:u_n(x)\geq 2\text{ and }\mathcal{D}_n(x)>\frac{n}{B_0}\cdot\delta\right\}$$

and Lemma 5.3 shows that the left hand subset of the above union has exponentially small measure. We now show that Proposition 5.4 does the same for the right hand subset.

Lemma 6.3. *For every $z\in\left(0,\frac{\pi}{2\beta}-5\frac{\zeta}{2}\right)$ there exists $k_1=k_1(z)$ such that if $k_0>k_1$ then $\int e^{z\cdot\sqrt{\mathcal{D}_n(x)}}\,d\lambda(x)\leq 1$.*

Proof. The integral in the statement equals the following series

$$\sum_{\substack{u\geq 1\\(\eta_1,\upsilon_1),\dots,(\eta_u,\upsilon_u)}}\exp\left(z\cdot\left(\sum_{k=1}^{u}(\eta_k+\upsilon_k)^2\right)^{1/2}\right)\cdot\lambda\left(A^u_{(\eta_1,\upsilon_1),\dots,(\eta_u,\upsilon_u)}(n)\right),$$

where $\upsilon_1 \geq 1$, $\upsilon_k \geq k_0$ and $\eta_k \geq k_0$ for $k = 1,\dots,u$, $u \geq 1$. Proposition 5.4 provides the bound

$$\sum_{\substack{u\geq 1\\(\eta_1,\upsilon_1),\dots,(\eta_u,\upsilon_u)}} e^{z\sum_{k=1}^{u}(\eta_k+\upsilon_k)} \cdot e^{(5\zeta-\pi/\beta)\sum_{k=1}^{u}(\eta_k+\upsilon_k)} = \sum_{\substack{u\geq 1\\(\eta_1,\upsilon_1),\dots,(\eta_u,\upsilon_u)}} \exp\Big(\big(z+5\zeta-\frac{\pi}{\beta}\big)\sum_{k=1}^{u}(\eta_k+\upsilon_k)\Big).$$

Now setting $D = \sum_{k=1}^{u}(\eta_k+\upsilon_k)$ and

$$S(u,D) = \#\Big\{((l_1,s_1),\dots,(l_u,s_u)) : \sum_{k=1}^{u}(l_k+s_k) = D \text{ and } l_k \geq k_0,\, s_k \geq 1, k = 1,\dots,u\Big\}$$

we may rewrite the last sum as $\sum_{u\geq 1}\sum_{D\geq k_0\cdot u} S(u,D)\cdot e^{(z+5\zeta-\pi/\beta)D}$. To estimate $S(u,D)$ we observe that

$$S(u,D) \leq \#\Big\{(n_1,\dots,n_{2u}) : \sum_{k=1}^{2u} n_k = D \text{ and } n_k \geq 0, k = 1,\dots,2u\Big\} = \binom{D+2u-1}{2u-1},$$

where $\binom{n}{k} = \frac{n!}{k!\cdot(n-k)!}$ is a binomial coefficient. By a standard application of Stirling's Formula we get

$$S(u,D) \leq \Big(K^{1/D}\cdot\big(1+\frac{2u-1}{D}\big)\cdot\big(1+\frac{D}{2u-1}\big)^{(2u-1)/D}\Big)^{D} \leq e^{z\cdot D},$$

since $D \geq k_0\cdot u \geq 2k_0$ ensures that the expression in parenthesis can be made arbitrarily close to 1 if k_0 is taken bigger than some constant $k_1 = k_1(z)$, where $0 < K < 1$ is a constant independent of k_0 and we assume that $z > 0$ is small. Hence we arrive at

$$\int e^{z\cdot\sqrt{\mathcal{D}_n(x)}}\,d\lambda(x) \leq \sum_{u\geq 1}\sum_{D\geq k_0\cdot u} e^{(2z+5\zeta-\pi/\beta)D} \leq \sum_{D\geq k_0}\frac{D}{k_0}\cdot e^{(2z+5\zeta-\pi/\beta)D} \leq 1$$

if $0 < z < (\pi/\beta-5\zeta)/2$ and $k_0 \geq k_1(z)$ is big enough, as stated. $\square$

As a consequence of this bound we can use Tcherbishev's inequality with z and k_0 as in the statement of Lemma 6.3 to obtain

$$\lambda\left(\Big\{\mathcal{D}_n \geq \frac{n}{B_0}\cdot\delta\Big\}\right) = \lambda\Big(\big\{e^{z\sqrt{\mathcal{D}_n}} > e^{z\sqrt{\delta\cdot n/B_0}}\big\}\Big) \leq e^{-z\sqrt{\delta\cdot n/B_0}}\int e^{z\cdot\sqrt{\mathcal{D}_n}}\,d\lambda \leq e^{-(z\sqrt{\delta/B_0})\sqrt{n}}.$$

As already explained, this together with Lemma 5.3 implies that there are $C,\xi > 0$, where $\xi = \xi(\delta)$, such that for all $n \geq 1$

$$\lambda\big(\{x\in I : \mathcal{C}_n(x) > \delta\}\big) \leq Ce^{-\xi\sqrt{n}}.$$

Then since $\{x\in I : \mathcal{R}(x) > n\} \subset \cup_{k>n}\{x \in I : \mathcal{C}_k(x) > \delta\}$ we conclude that there are constants $C_1,\xi_1 > 0$ such that Theorem B holds.

7. Fast expansion for most points

Here we use the results from the previous sections to prove Theorem C and as a consequence obtain Corollary D. We start by setting

$$E_n = \{x\in I : \exists k\in\{1,\dots,n\} \quad \text{such that} \quad \operatorname{dist}(f_\mu^k(x),\mathcal{C}) < e^{-\rho\cdot n}\}$$

and proving the following bound.

Lemma 7.1. *There are constants $C, \xi > 0$ dependent on $\hat{f}$, k_0, ζ, ρ, and τ only such that for all $n \geq 1$*

$$\lambda\big(E_n\big) \leq C \cdot e^{-\xi \cdot n}.$$

Proof. Let us take $x \in E_n$ and consider $k \in \{1, \dots, n\}$. We observe that by Remark 2.1 the constraint on the distance ensures that for every big enough n we have at every essential return time with host interval $I(l, s, j)$

$$-\rho n > \log \operatorname{dist}(f_\mu^k(x), \mathcal{C}) \geq \log a_2 - \frac{\pi}{\beta}(|l| + |s|) \quad \text{and so} \quad |l| + |s| \geq \frac{\beta\rho}{20\pi} \cdot n,$$

and the same holds for every inessential return time and every binding time before the next essential return, by the results in Subsection 5.1.

During free times after some essential return (we recall that every point x has an essential return either at time 0 or at time 1, see Remark 3.1 and Section 6) we have after Lemma 6.2 that

$$-\rho n > -\Big(3 + \frac{\pi}{\beta}\Big)(|l| + |s|) + 3 + \log(\tau a_2(\varepsilon - x_{k_0})) \quad \text{and so} \quad |l| + |s| \geq \frac{\beta\rho}{20\pi} \cdot n,$$

as along as n is big enough. Thus we have shown that we can control the distance to the critical set through the depth of the last essential return time.

Hence according to the definition of $A_{(\eta,\upsilon)}(n)$ from subsection 5.2

$$\begin{aligned} E_n &\subset \bigcup \Big\{A_{(\eta,\upsilon)}(n) : (\eta,\upsilon) \text{ is such that } \eta + \upsilon \geq \frac{\beta\rho}{20\pi} \cdot n\Big\}, \quad \text{thus by Corollary 5.5} \\ \lambda\big(E_n\big) &\leq \frac{n(n+1)}{2} \sum_{\substack{\eta+\upsilon \geq \beta\rho n/(20\pi) \\ \eta \geq k_0, \upsilon \geq s(\tau)}} e^{(5\zeta - \pi/\beta)(\eta+\upsilon)} \leq \frac{n(n+1)}{2} \sum_{k \geq \beta\rho n/(20\pi)} k \cdot e^{(5\zeta-\pi/\beta)k} \\ &\leq C_2 \cdot \exp\Big(-\frac{\rho}{20}\big(1 - \frac{5\zeta\beta}{\pi}\big) \cdot n\Big) \quad \text{for some constant } C_2 > 0\text{, as stated.} \end{aligned}$$

□

Lemma 7.2. *If n is big enough, ρ small enough (depending only on σ and A from Lemma 4.1) and $x \in I \setminus E_n$, then $\big|(f_\mu^n)'(x)\big| \geq \sigma^{n/3}$.*

Proof. Let us take $x \in I \setminus E_n$ and let $0 < r_1 < \cdots < r_k < n$ be the consecutive returns (either essential or inessential) of the first n iterates of the orbit of x, and $p_1, p_2, \dots, p_k$ the respective binding periods. We also set $q_i = r_{i+1} - (r_i + p_i)$ the free periods between consecutive returns, for $i = 1, \dots, k$, and $q_{k+1} = n - (r_k + p_k)$ if $n > r_k + p_k$ or $q_{k+1} = 0$ otherwise.

We split the argument in the following two cases. If $n \geq r_k + p_k$ then

$$\big|(f_\mu^n)'(x)\big| = \prod_{i=0}^{k} \Big(\big|(f_\mu^{q_i})'(f_\mu^{r_i+p_i}(x))\big| \cdot \big|(f_\mu^{p_i})'(f_\mu^{r_i}(x))\big|\Big) \geq \sigma_0^{\sum_{i=1}^{k+1} q_i} \cdot A_0^k \cdot \sigma^{\sum_{i=1}^{k} p_i} \geq \sigma^n,$$

since $\sigma_0 > \tilde{\sigma} > \sigma$ and $A_0 > 1$ by Lemma 4.4(c).

On the other hand, if $n < r_k + p_k$ then using Lemma 4.1 and that $x \in I \setminus E_n$

$$\begin{aligned}
\big|(f_\mu^n)'(x)\big| &= \big|(f_\mu^{r_k})'(x)\big| \cdot \big|(f_\mu)'(f_\mu^{r_k}(x))\big| \cdot \big|(f_\mu^{n-r_k-1})'(f_\mu^{r_k+1}(x))\big| \\
&\geq \big|(f_\mu^{r_k})'(x)\big| \cdot e^{-\rho r_k} \cdot \frac{1}{A} \cdot \big|(f_\mu^{n-r_k-1})'(x_l)\big| \\
&\geq A^{-1} \cdot \sigma^{r_k} \cdot e^{-\rho r_k} \cdot \sigma^{n-r_k-1} \\
&\geq \exp\Big((n-1)\big(\log\sigma - \rho\frac{n_k}{n} - \frac{\log A}{n}\big)\Big) \geq \sigma^{n/3},
\end{aligned}$$

for $\rho > 0$ small enough and n big enough since $n_k \leq n$, where x_l is the critical point associated to $f_\mu^{r_k}(x)$ and we have used also the calculation for the previous case to estimate $|(f_\mu^{r_k})'(x)|$. □

Finally since $\{x \in I : \mathcal{E}(x) > n\} \subset \cup_{k>n}\{x \in I : \big|(f_\mu^k)'(x)\big| < \sigma^{k/3}\}$ we conclude by Lemmas 7.1 and 7.2 that there are $C_2, \xi_2 > 0$ satisfying

$$\lambda\Big(\big\{x \in I : \mathcal{E}(x) > n\big\}\Big) \leq \sum_{k>n} \lambda(E_n) \leq C_2 \cdot e^{-\xi_2 \cdot n},$$

concluding the proof of Theorem C.

8. Constants depend uniformly on initial parameters

We finally complete the proof of Corollary E by explicitly showing the dependence of the constants used in the estimates on Sections 2 to 7.

In the statements of the lemmas and propositions in the aforementioned sections we stated explicitly the direct dependence of the constants appearing in each claim from earlier statements. For constants which depend only on $\hat{f}$ we used the plain letter C.

It is straightforward to see that every constant depends on values that ultimately rest on the choice of initial values for σ, σ_0 and ε or k_0 (which are related) and on the the choice of ρ and τ, which are taken to be small enough and where $0 < \rho < \tau$ is the unique restriction, used solely in the proof of Proposition 3.4.

Hence, by choosing $1 < \sigma < \sqrt{\tilde{\sigma}} < \sigma_0$, we may then take $0 < \rho < \tau$ as small as we need to obtain a small $\varepsilon > 0$ (and k_0 big enough, as a consequence, see Remark 3.2), so that the constants C_1, C_2, ξ_1, ξ_2, and consequently C_3, ξ_3 on the statements of Section 1, depend only on α, β, which depend only on $\hat{f}$, and on σ, σ_0, ρ and τ, but *do not depend on* $\mu \in S$.

This concludes the proof of Corollary E.

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