L_p-BOUNDEDNESS OF THE GENERAL INDEX TRANSFORMS

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Abstract

We establish the boundedness properties in L_p for a class of integral transformations with respect to an index of hypergeometric functions. In particular, by using the Riesz-Thorin interpolation theorem we get the corresponding results in $L_p(\mathbf{R}_+),\ 1 \leq p \leq 2$ for the Kontorovich-Lebedev, Mehler-Fock and Olevskii index transforms. An inversion theorem is proved for general index transformation. The case $p = 2$ is known as the Plancherel type theory for this class of transformations.

Key words: Kontorovich-Lebedev transform, Hausdorff-Young inequality, Fourier transform, Mellin transform, Mehler-Fock transform, Olevskii transform, Plancherel theory.

AMS subject classification. 44A15

1 Introduction

We deal with a special class of integral transforms over semi-axis $\mathbf{R}_+$ of the form

$$(\mathcal{G}_{H_\varphi} f)(x) = \int_0^\infty H_\varphi(x,\tau) f(\tau)\tau d\tau. \tag{1.1}$$

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This class was introduced in [10] and studied in L_2-weighted spaces. Here we assume that $f : \mathbf{R}_+ \to \mathbf{C}$ is a measurable function, which generally belongs to the space $L_p(\mathbf{R}_+; d\tau),\ 1 \le p \le \infty$ over the measure $d\tau$

$$||f||_{L_p(\mathbf{R}_+;d\tau)} = \left(\int_0^\infty |f(\tau)|^p d\tau\right)^{1/p} < \infty, \tag{1.2}$$

$$||f||_{L_\infty(\mathbf{R}_+;d\tau)} = \text{ess sup}_{\tau\in\mathbf{R}_+}|f(\tau)| < \infty. \tag{1.3}$$

The kernel $H_\varphi(x,\tau)$ is given as a Mellin type integral

$$H_\varphi(x,\tau) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma\left(s+\frac{i\tau}{2}\right)\Gamma\left(s-\frac{i\tau}{2}\right)\varphi(s)x^{-s}ds,\ x>0, \tag{1.4}$$

where $\Gamma(z)$ is Euler's Gamma-function (cf. [1, Vol. I]), $\varphi : \mathbf{C} \to \mathbf{C}$ is a multiplier of the kernel and $s = \gamma + it,\ \gamma > 0, t \in \mathbf{R}$ is a complex number. A multiplier $\varphi(s)$ is a complex-valued continuous function, which is defined on a vertical strip or semi-plane of the complex plane s and satisfies some conditions of the integrability, which give a definite sense of the convergence in (1.4). It is essentially known, that integrals of type (1.4) define a class of hypergeometric functions, when φ is a ratio of products of shifted Gamma-functions. In this case integral (1.4) is called the Mellin-Barnes integral. Such type of integral operators (1.1) in a slightly different form has been considered in [7], [9]. As we will see below all familiar integral transforms as the Kontorovich-Lebedev (KL), Mehler-Fock, Olevskii, Lebedev transforms [4], [7], [11] belong to this class.

Our main goal in this paper is to prove an analog of the Parseval equality for the KL-transform and the Hausdorff-Young inequality for the general operator (1.1) in order to establish the corresponding boundedness and inversion properties in L_p. We will appeal to the classical Hausdorff-Young inequality for the Fourier transform [5] and the Riesz-Thorin interpolation theorem. We note that for the KL-transform the Hausdorff-Young inequality has been proved recently in [12]. Another approach was given in [6], [8] to study the L_p- boundedness of the Mehler-Fock transform, basing on its composition structure and relationship with the Kontorovich-Lebedev and the Hankel transforms [5].

If we put in (1.4) $\varphi \equiv 1$ then by using the table of the Mellin transform from [3] we obtain

$$\frac{1}{4\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma\left(s+\frac{i\tau}{2}\right)\Gamma\left(s-\frac{i\tau}{2}\right)x^{-s}ds = K_{i\tau}(2\sqrt{x}),\ x>0, \tag{1.5}$$

where $K_\nu(z)$ is the modified Bessel function of the second kind or the Macdonald function [1, Vol. II]. The corresponding operator (1.1) is the modified KL-transform and it can be written in the form

$$(KLf)(x) = 2\int_0^\infty K_{i\tau}(2\sqrt{x})f(\tau)\tau d\tau. \tag{1.6}$$

Letting in (1.4) $\varphi(s)=\Gamma(1/2-\mu-s)/\Gamma(1/2+s), \mu\in\mathbf{R}$, under the condition $0<\gamma<1/2-\mu$, we obtain the integral representation for the generalized Legendre function [1, Vol. I], namely

$$|\Gamma((1+i\tau)/2-\mu)|^2 x^{-1/2}(1+x)^{\mu/2}P^{\mu}_{(i\tau-1)/2}\left(\frac{2}{x}+1\right)$$

$$=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma\left(s+\frac{i\tau}{2}\right)\Gamma\left(s-\frac{i\tau}{2}\right)\frac{\Gamma(1/2-\mu-s)}{\Gamma(1/2+s)}x^{-s}ds. \tag{1.7}$$

The corresponding transform (1.1) is the generalized Mehler-Fock operator

$$[MFf](x)=x^{-1/2}(1+x)^{\mu/2}\int_0^\infty |\Gamma((1+i\tau)/2-\mu)|^2 P^{\mu}_{(i\tau-1)/2}\left(\frac{2}{x}+1\right)f(\tau)\tau d\tau. \tag{1.8}$$

The classical Mehler-Fock transform [4] is the one with $\mu=0$. We note, that the Mehler-Fock transform is quite important in the theory of elasticity, in particular in the analysis of stress in the vicinity of an external crack.

If we take the multiplier function $\varphi(s)=\Gamma(c-a-s)/\Gamma(s+a), a<c, 0<\text{Re}s=\gamma<c-a, c\neq 0,-1,-2,\dots$, then appealing to the table of the Mellin transform [3] we obtain from (1.4) the kernel of the Olevskii transformation [7], [10]

$$H_\varphi(x,\tau)=\frac{|\Gamma(c-a+i\tau/2)|^2}{\Gamma(c)}x^{-a}(1+x)^{2a-c}{}_2F_1\left(a+\frac{i\tau}{2},a-\frac{i\tau}{2};c;-\frac{1}{x}\right), \tag{1.9}$$

where ${}_2F_1(a,b;c;z)$ is the Gauss hypergeometric function [1, Vol. I]. Putting $a=1/2$ we immediately arrive at the generalized Mehler-Fock transform (1.8).

In the next section we will study the L_p-boundedness of the KL- transform (1.6). We will see that it generalizes its L_2-properties, which are proved in [10]. Indeed, it forms a bijection $KL: L_2\left(\mathbf{R}_+;\frac{\tau}{\sinh\pi\tau}d\tau\right)\leftrightarrow L_2\left(\mathbf{R}_+;x^{-1}dx\right)$ with the Parseval equality

$$\int_0^\infty |(KLf)(x)|^2\,\frac{dx}{x}=4\pi^2\int_0^\infty\frac{\tau}{\sinh\pi\tau}|f(\tau)|^2d\tau. \tag{1.10}$$

When $f\in L_2\left(\mathbf{R}_+;d\tau\right)\subset L_2\left(\mathbf{R}_+;\frac{\tau}{\sinh\pi\tau}d\tau\right)$ then by the elementary inequality $\sinh\tau\geq\tau, \tau\geq 0$ we have from (1.10) that

$$\int_0^\infty |(KLf)(x)|^2\,\frac{dx}{x}\leq 4\pi\int_0^\infty |f(\tau)|^2d\tau, \tag{1.11}$$

which means that the KL-operator (1.6) is of type $(2,2)$ with the norm $||KL||\leq 2\sqrt{\pi}$.

2 The Kontorovich-Lebedev transform

We will prove that main theorem of this section, which stays the boundedness and inversion properties in L_p of the KL-operator (1.6). These results will be used to establish the corresponding properties of the general index transformation (1.1).

We have

Theorem 1. *Let $f \in L_p(\mathbf{R}_+; d\tau), 1 \le p \le \infty$. Then integral* (1.6) *exists for all $x > 0$ as a Lebesgue integral. For $1 \le p \le 2$ the KL-transformation is of type (p, p')*

$$L_p(\mathbf{R}_+; d\tau) \to L_{p'}\left(\mathbf{R}_+; x^{-1}dx\right), \ p^{-1} + {p'}^{-1} = 1$$

and when $1 < p \le 2$ an analog of the Hausdorff-Young inequality takes place

$$\left(\int_0^\infty |(KLf)(x)|^{p'} \frac{dx}{x}\right)^{1/p'} \le 2\pi^{1/p'} \left(\int_0^\infty |f(\tau)|^p d\tau\right)^{1/p}. \tag{2.1}$$

The inversion formula in this case is given for almost all $\tau \in \mathbf{R}_+$ by the integral

$$f(\tau) = \frac{1}{2\pi^2} \frac{\sinh(\pi\tau)}{\tau} \frac{d}{d\tau} \int_0^\infty \hat{K}(\tau, x)(KLf)(x)\frac{dx}{x}, \tag{2.2}$$

where

$$\hat{K}(\tau, x) = \int_0^\tau y K_{iy}(2\sqrt{x})dy. \tag{2.3}$$

Finally, if $f \in L_p(\mathbf{R}_+; d\tau), \ g \in L_{p'}(\mathbf{R}_+; d\tau), 1 \le p \le 2$, then the following generalized Parseval equality holds

$$\lim_{\varepsilon \to 0+} \int_0^\infty (KLf)(x)\overline{(KLg)(x)}\frac{dx}{x^{1-\varepsilon}} = 4\pi^2 \int_0^\infty \frac{\tau}{\sinh \pi\tau} f(\tau)\overline{g(\tau)}d\tau. \tag{2.4}$$

Proof. We show the existence of the integral (1.6) invoking the Hölder inequality and the uniform estimate [7] for the Macdonald function

$$|K_{i\tau}(x)| \le e^{-\delta|\tau|}K_0(x\cos\delta), \ \delta \in (0, \pi/2). \tag{2.5}$$

Hence we obtain

$$|(KLf)(x)| \le 2\int_0^\infty |K_{i\tau}(2\sqrt{x})f(\tau)|\tau d\tau \le 2||K_{i\tau}(2\sqrt{x})\tau||_{p'}||f||_p$$

where accordingly

$$||K_{i\tau}(2\sqrt{x})\tau||_{p'} = \left(\int_0^\infty \tau^{p'}|K_{i\tau}(2\sqrt{x})|^{p'}d\tau\right)^{1/p'} \le K_0(2\sqrt{x}\cos\delta)\left(\int_0^\infty \tau^{p'}e^{-\delta p'\tau}d\tau\right)^{1/p'}$$

$$= C_{\delta,p'} K_0(2\sqrt{x}\cos\delta) < \infty, p' \neq \infty,$$

where $x > 0, \delta \in (0, \pi/2)$ and $C_{\delta,p'}$ is a constant depending on δ, p'. When $p' = \infty$ we use the following representation of the Macdonald function (cf. [1, Vol. II]

$$\tau K_{i\tau}(x) = x \int_0^\infty e^{-x\cosh u} \sinh u \sin \tau u du, \tag{2.6}$$

and we estimate the corresponding norm as

$$||K_{i\tau}(2\sqrt{x})\tau||_\infty = \sup_{\tau\in\mathbf{R}_+} (\tau K_{i\tau}(2\sqrt{x})) \leq 2\sqrt{x} \int_0^\infty e^{-2\sqrt{x}\cosh u} \sinh u du = e^{-2\sqrt{x}} < 1, x > 0.$$

Thus the integrand in (1.6) is summable for any $f \in L_p(\mathbf{R}_+),\ 1 \leq p \leq \infty$.

Let $1 \leq p \leq 2$. In order to demonstrate the boundedness in L_p of the KL-operator (1.6) and to prove the inequality (2.1) we first show that (1.6) is an operator of type $(1, \infty)$ and then we apply the Riesz-Thorin interpolation theorem (cf. [12]). In fact, by using (2.6) we substitute the corresponding integral in (1.6) and we change the order of integration via Fubini's theorem. As a result we find

$$(KLf)(x) = 2\sqrt{2\pi x} \int_0^\infty e^{-2\sqrt{x}\cosh u} \sinh u (F_s f)(u) du, \tag{2.7}$$

where $(F_s f)(u)$ is the sine Fourier transform of f [5]. Consequently,

$$|(KLf)(x)| \leq 2\sqrt{2\pi x} \sup_{u\in\mathbf{R}_+} |(F_s f)(u)| \int_0^\infty e^{-2\sqrt{x}\cosh u} \sinh u du$$

$$= \sqrt{2\pi} e^{-2\sqrt{x}} \sup_{u\in\mathbf{R}_+} |(F_s f)(u)| \leq 2||f||_1.$$

Therefore, the KL-transform is of type $(1, \infty)$ and we have

$$||KLf||_{L_\infty(\mathbf{R}_+;x^{-1}dx)} \leq 2||f||_1,$$

where we put

$$||KLf||_{L_p(\mathbf{R}_+;x^{-1}dx)} = \left(\int_0^\infty |(KLf)(x)|^p \frac{dx}{x}\right)^{1/p}, 1 \leq p < \infty, \tag{2.8}$$

$$||KLf||_{L_\infty(\mathbf{R}_+;x^{-1}dx)} = \text{ess sup}_{x\in\mathbf{R}_+} |(KLf)(x)|. \tag{2.9}$$

Taking into account inequality (1.11) we immediately obtain via the Riesz-Thorin interpolation theorem the boundedness of the KL-transform (1.6) as an operator $KL : L_p(\mathbf{R}_+; d\tau) \to L_{p'}(\mathbf{R}_+; x^{-1}dx),\ 1 \leq p \leq 2$ and we prove inequality (2.1). It can be written in terms of the norms (1.2), (2.8), (2.9) for all $p \in [1, 2]$

$$||KLf||_{L_{p'}(\mathbf{R}_+;x^{-1}dx)} \leq 2\pi^{1/p'} ||f||_p.$$

Further, we derive the generalized Parseval equality (2.4). Indeed, since $K_{i\tau}(2\sqrt{x})$ is a real-valued function (see (2.6)) we have that $\overline{(KLg)(x)} = (KL\overline{g})(x)$. Hence we insert the value of $(KL\overline{g})(x)$ by the related integral (1.6) into the left-hand side of (2.4) and we change the order of integration. This gives the following equality for each $\varepsilon > 0$

$$\int_0^\infty (KLf)(x)\overline{(KLg)(x)}\frac{dx}{x^{1-\varepsilon}} = 2\int_0^\infty \overline{g(\tau)}\tau \int_0^\infty x^{\varepsilon-1}K_{i\tau}(2\sqrt{x})(KLf)(x)dxd\tau. \tag{2.10}$$

The change of the order of integration is motivated by using the absolute convergence of the iterated integral (2.10) and appealing to Fubini's theorem. In fact, with the Hölder inequality, the inequality (2.1) and the estimate (2.5) we obtain

$$\int_0^\infty |g(\tau)|\tau \int_0^\infty x^{\varepsilon-1}\left|K_{i\tau}(2\sqrt{x})(KLf)(x)\right| dxd\tau \le \int_0^\infty |g(\tau)|\tau \left(\int_0^\infty x^{p\varepsilon-1}\left|K_{i\tau}(2\sqrt{x})\right|^p dx\right)^{1/p} d\tau$$

$$\times||KLf||_{L_{p'}(\mathbf{R}_+;x^{-1}dx)} \le \int_0^\infty e^{-\delta\tau}|g(\tau)|\tau d\tau \left(\int_0^\infty x^{p\varepsilon-1}\left|K_0(2\sqrt{x}\cos\delta)\right|^p dx\right)^{1/p}$$

$$\times 2\pi^{1/p'}||f||_p \le \left(\int_0^\infty e^{-p\delta\tau}\tau^p d\tau\right)^{1/p}\left(\int_0^\infty x^{p\varepsilon-1}\left|K_0(2\sqrt{x}\cos\delta)\right|^p dx\right)^{1/p}$$

$$\times 2\pi^{1/p'}||f||_p||g||_{p'} = C_{p,\delta,\varepsilon}||f||_p||g||_{p'} < \infty,$$

since $\delta \in (0,\pi/2), \varepsilon > 0$ and the Macdonald function has the asymptotic behaviour [1, Vol. II]

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}[1+O(1/z)], \qquad z\to\infty, \tag{2.11}$$

and near the origin

$$K_\nu(z) = O\left(z^{-|\mathrm{Re}\nu|}\right),\ z\to 0, \tag{2.12}$$

$$K_0(z) = -\log z + O(1),\ z\to 0. \tag{2.13}$$

Returning to equality (2.10) and combining with (2.4) the problem is reduced to prove that

$$\lim_{\varepsilon\to 0+}\int_0^\infty \overline{g(\tau)}\left[\tau\int_0^\infty x^{\varepsilon-1}K_{i\tau}(2\sqrt{x})(KLf)(x)dx - \frac{2\pi^2\tau}{\sinh\pi\tau}f(\tau)\right]d\tau = 0. \tag{2.14}$$

However, the latter limit equality (2.14) is an immediate consequence of the theorem from [12], which says that

$$\left|\left|\tau\int_0^\infty x^{\varepsilon-1}K_{i\tau}(2\sqrt{x})(KLf)(x)dx - \frac{2\pi^2\tau}{\sinh\pi\tau}f(\tau)\right|\right|_p \to 0,$$

when $\varepsilon\to 0+$. Precisely, applying again the Hölder inequality we find

$$\int_0^\infty \left|\overline{g(\tau)}\left[\tau\int_0^\infty x^{\varepsilon-1}K_{i\tau}(2\sqrt{x})(KLf)(x)dx - \frac{2\pi^2\tau}{\sinh\pi\tau}f(\tau)\right]\right| d\tau$$

$$\leq ||g||_{p'} \left|\left| \tau \int_0^\infty x^{\varepsilon-1} K_{i\tau}(2\sqrt{x})(KLf)(x)dx - \frac{2\pi^2\tau}{\sinh \pi\tau} f(\tau) \right|\right|_p \to 0,$$

when $\varepsilon \to 0+$. Thus we obtain the relation (2.14) and we arrive at the equality (2.4). Now we put

$$g(y) = \begin{cases} 1, & \text{if } y \in [0, \tau], \\ 0, & \text{if } y \in (\tau, \infty), \end{cases}$$

and we get $(KLg)(x) = \hat{K}(\tau, x)$ (see (2.3)). We show that for each $\tau > 0$ $\hat{K}(\tau, x) \in L_p(\mathbf{R}_+; x^{-1}dx), 1 < p \leq 2$. Indeed, integrating by parts in (2.6) we easily find the representation

$$K_{i\tau}(x) = \int_0^\infty e^{-x \cosh u} \cos \tau u du, x > 0. \tag{2.15}$$

Hence $|K_{i\tau}(x)| \leq K_0(x)$ and from (2.3) we obtain $|\hat{K}(\tau, x)| \leq \frac{\tau^2}{2} K_0(2\sqrt{x})$. Taking into account the asymptotic formula (2.11) for the Macdonald function $K_0(2\sqrt{x})$ we conclude that $\hat{K}(\tau, x) \in L_p((a, \infty); x^{-1}dx)$ for any $a > 0$. So, choosing $0 < a < 1$ it is sufficient to show that $\hat{K}(\tau, x) \in L_p((0, a); x^{-1}dx), 1 < p \leq 2$. This fact follows from the asymptotic behavior of the kernel (2.3) when $x \to 0+$ (cf. [7], [10]). Precisely, we get $\hat{K}(\tau, x) = O([\log x]^{-1}), x \to 0+$ and the result follows. Now we write the equality (2.4) in the form

$$\lim_{\varepsilon \to 0+} \int_0^\infty (KLf)(x) \hat{K}(\tau, x) \frac{dx}{x^{1-\varepsilon}} = 2\pi^2 \int_0^\tau \frac{y}{\sinh \pi y} f(y) dy. \tag{2.16}$$

However, we can pass to the limit through the integral sign in the left-hand side of the equality (2.16). In fact, since for all $\varepsilon \in [0, 1]$ the integrand in the left-hand side of (2.16) is less than or equal to the majorant $\Phi(\tau, x)$, where

$$\Phi(\tau, x) = \begin{cases} x^{-1} |(KLf)(x) \hat{K}(\tau, x)|, & \text{if } x \in [0, 1], \\ |(KLf)(x) \hat{K}(\tau, x)|, & \text{if } x \in (1, \infty). \end{cases}$$

Furthermore, for each $\tau > 0$ $\Phi(\tau, x) \in L_1(\mathbf{R}_+; dx)$ since

$$\int_0^\infty \Phi(\tau, x) dx = \left(\int_0^1 + \int_1^\infty \right) \Phi(\tau, x) dx \leq ||KLf||_{L_{p'}(\mathbf{R}_+; x^{-1}dx)}$$

$$\times \left[\left(\int_0^1 |\hat{K}(\tau, x)|^p \frac{dx}{x} \right)^{1/p} + \left(\int_1^\infty x^{p/p'} |\hat{K}(\tau, x)|^p dx \right)^{1/p} \right]$$

$$\leq ||KLf||_{L_{p'}(\mathbf{R}_+; x^{-1}dx)} \left[||\hat{K}||_{L_p(\mathbf{R}_+; x^{-1}dx)} + \frac{\tau^2}{2} \left(\int_1^\infty x^{p/p'} K_0^p(2\sqrt{x}) dx \right)^{1/p} \right] < \infty.$$

Consequently, passing to the limit under the integral sign in (2.16) by virtue of the Lebesgue dominated convergence theorem we derive

$$\int_0^\infty (KLf)(x)\hat{K}(\tau,x)\frac{dx}{x} = 2\pi^2 \int_0^\tau \frac{y}{\sinh \pi y} f(y)dy.$$

Hence for almost all $\tau \in \mathbf{R}_+$ we arrive at the inversion formula (2.2) for the KL-transform (1.6) and we complete the proof of Theorem 1.

Remark 1. For the case $p = p' = 2$ we can pass to the limit with respect to $\varepsilon \to 0+$ under the integral sign in the left-hand side of (2.4). Then we immediately arrive at the Plancherel theorem for the KL-transformation (see in [10]). In particular, it gives the Parseval equality (1.10).

3 General index transform

In this section we establish boundedness properties in L_p-spaces for the general index transformation (1.1). We will prove that transformation (1.1) represents a bounded operator $\mathcal{G}_{H_\varphi} : L_p(\mathbf{R}_+; d\tau) \to L_{p'}\left(\mathbf{R}_+; x^{\gamma p'-1}dx\right),\ p^{-1} + {p'}^{-1} = 1$ for $1 \le p \le 2,\ \gamma > 0$, where the latter space $L_{p'}$ is over the measure $x^{\gamma p'-1}dx$ and under sufficient conditions for the multiplier $\varphi(s)$ of the kernel (1.4). An inversion theorem in L_p for this transformation will be proved in the next section.

We begin to treat the kernel (1.4) by using the following integral representation for the product of Gamma - functions (cf. formula (1.104) from [7] with integration by parts and the use of the inverse sine Fourier transform)

$$\tau\Gamma\left(s+\frac{i\tau}{2}\right)\Gamma\left(s-\frac{i\tau}{2}\right) = 2^{2(1-s)}\Gamma(2s+1)\int_0^\infty \frac{\sin(\tau y)\tanh y dy}{\cosh^{2s} y},\ \mathrm{Re}s > 0. \tag{3.1}$$

Hence we substitute this into (1.4) and invert the order of integration. As a result we find

$$\tau H_\varphi(x,\tau) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} 2^{2(1-s)}\Gamma(2s+1)\varphi(s)x^{-s}\int_0^\infty \frac{\sin(\tau y)\tanh y}{\cosh^{2s} y} dyds$$

$$= \int_0^\infty \Phi(x\cosh^2 y)\tanh y \sin(\tau y)dy, \tag{3.2}$$

where we denoted by

$$\Phi(z) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} 2^{2(1-s)}\Gamma(2s+1)\varphi(s)z^{-s}ds. \tag{3.3}$$

The change of the order of integration is motivated by the Fubini theorem if we suppose that $\varphi(\gamma+it) \in L_1(\mathbf{R};\Gamma(2(\gamma+it)+1)dt)$ or

$$\int_{\gamma-i\infty}^{\gamma+i\infty} |\Gamma(2s+1)\varphi(s)ds| < \infty. \tag{3.4}$$

Indeed, we have the estimate

$$\int_{\gamma-i\infty}^{\gamma+i\infty} \left|2^{2(1-s)}\Gamma(2s+1)\varphi(s)x^{-s}\right| \int_0^\infty \left|\frac{\sin(\tau y)\tanh y}{\cosh^{2s} y}dyds\right| \le 2^{2(1-\gamma)}x^{-\gamma}$$

$$\times \int_{\gamma-i\infty}^{\gamma+i\infty} |\Gamma(2s+1)\varphi(s)ds| \int_0^\infty \frac{\tanh y}{\cosh^{2\gamma} y}dy < \infty,\ \gamma > 0, \tag{3.5}$$

which guarantees the change of the order of integration. Now we are ready to prove that operator (1.1) $\mathcal{G}_{H_\varphi} : L_1(\mathbf{R}_+; d\tau) \to L_{\gamma,\infty}(\mathbf{R}_+; dx)$ is bounded, where the norm in the space $L_{\gamma,\infty}(\mathbf{R}_+; dx)$ is defined by (see (1.3))

$$||f||_{L_{\gamma,\infty}(\mathbf{R}_+;dx)} = \text{ess sup}_{x\in\mathbf{R}_+} x^\gamma |f(x)| < \infty,\ \gamma > 0. \tag{3.6}$$

Precisely, we substitute the latter integral in (3.2) into (1.1) and we change the order of integration appealing again to the estimate (3.5) and using the fact, that $f \in L_1(\mathbf{R}_+; d\tau)$. Hence invoking the definition of the sine Fourier transform we derive

$$(\mathcal{G}_{H_\varphi} f)(x) = \int_0^\infty f(\tau) \int_0^\infty \Phi(x\cosh^2 y)\tanh y \sin(\tau y)dyd\tau$$

$$= \sqrt{\frac{\pi}{2}} \int_0^\infty \Phi(x\cosh^2 y)\tanh y (F_s f)(y)dy. \tag{3.7}$$

Further with (3.3) we get

$$x^\gamma \left|(\mathcal{G}_{H_\varphi} f)(x)\right| \le \frac{2^{2(1-\gamma)}}{2\pi}\sqrt{\frac{\pi}{2}} \sup_{y\in\mathbf{R}_+} |(F_s f)(y)| \int_{\gamma-i\infty}^{\gamma+i\infty} |\Gamma(2s+1)\varphi(s)ds| \int_0^\infty \frac{\tanh y}{\cosh^{2\gamma} y}dy$$

$$\le C_\gamma ||f||_1,\ \gamma > 0,$$

where $C_\gamma > 0$ is a constant, which depends on $\gamma > 0$ and it is equal to

$$C_\gamma = \frac{2^{2(1-\gamma)}}{2\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} |\Gamma(2s+1)\varphi(s)ds| \int_0^\infty \frac{\tanh y}{\cosh^{2\gamma} y}dy = \frac{2^{-2\gamma}}{\pi\gamma} \int_{\gamma-i\infty}^{\gamma+i\infty} |\Gamma(2s+1)\varphi(s)ds|.$$

Thus we obtain

$$||\mathcal{G}_{H_\varphi} f||_{L_{\gamma,\infty}(\mathbf{R}_+;dx)} \le C_\gamma ||f||_1, \tag{3.8}$$

and we have proved the desired boundedness for the operator (1.1). It says that operator (1.1) is of type $(1,\infty)$. We will show that $\mathcal{G}_{H_\varphi}$ is also of type (2,2). However first we prove that for any $f \in L_p(\mathbf{R}_+; d\tau)$, $1 \le p \le 2$ integral (1.1) exists for all $x > 0$ as a Lebesgue integral.

Indeed, with the Hölder inequality we find

$$\left|(\mathcal{G}_{H_\varphi} f)(x)\right| \le \int_0^\infty |H_\varphi(x,\tau) f(\tau)\tau|\, d\tau \le ||f||_{L_p(\mathbf{R}_+;d\tau)} \left(\int_0^\infty \tau^{p'} |H_\varphi(x,\tau)|^{p'} d\tau\right)^{1/p'}. \tag{3.9}$$

Meantime, employing the generalized Minkowski inequality we derive

$$\left(\int_0^\infty \tau^{p'} |H_\varphi(x,\tau)|^{p'} d\tau\right)^{1/p'}$$

$$= \left(\int_0^\infty \tau^{p'} d\tau \left|\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma\left(s+\frac{i\tau}{2}\right)\Gamma\left(s-\frac{i\tau}{2}\right)\varphi(s) x^{-s} ds\right|^{p'} d\tau\right)^{1/p'}$$

$$\le \frac{x^{-\gamma}}{2\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} |\varphi(s) ds| \left(\int_0^\infty \tau^{p'} \left|\Gamma\left(s+\frac{i\tau}{2}\right)\Gamma\left(s-\frac{i\tau}{2}\right)\right|^{p'} d\tau\right)^{1/p'}.$$

In order to estimate the latter integral we appeal to the Hausdorff-Young inequality (cf. in [5]) for the sine Fourier transform

$$||F_s f||_{p'} \le \left(\frac{2}{\pi}\right)^{\frac{1}{p}-\frac{1}{2}} ||f||_p. \tag{3.10}$$

Hence invoking (3.1) we arrive at the inequality

$$\left(\int_0^\infty \tau^{p'} \left|\Gamma\left(s+\frac{i\tau}{2}\right)\Gamma\left(s-\frac{i\tau}{2}\right)\right|^{p'} d\tau\right)^{1/p'} \le \left(\frac{2}{\pi}\right)^{\frac{1}{p}-1} 2^{2(1-\gamma)} |\Gamma(2s+1)| \left(\int_0^\infty \frac{\tanh^p y dy}{\cosh^{2p\gamma} y}\right)^{1/p}$$

$$= C_{\gamma,p} |\Gamma(2s+1)|.$$

Combining with (3.9) we obtain

$$\int_0^\infty |H_\varphi(x,\tau) f(\tau)\tau d\tau| \le \frac{C_{\gamma,p} x^{-\gamma}}{2\pi} ||f||_{L_p(\mathbf{R}_+;d\tau)} \int_{\gamma-i\infty}^{\gamma+i\infty} |\Gamma(2s+1)\varphi(s) ds| < \infty$$

for all $x > 0$. This fact implies the existence of the Lebesgue integral (1.1).

Further we study transformation (1.1) in L_2. For this we use the L_2-theory of the Mellin transform [4], [5]. We observe that due to condition (3.4) $\varphi(\gamma+it) \in L_1(\mathbf{R}; \Gamma(2(\gamma+it)+1)dt)$ and since the integrand in (3.4) is continuous with respect to $t \in \mathbf{R}$ it follows that the product

$\varphi(\gamma+it)\Gamma(2(\gamma+it)+1)$ is bounded. Assuming that f belongs to the space $S(\mathbf{R}_+)$ of rapidly decreasing smooth functions and invoking (1.4), (3.1) we write $\mathcal{G}_{H_\varphi}$ in the form

$$(\mathcal{G}_{H_\varphi}f)(x) = \frac{1}{2\pi}\sqrt{\frac{\pi}{2}}\int_{-\infty}^{\infty} 2^{2(1-\gamma-it)}\Gamma(2(\gamma+it)+1)\varphi(\gamma+it)x^{-\gamma-it}$$

$$\times \int_0^\infty \frac{(F_s f)(y)\tanh y}{\cosh^{2(\gamma+it)} y} dydt, \tag{3.11}$$

where we change the order of integration by the Fubini theorem. Then making the substitution $e^\xi = \cosh^2 y$ in the latter integral with respect to y we appeal to the Parseval equalities [5] for the Mellin and the Fourier transforms to represent the L_2-norm for the operator $\mathcal{G}_{H_\varphi}$ as

$$||\mathcal{G}_{H_\varphi}f||^2_{L_2(\mathbf{R}_+;x^{2\gamma-1}dx)} = \int_0^\infty |(\mathcal{G}_{H_\varphi}f(x)|^2 x^{2\gamma-1}dx$$

$$= 2^{-4\gamma}\int_{-\infty}^{\infty} |\varphi(\gamma+it)\Gamma(2(\gamma+it)+1)|^2 \left|\int_0^\infty e^{-(\gamma+it)\xi}(F_s f)(\text{arccosh}e^{\xi/2})d\xi\right|^2 dt$$

$$\leq A\int_{-\infty}^{\infty} \left|\int_0^\infty e^{-(\gamma+it)\xi}(F_s f)(\text{arccosh}e^{\xi/2})d\xi\right|^2 dt$$

$$= 2\pi A\int_0^\infty e^{-2\gamma\xi}|(F_s f)(\text{arccosh}e^{\xi/2})|^2 d\xi \leq 2\pi A\int_0^\infty |(F_s f)(\text{arccosh}e^{\xi/2})|^2 d\xi$$

$$= 2\pi A\int_0^\infty |(F_s f)(y)|^2 \tanh y dy \leq 2\pi A\int_0^\infty |(F_s f)(y)|^2 dy$$

$$= 2\pi A\int_0^\infty |f(y)|^2 dy = 2\pi A||f||^2_{L_2(\mathbf{R}_+;d\tau)}, \tag{3.12}$$

where $A > 0$ is an absolute constant. Thus we obtain the norm inequality

$$||\mathcal{G}_{H_\varphi}f||_{L_2(\mathbf{R}_+;x^{2\gamma-1}dx)} \leq \sqrt{2\pi A}||f||_{L_2(\mathbf{R}_+;d\tau)}, \tag{3.13}$$

which takes place for the dense subspace $S(\mathbf{R}_+) \subset L_2(\mathbf{R}_+;d\tau)$. Let $f \in L_2(\mathbf{R}_+;d\tau)$. Hence for Cauchy's sequence $\{f\}_n \in S(\mathbf{R}_+)$, which converges to f we have

$$||\mathcal{G}_{H_\varphi}f_n - \mathcal{G}_{H_\varphi}f_m||_{L_2(\mathbf{R}_+;x^{2\gamma-1}dx)} \leq \sqrt{2\pi A}||f_n - f_m||_{L_2(\mathbf{R}_+;d\tau)} \to 0, n, m \to \infty.$$

Consequently, $\{(\mathcal{G}_{H_\varphi}f_n)(x)\}$ is a Cauchy sequence in the space $L_2\left(\mathbf{R}_+;x^{2\gamma-1}dx\right)$, which converges to the limit $(\mathcal{G}_{H_\varphi}f)(x)$ with respect to the norm. Therefore we get by the continuity of norms that (3.13) is true for any $f \in L_2(\mathbf{R}_+;d\tau)$. Moreover, the limit function $(\mathcal{G}_{H_\varphi}f)(x)$ coincides with transformation (1.1), since via (3.9) the corresponding sequence $\{(\mathcal{G}_{H_\varphi}f_n)(x)\}$ converges uniformly to the same limit. Precisely, we find

$$x^\gamma\left|(\mathcal{G}_{H_\varphi}f)(x) - (\mathcal{G}_{H_\varphi}f_n)(x)\right| = x^\gamma\left|(\mathcal{G}_{H_\varphi}(f-f_n))(x)\right| \leq \text{const.}||f-f_n||_{L_2(\mathbf{R}_+;d\tau)} \to 0, n \to \infty.$$

Thus we conclude that operator $\mathcal{G}_{H_\varphi}$ is of type (2,2). Taking into account that this transformation is also of type $(1,\infty)$ via the Riesz-Thorin interpolation theorem we arrive at the following

Theorem 2. *Let the multiplier function* $\varphi(s)$ *of the kernel* (1.4) *satisfy condition* (3.4) *and* $1 \le p \le 2$. *Then transformation* (1.1) *is a bounded operator* $\mathcal{G}_{H_\varphi} : L_p(\mathbf{R}_+; d\tau) \to L_{p'}\left(\mathbf{R}_+; x^{\gamma p'-1}dx\right),\ p^{-1} + p'^{-1} = 1,\ \gamma > 0$ *and*

$$||\mathcal{G}_{H_\varphi} f||_{L_{p'}\left(\mathbf{R}_+; x^{\gamma p'-1}dx\right)} \le B_{p,\gamma} ||f||_{L_p(\mathbf{R}_+;d\tau)}, \tag{3.14}$$

where $B_{p,\gamma} > 0$ *is a constant depending on* p, γ. *Moreover, for all* $x > 0$ *integral* (1.1) *exists in Lebesgue's sense.*

4 Inversion theorem

Here we prove an inversion theorem for the general transformation (1.1) basing on the Mellin transform theory in L_p (see [5], Chapter IV). We begin with a different integral representation for the operator (1.1). Indeed, by using the results of the previous section, precisely the estimate (3.9) under condition (3.4) we substitute integral (1.4) into (1.1) and we change the order of integration. Thus we arrive at the representation

$$(\mathcal{G}_{H_\varphi} f)(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \varphi(s) \Theta_f(s) x^{-s} ds, \tag{4.1}$$

where (see (3.1))

$$\Theta_f(w) = \int_0^\infty \Gamma\left(w + \frac{i\tau}{2}\right) \Gamma\left(w - \frac{i\tau}{2}\right) \tau f(\tau) d\tau$$
$$= \frac{\sqrt{\pi}\Gamma(2w+1)}{2^{2w-3/2}} \int_0^\infty \frac{(F_s f)(y) \tanh y}{\cosh^{2w} y} dy dt. \tag{4.2}$$

Hence if we prove that $\varphi(\gamma + it)\Theta_f(\gamma + it) \in L_p(\mathbf{R}; dt),\ 1 < p \le 2$ then from (4.1) via Th. 86 from [5] we obtain that functions $\varphi(\gamma + it)\Theta_f(\gamma + it),\ (\mathcal{G}_{H_\varphi} f)(x)$ realize a Mellin transform pair from $L_p(\mathbf{R}; dt)$ into $L_{p'}\left(\mathbf{R}_+; x^{\gamma p'-1}dx\right),\ p^{-1} + p'^{-1} = 1,\ \gamma > 0$. To do this we use the generalized Minkowski inequality, the Hölder inequality, inequality (3.10) and representation (3.11), (4.2) for the dense set of $S(\mathbf{R}_+)$- functions. But first we observe that due to condition (3.4) and since $\varphi(s)$ is continuous it follows that $\Gamma(2(\gamma+it)+1)\varphi(\gamma+it)$ is bounded by $t \in \mathbf{R}$.

We have

$$\left(\int_{-\infty}^\infty |\varphi(\gamma + it)\Theta_f(\gamma + it)|^p dt\right)^{1/p} = \frac{\sqrt{\pi}}{2^{2\gamma - 3/2}} \left(\int_{-\infty}^\infty |\varphi(\gamma + it)\Gamma(2(\gamma + it) + 1)|^p\right.$$
$$\left.\times \left|\int_0^\infty \frac{(F_s f)(y) \tanh y}{\cosh^{2(\gamma + it)} y} dy\right|^p dt\right)^{1/p} \le \text{const.} \left(\int_{-\infty}^\infty |\varphi(\gamma + it)\Gamma(2(\gamma + it) + 1)|\right.$$

$$\times \left| \int_0^\infty \frac{(F_s f)(y) \tanh y}{\cosh^{2(\gamma+it)} y} dy \right|^p dt \Bigg)^{1/p} = \text{const.} \left(\int_{-\infty}^\infty |\varphi(\gamma+it)\Gamma(2(\gamma+it)+1)| \right.$$

$$\times \left| \int_0^\infty e^{-(\gamma+it)\xi} (F_s f)(\text{arccosh} e^{\xi/2}) d\xi \right|^p dt \Bigg)^{1/p} \leq \text{const.} \int_0^\infty e^{-\gamma\xi} \left| (F_s f)(\text{arccosh} e^{\xi/2}) \right| d\xi$$

$$\times \left(\int_{-\infty}^\infty |\varphi(\gamma+it)\Gamma(2(\gamma+it)+1)| \, dt \right)^{1/p} \leq \text{const.} \left(\int_0^\infty e^{-p\gamma\xi} d\xi \right)^{1/p}$$

$$\times \left(\int_0^\infty \left| (F_s f)(\text{arccosh} e^{\xi/2}) \right|^{p'} d\xi \right)^{1/p'} = \text{const.} \left(\int_0^\infty |(F_s f)(y)|^{p'} \tanh y dy \right)^{1/p'}$$

$$\leq \text{const.} \left(\int_0^\infty |(F_s f)(y)|^{p'} dy \right)^{1/p'} \leq \text{const.} ||f||_{L_p(\mathbf{R}_+; d\tau)} < \infty,$$

where all constants depend on p, $1 < p \leq 2$ and $\gamma > 0$. Thus we obtain

$$\left(\int_{-\infty}^\infty |\varphi(\gamma+it)\Theta_f(\gamma+it)|^p dt \right)^{1/p} \leq \text{const.} ||f||_{L_p(\mathbf{R}_+; d\tau)}, \tag{4.3}$$

and this is true for all $f \in L_p(\mathbf{R}_+; d\tau), 1 < p \leq 2$ by the continuity of norms. Further, we prove that inequality (3.14) keeps true for $\gamma = 0$ and the corresponding constant is depending on p. Indeed, from (3.13) and the Fatou lemma we immediately obtain that

$$||\mathcal{G}_{H_\varphi} f||_{L_2(\mathbf{R}_+; x^{-1}dx)} \leq \sqrt{2\pi A} ||f||_{L_2(\mathbf{R}_+; d\tau)}. \tag{4.4}$$

Meantime, returning to the representation (3.7) we assume that the function $\Phi(x)$, which is defined by (3.3) satisfies the condition $\Phi(x) \in L_1((0,1); x^{-1}dx)$. Moreover due to condition (3.4) we immediately find that $\Phi(x) \in L_1((1,\infty); x^{-1}dx)$ since $x^\gamma \Phi(x)$ is bounded, $\gamma > 0$. Thus $\Phi(x) \in L_1(\mathbf{R}_+; x^{-1}dx)$. Hence with elementary substitutions we find the estimate

$$\left| (\mathcal{G}_{H_\varphi} f)(x) \right| \leq \sqrt{\frac{\pi}{2}} \sup_{y \in \mathbf{R}_+} |(F_s f)(y)| \int_0^\infty |\Phi(x \cosh^2 y)| \tanh y dy$$

$$= \frac{1}{2} \sqrt{\frac{\pi}{2}} \sup_{y \in \mathbf{R}_+} |(F_s f)(y)| \int_x^\infty |\Phi(u)| \frac{du}{u} \leq \frac{1}{2} \sqrt{\frac{\pi}{2}} ||f||_{L_1(\mathbf{R}_+; d\tau)} ||\Phi||_{L_1(\mathbf{R}_+; x^{-1}dx)} < \infty. \tag{4.5}$$

Thus combining with (4.4) via Riesz-Thorin theorem we arrive at the inequality

$$||\mathcal{G}_{H_\varphi} f||_{L_{p'}(\mathbf{R}_+; x^{-1}dx)} \leq B_p ||f||_{L_p(\mathbf{R}_+; d\tau)}, \tag{4.6}$$

where the constant B_p depends only on p.

Now let us introduce for each $\tau \in \mathbf{R}_+$ similarly to (2.3) the following kernel

$$\mathcal{H}(\tau, x) = \int_0^\tau y H_\psi(x, y) dy, \tag{4.7}$$

which contains a different multiplier function ψ. We note that the corresponding function (3.3) $\Psi(z)$ is given by the integral

$$\Psi(z) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} 2^{2(1-s)}\Gamma(2s+1)\psi(s)z^{-s}ds, \tag{4.8}$$

where we assume the multiplier condition $\psi(\gamma+it) \in L_1(\mathbf{R};\Gamma(2(\gamma+it)+1)dt)$.

Hence as in (3.2) we can write correspondingly

$$\tau H_\psi(\tau,x) = \int_0^\infty \Psi(x\cosh^2 y)\tanh y\sin(\tau y)dy. \tag{4.9}$$

Under condition above on the multiplier ψ it is not difficult to show that the latter integral converges uniformly with respect to $\tau \in [0,T],\ T>0$. Consequently, we can integrate through in (4.9) with respect to τ and invoking (4.7) we obtain

$$\mathcal{H}(\tau,x) = \int_0^\infty \Psi(x\cosh^2 y)\tanh y\frac{1-\cos(\tau y)}{y}dy. \tag{4.10}$$

On the other hand (see (4.1)) we have

$$\mathcal{H}(\tau,x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \psi(s)\Theta(\tau,s)x^{-s}ds, \tag{4.11}$$

where

$$\Theta(\tau,s) = \int_0^\tau \Gamma\left(s+\frac{iy}{2}\right)\Gamma\left(s-\frac{iy}{2}\right)ydy.$$

Thus if we show that for each $\tau \in \mathbf{R}_+$ the kernel (4.7) $\mathcal{H}(\tau,x) \in L_p\left(\mathbf{R}_+;x^{\gamma p-1}dx\right),\ 1<p<2,\ \gamma>0$, then from (4.11) via Th. 86 from [5] we obtain that functions $\mathcal{H}(\tau,x), \psi(\gamma-it)\Theta(\tau,\gamma-it)$ realize a Mellin transform pair from $L_p\left(\mathbf{R}_+;x^{\gamma p-1}dx\right)$ into $L_{p'}(\mathbf{R};dt)$ $p^{-1}+p'^{-1}=1,\ \gamma>0$. Making the substitution $x\cosh^2 y = w$ in (4.10) we majorate the kernel $\mathcal{H}(\tau,x)$ as follows

$$|\mathcal{H}(\tau,x)| \le 2\int_0^\infty \left|\Psi(x\cosh^2 y)\right|\frac{\tanh y}{y}dy = 2\int_x^\infty |\Psi(w)|\frac{dw}{w\operatorname{arccosh}\sqrt{w/x}}.$$

Therefore appealing again to the generalized Minkowski inequality we obtain

$$\left(\int_0^\infty |\mathcal{H}(\tau,x)|^p x^{\gamma p-1}dx\right)^{1/p} \le 2\left(\int_0^\infty x^{\gamma p-1}dx\left|\int_x^\infty |\Psi(w)|\frac{dw}{w\operatorname{arccosh}\sqrt{w/x}}\right|^p\right)^{1/p}$$

$$\le 2\int_0^\infty |\Psi(w)|\frac{dw}{w}\left(\int_0^w \frac{x^{\gamma p-1}}{\operatorname{arccosh}^p\sqrt{w/x}}dx\right)^{1/p} = 2^{1+1/p}\int_0^\infty |\Psi(w)|\,w^{\gamma-1}dw$$

$$\times\left(\int_0^\infty \frac{\tanh u}{u^p\cosh^{2\gamma p}u}du\right)^{1/p} < \infty,\ 1<p<2, \gamma>0, \tag{4.12}$$

if we assume that $\Psi(w)\in L_1(\mathbf{R}_+; w^{\gamma-1}dw)$. If also $\Psi(w)\in L_1((0,1); w^{-1}dw)$ then from (4.12) it easily follows that for each $\tau\in\mathbf{R}_+$ the kernel $\mathcal{H}(\tau,x)\in L_1(\mathbf{R}_+; x^{-1}dx)$. Indeed, under condition on the multiplier function $\psi(s)$ we get $\Psi(w)\in L_1(\mathbf{R}_+; w^{-1}dw)$ and it attains the value $\gamma=0$ in the estimates (4.12). These preliminary discussions lead us to the inversion theorem for the transformation (1.1).

Theorem 3. *Let* $1<p<2,\ f(\tau)\in L_p(\mathbf{R}_+; d\tau)$. *Let also* $\varphi(\gamma+it)$ *and* $\psi(\gamma+it)$ *from the space* $L_1(\mathbf{R};\Gamma(2(\gamma+it)+1)dt), \gamma>0$ *be two multipliers, which define* $\Phi(x)$ *and* $\Psi(x)$ *by formulas* (3.3), (4.8), *correspondingly and satisfy the equality*

$$\lim_{\varepsilon\to 0+}\sup_{t\in\mathbf{R}}|\varphi(\gamma+it)\psi(\varepsilon-\gamma-it)-1| = 0. \tag{4.13}$$

If $\Phi(x)\in L_1((0,1); x^{-1}dx)$ *and* $\Psi(x)\in L_1(\mathbf{R}_+; x^{\kappa-1}dx)\ \cap L_1((0,1); x^{-1}dx),\ \kappa\in\left(0,\frac{1}{p}\right]$, *then for almost all* $\tau\in\mathbf{R}_+$ *the inversion formula for the general index transform* (1.1) *holds true*

$$f(\tau) = \frac{1}{4\pi^2}\frac{\sinh(\pi\tau)}{\tau}\frac{d}{d\tau}\int_0^\infty \mathcal{H}(\tau,x)\left(\mathcal{G}_{H_\varphi}f\right)(x)\frac{dx}{x}, \tag{4.14}$$

where the kernel $\mathcal{H}(\tau,x)$ *is defined by* (4.7) *and the latter integral is a Lebesgue one.*

Proof. In fact, as we concluded above under conditions of the theorem functions $\varphi(\gamma+it)\Theta_f(\gamma+it)$, $(\mathcal{G}_{H_\varphi}f)(x)$ realize a Mellin transform pair from $L_p(\mathbf{R}; dt)$ into $L_{p'}\left(\mathbf{R}_+; x^{\gamma p'-1}dx\right)$ with $1<p<2, \gamma>0$. Meantime for any $\varepsilon>0$ we choose $\gamma>0$ such that $\varepsilon-\gamma\in\left(0,\frac{1}{p}\right)$. Thus under conditions of the theorem we find that functions $\mathcal{H}(\tau,x), \psi(\varepsilon-\gamma-it)\Theta(\tau,\varepsilon-\gamma-it)$ form a Mellin transform pair from $L_p\left(\mathbf{R}_+; x^{(\varepsilon-\gamma)p-1}dx\right)$ into $L_{p'}(\mathbf{R}; dt)$. Consequently, due to Th. 88 from [5] the following relation takes place

$$\int_0^\infty (\mathcal{G}_{H_\varphi}f)(x)\mathcal{H}(\tau,x)x^{\varepsilon-1}dx = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Theta_f(s)\Theta(\tau,\varepsilon-s)\varphi(s)\psi(\varepsilon-s)ds$$

$$= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Theta_f(s)\Theta(\tau,\varepsilon-s)\left[\varphi(s)\psi(\varepsilon-s)-1\right]ds + \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Theta_f(s)\Theta(\tau,\varepsilon-s)ds$$

$$= I_1(\tau,\varepsilon)+I_2(\tau,\varepsilon). \tag{4.15}$$

Hence by using Theorem 1 and relations (1.4), (1.5), (1.6), (2.3), (4.2), (4.11) in the same manner via Th. 88 from [5] we obtain that

$$I_2(\tau,\varepsilon) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Theta_f(s)\Theta(\tau,\varepsilon-s)ds = 2\int_0^\infty (KLf)(x)\hat{K}(\tau,x)x^{\varepsilon-1}dx. \tag{4.16}$$

Hence invoking equality (2.16) and making $\varepsilon \to 0+$ we find

$$\lim_{\varepsilon\to 0+} I_2(\tau,\varepsilon) = 4\pi^2 \int_0^\tau \frac{y}{\sinh \pi y} f(y)dy. \tag{4.17}$$

Meanwhile with the Hölder inequality we estimate $I_1(\tau,\varepsilon)$ as follows

$$|I_1(\tau,\varepsilon)| \le \frac{1}{2\pi} \sup_{t\in\mathbf{R}} |\varphi(\gamma+it)\psi(\varepsilon-\gamma-it)-1| \int_{\gamma-i\infty}^{\gamma+i\infty} |\Theta_f(s)\Theta(\tau,\varepsilon-s)ds|$$

$$\le \frac{1}{2\pi} \sup_{t\in\mathbf{R}} |\varphi(\gamma+it)\psi(\varepsilon-\gamma-it)-1| \left(\int_{\gamma-i\infty}^{\gamma+i\infty} |\Theta_f(s)|^p \, |ds|\right)^{1/p}$$

$$\times \left(\int_{\varepsilon-\gamma-i\infty}^{\varepsilon-\gamma+i\infty} |\Theta(\tau,z)|^{p'} \, |dz|\right)^{1/p'},$$

where the norm $||\Theta_f||_{L_p(\mathbf{R};dt)}$ is finite for any $f\in L_p(\mathbf{R};d\tau)$ due to (4.3). Moreover, employing an analog of the Hausdorff-Young inequality for the Mellin transform (cf. Th. 74, 86 in [5]) we establish that the latter integral is also finite . Indeed, since via (1.5) the kernel (2.3) and function $\Theta(\tau,s)$ realize a Mellin transform pair then by taking into account asymptotic properties of the kernel $\hat{K}(\tau,x)$ (see the proof of Theorem 1) we deduce

$$\left(\int_{\varepsilon-\gamma-i\infty}^{\varepsilon-\gamma+i\infty} |\Theta(\tau,z)|^{p'} \, |dz|\right)^{1/p'} \le C_p \left(\int_0^\infty \left|\hat{K}(\tau,x)\right|^p x^{(\varepsilon-\gamma)p-1} dx\right)^{1/p}$$

$$\le C_p \left(\int_0^1 \left|\hat{K}(\tau,x)\right|^p \frac{dx}{x} + \int_1^\infty \left|\hat{K}(\tau,x)\right|^p dx\right)^{1/p} < \infty.$$

Therefore appealing to (4.13) we derive

$$|I_1(\tau,\varepsilon)| \le \text{const.} \sup_{t\in\mathbf{R}} |\varphi(\gamma+it)\psi(\varepsilon-\gamma-it)-1| \to 0, \ \varepsilon\to 0+\,.$$

Combining with (4.15), (4.17) we arrive at the equality

$$\lim_{\varepsilon\to 0+} \int_0^\infty (\mathcal{G}_{H_\varphi} f)(x)\mathcal{H}(\tau,x)x^{\varepsilon-1}dx = 4\pi^2 \int_0^\tau \frac{y}{\sinh \pi y} f(y)dy, \tau\in\mathbf{R}_+. \tag{4.18}$$

On the other hand we will motivate the passage to the limit under the integral sign in the left-hand side of (4.18) via the dominated convergence theorem. In fact, the integrand is less than or equal to the majorant $M(\tau,x)$

$$M(\tau,x) = \begin{cases} x^{-1}|(\mathcal{G}_{H_\varphi} f)(x)\mathcal{H}(\tau,x)|, & \text{if } x\in[0,1], \\ x^{\gamma+1/p-1}|(\mathcal{G}_{H_\varphi} f)(x)\mathcal{H}(\tau,x)|, & \text{if } x\in(1,\infty), \end{cases}$$

which belongs to $L_1(\mathbf{R}_+; dx)$. The summability of $M(\tau, x)$ can be verified by using (3.14) the chain of estimates (4.12) and the condition $\Psi(x) \in L_1(\mathbf{R}_+; x^{\kappa-1}dx)$ with $\kappa = \frac{1}{p}$. Precisely, we have

$$\int_0^\infty M(\tau,x)dx = \int_0^1 |(\mathcal{G}_{H_\varphi} f)(x)\mathcal{H}(\tau,x)|\frac{dx}{x} + \int_1^\infty x^{\gamma+1/p-1}|(\mathcal{G}_{H_\varphi} f)(x)\mathcal{H}(\tau,x)|dx$$

$$\leq \left(\int_0^1 |(\mathcal{G}_{H_\varphi} f)(x)|^{p'}\frac{dx}{x}\right)^{1/p'} \left(\int_0^1 |\mathcal{H}(\tau,x)|^{p}\frac{dx}{x}\right)^{1/p} + \left(\int_1^\infty |(\mathcal{G}_{H_\varphi} f)(x)|^{p'} x^{\gamma p'-1}dx\right)^{1/p'}$$

$$\times \left(\int_1^\infty |\mathcal{H}(\tau,x)|^{p}dx\right)^{1/p} \leq ||\mathcal{G}_{H_\varphi} f||_{L_{p'}\left(\mathbf{R}_+; x^{\gamma p'-1}dx\right)} \left[||\mathcal{H}||_{L_p(\mathbf{R}_+; x^{-1}dx)}\right.$$

$$\left. +\text{const.} \int_0^\infty |\Psi(w)|\, w^{1/p-1}dw\right] < \infty.$$

Hence equality (4.18) becomes as

$$\int_0^\infty (\mathcal{G}_{H_\varphi} f)(x)\mathcal{H}(\tau,x)\frac{dx}{x} = 4\pi^2 \int_0^\tau \frac{y}{\sinh \pi y} f(y)dy, \tau \in \mathbf{R}_+. \qquad (4.18)$$

Differentiating through this equality with respect to τ we finally arrive at the inversion formula (4.14), which takes place for almost all $\tau \in \mathbf{R}_+$. Theorem 3 is proved.

5 Examples

In this final section we consider some classical index transformations in L_p, which are particular cases of the general transform (1.1). Namely, we will mention the results of Section 2 for the Kontorovich-Lebedev transform (1.6) and we will establish boundedness and inversion properties for the generalized Mehler-Fock transform (1.8) and for the Olevskii transform (1.9). The L_2 case for these transformations is considered in [10].

1. The Kontorovich-Lebedev transform. We put in (1.2) $\varphi(s) = \psi(s) \equiv 1$. By using the Stirling asymptotic formula for the Gamma - function (cf. [1, Vol. I]) we see that these multipliers evidently satisfy condition (3.4). So, via (1.5) we obtain the modified KL-transform (1.6), for which Theorem 1 holds true. Calculating the corresponding Mellin integrals (3.3), (4.8) we invoke relation (8.4.3.1) from [3] and we get $\Phi(x) = \Psi(x) = 4\sqrt{x}e^{-2\sqrt{x}}$. We note that all conditions of Theorem 3 are clearly satisfied and we arrive at the inversion formula (4.14), which coincides in this case with (2.2).

2. The generalized Mehler -Fock transform. Let us consider transformation (1.8), where the corresponding multiplier function is equal to $\varphi(s) = \Gamma(1/2-\mu-s)/\Gamma(1/2+s), \mu <$

1/2. It satisfy the condition (3.4). Substituting the value of $\varphi(s)$ into (3.3) and employing the duplication formula for the Gamma - function we derive

$$\Phi(x)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}2^{2(1-s)}\Gamma(2s+1)\frac{\Gamma(1/2-\mu-s)}{\Gamma(1/2+s)}x^{-s}ds=\frac{4}{\sqrt{\pi}}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(s+1)$$

$$\times\Gamma(1/2-\mu-s)x^{-s}ds=\frac{4}{\sqrt{\pi}}\Gamma(3/2-\mu)\frac{x}{(1+x)^{3/2-\mu}},$$

where the latter integral is calculated via relation (8.4.2.5) from [3]. It is clear that the function $\Phi(x)\in L_1(\mathbf{R}_+;x^{-1}dx),\mu<1/2$. Thus according to Theorem 2 and inequality (4.6) the generalized Mehler-Fock transform (1.8) is a bounded operator $MF:L_p\left(\mathbf{R}_+;d\tau\right)\to L_{p'}\left(\mathbf{R}_+;x^{\gamma p'-1}dx\right),\ p^{-1}+p'^{-1}=1,\ 1\le p\le 2,\ 0\le\gamma<1/2-\mu$. In order to establish the corresponding inversion formula (4.14) we look for the kernel (4.7) for this case with $\psi(s)=\Gamma(1/2-s)/\Gamma(1/2-\mu+s)$. Appealing to the relation (8.4.41.12) from [3] we obtain

$$\mathcal{H}(\tau,x)=\pi x^{-1/2}(1+x)^{-\mu/2}\int_0^\tau\frac{y}{\cosh(\pi y/2)}P^{\mu}_{(iy-1)/2}\left(\frac{2}{x}+1\right)dy.$$

Furthermore, the multipliers plainly satisfy condition (4.13). From (4.8) we have for this case

$$\Psi(x)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}2^{2(1-s)}\Gamma(2s+1)\frac{\Gamma(1/2-s)}{\Gamma(1/2-\mu+s)}x^{-s}ds$$

$$=\frac{4}{\sqrt{\pi}}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(s+1/2)\Gamma(s+1)\Gamma(1/2-s)}{\Gamma(1/2-\mu+s)}x^{-s}ds.$$

The latter integral via relation (8.4.49.14) from [3] can be expressed in terms of the Gauss hypergeometric function. Namely, we get

$$\Psi(x)\equiv\Psi_>(x)=\frac{4}{\sqrt{\pi}}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(s+1/2)\Gamma(s+1)\Gamma(1/2-s)}{\Gamma(1/2-\mu+s)}x^{-s}ds$$

$$=\frac{2}{\sqrt{x}\Gamma(1-\mu)}{}_2F_1\left(1,\frac{3}{2};1-\mu;-\frac{1}{x}\right),\ x>1,$$

$$\Psi(x)\equiv\Psi_<(x)=\frac{4\cos\pi\mu\Gamma(3/2+\mu)}{\sqrt{\pi}}\frac{x}{(1+x)^{3/2+\mu}}-\frac{4\sqrt{x}\sin\pi\mu}{\pi}\Gamma(1+\mu){}_2F_1\left(1,1+\mu;\frac{1}{2};-x\right),$$

where $0<x<1$. Moreover, it satisfies $\Psi_>(1)=\Psi_<(1)$. Thus, $\Psi(x)\in L_1(\mathbf{R}_+;x^{\kappa-1}dx)\ \cap L_1((0,1);x^{-1}dx)$, where $\kappa\in(0,1/2)\subset\left(0,\frac{1}{p}\right]$ and the corresponding inversion formula for the generalized Mehler-Fock transform can be written in the form

$$f(\tau)=\frac{1}{4\pi}\frac{\sinh(\pi\tau)}{\tau}\frac{d}{d\tau}\int_0^\infty\int_0^\tau\frac{y}{\cosh(\pi y/2)}P^{\mu}_{(iy-1)/2}\left(\frac{2}{x}+1\right)x^{-3/2}(1+x)^{-\mu/2}[MFf](x)dydx.$$

3. The Olevskii tranformation. Our final example is the Olevskii transformation with the kernel (1.9), which generalizes the Mehler-Fock transform (1.8)

$$[\mathcal{O}f](x) = \frac{x^{-a}(1+x)^{2a-c}}{\Gamma(c)} \int_0^\infty \left|\Gamma\left(c-a+\frac{i\tau}{2}\right)\right|^2 {}_2F_1\left(a+\frac{i\tau}{2}, a-\frac{i\tau}{2}; c; -\frac{1}{x}\right) f(\tau)\tau d\tau. \quad (5.1)$$

As we could see in Section 1 in this case $\varphi(s) = \Gamma(c-a-s)/\Gamma(s+a), 0 < a < c, 0 < \text{Re}s = \gamma < c-a, c \neq 0, -1, -2, \ldots$. Hence we observe that condition (3.4) is guaranteed. Analogously we write the function (3.3) for this case

$$\Phi(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} 2^{2(1-s)}\Gamma(2s+1)\frac{\Gamma(c-a-s)}{\Gamma(a+s)}x^{-s}ds = \frac{4}{\sqrt{\pi}}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(1/2+s)\Gamma(s+1)$$

$$\times\frac{\Gamma(c-a-s)}{\Gamma(a+s)}x^{-s}ds.$$

Omitting calculations of the latter Mellin-Barnes integral for $0 < x \leq 1$ and $x > 1$ we use its asymptotic behavior (or the asymptotic behavior of the corresponding Meijer G-function) [3] when $x \to 0$ and $x \to \infty$. For other values of $x > 0$ it converges uniformly and represents a continuous function. So we find that $\Phi(x) = O(x^{1/2}), x \to 0$ and $\Phi(x) = O(x^{a-c}), x \to \infty$. Therefore it plainly belongs to the space $L_1(\mathbf{R}_+; x^{-1}dx)$. Thus by Theorem 2 we conclude that the Olevskii transform (5.1) is a bounded operator $\mathcal{O} : L_p(\mathbf{R}_+; d\tau) \to L_{p'}\left(\mathbf{R}_+; x^{\gamma p'-1}dx\right),\ p^{-1} + p'^{-1} = 1,\ 1 \leq p \leq 2,\ 0 \leq \gamma < c-a.$

Similarly by taking $\psi(s) = \Gamma(a-s)/\Gamma(c-a+s)$, we satisfy condition (4.13) and via relation (8.4.49.14) we arrive at the following kernel (4.7) for this case

$$\mathcal{H}(\tau, x) = \frac{x^{-a}}{\Gamma(c)}\int_0^\tau \left|\Gamma\left(a+\frac{iy}{2}\right)\right|^2 {}_2F_1\left(a+\frac{iy}{2}, a-\frac{iy}{2}; c; -\frac{1}{x}\right) dy.$$

The related function $\Psi(x)$ is continuous for $x > 0$ and behaves as $\Psi(x) = O(x^{1/2}), x \to 0$ and $\Psi(x) = O(x^{-a}), x \to \infty$. Consequently, if $a < \frac{1}{p}$ then $\Psi(x) \in L_1(\mathbf{R}_+; x^{\kappa-1}dx) \cap L_1((0,1); x^{-1}dx)$, where $\kappa \in (0, a) \subset \left(0, \frac{1}{p}\right]$. Hence via Theorem 3 the inversion formula for the Olevskii transform

$$f(\tau) = \frac{1}{4\pi^2\Gamma(c)}\frac{\sinh(\pi\tau)}{\tau}\frac{d}{d\tau}\int_0^\infty\int_0^\tau \left|\Gamma\left(a+\frac{iy}{2}\right)\right|^2 {}_2F_1\left(a+\frac{iy}{2}, a-\frac{iy}{2}; c; -\frac{1}{x}\right)$$

$$\times x^{-a-1}[\mathcal{O}f](x)dydx$$

holds valid for almost all $\tau \in \mathbf{R}_+$.

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