SEMI-COMPLETE VECTOR FIELDS OF SADDLE-NODE TYPE IN $\mathbb{C}^n$

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Abstract. We classify the foliations associated to semi-complete vector fields at an isolated singularity of saddle-node type in $\mathbb{C}^n$, i.e., associated to semi-complete vector fields in $\mathbb{C}^n$, whose linear part is diagonalizable and with an isolated singularity, where the $n-1$ non-vanishing eigenvalues of its linear part are non-resonant and in the Poincaré Domain. We will also analyse the semi-completude of a vector field with a diagonal linear part associated to a saddle-node foliation, whose set of singularities is the holomorphic invariant hypersurface transverse to the weak direction.

1. Introduction

The definition of a semi-complete vector field relatively to a (relatively compact) open set U is introduced in [8]. The importance of that definition is that:

Proposition. [8] *Let X be a complete holomorphic vector field on a complex manifold M. The restriction of X to any connected, (relatively compact) open set U ($U \subseteq M$) is a semi-complete vector field relatively to U.*

Therefore, if a holomorphic vector field in an open set U is not semi-complete it cannot be extended to a compact manifold containing U.

In [9] Rebelo classifies the semi-complete singularities of saddle-node type in $\mathbb{C}^2$, such that $(0,0) \in \mathbb{C}^2$ is an isolated singularity. There, he proves:

Theorem. [9] *Let $\mathcal{F}$ be a saddle-node defined in a neighbouhood of $(0,0) \in \mathbb{C}^2$ and w a differential 1-form, with an isolated singularity at the origin, defining $\mathcal{F}$. The foliation $\mathcal{F}$ is associated to a semi-complete vector field iff w admits*

$$x(1+\lambda y)dy - y^2 dx$$

as normal form, with $\lambda \in \mathbb{Z}$.

Here we classify the semi-complete vector fields of saddle-node type in $\mathbb{C}^n$, at an isolated singularity. By a saddle-node, with an isolated singularity at p, we mean a holomorphic vector field X such that $X(p) = 0$ and $DX(p)$ is diagonalizable and has one and exactly one eigenvalue equal to zero, beeing the other eigenvalues non-resonant and in the Poincaré Domain.

In both problems the vector fields are 1-resonant: we say that a vector field is 1-resonant if $\dim\{m \in \mathbb{Z}^n : (m, \lambda) = 0\} = 1$, where λ is the vector constitued by the eigenvalues of the linear part of the vector field.

The result obtained here is similar to the one obtained by Rebelo. We prove:

Theorem. *Let $\mathcal{F}$ be a foliation of a saddle-node, in a neighbourhood of the origin, with an isolated singularity at the origin. Then $\mathcal{F}$ is associated to a semi-complete vector field iff it admits the normal form*

$$\begin{cases} \dot{x} = x_1^2 \\ \dot{y} = x_2(\lambda_2 + \alpha_2 x_1) \\ \vdots \\ \dot{z} = x_n(\lambda_n + \alpha_n x_n) \end{cases}$$

with $(\alpha_2, \dots, \alpha_n) \in \mathbb{Z}^{n-1}$.

At the end of the article we study the semi-completude of vector fields with diagonal linear part associated to saddle-node foliations, whose set of singularities coincides with the invariant manifold transverse to the weak axis at the origin, whose existence is guaranteed in [1].

Let $Y_p(x) = (x_1^{p+1}, \lambda_2 x_2 + x_1 a_2(x), \dots, \lambda_n x_n + x_1 a_n(x))$, $p \in \mathbb{N}$, be a holomorphic vector field of saddle-node type such that $a_i(0) = 0$, $\forall i = 2, \dots, n$. We obtain:

Proposition. *Let X be a vector field of type $f x_1^{-k} Y_p$, where f is a holomorphic function such that $f(0) \neq 0$, $k \in \mathbb{Z}$ and X defined in an open neighbourhood $U \subseteq \mathbb{C}^n$ of the origin. Suppose that X is semi-complete in a neighbourhood of the origin, then $k \in \{p-1, p, p+1\}$.*

Consequently we conclude that:

Corollary. *Let X be a holomorphic vector field of saddle-node type, with an isolated singularity at the origin, and M the invariant hypersurface transverse to the weak direction of X. If F is a holomorphic function such that $F(x) = 0 \Leftrightarrow x \in M$, then the holomorphic vector field FX is not semi-complete in any neighbourhood of the origin.*

I would like to thank my Ph. D. advisor, José Basto Gonçalves, for his support. I also wish to thank Júlio Rebelo, who sugested me the problem, and J. C. Canille Martins for their valuable conversations on the subject.

2. Premilinaries - Definitions and Basic Results

In this section we introduce the definitions and the basic and most important results related and necessary to solve the problem.

Let $X : U \subseteq \mathbb{C}^n \to \mathbb{C}^n$ and $Y : V \subseteq \mathbb{C}^n \to \mathbb{C}^n$ be holomorphic vector fields with a singularity at the origin. We say that X is analytically (formally, C^∞, C^k) conjugated to Y in a neighbourhood of the origin if there exists a holomorphic (formal, C^∞, C^k) diffeomorphism $H : V_1 \to U_1$, where $0 \in U_1 \subseteq U$, $0 \in V_1 \subseteq V$, such that $H(0) = 0$ and

$$Y = (DH)^{-1} X \circ H.$$

We say that X and Y are analytically (formally, C^∞, C^k) equivalent if X is analytically (formally, C^∞, C^k) conjugated to a multiple of Y.

Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be the vector of the eigenvalues of $DX(0)$. We say that the eigenvalues are resonant if, for some i, there exists $I = (i_1, \dots, i_n) \in \mathbb{N}_0^n$ with $\sum_{j=1}^n i_j \geq 2$ and such that

$$\lambda_i = I.\lambda = i_1\lambda_1 + \ldots + i_n\lambda_n.$$

Finally, we say that λ is in the Poincaré Domain if the origin is not in the convex hull of the points $\{\lambda_i : i = 1, \dots, n\}$ (in the complex plane). Otherwise we say that they are in the Siegel Domain.

Definition 1. Let X be a holomorphic vector field defined in a complex manifold M. We say that X is complete if there is a holomorphic application

$$\Phi : \mathbb{C} \times M \to M$$

such that

a) $\Phi(0, x) = x \qquad \forall x \in M$
b) $\Phi(T_1 + T_2, x) = \Phi(T_2, \Phi(T_1, x)) \qquad \forall x \in M, \forall T_1, T_2 \in \mathbb{C}$
c) $X(x) = \dfrac{d}{dT}|_{T=0} \Phi(T, x)$

The orbits of a complete vector field are topologically the complex plane $\mathbb{C}$, the cylinder $\mathbb{C}/\mathbb{Z}$ or the torus $\mathbb{C}/\Lambda$. The orbits of a non-complete vector field also define a singular foliation of M, where each leaf is also a Riemann surface, but its topology can be much more complex.

Definition 2. Let X be a holomorphic vector field defined in an open set U, $U \subseteq M$, where M is a complex manifold. We say that X is semi-complete relatively to U if there exists a holomorphic application

$$\Phi : \Omega \subseteq \mathbb{C} \times U \to U$$

where Ω is an open set containing $\{0\} \times U$ such that

a) $X(x) = \dfrac{d}{dT}|_{T=0} \Phi(T, x)$
b) $\Phi(T_1 + T_2, x) = \Phi(T_2, \Phi(T_1, x))$, when the two members are defined
c) $(T_i, x) \in \Omega$ and $(T_i, x) \to \partial\Omega \quad \Rightarrow \quad \Phi(T_i, x) \to \partial U$

We call Φ the semi-complete flow associated to the vector field X.

In [8] and [9], Rebelo presents sufficient and necessary conditions for a vector field to be semi-complete in an open set U. The regular orbits of a vector field X ($X \not\equiv 0$) are Riemann surfaces. To each one of its orbits (leaves), L, we can associate a holomorphic differential 1-form, dT_L, such that $dT_L(X) = 1$. In this way, we can define an application $D : \tilde{L} \to \mathbb{C}$, where $\tilde{L}$ is the universal covering of L, such that its differential is the lift of dT_L to $\tilde{L}$.

Proposition 1. [8] *Let X be a semi-complete vector field relatively to an open set U ($U \subseteq M$). If L is a non-singular orbit of X in U, then the integral of dT_L over any one-to-one embedded curve in L is non zero.*

Proposition 2. [9] *Let X be a holomorphic vector field defined in a neighbourhood U of the origin of $\mathbb{C}^n$. Suppose that for all regular orbits L of X and every $c : [0,1] \to L$ such that $c(0) \neq c(1)$ the integral of dT_L over c is non zero. Then the vector field X is semi-complete in U.*

Those propositions are essential in the classification of the semi-complete singularities of saddle-node type in $\mathbb{C}^2$, as it will also be important in our case.

We also have a very easy to prove result, which enables us to obtain semi-complete vector fields from other semi-complete vector fields:

Proposition 3. [3] *Let X be a semi-complete vector field relatively to an open set U and h a first integral of X. Then, the vector field hX is also semi-complete relatively to U. In particular, if X is semi-complete, then cX is semi-complete for any constant c.*

As we said in the introduction, in the $\mathbb{C}^2$ case Rebelo shows:

Theorem 1. [9] *Let $\mathcal{F}$ be a saddle-node defined in a neighbouhood of $(0,0) \in \mathbb{C}^2$ and w a differential 1-form, with an isolated singularity at the origin, defining $\mathcal{F}$. The foliation $\mathcal{F}$ is associated to a semi-complete vector field iff w admits*

$$x(1 + \lambda y)dy - y^2 dx$$

as normal form, with $\lambda \in \mathbb{Z}$.

In fact, by the Dulac's Theorem, as $w = 0$ is a germ of a saddle-node defined in a neighbourhood of $(0,0) \in \mathbb{C}^2$, there exist $p \geq 1$, $\lambda \in \mathbb{C}$ and a systems of coordinates where $w = 0$ is written, in a neighbourhood of the origin, as

$$[x(1 + \lambda y^p) + yR(x,y)]dy - y^{p+1}dx = 0$$

where R has multiplicity greater or equal to $p+1$ at $(0,0)$.

Let

$$X : \begin{cases} \dot{x} = x(1 + \lambda y^p) + yR(x,y) \\ \dot{y} = y^{p+1} \end{cases}$$

be (one of) the vector field(s) whose foliation coincides with $\mathcal{F}$.

Let $\hat{G}_0$ represents the group of formal diffeomorphisms of the form $\hat{H} = (x + \sum_{k=1}^{\infty} h_k(x)y^n, y)$, with h_n holomorphic in a neighbourhood of the origin.

Each element X is equivalent, by a unique element of $\hat{G}_0$, to its formal normal form

$$\begin{cases} \dot{x} = x(1 + \lambda y^p) \\ \dot{y} = y^{p+1} \end{cases}$$

where p and λ are formal invariants.

In this context Theorem 1 means that there exists a holomorphic function f, with $f(0,0) \neq 0$, such that fX is semi-complete if and only if X is analytically conjugated to its formal normal form with $\lambda \in \mathbb{Z}$ and $p = 1$.

Our main objective is to prove a similar result for saddle-nodes in $\mathbb{C}^n$.

In [8] is proved that a 1-dimensional holomorphic vector field X : $\dot{x} = f(x)$ such that $f(0) = f^{(1)}(0) = f^{(2)}(0) = 0$ is not semi-complete in any neighbourhood of the origin. Here we also prove that if the origin is a pole for X, and consequently an isolated singularity, then X is not semi-complete in any neighbourhood of the origin.

Proposition 4. *Consider the* 1*-dimensional vector field given by* X : $\dot{x} = x^k f(x)$*, such that* f *is a holomorphic function verifying* $f(0) \neq 0$ *and* $k \in \mathbb{Z}$*. If* $k \geq 3$ *or* $k \leq -1$ *the vector field is interdict, i.e., is not semi-complete relatively to any small neighbourhood of the origin.*

Remark 1. If $k \leq -1$, the vector field considered is not holomorphic in $\mathbb{C}$, but is holomorphic in $\mathbb{C} \setminus \{0\}$. In this case the origin is a singularity in the sense that the vector field is not defined there.

The result of Rebelo is inclued in this proposition. To prove this result we need the following lemma:

Lemma 1. *Let* $f : U \subseteq \mathbb{C} \to \mathbb{C}$ *be a function of the form*

$$f(x) = ax^k + g(x, \lambda)$$

where g *is a holomorphic function in* U *such that* $g(x,0) = 0$*,* $k \in \mathbb{Z}$ *and* $a \in \mathbb{C} \setminus \{0\}$ *(*$\lambda \in \mathbb{C}^n$*,* $n \geq 1$*, is a parameter). Let* W *be a simply connected open set of* U *such that* $0 \notin W$ *and* f *is never zero in* W*,* $\forall \lambda : \|\lambda\| \leq \varepsilon_0$*. Consider the function*

$$\begin{aligned} I_\lambda &: W \to \mathbb{C} \\ p &\mapsto \int_{c_p} \frac{dx}{ax^k + g(x,\lambda)} \end{aligned}$$

where $c_p \subseteq W$ *is a curve joining a fixed* $x_0 \in W$ *to* p*. Then, there exist real and positive numbers* θ *and* λ_0 *such that*

$$\forall \lambda : \|\lambda\| \leq \lambda_0, \quad B(0,\theta) \subseteq I_\lambda(W)$$

Remark 2. As $0 \notin W$ and f is non zero in W the integral does not depend on the choosen curve.

Proof of the Lemma. The proof of this lemma is based in the following theorem:

Theorem 2. [6] *Let $f : \mathbb{C} \to \mathbb{C}$ be analytic and not constant on a region A. Let $z_0 \in A$ and $w_0 = f(z_0)$. Suppose that $f(z) - w_0$ has a zero of order $k \geq 1$ at z_0. Then there is an $\eta > 0$ such that for any $\tau \in]0, \eta]$ there is a $\delta > 0$ such that, if $|w - w_0| < \delta$, then $f(z) - w$ has exactly k distinct roots in the disc $|z - z_0| < \tau$.*

We have $I_\lambda(x_0) = 0, \forall \lambda \in \mathbb{C}$, in particular $I_0(x_0) = 0$. As $g(x, 0) = 0$, $I_0'(x_0) = \dfrac{1}{x_0^k} \neq 0$. Thus I_0 has a zero of order 1 at x_0. However, as I_λ is a continuous function of λ, there exists $0 < \lambda_0 \leq \varepsilon_0$ such that

$$I_\lambda'(x_0) \in B(\frac{1}{x_0^k}, \frac{1}{2|x_0^k|}) \qquad \forall \lambda : \|\lambda\| \leq \lambda_0.$$

In particular, $I_\lambda'(x_0) \neq 0$, $\forall \lambda : \|\lambda\| \leq \lambda_0$, and so I_λ as a zero of order 1 at x_0, $\forall \lambda : \|\lambda\| \leq \lambda_0$.

Thus, by Theorem 2, for each $\lambda \in B(0, \lambda_0) \subseteq \mathbb{C}^n$ there exists $\eta_\lambda > 0$ such that $\forall \tau \in]0, \eta_\lambda]$ exists $\delta_\lambda^\tau > 0$ such that if $|w - 0| < \delta_\lambda^\tau$, then $I_\lambda(p) = w$ has exactily one solution in the disc $|p - x_0| < \tau$.

Consider the application

$$\begin{aligned} T_\eta : D(0, \lambda_0) &\to \mathbb{R} \\ \lambda \quad &\mapsto \eta_\lambda \end{aligned}$$

As 0 does not belong to the image of $D(0, \lambda_0)$ by T_η and $D(0, \lambda_0)$ is compact, its image has a minimum μ. Then $0 < \mu \leq \eta_\lambda$, $\forall \lambda : \|\lambda\| \leq \lambda_0$.

Consider now the application

$$\begin{aligned} T_\delta : D(0, \lambda_0) &\to \mathbb{R} \\ \lambda \quad &\mapsto \delta_\lambda^\mu \end{aligned}$$

Denote by θ the minimum of $T_\delta(D(0, \lambda_0))$ $(\theta > 0)$.

Remark that

$$|w - 0| < \delta_\lambda^\tau \quad \Rightarrow \quad D_\lambda(p) = w \text{ has exactily one solution in } |p - x_0| < \tau$$

In our case

$$\begin{aligned} |w - 0| < \theta \quad &\Rightarrow \quad |w - 0| < \delta_\lambda^\mu \quad \forall \lambda : \|\lambda\| \leq \lambda_0 \\ &\Rightarrow \quad D_\lambda(p) = w \text{ has exactly one solution in } |p - x_0| < \mu \\ &\qquad \forall \lambda : \|\lambda\| \leq \lambda_0 \end{aligned}$$

As μ is the minimum of T_η we can conclude that

$$B(0, \theta) \subseteq D_\lambda(W), \quad \forall \lambda : \|\lambda\| \leq \lambda_0$$

$\square$

Proof of the Proposition. Suppose that $k \geq 3$ or $k \leq -1$ and that the vector field X is semi-complete relatively to $B(0,\varepsilon) \subseteq \mathbb{C}$. We can assume ε so small that the origin is the only singularity in $B(0,\varepsilon)$ (this is possible because $f(0) \neq 0$ and so is non zero in a small neighbourhood of the origin).

Let $f(0) = a_0$. Thus $g(x) = x^k f(x) = a_0 x^k + a_1 x^{k+1} + a_2 x^{k+2} + \ldots$.

Fixed $\lambda \in \mathbb{C} \setminus \{0\}$ the vector field X is semi-complete relatively to $B(0,\varepsilon)$ iff $Y_\lambda = \lambda^{-1} X(\lambda x)$ is semi-complete relatively to $B(0, \frac{\varepsilon}{|\lambda|})$. This is so, because X and Y_λ are analytically conjugated by the homothety $H_\lambda(x) = \lambda x$. If we take $\lambda = \varepsilon$ it is sufficient to analyse if Y_λ is semi-complete in $B(0,1)$.

We have that the 1-form dT^X such that $dT^X(X) = 1$ is given by

$$dT^X = \frac{dx}{x^k f(x)}$$

As the 1-form $dT^{Y_\lambda} = H_\lambda^*(dT^X)$ we have that

$$dT^{Y_\lambda} = \frac{d(\lambda x)}{(\lambda x)^k f(\lambda x)} = \frac{dx}{\lambda^{k-1} x^k f(\lambda x)}$$

As we are assuming $k \geq 3$ or $k \leq -1$, then $k-1 \geq 2$ or $k-1 \leq -2$, i.e., $|k-1| \geq 2$. As λ^{k-1} is a constant, Y^λ is semi-complete in $B(0,1)$ iff $Z^\lambda = \dfrac{Y^\lambda}{\lambda^{k-1}}$ is semi-complete in the same ball.

We only need to verify the existence of an one-to-one embedded curve $c : [0,1] \to B(0,1) \setminus \{0\}$ such that

$$\int_c \frac{dx}{x^k f(\lambda x)} = 0$$

to contradict the hypothesis.

Consider the curve $c(t) = re^{2\pi i t/(k-1)}$, $t \in [0,1]$ and $0 < r < 1$. Since $|k-1| \geq 2$, this curve is an one-to-one embedded curve.

Let $W \subseteq B(0,1)$ be a simply connected neighbourhood of $c(1)$ containing neither the origin nor $c(0)$. We choose, for example, $W = B(c(1),\delta) \subseteq B(0,1)$, where $0 < \delta < \dfrac{|c(1)-0|}{2}$ and $0 < \delta < |c(1)-c(0)|$.

Denote by

$$\begin{aligned} I_\lambda : W &\to \mathbb{C} \\ p &\mapsto \int_{c_p} \frac{dx}{x^k f(\lambda x)} \end{aligned}$$

the application that associates to every point $p \in W$ the integral of the 1-form $dT^{\frac{Z_\lambda}{\lambda^{k-1}}}$ through a curve c_p joining $c(1)$ to p, inside W. It is obvious that the value of the integral does not depend on the choosen curve.

As $x^k f(\lambda x) = a_0 x^k + a_1 \lambda x^{k+1} + a_2 \lambda^2 x^{k+2} + \ldots$, we can write this function as $a_0 x^k + h(x,\lambda)$, where $h(x,\lambda)$ is holomorphic in W and satisfies $h(x,0)=0$. By lemma 1 there exist real and positive numbers θ and λ_0 such that $B(0,\theta) \subseteq I_\lambda(W)$, $\forall \lambda : |\lambda| \leq \lambda_0$.

As

$$\int_c \frac{dx}{x^k f(\lambda x)} = \int_c \frac{dx}{a_0 x^k + h(x,\lambda)} \xrightarrow{\lambda \to 0} \int_c \frac{dx}{a_0 x^k} = 0$$

we can choose λ so small (in particular, λ smaller than λ_0) such that

$$\int_c \frac{dx}{x^k f(\lambda x)} = \alpha$$

with $|\alpha| < \theta$. Let $p \in W$ be such that

$$\int_{c_p} \frac{dx}{x^k f(\lambda x)} = -\alpha$$

Obviously $p \neq c(0)$, because $c(0) \notin W$. We can always choose c_p in such a manner that c does not intersect c_p except when $p = c(t)$ for some $t \in [0,1]$. In the first case the curve $\tilde{c}$ joining $c(0)$ to p obtained by concatenating c to c_p is an one-to-one embedded curve such that

$$\int_{\tilde{c}} \frac{dx}{x^k f(\lambda x)} = 0$$

If $p = c(t)$, for some $t \in [0,1]$ we consider $\tilde{c}$ as the subset of c joining $c(0)$ to p.

Thus, the vector field X is not semi-complete relatively to any neighbourhood of the origin.

$\square$

Remark 3. The proof of the last proposition also implies that X is not semi-complete in any sector with vertex at the origin and angle greater than $\dfrac{2\pi}{|k-1|}$.

3. A necessary condition for the semi-completude of a saddle-node in $\mathbb{C}^n$

Consider the set of vector fields of saddle-node type defined in a neighbourhood of the origin and with an isolated singularity there. We are considering only vector fields whose linear part is diagonalizable. So the vector field is analytically conjugated, by a linear transformation, to another one where the linear part is diagonal. So, let

$$\begin{aligned}\mathfrak{X} = \{X : (\mathbb{C}^n,0) \to (\mathbb{C}^n,0), &\,\text{holomorphic} : DX(0) = \operatorname{diag}(\lambda_1,\ldots,\lambda_n),\\ &\lambda_1 = 0, 0 \notin \mathcal{H}(\lambda_2,\ldots,\lambda_n), \text{there are no resonance}\\ &\text{relations between the non-vanishing eigenvalues}\}\end{aligned}$$

Our objective is to classify the foliations associated to a semi-complete vector field of saddle-node type in $\mathbb{C}^n$. To do this classification we need first to classify the foliations associated to semi-complete vector fields in $\mathfrak{X}$. So, from now on we are going to consider vector fields in $\mathfrak{X}$.

Proposition 5. [1] *Let $X \in \mathfrak{X}$. Then X is analytically conjugated to a vector field of the form*

$$\begin{cases} \dot{x_1} = x_1^{p+1}(a + u(x)) \\ \dot{x_2} = \lambda_2 x_2 + x_1 g_2(x) \\ \vdots \\ \dot{x_n} = \lambda_n x_n + x_1 g_n(x) \end{cases} \tag{1}$$

where $a \in \mathbb{C} \setminus \{0\}$ (a constant), $x = (x_1, \dots , x_n)$ and u, g_2, ... ,g_n are holomorphic functions such that $u(0) = g_2(0) = \dots = g_n(0) = 0$.

Dividing the analytical normal form (1) by $1 + a^{-1}u(x)$ and substituting x_1 by bx_1 where b is such that $b^p a = 1$, we obtain that X is analytically equivalent to

$$Y_p : \begin{cases} \dot{x_1} = x_1^{p+1} \\ \dot{x_2} = \lambda_2 x_2 + x_1 a_2(x) \\ \vdots \\ \dot{x_n} = \lambda_n x_n + x_1 a_n(x) \end{cases}$$

where a_i are holomorphic functions such that $a_i(0) = 0$, $\forall i = 2, \dots , n$. This is the Dulac's normal form for a saddle-node in $\mathbb{C}^n$, under the conditions described before [1].

Remark 4. Given a vector field X and denoting by $\mathcal{F}$ its foliation, the holomorphic vector fields whose foliation coincides with $\mathcal{F}$ in a neighbourhood of the origin are written as fX, where f is a non-vanishing holomorphic function in that neighbourhood.

In this way, any holomorphic vector field in $\mathfrak{X}$, with an isolated singularity at the origin, can always be written in the form fY_p, for some p and some holomorphic function f such that $f(0) \neq 0$.

In the $\mathbb{C}^n$ case we have a result analogous to Theorem 1, which has already been enunciated in the introduction:

Theorem 3. *Let $\mathcal{F}$ be a foliation of a saddle-node, in a neighbourhood of the origin, with an isolated singularity at the origin. Then $\mathcal{F}$ is associated to a semi-complete vector field iff it admits the normal form*

$$\begin{cases} \dot{x} = x_1^2 \\ \dot{y} = x_2(\lambda_2 + \alpha_2 x_1) \\ \vdots \\ \dot{z} = x_n(\lambda_n + \alpha_n x_n) \end{cases}$$

with $(\alpha_2, \dots , \alpha_n) \in \mathbb{Z}^{n-1}$.

Our objective is to prove Theorem 3. In this section we exibit a necessary condition for a saddle-node to be semi-complete.

Proposition 6. *Let X be a semi-complete holomorphic vector field, defined in a neighbourhood U of the origin, of the type fY_p for some holomorphic function f such that $f(0) \neq 0$. Then $p = 1$.*

Proof. Suppose that $p \geq 2$.

Let Π_1 be the projection of U on the first axis ($\Pi_1(x) = x_1$) and $\mathcal{F}$ be the foliation associated to the vector field fY_p. There is a neighbourhood of the origin in which the fibres of Π_1 are transverse to the leaves of the foliation $\mathcal{F}$, except to the invariant manifold $\{x_1 = 0\}$:

$$D\Pi_1(x).X(x) = f(x)x_1^{p+1}$$

because $f(0) \neq 0$, and, consequently, non zero in a sufficiently small neighbourhood of the origin.

Fix a disc $B(0, \varepsilon) \subseteq \mathbb{C}^n$ centered at $0 \in \mathbb{C}^n$ with radius $\varepsilon > 0$ in which X is semi-complete.

Remark 5. If X is semi-complete relatively to an open set U, then X is semi-complete relatively to any relatively compact open set contained in U ([8]). So, there is always $\varepsilon > 0$ such that U is semi-complete relatively to $B(0, \varepsilon)$ (it is enough to take ε as small as we want). In reality it is semi-complete relatively to any open subset of U.

The proof for $f \equiv k$ ($k \in \mathbb{C}$) is very simple: consider the curve $c(t) = (re^{2\pi it/p}, 0, \ldots, 0) \subseteq B(0, \varepsilon)$, $t \in [0, 1]$. As $p \geq 2$, c is an one-to-one embedded curve.

As $\Pi_1(c(t)) \neq 0$, $\forall t \in [0, 1]$, for each $(r, x_2^0, \ldots, x_n^0)$ sufficiently close to $0 \in \mathbb{C}^n$, we can lift the curve c to a curve c_L contained in $L \cap B(0, \varepsilon)$ through $(r, x_2^0, \ldots, x_n^0)$.

We are assuming $f \equiv k$, so

$$\int_{c_L} dT_L = \int_c \frac{dx_1}{kx_1^{p+1}} = 0$$

where dT_L is the differential 1-form described before (i.e., such that $dT_L(X) = 1$). As c is an one-to-one embedded curve, so is c_L. This contradicts the fact that X is semi-complete.

We are going to treat now the case $f \not\equiv const$. In this case, we obtain the differential 1-form

$$dT_L^X = \frac{dx_1}{x_1^{p+1} f(x)}$$

with $f(0) \neq 0$.

Consider $S \subseteq \mathbb{C}$, an angular sector with vertex at the origin and angle greater than $\frac{2\pi}{p}$ and less than 2π. As, in a neighbourhood of the origin, Π_1 is transverse to the leaves, except to those contained in the invariant manifold $\{x_1 = 0\}$, for each leaf L in $S \setminus \{0\} \times (\mathbb{C}^{n-1}, 0)$, we can write

$$\begin{cases} x_2 = x_2^L(x) \\ \vdots \\ x_n = x_n^L(x) \end{cases}$$

univocaly.

Let c_L be an one-to-one embedded curve in L. We have that

$$\int_{c_L} dT_L = \int_{c_L} \frac{dx_1}{x_1^{p+1} f(x)} = \int_{\Pi_1(c_L)} \frac{dx_1}{x_1^{p+1} f(x_1, x_2^L(x_1), \dots, x_n^L(x_1))}$$

As c_L is an one-to-one embedded curve, so is $\Pi_1(c_L)$ (because we are in a sector where x_i is a function of x_1 for all $i = 2, \dots, n$). So we reduced the study of the semi-completude of a vector field in $S \times (\mathbb{C}^{n-1}, 0)$ to the study of the semi-completude of a vector field in S.

Remark that if fY_p is not semi-complete in any neighbourhood of the origin of the type $(S \times (\mathbb{C}^{n-1}, 0)) \cap B(0, \varepsilon)$, then it canot be semi-complete in any neighbourhood $B(0, \varepsilon)$ of the origin (remark 5).

We know, by remark 3 and also from [8], that any holomorphic uni-dimensional vector field of order k, $k \geq 3$, (i.e., a vector field $\dot{x} = f(x)$ such that $j^l f = 0$, $\forall 0 \leq l < k$ and $j^k f \neq 0$) is not semi-complete relatively to any sector of amplitude greater than $\dfrac{2\pi}{k-1}$.

As $f(0) \neq 0$, the vector field

$$X: \quad \dot{x_1} = x_1^{p+1} f(x_1, x_2^L(x_1), \dots, x_n^L(x_1))$$

has order equal to $p + 1$. As the sector S has amplitude greater than $\dfrac{2\pi}{p}$, X is not semi-complete relatively to any neighbourhood of the origin of the type $S \cap B(0, \varepsilon)$. Thus fY_p is not semi-complete relatively to any neighbourhood of the origin.

□

We are going to study now the case $p = 1$.

Suppose $p = 1$. To simplify the study of the semi-complete vector fields in $\mathfrak{X}$ we can observe that:

Proposition 7. *A vector field X of type Y_1 is analytically equivalent to the vector field*

$$Y: \begin{cases} \dot{x_1} = x_1^2 \\ \dot{x_2} = x_2 + x_1 f_2(x) \\ \dot{x_3} = \gamma_3 x_3 + x_1 f_3(x) \\ \vdots \\ \dot{x_n} = \gamma_n x_n + x_1 f_n(x) \end{cases}$$

where $\gamma_i = \dfrac{\lambda_i}{\lambda_2}$, $i = 2, \dots, n$ *and the coefficients of* $x_1 x_i$ *on the* i^{th} *equation of* X *and* Y *are equal.*

Proof. Consider the vector field

$$\begin{cases} \dot{x_1} = x_1^2 \\ \dot{x_2} = \lambda_2 x_2 + \alpha_2 x_1 x_2 + x_1 h_2(x) \\ \vdots \\ \dot{x_n} = \lambda_n x_n + \alpha_n x_1 x_n + x_1 h_n(x) \end{cases}$$

where $\dfrac{\partial h_i}{\partial x_i}|_0 = 0$, $\forall i = 2, \dots, n$.

Dividing by λ_2 we obtain

$$\begin{cases} \dot{x_1} = \frac{1}{\lambda_2} x_1^2 \\ \dot{x_2} = x_2 + \frac{\alpha_2}{\lambda_2} x_1 x_2 + \frac{1}{\lambda_2} x_1 h_2(x) \\ \vdots \\ \dot{x_n} = \frac{\lambda_n}{\lambda_2} x_n + \frac{\alpha_n}{\lambda_2} x_1 x_n + \frac{1}{\lambda_2} x_1 h_n(x) \end{cases}$$

Substituting $\lambda_2 \tilde{x_1}$ for x_1 we have

$$\dot{\tilde{x_1}} = \frac{1}{\lambda_2} \dot{x_1} = \frac{1}{\lambda_2^2} x_1^2 = \frac{1}{\lambda_2^2} \lambda_2^2 \tilde{x_1}^2 = \tilde{x_1}^2$$

So, we obtain

$$\begin{cases} \dot{\tilde{x_1}} = \tilde{x_1}^2 \\ \dot{x_2} = x_2 + \alpha_2 \tilde{x_1} x_2 + \tilde{x_1} \tilde{h}_2(\tilde{x_1}, x_2, \dots, x_n) \\ \vdots \\ \dot{x_n} = \gamma_n x_n + \alpha_n \tilde{x_1} x_n + \tilde{x_1} \tilde{h}_n(\tilde{x_1}, x_2, \dots, x_n) \end{cases}$$

where it is obvious that $\dfrac{\partial \tilde{h}_i}{\partial x_i}|_0 = 0$, $\forall i = 2, \dots, n$.

□

Remark 6. From now on, when we refer to an element of type Y_1 we mean an element of type Y_1 such that the eigenvalues are 0, 1 and γ_i without resonances between the non-vanishing eigenvalues and in the Poincaré domain. For example, in the $\mathbb{C}^3$ case we are speaking about an element of type Y_1 such that the eigenvalues are 0, 1 and $\gamma \notin \mathbb{R}_0^- \cup \mathbb{N}$. From the proof of the last proposition and proposition 3 we can easily conclude that X is semi-complete iff its equivalent element of the last proposition is semi-complete.

Notation 1. *We denote by $Y_{1,\alpha}$ a vector field of the type*

$$Y_{1,\alpha}: \begin{cases} \dot{x_1} = x_1^2 \\ \dot{x_2} = x_2(1+\alpha_2 x_1) + x_1 h_2(x) \\ \vdots \\ \dot{x_n} = x_n(\gamma_n + \alpha_n x_1) + x_1 h_n(x) \end{cases}$$

where $\alpha = (\alpha_2, \dots, \alpha_n)$ and h_i are holomorphic functions such that $\dfrac{\partial h_i}{\partial x_i}|_0 = 0$, $\forall i = 2, \dots, n$.

In [1], it is proved that a formal change of coordinates of the form

$$H(x) = (x_1, x_2 + \sum_{k=1}^{\infty} f_{2k}(\bar{x})x_1^k, \dots, x_n + \sum_{k=1}^{\infty} f_{nk}(\bar{x})x_1^k) \tag{2}$$

with $\bar{x} = (x_2, \dots, x_n)$ and f_{ik} holomorphic in a neighbourhood of $0 \in \mathbb{C}^{n-1}$ such that $f_{i1}(0) = 0$ for all $i \in \{2, \dots, n\}$, conjugates $Y_{1,\alpha}$ with

$$Z_{\alpha}: \begin{cases} \dot{x_1} = x_1^2 \\ \dot{x_2} = x_2(1+\alpha_2 x_1) \\ \vdots \\ \dot{x_n} = x_n(\gamma_n + \alpha_n x_1) \end{cases}$$

Let $\hat{G}_0 = \{H(x) = (x_1, x_2 + \sum_{k=1}^{\infty} f_{2k}(\bar{x})x_1^k, \dots, x_n + \sum_{k=1}^{\infty} f_{nk}(\bar{x})x_1^k) :$ $\bar{x} = (x_2, \dots, x_n)$, f_{ik} are holomorphic in a neighbourhood of the origin, $f_{i1}(0) = 0, \forall i \in \{2, \dots, n\}\}$

It is also implicit in the proof that α_i is exactly the coefficient of $x_1 x_i$ on the i^{th} equation of $Y_{1,\alpha}$.

The vector fields $Y_{1,\alpha}$ and Z_α are not necessary analytically conjugated. However there are sectors $U \subseteq \mathbb{C}$ with radius r, of angle less than 2π and with vertex at $0 \in \mathbb{C}$, such that $Y_{1,\alpha}$ and Z_α are analytically conjugated in $U \times (\mathbb{C}^{n-1}, 0)$. This is the Theorem of Malmquist.

Theorem of Malmquist. [4, 1] *Let $\hat{H}$ be the unique formal tansformation of type (2) that conjugates $Y_{1,\alpha}$ and Z_α. Then there exists a holomorphic transformation H defined in $U \times (\mathbb{C}^{n-1}, 0)$, U a sector as described before, such that*

a) *$dH(Y_{1,\alpha}) = Z_\alpha(H)$, in $U \times (\mathbb{C}^{n-1}, 0)$*
b) *$H \tilde{\rightarrow} \hat{H}$ in U, as $x_1 \to 0$*

The vector fields are analytically conjugated if and only if $H_i = H_j$ in $U_i \cap U_j$, $\forall i \neq j$. Each holomorphic transformation H_i is called a normalizing application.

Let f be a holomorphic function in $U \times T$, where $U \subseteq \mathbb{C}$ is a sector with vertex at the origin and T is a (compact) subset of $\mathbb{C}^{n-1}$.

Definition 3 ([2, 7])**.** We say that the function f is assymptotic to $\hat{f}(x) = \sum_{r=0}^{\infty} a_r(\bar{x})x_1^r$, $a_r(\bar{x})$ holomorphic in T, as $x_1 \to 0$ and $x_1 \in U$ if

$$\text{(3)}\quad \forall \bar{x} \in T, m \in \mathbb{N}, \exists A_m(\bar{x}) > 0 : \quad |f(x) - \sum_{r=0}^{m-1} a_r(\bar{x})x_1^r| \leq A_m(\bar{x})x_1^m$$

In this case we write $f \tilde{\to} \hat{f}$ in U, as $x_1 \to 0$.

Remark 7. Equation 3 is equivalent to

$$f(x) = \sum_{r=0}^{m} a_r(\bar{x})x_1^r + x_1^m \varepsilon_m(x) \quad , \quad \lim_{\substack{x_1 \to 0 \\ x_1 \in U}} \varepsilon_m(x) = 0$$

On the other hand, given a sector $U \subseteq \mathbb{C}$ with vertex at the origin and an open subset $T \subseteq \mathbb{C}^{n-1}$, if $\hat{f}(x) = \sum_{r=0}^{\infty} a_r(\bar{x})x_1^r \in \mathbb{C}\{\bar{x}\}[[x_1]]$ there exists a holomorphic function f in $U \times T$ such that $f \tilde{\to} \hat{f}$ in U, as $x_1 \to 0$.

Before we describe the sectors U where the Theorem of Malmquist is valid we are going to present some definitions and results that will be necessary to describe those sectors.

4. Study of the Sectorial Isotropy of the formal normal form

The solutions of the formal normal form

$$Z_\alpha : \begin{cases} \dot{x_1} = x_1^2 \\ \dot{x_2} = x_2(1 + \alpha_2 x_1) \\ \vdots \\ \dot{x_n} = x_n(\gamma_n + \alpha_n x_1) \end{cases}$$

out of $\{0\} \times (\mathbb{C}^{n-1}, 0)$ are parametrized by

$$\begin{cases} x_2(x_1) = c_2 x_1^{\alpha_2} e^{-\frac{1}{x}} \\ \vdots \\ x_n(x_1) = c_n x_1^{\alpha_n} e^{-\frac{\gamma_n}{x}} \end{cases}$$

with $(c_2, \dots, c_n) \in \mathbb{C}^{n-1}$.

Our objective in this subsection is to relate the solutions of Z_α with the solutions of $Y_{1,\alpha}$, by sectors.

Denote by φ_i the argument of the eigenvalue γ_i (in particular $\gamma_2 = 1$ and so $\varphi_2 = 0$), for $i = 2, \dots, n$.

The behaviour of $x_j(x_1)$ along the curve $x_1 = re^{i\theta}$ as $r \to 0$, for a fixed θ, is given by the term $\dfrac{|\gamma_j|}{r}\cos(\varphi_j - \theta)$:

$$\begin{cases} x_2(re^{i\theta}) = c_2 r^{\alpha_2} e^{i\theta\alpha_2} e^{-\frac{1}{r}(\cos(-\theta)+i\sin(-\theta))} \\ \vdots \\ x_n(re^{i\theta}) = c_n r^{\alpha_n} e^{i\theta\alpha_n} e^{-\frac{|\gamma_n|}{r}(\cos(\varphi_n-\theta)+i\sin(\varphi_n-\theta))} \end{cases}$$

The sector such that $(x_2(x_1), \dots, x_n(x_1)) \to (0, \dots, 0)$ as $r \to 0$ is called attractor. This sector corresponds exactly to the points such that $\cos(\varphi_j - \theta) > 0$, $\forall j = 2, \dots, n$. The sector where $\cos(\varphi_j - \theta) < 0$, $\forall j = 2, \dots, n$, is called saddle (in this case $|x_i(x_1)| \to \infty$, $\forall j = 2, \dots, n$).

Contrary to the case of the saddle-node in $\mathbb{C}^2$, if $\frac{\gamma_i}{\gamma_j} \notin \mathbb{R}$, i.e, if $\varphi_i \neq \varphi_j$ for some $i \neq j$ we have sectors that are neither atractors nor saddles. They are called mixed. The mixed sectors are those where $\cos(\varphi_i - \theta)\cos(\varphi_j - \theta) < 0$, for some $i \neq j$.

The directions for which there exists j such that $\cos(\varphi_j - \theta) = 0$ are called singular directions of the solution. Those are given by $\theta = \varphi_i \pm \dfrac{\pi}{2}$, $j = 2, \dots, n$. In particular, as in our case $\varphi_2 = 0$, $\theta = \pm\frac{\pi}{2}$ are always singular directions of the solution.

Remark 8. For simplicity in the notation we sometimes say that $\theta \in U$ in the sense that $x = re^{i\theta} \in U$.

The study of the Sectorial Isotropy of the formal normal form, for any dimension, is done in [1] and will be important to prove Theorem 3 (there, all the theory is presented for vector fields whose linear part is diagonal; thats why we are going to consider only vector fields with a diagonal linear part). We will present the most important ideas and results.

4.1. **The sectors where the Theorem of Malmquist is valid.** Let U be a sector as described before. Denote by $\Lambda_{Z_\alpha}(U)$ the group of holomorphic transformations $H : U \times (\mathbb{C}^{n-1}, 0) \to U \times (\mathbb{C}^{n-1}, 0)$ such that:

a) $dH(Z_\alpha) = Z_\alpha(H)$
b) H is assymptotic to the identity of $\{0\} \times \mathbb{C}^{n-1}$, as $x_1 \to 0$, $x_1 \in U$

Remark 9. $\Lambda_{Z_\alpha}(U)$ is a presheaf. We denote by Λ_{Z_α} the sheaf associated to the presheaf.

An element H of $\Lambda_{Z_\alpha}(U)$ is of the form ([1])

$$\begin{aligned} H(x) = (x_1, x_2 + a_{20}(x_1) + &\sum_{|Q|\geq 1} a_{2Q}(x_1)\bar{x}^Q, \dots, \\ & x_n + a_{n0}(x_1) + \sum_{|Q|\geq 1} a_{nQ}(x_1)\bar{x}^Q) \end{aligned} \tag{4}$$

where $Q = (q_2, \dots, q_n)$, $|Q| = q_2 + \dots + q_n$ and $\bar{x}^Q = x_2^{q_2} \dots x_n^{q_n}$.

From $a)$ it follows that ([1])

$$a_{jQ}(x_1) = a_{jQ} x_1^{-((Q,\alpha)-\alpha_j)} e^{\frac{(Q,\gamma)-\gamma_j}{x_1}} \tag{5}$$

where $\gamma = (\gamma_2, \dots, \gamma_n)$.

Condition b) says that $H \tilde{\rightarrow} Id|_{\{0\}\times\mathbb{C}^{n-1}}$ for $x_1 \to 0$, $x_1 \in U$. This is equivalent to $a_{jQ}(x_1) \tilde{\rightarrow} 0$, for $x_1 \in U$, $x_1 \to 0$, $\forall j \in \{2, \dots, n\}$, $\forall Q \in \mathbb{N}_0^{n-1}$.

Denote by φ_{jQ} the argument of the complex number $(Q,\gamma) - \gamma_j$. The behaviour of $a_{jQ}(x_1)$ along $x_1 = re^{i\theta}$, as $r \to 0$, is given by $\cos(\varphi_{jQ} - \theta)$, (5).

The directions θ for which $\cos(\varphi_{jQ} - \theta) = 0$ for some j and Q, are called singular directions of the sheaf Λ_{Z_α}. In our particular case the singular directions of the sheaf are given by $\theta_{jQ}^{\pm} = \varphi_{jQ} \pm \frac{\pi}{2}$, $i = 2, \dots, n$.

We should remark that if $\varphi_2 = \dots = \varphi_n$ then the number of singular directions of the sheaf are finite. More specifically, they coincide with the singular directions of the solutions.

Remark 10. The singular directions of the solution are always singular directions of the sheaf. They correspond to $Q = 0 \in \mathbb{C}^{n-1}$.

To study the behaviour of the arguments of $(Q,\gamma) - \gamma_j$, $Q \in \mathbb{N}_0^{n-1}$, we represent all these points in the complex plane (figure 1).

Although all results are true in $\mathbb{C}^n$, $\forall n$, in this section all figures are represented for the $\mathbb{C}^3$ case. In figure 1 we are assuming that $\gamma (= \gamma_3) \notin \mathbb{R}$. At the end of the section we analyse the case $\gamma \in \mathbb{R}^+ \setminus \mathbb{N}$.

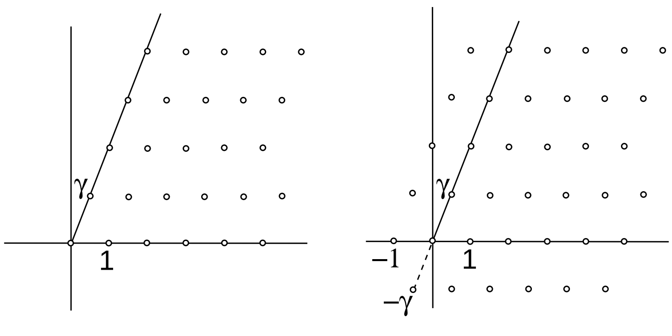


FIGURE 1. In the left the set $\{q_2 + q_3\gamma : (q_2, q_3) \in \mathbb{N}_0 \times \mathbb{N}_0\}$. In the right the set $\{q_2 + q_3\gamma - 1 : (q_2, q_3) \in \mathbb{N}_0 \times \mathbb{N}_0\} \cup \{q_2 + q_3\gamma - \gamma : (q_2, q_3) \in \mathbb{N}_0 \times \mathbb{N}_0\}$.

As the singular directions of the sheaf are given by $\theta_{jQ}^{\pm} = \varphi_{jQ} \pm \frac{\pi}{2}$, we can easily observe that the singular directions of the sheaf are dense in the mixed sectors, while they are discrete in the attractor and saddle sectors. The singular directions of the solution are points of accumulation (figures 1, 2).

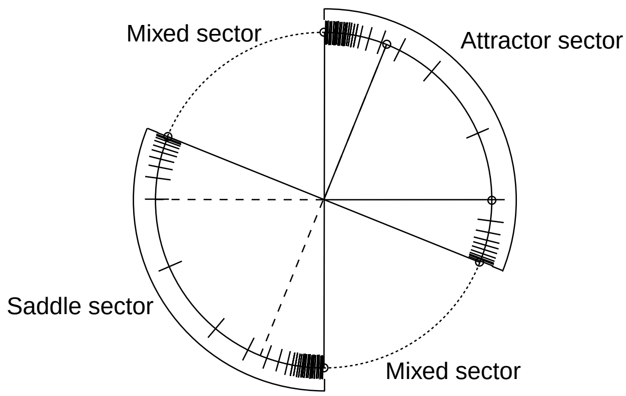


FIGURE 2. Singular directions of the sheaf Λ_{Z_α}. Singular directions are dense in the mixed sectors

We can describe now the sectors U where the Theorem of Malmquist is valid. Let us consider a direction φ_0 in the attractor sector that is not a singular direction of the sheaf Λ_{Z_α}: the sectors where the Theorem of Malmquist is valid are the sectors obtained by extending the sectors between the angles φ_0 and $\varphi_0 \pm \pi$ till reach a singular direction of the sheaf Λ_{Z_α} (figure 3). Remark that those sectors have amplitude greater than π.

Denote each one of this sectors by U_1 and U_2. They are well defined because, as we said before, the singular directions of the sheaf are discret in the attractor and in the saddle sectors.

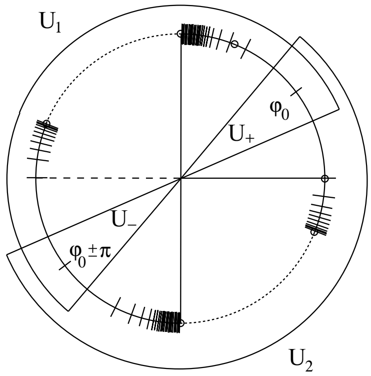


FIGURE 3. How to construct sectors where the Theorem of Malmquist is valid

By the definition of U_1 and U_2 we have that $U_1 \cap U_2$ is the union of two open sets U_+ and U_-, cointained in the attractor sector and in the saddle sector, respectively (and so $U_+ \cap U_- = \emptyset$) (figure 3). The saddle sector and the attractor sector are antipodes (this means that $S = A + \pi = \{e^{\pi i} a : a \in A\}$, where S and A represent, respectively, the saddle and the attractor sectors of the solution) and so are U_+ and U_-.

4.2. **The importance of the pre-sheaves $\Lambda_{Z_\alpha}(U_+)$ and $\Lambda_{Z_\alpha}(U_-)$.** As we have already said, there exists only one element $\hat{H} \in \mathbb{C}\{\bar{x}\}[[x_1]]$ of type (2) conjugating $Y_{1,\alpha}$ and Z_α.

Proposition 8. [1] *Let U_1 and U_2 be the sectors where the Theorem of Malmquist is valid. Let H_1 and H_2 be the normalizing applications defined in U_1 and U_2, respectively, i.e., H_i, $i = 1, 2$, are holomorphic applications defined in $U_i \times (\mathbb{C}^{n-1}, 0)$ and such that*

a) $dH_i(X) = Z_\alpha(H_i)$

b) $H_i \tilde{\to} \hat{H}$ *in* $U_i \times (\mathbb{C}^{n-1}, 0)$

Then, $H_j \circ H_i^{-1}$, $i \neq j$ belongs to $\Lambda_{Z_\alpha}(U_+)$ and $\Lambda_{Z_\alpha}(U_-)$.

Proof. We know that $U_1 \cap U_2 = U_+ \cup U_-$, where U_+ and U_- are open sets contained in the attractor and saddle sectors, respectively.

The change of coordinates in U_+ and U_-, given by $H_j \circ H_i^{-1}$, $i \neq j$, verifies

$$H_j \circ H_i^{-1} \tilde{\to} \hat{H} \circ \hat{H}^{-1} = Id$$

and

$$\begin{aligned} & d(H_j \circ H_i^{-1})(Z_\alpha)(H_j \circ H_i^{-1})^{-1} \\ =& dH_j \circ dH_i^{-1}(Z_\alpha) H_i \circ H_j^{-1} \\ =& dH_j(X) H_j^{-1} \\ =& Z_\alpha \end{aligned}$$

Thus the result follows.

□

Remark 11. The holomorphic functions $H_2 \circ H_1^{-1}$ and $H_1 \circ H_2^{-1}$ define the gluing of the leaves. As $H_1 \circ H_2^{-1}$ is the inverse of $H_2 \circ H_1^{-1}$, it is enought to analyse the behaviour of only one of them.

Let $g_+ = H_2 \circ H_1^{-1}|_{U_+}$ and $g_- = H_2 \circ H_1^{-1}|_{U_-}$. The functions g_+ and g_- are both the identity iff $H_2 \circ H_1^{-1}$ is the identity in U_+ and in U_-. This means that $H_1 = H_2$ in $U_1 \cap U_2$ and so, there is a holomorphic function defined in $U_1 \cup U_2$ that coincide with H_1 in U_1 and with H_2 in U_2.

So, the vector field $Y_{1,\alpha}$ is analytically conjugated to its formal normal form Z_α iff g_+ and g_- are both the identity function.

We know that $H_2 \circ H_1^{-1}$ has the form (4).

$$\begin{aligned} H(x) = (x_1, x_2 + a_{20}(x_1) + & \sum_{|Q| \geq 1} a_{2Q}(x_1) \bar{x}^Q, \dots, \\ & x_n + a_{n0}(x_1) + \sum_{|Q| \geq 1} a_{nQ}(x_1) \bar{x}^Q) \end{aligned}$$

where

$$a_{jQ}(x_1) = a_{jQ} x_1^{-((Q,\alpha)-\alpha_j)} e^{\frac{(Q,\gamma)-\gamma_j}{x_1}}$$

As we said before, the behaviour of $a_{jQ}(x_1)$ along $x_1 = re^{i\theta}$, as $r \to 0$, is given by $\cos(\varphi_{jQ} - \theta)$. So, if U and V are two open sectors contained in a sector not containing singular directions of the sheaf then $\Lambda_{Z_\alpha}(U) = \Lambda_{Z_\alpha}(V)$ because $\cos(\varphi_{jQ} - \theta)$ has the same sign for all $j = 2, \dots, n$, $Q \in \mathbb{Z}^{n-1}$, $|Q| \geq 2$ and $\theta \in U \cup V$.

4.3. **Gluing of the leaves.** To know how the gluing of the leaves is done, is important to know the behaviour of the applications (4) in $\Lambda_{Z_\alpha}(U_+)$ and $\Lambda_{Z_\alpha}(U_-)$.

Proposition 9. *If $a_{jQ} \neq 0$ in U then $\cos(\varphi_{jQ} - \theta) < 0$, $\forall \theta : re^{i\theta} \in U$.*

Proof. We must have $a_{jQ}(x_1) \tilde{\longrightarrow} 0$ as $x_1 \to 0$. Studing $x_1 = re^{i\theta}$, for θ fixed and $r \to 0$, the behaviour of $a_{jQ}(x_1)$ is given by the real part of $\dfrac{(Q,\gamma)-\gamma_j}{x_1}$:

$$Re(\frac{(Q,\gamma)-\gamma_j}{x_1}) = \frac{|(Q,\gamma)-\gamma_j|}{r} \cos(\varphi_{jQ} - \theta)$$

Suppose that $a_{jQ} \neq 0$. If

$$\begin{aligned} \exists \theta \in U : \cos(\varphi_{jQ} - \theta) > 0 &\Rightarrow \frac{|(Q,\gamma)-\gamma_j|}{r} \cos(\varphi_{jQ} - \theta) \overset{r\to 0}{\longrightarrow} +\infty \\ &\Rightarrow a_{jQ}(x) \tilde{\not\longrightarrow} 0 \end{aligned}$$

□

Proposition 10. [1] *There exists duality between $\Lambda_{Z_\alpha}(U)$ and $\Lambda_{Z_\alpha}(U+\pi)$, where $U + \pi = \{e^{\pi i} u : u \in U\}$, in the following sense: if $a_{jQ} \neq 0$ in U then $a_{jQ} = 0$ in $U + \pi$.*

Remark 12. In particular there exists duality between $\Lambda_{Z_\alpha}(U_+)$ and $\Lambda_{Z_\alpha}(U_-)$.

Proof. By proposition 9, if $a_{jQ} \neq 0$ in U then $\cos(\varphi_{jQ} - \theta) < 0$, $\forall \theta \in U$.

$$\begin{aligned} \cos(\varphi_{jQ} - \theta) < 0, \forall \theta \in U &\Rightarrow \cos(\varphi_{jQ} - (\theta + \pi)) > 0, \forall \theta \in U \\ &\Rightarrow \cos(\varphi_{jQ} - \eta) > 0, \forall \eta \in U + \pi \quad (\eta = \theta + \pi) \\ &\Rightarrow \frac{|(Q,\gamma)-\gamma_j|}{r} \cos(\varphi_{jQ} - \eta) \overset{r\to 0}{\longrightarrow} +\infty \\ &\Rightarrow a_{jQ}(e^{\pi i} x) \tilde{\to} 0 \Leftrightarrow a_{jQ} = 0 \quad \text{in} \quad U + \pi \end{aligned}$$

Thus $a_{jQ} = 0$ in $U + \pi$.

□

We only have to know the behaviour of the applications in $\Lambda_{Z_\alpha}(U_+)$ and $\Lambda_{Z_\alpha}(U_-)$, because $g_+ \in \Lambda_{Z_\alpha}(U_+)$ and $g_- \in \Lambda_{Z_\alpha}(U_-)$. By the duality property we only need to know the constants that can be non zero in $\Lambda_{Z_\alpha}(U_+)$, i.e., we need to know for which (j, Q)

$$\cos(\varphi_{jQ} - \theta) < 0 \quad , \quad \forall \theta : re^{i\theta} \in U_+ \quad (r \in \mathbb{R}^+)$$

The next result expresses how the gluing of the leaves is done.

Proposition 11. [1] *Take an element of the sheaf Λ_{Z_α}.*

$$\begin{aligned} H(x) = (x_1, x_2 + a_{20}(x_1) + & \sum_{|Q|\geq 1} a_{2Q}(x_1)\bar{x}^Q, \dots , \\ & x_n + a_{n0}(x_1) + \sum_{|Q|\geq 1} a_{nQ}(x_1)\bar{x}^Q) \end{aligned}$$

with $a_{jQ}(x_1)$ given by (5). Thus H transforms the solution, of the diferential equation associated to the formal normal form, given by

$$\begin{cases} x_2(x_1) = c_2 x_1^{\alpha_2} e^{-\frac{1}{x_1}} \\ \vdots \\ x_n(x_1) = c_n x_1^{\alpha_n} e^{-\frac{\gamma_n}{x_1}} \end{cases}$$

into the solution, of the same equation, given by

$$\begin{cases} x_2(x_1) = (c_2 + a_{20} + \sum_{|Q|\geq 1} a_{2Q} c^Q) x_1^{\alpha_2} e^{-\frac{1}{x_1}} \\ \vdots \\ x_n(x_1) = (c_n + a_{n0} + \sum_{|Q|\geq 1} a_{nQ} c^Q) x_1^{\alpha_n} e^{-\frac{\gamma_n}{x_1}} \end{cases}$$

where $c^Q = c_2^{q_2} \dots c_n^{q_n}$.

Proof. Consider the solution

$$\begin{cases} x_2(x_1) = c_2 x_1^{\alpha_2} e^{-\frac{1}{x_1}} \\ \vdots \\ x_n(x_1) = c_n x_1^{\alpha_n} e^{-\frac{\gamma_n}{x_1}} \end{cases}$$

Thus

$$\begin{aligned} & H(x_1, x_2(x_1), \dots , x_n(x_1)) \\ & \quad = (x_1, c_2 x_1^{\alpha_2} e^{-\frac{1}{x_1}} + a_{20}(x_1) + \sum_{|Q|\geq 1} a_{2Q}(x_1)\bar{x}(x_1)^Q, \dots , \\ & \qquad\qquad c_n x_1^{\alpha_n} e^{-\frac{\gamma_n}{x_1}} + a_{n0}(x_1) + \sum_{|Q|\geq 1} a_{nQ}(x_1)\bar{x}(x_1)^Q) \end{aligned}$$

where $\bar{x}(x_1)^Q = x_2(x_1)^{q_2} \dots x_n(x_1)^{q_n} = c^Q x_1^{(Q,\alpha)} e^{-\frac{(Q,\gamma)}{x_1}}$. Thus substituting the expression of $a_{jQ}(x_1)$ in the last expression we obtain

$$
\begin{aligned}
&= (x_1, c_2 x_1^{\alpha_2} e^{-\frac{1}{x_1}} + a_{20} x_1^{\alpha_2} e^{-\frac{1}{x_1}} + \\
&\qquad \sum_{|Q|\geq 1} a_{2Q} x_1^{-((Q,\alpha)-\alpha_2)} e^{\frac{(Q,\gamma)-\gamma_2}{x_1}} c^Q x_1^{(Q,\alpha)} e^{-\frac{(Q,\gamma)}{x_1}}, \dots, c_n x_1^{\alpha_2} e^{-\frac{\gamma_n}{x_1}} + \\
&\qquad a_{n0} x_1^{\alpha_n} e^{-\frac{\gamma_n}{x_1}} + \sum_{|Q|\geq 1} a_{nQ} x_1^{-((Q,\alpha)-\alpha_n)} e^{\frac{(Q,\gamma)-\gamma_n}{x_1}} c^Q x_1^{(Q,\alpha)} e^{-\frac{(Q,\gamma)}{x_1}})
\end{aligned}
$$

$$
\begin{aligned}
&= (x_1, (c_2 + a_{20} + \sum_{|Q|\geq 1} a_{2Q} c^Q) x_1^{\alpha_2} e^{-\frac{1}{x_1}}, \dots, \\
&\qquad\qquad (c_n + a_{n0} + \sum_{|Q|\geq 1} a_{nQ} c^Q) x_1^{\alpha_n} e^{-\frac{\gamma_n}{x_1}})
\end{aligned}
$$

which represents the solution given by

$$
\begin{cases}
x_2(x_1) = (c_2 + a_{20} + \sum_{|Q|\geq 1} a_{2Q} c^Q) x_1^{\alpha_2} e^{-\frac{1}{x_1}} \\
\vdots \\
x_n(x_1) = (c_n + a_{n0} + \sum_{|Q|\geq 1} a_{nQ} c^Q) x_1^{\alpha_n} e^{-\frac{\gamma_n}{x_1}}
\end{cases}
$$

$\square$

In the sector U_+, each $c = (c_2, \dots, c_n) \in (\mathbb{C}^{n-1}, 0)$ determines a leaf of the foliation of the formal normal form in $Z_\alpha|_{U_+\times(\mathbb{C}^{n-1},0)}$, i.e., c works like a parametrization of the leaves of $Z_\alpha|_{U_+\times(\mathbb{C}^{n-1},0)}$. So, as U_+ is a sector that does not contain singular directions of the sheaf we can identify $\Lambda_{Z_\alpha}(U_+)$:

$$\{x \mapsto (x_1, x_2 + a_{20}(x_1) + \sum_{|Q|\geq 1} a_{2Q}(x_1)\bar{x}^Q, ..., x_n + a_{n0}(x_1) + \sum_{|Q|\geq 1} a_{nQ}(x_1)\bar{x}^Q)\}$$

with the set of the transformations, in the space of the leaves, given by

$$\{c \mapsto (c_2 + a_{20} + \sum_{|Q|\geq 1} a_{2Q} c^Q, \dots, c_n + a_{n0} + \sum_{|Q|\geq 1} a_{nQ} c^Q)\}$$

and also denoted by $\Lambda_{Z_\alpha}(U_+)$.

More specificaly, the presheaf $\Lambda_{Z_\alpha}(U_+)$ expresses that the leaf of $Z_\alpha|_{U_+\times(\mathbb{C}^{n-1},0)}$ parametrized by $(c_2, \dots, c_n)$ is taken into the leaf parametrized by $(c_2 + a_{20} + \sum_{|Q|\geq 1} a_{2Q} c^Q, \dots, c_n + a_{n0} + \sum_{|Q|\geq 1} a_{nQ} c^Q)$. The same happens to the presheaf $\Lambda_{Z_\alpha}(U_-)$.

We should remark that the elements $\Lambda_{Z_\alpha}(U_-)$ are all tangent to the identity, i.e, the terms a_{i0} are equal to 0 in U_- for all $i = 2, \dots, n$. This is so by the definition of the saddle sector: $\cos(\varphi_j - \theta) < 0$ for all $\theta \in S$; as $\varphi_{j0} = \arg(-\gamma_j) = \arg(\gamma_j) + \pi = \varphi_j + \pi$ then $\cos(\varphi_{j0} - \theta) > 0$ for all $\theta \in S$ and so $a_{j0} = 0$ in S from proposition 9.

4.4. **Determination of $\Lambda_{Z_\alpha}(U_+)$ for a given sector U_+.** In this subsection we are going to present how to determine $\Lambda_{Z_\alpha}(U_+)$ for a given sector U_+.

We will explain here only the $\mathbb{C}^3$ case. In this case we can determine explicitly $\Lambda_{Z_\alpha}(U)$ for any sector U.

The $\mathbb{C}^n$ case, with $n \geq 4$, will be explained only in the next section.

4.4.1. *The $\gamma \notin \mathbb{R}$ case.* We are going to explain the case $0 < arg(\gamma) \leq \frac{\pi}{2}$ ($\gamma = \gamma_3$). The other cases are similar, as we will see later, geometrically (there exist differences only in the inequalities and in the signs of $\frac{\pi}{2}$).

The first step is to choose the sectors U_1 and U_2 or, equivalently, the sectors U_- and U_+ we are going to work on.

Here, there is a difference between the case $\gamma \notin \mathbb{R}$ and the case $\gamma \in \mathbb{R}^+ \setminus \mathbb{N}$, as we will see later.

The attractor and saddle sectors increase as φ_3 (φ_3 = argument of $\frac{\lambda_3}{\lambda_2}$) decreases.

We represent again the complex numbers $q_2+q_3\gamma-1$ and $q_2+q_3\gamma-\gamma$ in the complex plane (figure 4).

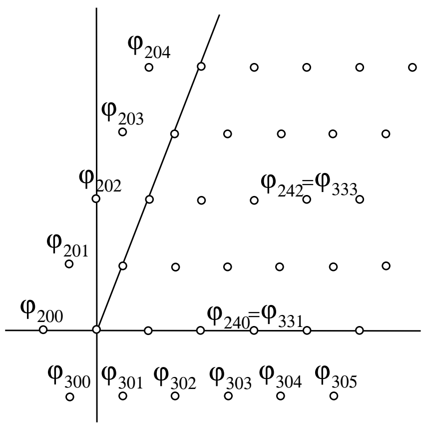


FIGURE 4. The set $\{j+k\gamma-1 : (j,k) \in \mathbb{N}_0 \times \mathbb{N}_0\} \cup \{j+k\gamma-\gamma : (j,k) \in \mathbb{N}_0 \times \mathbb{N}_0\}$. The term φ_{ijk} denotes the argument of the corresponding point $j+k\gamma-q$, where $q=1$ if $i=2$ and $q=\gamma$ if $i=3$.

Note that

$$\varphi_{201} - \frac{\pi}{2} = \varphi_{310} + \frac{\pi}{2} \text{ or equivalently } \theta^-_{201} = \theta^+_{310} \tag{6}$$

because

$$\varphi_{201} = arg(\gamma-1) = arg(-(1-\gamma)) = arg(1-\gamma) + \pi = \varphi_{310} + \pi$$

We also have that

$$\begin{aligned}(7)\quad & arg(\gamma) - \frac{\pi}{2} < \varphi_{20(j+1)} - \frac{\pi}{2} < \varphi_{20j} - \frac{\pi}{2} < \varphi_{201} - \frac{\pi}{2} = \theta^-_{201} \\ & = \theta^+_{310} = \varphi_{310} + \frac{\pi}{2} < \varphi_{3j0} + \frac{\pi}{2} < \varphi_{3(j+1)0} + \frac{\pi}{2} < \frac{\pi}{2} \quad , \quad \forall j \geq 2\end{aligned}$$

Figure 5 expresses these inequalities geometrically

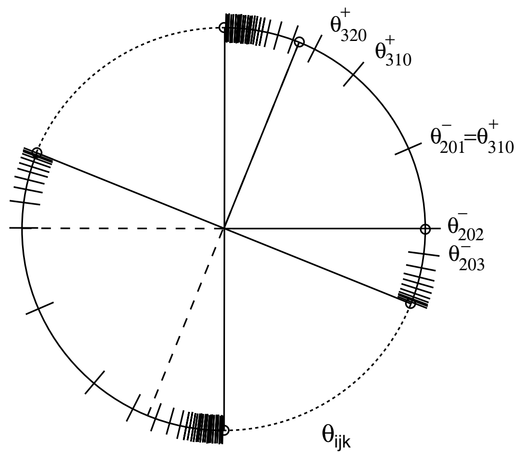


FIGURE 5. Singular directions of Λ_{Z_α}

The value of a_{201} can be non zero, in the attractor sector, for all θ such that

$$arg(\gamma) - \frac{\pi}{2} < \theta < \theta^-_{201} = \varphi_{201} - \frac{\pi}{2}$$

because $\cos(\varphi_{201} - \theta) < 0$, or equivalently, because $|\varphi_{201} - \theta| > \frac{\pi}{2}$ for those values of θ. It is very easy to see this because $\theta^-_{201} = \varphi_{201} - \frac{\pi}{2}$ (figures 4, 5).

As $\theta^-_{20k} = \varphi_{20k} - \frac{\pi}{2}$, a_{20k}, for $k \geq 2$, is non zero, in the attractor sector, only for θ such that

$$arg(\gamma) - \frac{\pi}{2} < \theta < \theta^-_{20k}$$

Denote by U_+ the sector, with vertex at the origin and radius r, whose elements have arguments between θ^-_{202} and θ^-_{201}. In U_+, $a_{20k} = 0$, $\forall k \geq 2$.

By (6) and (7) it is easy to see that

$$\cos(\varphi_{3j0} - \theta) > 0 \quad , \quad \forall \theta : re^{i\theta} \in U_+$$

So, $a_{3j0} = 0$ in U_+.

Consider now (j,k) with $j \neq 0$ and $k \neq 0$. For each (j,k) we need to determine the points in U_+ whose argument θ satisfy $\cos(\varphi_{ijk} - \theta) < 0$. Geometrically it is easy to verify that

$$jk \neq 0 \quad \Rightarrow \cos(\varphi_{ijk} - \theta) > 0$$

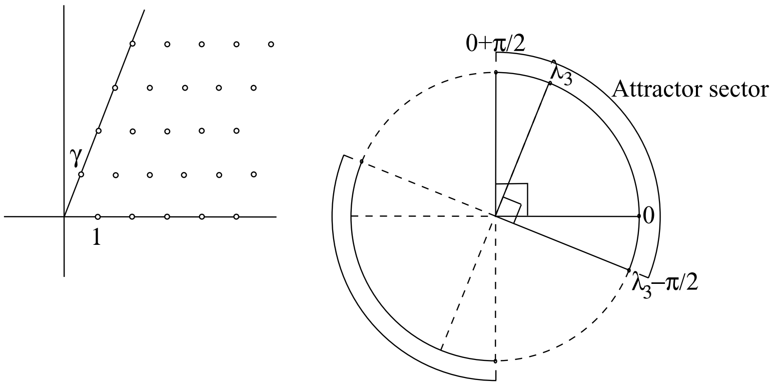


FIGURE 6

i.e., $a_{ijk} = 0$ if $j \neq 0$ and $k \neq 0$ (figure 6).

Remark that $K = \{j + k\gamma - 1 : jk \neq 0\} \cup \{j + k\gamma - \gamma : jk \neq 0\}$ is contained in the angular sector defined by the points 1 and γ and with vertex at 0. The arguments of the extrema of K are 0 and φ_3. We are assuming $0 < \varphi_3 < \frac{\pi}{2}$, then the arguments of the extrema of the attractor sector are $\varphi_3 - \frac{\pi}{2}$ and $\frac{\pi}{2}$. So $|\varphi_{ijk} - \theta| \leq \frac{\pi}{2}$ for all (j,k) with $jk \neq 0$ and $\theta \in [\varphi_3 - \frac{\pi}{2}, \frac{\pi}{2}] \supseteq U_+$. Furthermore, $|\varphi_{ijk} - \theta|$ can only assume the value $\frac{\pi}{2}$ when θ coincides with the singular directions of the solution: $\varphi - \frac{\pi}{2}$ and $\frac{\pi}{2}$.

As

$$\cos(\varphi_{ijk} - \theta) > 0 \quad \Leftrightarrow \quad |\varphi_{ijk} - \theta| < \frac{\pi}{2}$$

a_{2jk} and a_{3jk} are zero in any sector contained in the attractor sector, for all (j,k) with $jk \neq 0$. In particular they are zero in U_+. The same argument is valid for the cases $\frac{\pi}{2} < \varphi_3 < \pi$, $-\pi < \varphi_3 < -\frac{\pi}{2}$ and $\frac{\pi}{2} < \varphi_3 < 0$, as figure 9 ilustrates.

In this way we conclude that

$$\Lambda_{Z_\alpha}(U_+) = \{(x,y,z) \mapsto (x, y + a_{200} + a_{201}z, z + a_{300}\}.$$

Contrary to the $\mathbb{C}^2$ case, there is no sector U, contained in the attractor sector , limited by singular directions of the sheaf and with no singular direction of the sheaf in its interior, such that the elements of $\Lambda_{Z_\alpha}(U)$ are constitued only by the sum of the identity with a translation.

Defined U_+ we have that $U_- = U_+ + \pi$ and so the sectors U_1 and U_2 are well defined up to a change between them (figure 7).

We have already said that there is duality between $\Lambda_{Z_\alpha}(U_+)$ and $\Lambda_{Z_\alpha}(U_-)$. So

$$\Lambda_{Z_\alpha}(U_-) = \{(x,y,z) \mapsto (x, y + \sum_{\substack{j+k\geq 1 \\ (j,k)\neq(1,0) \\ (j,k)\neq(0,1)}} a_{2jk}y^j z^k, z + \sum_{\substack{j+k\geq 1 \\ (j,k)\neq(0,1)}} a_{3jk}y^j z^k\}$$

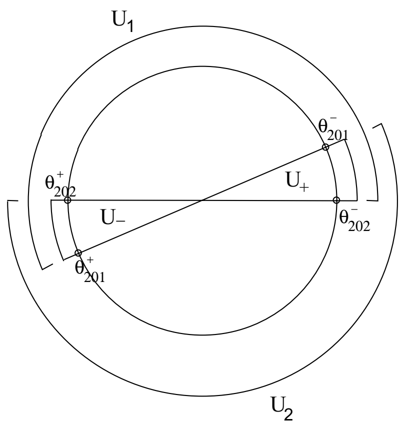


FIGURE 7. The sectors U_1, U_2, U_- and U_+

The constructions made before are valid in the other cases: $\pi < arg(\gamma) < \frac{\pi}{2}$, $-\frac{\pi}{2} < arg(\gamma) < 0$ and $-\pi < arg(\gamma) < -\frac{\pi}{2}$, because $\varphi_{201} = \varphi_{310} + \pi$ in all cases (figures 8 and 9).

4.4.2. *The* $\gamma \in \mathbb{R}^+ \setminus \mathbb{N}$ *case.* Suppose now that $\gamma \in \mathbb{R}^+ \setminus \mathbb{N}$. Then $\varphi_2 = \varphi_3 = 0$ and, consequently, the singular directions of the solution are given by $\theta = \pm\frac{\pi}{2}$.

The argument of the complex numbers $j + k\gamma - 1$ and $j + k\gamma - \gamma$ are equal to zero or π. So, the singular directions of the sheaf coincide with the singular directions of the solution: $\theta = \pm\frac{\pi}{2}$.

In this way the attractor sector is given by $\{x : |x| < r \wedge \Re(x) > 0\}$ and the saddle sector is given by $\{x : |x| < r \wedge \Re(x) < 0\}$. We can choose

$$U_1 = B(0, r) \setminus \{x : \Re(x) = 0 \wedge \Im(x) > 0\}$$

and

$$U_2 = B(0, r) \setminus \{x : \Re(x) = 0 \wedge \Im(x) < 0\}$$

Then, U_+ can be the attractor sector and U_- the saddle sector.

We represent the points $j + k\gamma - 1$ and $j + k\gamma - \gamma$ in figure 10. We have to distinguish between the cases $\gamma > 1$ and $0 < \gamma < 1$. If $\gamma > 1$ let $l \in \mathbb{N}$ be such that $l < \gamma < l + 1$, and if $0 < \gamma < 1$ let $p \in \mathbb{N}$ be such that $p\gamma < 1 < (p + 1)\gamma$.

The coeficients a_{ijk} of $\Lambda_{Z_\alpha}(U_+)$ can be non zero if and only if φ_{ijk} is equal to π.

So, if $\gamma > 1$ we have

$$\Lambda_{Z_\alpha}(U_+) = \{(c, d) \mapsto (c + a_{200}, d + a_{300} + \sum_{j=1}^{l} a_{3j0}c^j)\}$$

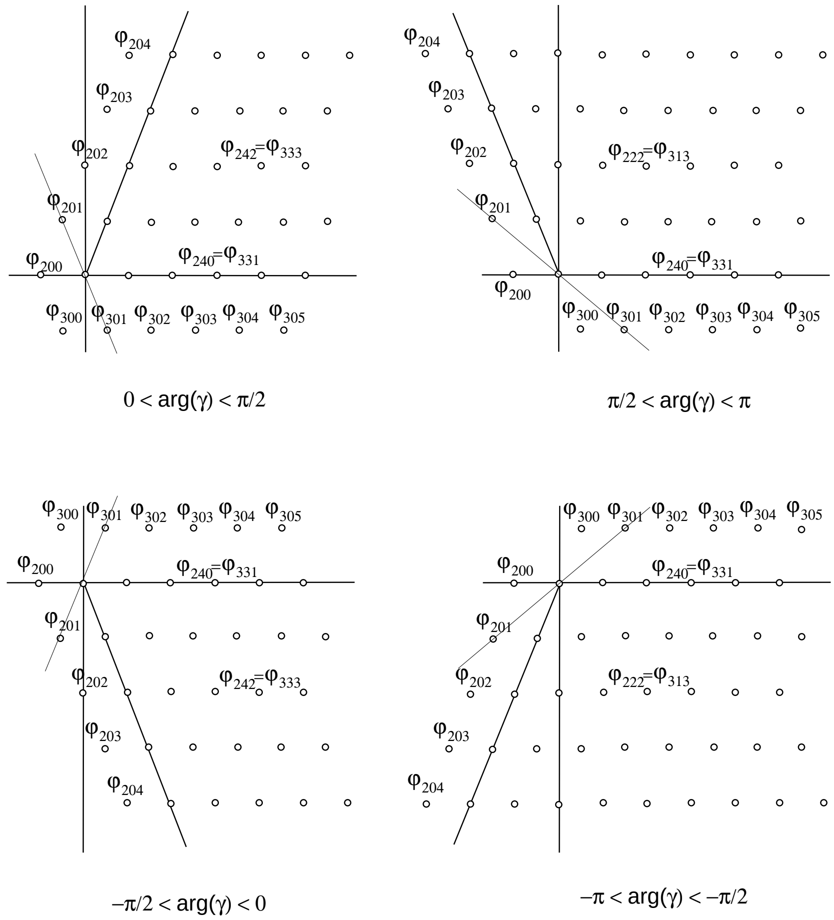


FIGURE 8

and

$$\Lambda_{Z_\alpha}(U_-) = \{(c,d) \mapsto (c + \sum_{j+k\geq 1} a_{2jk}c^j d^k, d + \sum_{\substack{j+k\geq 1 \\ (j,k)\neq(0,1) \\ (j,k)\neq(1,0),\ldots,(l,0)}} a_{3jk}c^j d^k)\}$$

If $0 < \gamma < 1$ the presheaves are given by

$$\Lambda_{Z_\alpha}(U_+) = \{(c,d) \mapsto (c + a_{200} + \sum_{j=1}^{p} a_{20j}d^j, d + a_{300})\}$$

and

$$\Lambda_{Z_\alpha}(U_-) = \{(c,d) \mapsto (c + \sum_{\substack{j+k\geq 1 \\ (j,k)\neq(1,0) \\ (j,k)\neq(0,1),\ldots,(0,p)}} a_{2jk}c^j d^k, d + \sum_{j+k\geq 1} a_{3jk}c^j d^k)\}$$

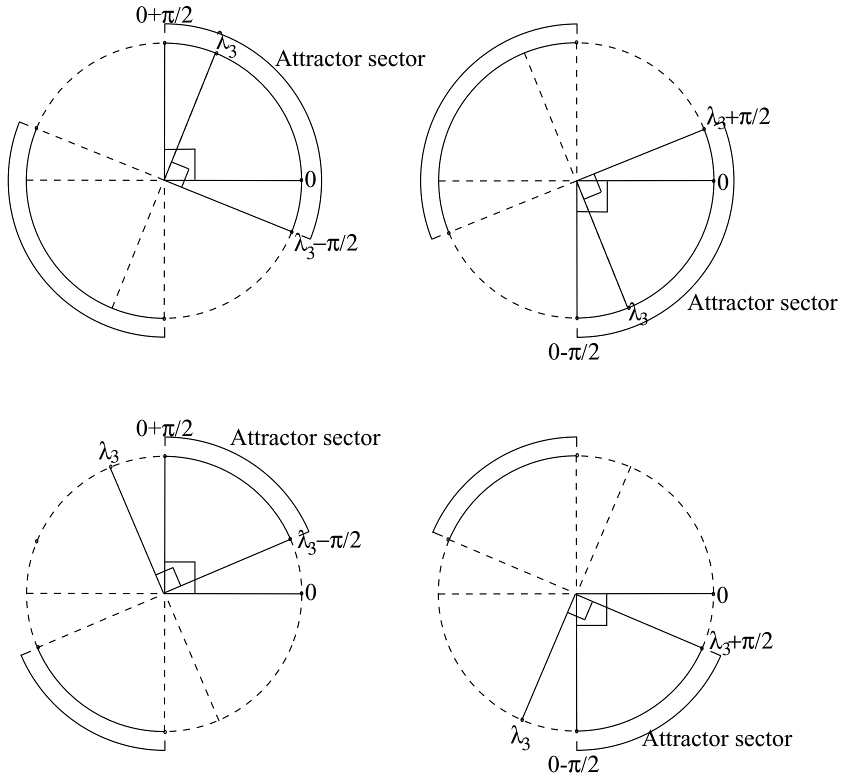


Figure 9

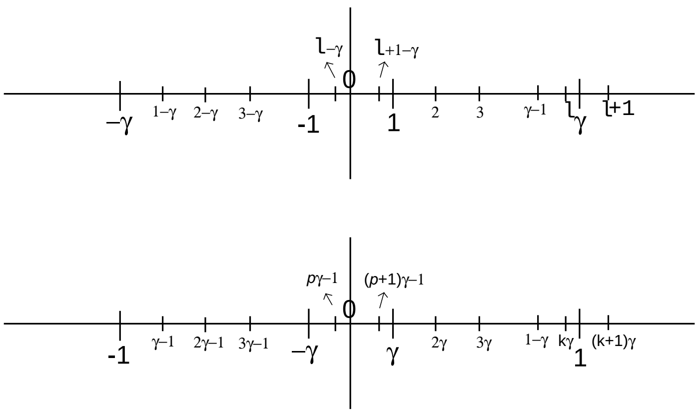


Figure 10. The complex numbers $j + k\gamma - 1$ and $j + k\gamma - \gamma$, for $(j, k) \in \mathbb{N}_0 \times \mathbb{N}_0$.

5. Semi-complete saddle-node foliations in $\mathbb{C}^n$

We return now to the study of the semi-complete saddle-node foliations.

We are going to treat first the case $f \equiv k \in \mathbb{C}$. By proposition 3 it is sufficient to study the case $f = 1$.

We can easily verify that the x_i-axis (that corresponds to the manifold $\{x_j = 0, j \neq i\}$) is an invariant manifold of the vector field for all $i = 2, \ldots, n$. We do not know if the x_1-axis is an invariant manifold, nor if there is a holomorphic invariant manifold tangent to the x_1-axis.

However we have necessary and sufficient conditions for the existence of that invariant manifold.

Proposition 12. *A vector field belonging to the $\hat{G}_0$-orbit of $Y_{1,\alpha}$ has invariant central manifold if and only if the associated sheaf has no translation, i.e, iff $a_{i0\ldots0} = 0$ for all $i = 2, \ldots, n$.*

Proof. Suppose that $a_{i0\ldots0} = 0$ for all $i = 2, \ldots, n$.

Let L be the leaf containing $H_1^{-1}(U_1 \setminus \{0\} \times \{0, \ldots, 0\})$, where H_1 is the normalizing application.

Consider the curve $c(t) = (re^{2\pi it}, 0, \ldots, 0)$ where r is such that $\Pi_1(c(0)) = r \in U_+$ and let c_L be its lift to L. As $H_1(L \cap (U_1 \times (\mathbb{C}^{n-1}, 0))) \subseteq \{x_i = 0, i = 2, \ldots, n\}$, the leaf L is parametrized by $(0, \ldots, 0)$.

As $\Pi_1(c(\frac{1}{2})) \in U_-$ and the application g_- is tangent to the identity $(0, \ldots, 0)$ is taken into $(0, \ldots, 0)$ by g_-. The leaf L is also parametrized by $(0, \ldots, 0)$ in U_2; this means that $H_2(L \cap (U_2 \times (\mathbb{C}^{n-1}, 0))) \subseteq \{x_i = 0, i = 2, \ldots, n\}$.

On the other hand, $\Pi_1(c_L(1)) = \Pi_1(c_L(0))$ belongs to U_+. So g_+ takes $(0, \ldots, 0)$ into $(a_{20\ldots0}, \ldots, a_{n0\ldots0})$. As, by hypothesis, $a_{i0\ldots0} = 0$, for all $i = 2, \ldots, n$, we have that $\Lambda_{Y_{1,\alpha}}(U_+)$ and $\Lambda_{Y_{1,\alpha}}(U_-)$ restricted to the leaf $\{x_i = 0, i = 2, \ldots, n\}$ is given by

$$(x, 0, \ldots, 0) \mapsto (x, 0, \ldots, 0)$$

i.e., is the identity.

Thus, as $\{x_i = 0, i = 2, \ldots, n\}$ is a holomorphic central manifold for the formal normal form, the leaf L is a holomorphic central manifold for $Y_{1,\alpha}$.

Suppose now that $Y_{1,\alpha}$ has a holomorphic central invariant manifold. Denote this leaf by L.

Consider the image of L by H_1. The normalizing application H_1 is defined in $U_1 \times (\mathbb{C}^{n-1}, 0)$.

However the intersection of U_1 with the saddle sector is not empty. As in the formal normal form the only leaf in $U_1 \times (\mathbb{C}^{n-1}, 0)$ such that $x_i(x_1) \to 0$ as $x_1 \to 0$, for all $i \in \{2, \ldots, n\}$, is the invariant manifold $\{x_i = 0, i = 2, \ldots, n\}$ we have that

$$H_1(L \cap (U_1 \times (\mathbb{C}^{n-1}, 0))) = U_1 \times \{x_i = 0, i = 2, \ldots, n\}$$

In the same way we can deduce that

$$H_2(L \cap (U_2 \times (\mathbb{C}^{n-1}, 0))) = U_2 \times \{x_i = 0, i = 2, \ldots, n\}$$

and so we can conclude that $(a_{20\ldots0}, \ldots, a_{n0\ldots0}) = (0, \ldots, 0)$.

□

The next lemma enable us to guarantee that semi-completude implies the existence of a holomorphic invariant manifold tangent to the x_1-axis.

Lemma 2. *Let X be a field of type $Y_{1,\alpha}$ and suppose that X is semi-complete in a neighbourhood of its isolated singularity. Then there is no translation in the sheaf, i.e., $a_{i0\ldots0} = 0$, $\forall i = 2, \ldots, n$.*

Proof. We can write X as

$$\begin{cases} \dot{x_1} = x_1^2 \\ \dot{x_2} = x_2(1 + \alpha_2 x_1) + x_1 R_2(x) \\ \vdots \\ \dot{x_n} = x_n(\gamma_n + \alpha_n x_1) + x_1 R_n(x) \end{cases} \tag{8}$$

where $\dfrac{\partial R_i}{\partial x_i}|_0 = 0$ for all $i = 2, \ldots, n$. Let $\mathcal{F}$ be the foliation associated to (8).

We have that, in a neighbourhood of the origin, Π_1 is transverse to the leaves of $\mathcal{F}$, except to those contained in the invariant manifold $\{x_1 = 0\}$.

Consider the curve $c(t) = (re^{2\pi i t}, 0, \ldots, 0)$,with $t \in [0,1]$ and r such that $r \in U_+$. Let L be the leaf containing $H_1^{-1}(U_1 \setminus \{0\} \times \{(0, \ldots, 0)\})$, where H_1 is the normalizing application, and c_L be the lift of the curve c to the leaf L.

Denoting by dT_L the 1-form such that $dT_L(X) = 1$ we have that

$$\int_{c_L} dT_L = \int_c \frac{dx_1}{x_1^2} = 0$$

As the vector field is semi-complete we conclude that the curve c_L is closed.

As $H_1^{-1}(U_1 \setminus \{0\} \times \{(0, \ldots, 0)\}) \subseteq L$, $H_1(L \cap (U_1 \times (\mathbb{C}^{n-1}, 0))) \subseteq U_1 \times (\mathbb{C}^{n-1}, 0)$ is given by

$$\begin{cases} x_2(x_1) = 0 \\ \vdots \\ x_n(x_1) = 0 \end{cases} \tag{9}$$

and so the leaf $H_1(L \cap (U_1 \times (\mathbb{C}^{n-1}, 0)))$ is parametrized by $(0, \ldots, 0)$ (9).

As $\Pi_1(c_L(0)) \in U_+$, and $U_- = U_+ + \pi$, $\Pi_1(c_L(\frac{1}{2})) \in U_-$. The application g_- is tangent to the identity so, $(0, \ldots, 0)$ is transformed into $(0, \ldots, 0)$, by g_-. This means that $H_2(L \cap (U_2 \times (\mathbb{C}^{n-1}, 0))) \subseteq \{x_i = 0, i = 2, \ldots, n\}$.

Then, by g_+, $(0, \ldots, 0)$ is taken into $(a_{20\ldots0}, \ldots, a_{n0\ldots0})$. As c_L is closed there is no translation, i.e, $(a_{20\ldots0}, a_{n0\ldots0}) = (0, \ldots, 0)$.

□

In this result there is a great difference between the $\mathbb{C}^2$ and $\mathbb{C}^3$ cases. In $\mathbb{C}^2$, g_+ is the identity plus a translation. So, the semi-completude of X implies that g_+ is the identity. Here this is not possible. We do not still know if the semi-completude of X implies that g_+ is the identity: for the set U_+ choosen in the last section, lemma 2 allows only to say that, if $\gamma \notin \mathbb{R}$, g_+ is of type

$$(y, z) \mapsto (y + a_{201}z, z).$$

Lemma 3. *Let X be a semi-complete vector field as in lemma 2. Then the holonomy relative to the invariant manifold tangent to the x_1-axis is the identity.*

Proof. Consider the curve $c(t) = (re^{2\pi it}, 0, \ldots, 0)$ such that $r \in U_-$, r sufficiently close to 0, and let c_0 be the lift of c to the invariant manifold tangent to the x_1-axis (whose existence is guaranteed in the last lemma).

Let Σ be the transversal section to the curve c_0 at $c_0(0)$ given by $\{c_0(0) + (0, \tau_2, \ldots, \tau_n) : 0 \leq \sum_{i=2}^n |\tau_i|^2 < \varepsilon\}$ and c_L be the lift of c to the leaf through the point $c_0(0) + (0, \tau_2, \ldots, \tau_n) \in \Sigma$. Then

$$\int_{c_L} dT_L = \int_c \frac{dx_1}{x_1^2} = 0$$

As X is semi-complete we conclude that c_L is closed. But c_L is closed for all $(\tau_2, \ldots, \tau_n)$ with norm less then ε. This means that the holonomy is the identity.

□

The next proposition is valid for $\mathbb{C}^n$, but is first represented for $\mathbb{C}^3$ because it is much easier.

Proposition 13. *Let X be a vector field, in $\mathbb{C}^3$, of type $Y_{1,\alpha}$. Suppose that X has a holomorphic invariant manifold tangent to the x_1-axis and the holonomy relative to that invariant manifold is the identity. Then X is analytically conjugated to its formal normal form.*

Proof. To prove that X is analytically conjugated to its formal normal form we need to prove that g_- and g_+ are the identity or, equivalently, that $a_{2jk} = a_{3jk} = 0$ for all $(j, k) \in \mathbb{N}_0 \times \mathbb{N}_0$.

Denote (x_1, x_2, x_3) by (x, y, z) and (α_2, α_3) by (α, β).

By hypothesis, X has an invariant manifold tangent to the x-axis and the holonomy relative to this invariant manifold is the identity.

We are going to translate this in terms of the presheaves $\Lambda_{Z_{\alpha,\beta}}(U_+)$ and $\Lambda_{Z_{\alpha,\beta}}(U_-)$.

We treat first the case $\gamma (= \gamma_3) \notin \mathbb{R}$. In this case, by proposition 12, we have that

$$\Lambda_{Z_{\alpha,\beta}}(U_+) : \{(y, z) \mapsto (y + a_{201}z, z)\}$$

and

$$\Lambda_{Z_{\alpha,\beta}}(U_-) : \{(y,z) \mapsto (y + \sum_{\substack{j+k\geq 1\\(j,k)\neq(1,0)\\(j,k)\neq(0,1)}} a_{2jk} y^j z^k, z + \sum_{\substack{j+k\geq 1\\(j,k)\neq(0,1)}} a_{3jk} y^j z^k)\}$$

Consider the curve $c(t) = (re^{2\pi i t}, 0, 0)$, $t \in [0,1]$, with r sufficiently close to zero and such that $r \in U_-$.

Let L be the invariant manifold tangent to the x-axis and c_L the lift of c to L. For each (τ,η) such that $0 \leq |\tau|^2 + |\eta|^2 < \varepsilon$ let $L_{\tau,\eta}$ be the leaf through $c_L(0) + (0,\tau,\eta)$ and $c_{L_{\tau,\eta}}$ the lift of c to $L_{\tau,\eta}$.

The image of the leaf $L_{\tau,\eta}$ by H_2 is given by

$$\begin{cases} y(x) = cx^\alpha e^{-\frac{1}{x}} \\ z(x) = dx^\beta e^{-\frac{\gamma}{x}} \end{cases}$$

in $U_2 \times \mathbb{C}^2$ (where $c = c(\tau,\eta)$ and $d = d(\tau,\eta)$), and so has coordinates $(c,d) \in \mathbb{C}^2$. By proposition 12, $c(0,0) = 0 = d(0,0)$. As $\Pi_1(c(0)) \in U_-$ then $\Pi_1(c(\frac{1}{2})) \in U_+$. The change of the leaves in U_+ is of the type

$$(c,d) \mapsto (c + a_{201}d, d)$$

(as the translation is zero). So the image of $L_{\tau,\eta}$ by H_1 corresponds to the leaf of the formal normal form with coordinates $(c + a_{201}d, d)$, i.e., whose solution is given by

$$\begin{cases} y(x) = (c + a_{201}d)x^\alpha e^{-\frac{1}{x}} \\ z(x) = dx^\beta e^{-\frac{\gamma}{x}} \end{cases}$$

in $U_1 \times \mathbb{C}^2$.

As $\Pi_1(c(1)) = \Pi_1(c(0)) \in U_-$ and in U_- the change of the leaves is given by

$$(c,d) \mapsto (c + \sum_{\substack{j+k\geq 1\\(j,k)\neq(1,0)\\(j,k)\neq(0,1)}} a_{2jk} c^j d^k, d + \sum_{\substack{j+k\geq 1\\(j,k)\neq(0,1)}} a_{3jk} c^j d^k)$$

the image by H_2 of the leaf $L_{\tau,\eta}$ through $c_{L_{\tau,\eta}}(1)$ is the leaf with coordinates

$$(c + a_{201}d + \sum_{\substack{j+k\geq 1\\(j,k)\neq(1,0)\\(j,k)\neq(0,1)}} a_{2jk}(c + a_{201}d)^j d^k, d + \sum_{\substack{j+k\geq 1\\(j,k)\neq(0,1)}} a_{3jk}(c + a_{201}d)^j d^k)$$

But the holonomy is the identity. This means that $c_L(0) = c_L(1)$ and, consequently,

$$\begin{cases} c = c + a_{201}d + \sum_{\substack{j+k\geq 1\\(j,k)\neq(1,0)\\(j,k)\neq(0,1)}} a_{2jk}(c + a_{201}d)^j d^k \\ d = d + \sum_{\substack{j+k\geq 1\\(j,k)\neq(0,1)}} a_{3jk}(c + a_{201}d)^j d^k \end{cases}$$

$$\Leftrightarrow \begin{cases} a_{201}d + \sum_{\substack{j+k\geq 1\\(j,k)\neq(1,0)\\(j,k)\neq(0,1)}} a_{2jk}(c+a_{201}d)^j d^k = 0 \\ \sum_{\substack{j+k\geq 1\\(j,k)\neq(0,1)}} a_{3jk}(c+a_{201}d)^j d^k = 0 \end{cases}$$

$\forall (c,d) \in B(0,\varepsilon_1)$, with ε_1 sufficiently small (because $c(0,0) = 0 = d(0,0)$).

A series is zero in a small ball centered at the origin iff the coeficients of the powers of the variables are all zero. In the first equation, the coefficient of the term d ($d = c^0 d^1$) is a_{201}, thus $a_{201} = 0$. In this way, the system reduces to

$$\begin{cases} \sum_{\substack{j+k\geq 1\\(j,k)\neq(1,0)\\(j,k)\neq(0,1)}} a_{2jk}c^j d^k = 0 \\ \sum_{\substack{j+k\geq 1\\(j,k)\neq(0,1)}} a_{3jk}c^j d^k = 0 \end{cases}$$

Each equation of the system is a series in two variables. Those series are independent, in the sense that coefficients of each series are independent of the other. So, as the series are zero in a small ball centered at the origin we conclude that $a_{2jk} = 0$ and $a_{3jk} = 0$, $\forall (j,k) \in \mathbb{N}_0 \times \mathbb{N}_0$. So, in this case, $g_+ \equiv 0$ and $g_- \equiv 0$.

Suppose now that $\gamma \in \mathbb{R}^+ \setminus \mathbb{N}$. We will only analize the case $\gamma > 1$. The case $0 < \gamma < 1$ is analogous.

If $\gamma > 1$ we know that, by g_+

$$(c,d) \mapsto (c, d + \sum_{j=1}^{l} a_{3j0}c^j)$$

When we return back to $c_L(1)$, by g_-

$$\begin{aligned}(c, d + \sum_{j=1}^{l} a_{3j0}c^j) \mapsto (c + &\sum_{j+k\geq 1} a_{2jk}c^j (d + \sum_{p=1}^{l} a_{3p0}c^p)^k, \\ &d + \sum_{p=1}^{l} a_{3p0}c^p + \sum_{\substack{j+k\geq 1\\(j,k)\neq(0,1)\\(j,k)\neq(1,0),\dots,(l,0)}} a_{3jk}c^j (d + \sum_{p=1}^{l} a_{3p0}c^p)^k)\end{aligned}$$

As the holonomy is the identity

$$\begin{cases} \sum_{j+k\geq 1} a_{2jk}c^j (d + \sum_{p=1}^{l} a_{3p0}c^p)^k = 0 \\ \sum_{p=1}^{l} a_{3p0}c^p + \sum_{\substack{j+k\geq 1\\(j,k)\neq(0,1)\\(j,k)\neq(1,0),\dots,(l,0)}} a_{3jk}c^j (d + \sum_{p=1}^{l} a_{3p0}c^p)^k = 0 \end{cases}$$

In the second equation the coeficient of $c(= c^1d^0)$ is equal to a_{310}. Thus $a_{310} = 0$ and the equation is reduced to

$$\sum_{p=2}^{l} a_{3p0}c^p + \sum_{\substack{j+k\geq 1\\ (j,k)\neq(0,1)\\ (j,k)\neq(1,0),\dots,(l,0)}} a_{3jk}c^j(d + \sum_{p=2}^{l} a_{3p0}c^p)^k = 0$$

In the same way, because of the restrictions imposed on the second sum, the coefficient of c^2 is equal to a_{320}. So $a_{320} = 0$. Proceeding successively in the same manner, we can deduce that $a_{3p0} = 0$, $\forall p = 1, \dots, l$. Thus the system, is reduced to

$$\begin{cases} \sum_{j+k\geq 1} a_{2jk}c^jd^k = 0 \\ \sum_{\substack{j+k\geq 1\\ (j,k)\neq(0,1)\\ (j,k)\neq(1,0),\dots,(l,0)}} a_{3jk}c^jd^k = 0 \end{cases}$$

and, consequently, $a_{2jk} = a_{3jk} = 0$, $\forall (j,k) \in \mathbb{N}_0 \times \mathbb{N}_0$, i.e., $g_+ \equiv 0$ and $g_- \equiv 0$.

□

We are going to analyse now what happens in the $\mathbb{C}^n$ case, for $n \geq 4$.

First of all I am going to explain, geometrically, how to determine $\Lambda_{Z_\alpha}(U_+)$ for a given U_+:

We represent the set of complex numbers $\{(Q, \gamma) - \gamma_i : i = 2, \dots, n, Q \in \mathbb{N}_0^{n-1}\}$. We can assume that $0 = \arg(\gamma_2) \leq \dots \leq \arg(\gamma_n) < \pi$.

Denote by K the sector with vertex at the origin whose elements have arguments between $0 = \arg(\gamma_2)$ and $\arg(\gamma_n)$.

Then we choose two straight lines, not contained in K, with arguments equal to φ_{jQ}, for some $j = 2, \dots, n$ and $Q \in \mathbb{N}_0^{n-1}$, and such that the two sectors defined by those straight lines do not contain any complex number of the type $(Q, \gamma) - \gamma_i$ (figure 11).

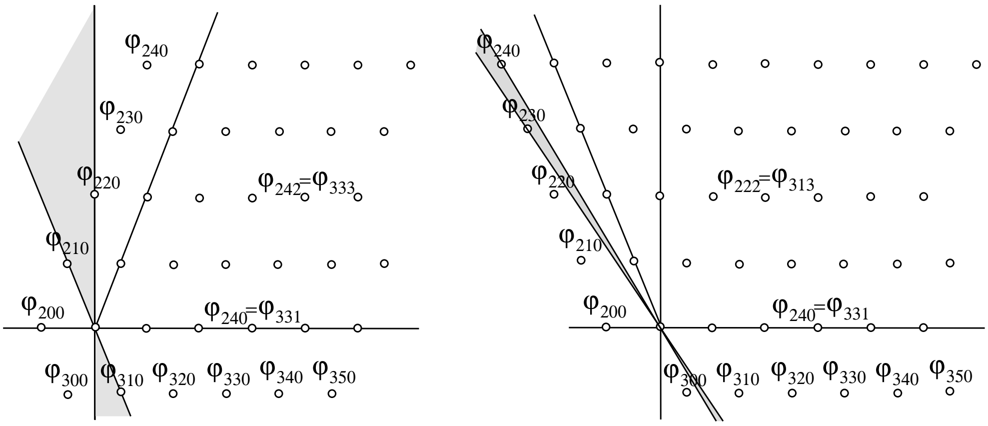
Fix one of those sectors: U. Then, if $U + \frac{\pi}{2}$ is contained in the attractor sector we take $U_+ = U + \frac{\pi}{2}$, otherwise we take $U_+ = U - \frac{\pi}{2}$.

The constants a_{jQ} that can be non zero in $\Lambda_{Z_\alpha}(U_+)$ are those such that $(Q, \gamma) - \gamma_j$ is on the opposite side of U_+ relativelly to the choosen straight lines. If φ_{jQ} is over the boundary of $U \cap (U + \pi) \cap \{$ half plane defined by the bisectrix of U not containing $U_+\}$ then a_{jQ} can also be non zero in U_+. For example, if we look to figure 11, on the left case $\Lambda_{Z_\alpha}(U_+)$ is given by

$$\{(y, z) \mapsto (y + a_{200} + a_{201}z, z + a_{300})\}$$

while on the right one $\Lambda_{Z_\alpha}(U_+)$ is given by

$$\{(y, z) \mapsto (y + a_{200} + a_{201}z + a_{202}z^2 + a_{203}z^3, z + a_{300})\}.$$



$0 < \arg(\gamma) < \pi/2$ $\qquad$ $\pi/2 < \arg(\gamma) < \pi$

FIGURE 11

In the general case we can write $\Lambda_{Z_\alpha}(U_+)$ as

$$\{c \mapsto (c_2 + \sum_{j=1}^{k_2} a_{2Q_{2j}} c^{Q_{2j}}, \ldots , c_n + \sum_{j=1}^{k_n} a_{nQ_{nj}} c^{Q_{nj}})\}$$

where Q_{ij}, for $j = 1, \ldots , k_i$ are all the $(n-1)$-tuples in $\mathbb{N}_0^{n-1}$ such that $(Q_{ij}, \gamma) - \gamma_i \in R$ where R is the sector $\{$ half plane defined by the bisectrix of U not containing $U_+\}$.

This construction is valid for any dimension, except when $0 = \arg(\gamma_2) = \ldots = \arg(\gamma_n)$. In this case we proceed as in the subsubsection 4.4.2: the constants a_{jQ} that can be non zero in $\Lambda_{Z_\alpha}(U_+)$ are those such that $\arg((Q, \gamma) - \gamma_j) = \pi$.

We first demonstrate a property of the elements a_{jQ}.

Proposition 14. *Suppose that $(Q, \gamma) - \gamma_j \in R$. Then $(P, \gamma) - \gamma_j \in R$, $\forall P : p_i \leq q_i$.*

Remark 13. As we are looking for complex numbers not in K, we should remark that $(Q, \gamma) - \gamma_j \notin K$ implies that $q_j = 0$.

Proof. We can easily prove this result geometrically. Suppose that $(Q, \gamma) - \gamma_j \in R$.

$$(Q, \gamma) - \gamma_j = (P + (Q - P), \gamma) - \gamma_j = (P, \gamma) - \gamma_j - (Q - P, \gamma)$$

Thus

$$(P, \gamma) - \gamma_j = (Q, \gamma) - \gamma_j - (Q - P, \gamma)$$

But $(Q, \gamma) - \gamma_j \in R$, by hypothesis, and $Q - P \in \mathbb{N}_0^{n-1}$ because $p_i \leq q_i$. As to an element of R we are subtracting a linear positive combination of the eigenvalues (i.e., an element of K), we still remain in R (figure 12).

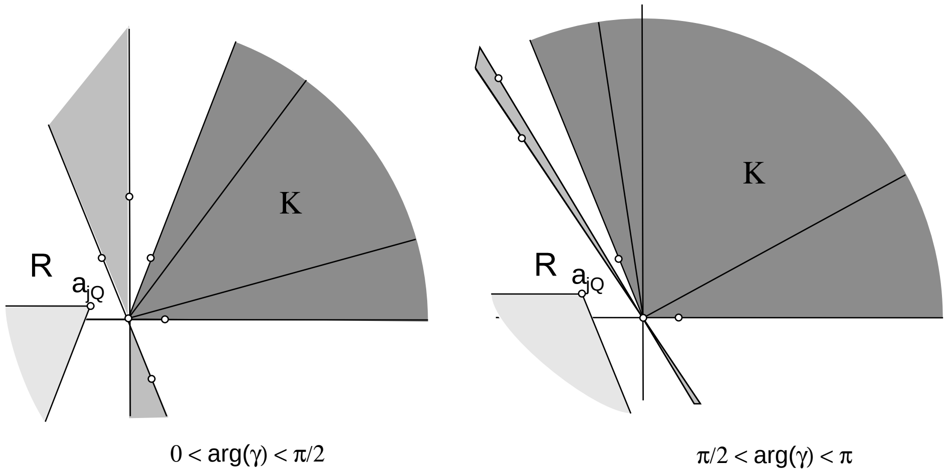


FIGURE 12. The elements $(P,\gamma)-\gamma_j$ belongs to the lighter shaded region, $\forall P : p_i \leq q_i$, $\forall i = 2,\dots n$.

□

Proposition 15. *Let X be a vector field of type $Y_{1,\alpha}$. Suppose that X has a holomorphic invariant manifold tangent to the x_1-axis and the holonomy relative to that invariant manifold is the identity. Then X is analytically conjugated to its formal normal form.*

Proof. The idea of the proof is to use induction over the degree of Q, i.e., over $|Q|$.

More specifically by proposition 12 we know that $a_{jQ} = 0$ for all $j = 2,\dots,n$, with $Q = 0$. We will prove that if $\forall Q : |Q| \leq q$ we have $a_{iQ} = 0$, $\forall i : a_{iQ} \in \Lambda_{Z_\alpha}(U_+)$, then $\forall Q : |Q| = q+1$ we also have $a_{iQ} = 0$, $\forall i : a_{iQ} \in \Lambda_{Z_\alpha}(U_+)$.

Abusing notation, we say that $a_{iQ} \in \Lambda_{Z_\alpha}(U_+)$ if a_{iQ} can be non zero in U_+, i.e., $(Q,\gamma)-\gamma_i \in R$.

Consider the curve $c(t) = (re^{2\pi i t}, 0, \dots, 0)$, $t \in [0,1]$, with r sufficiently close to zero and such that $r \in U_-$.

For each $\tau = (\tau_2,\dots,\tau_n)$ sufficiently close to $0 \in \mathbb{C}^{n-1}$, let L_τ be the leaf through $c_L(0)+(0,\tau)$ where c_L is the lift of c to the invariant leaf tangent to the x_1-axis, L. Denote by c_τ the lift of c to the leaf L_τ.

The image of the leaf L_τ by H_2 has coordinates $c = (c_2,\dots,c_n) \in \mathbb{C}^{n-1}$. In particular, by the proof of proposition 12, the leaf $L = L_0$ is parametrized by $0 \in \mathbb{C}^{n-1}$.

As $\Pi_1(c_\tau(0)) \in U_-$ then $\Pi_1(c_\tau(\frac{1}{2})) \in U_+$. The change of the leaves in U_+, g_+, is polynomial. The image of the same leaf by H_1 corresponds to the leaf of the formal normal form with coordinates $g_+(c)$ in $U_1 \times \mathbb{C}^{n-1}$.

As $\Pi_1(c_\tau(1)) = \Pi_1(c_\tau(0)) \in U_-$ and in U_- the change of the leaves is given by g_-, the image of the leaf through $c_\tau(1)$ by H_2 is the leaf with coordinates $g_-(g_+(c))$.

But the holonomy is the identity. This means that $c_\tau(0) = c_\tau(1)$ and, consequently, $g_-(g_+(c)) = id$.

As X has a holomorphic invariant manifold tangent to the x_1-axis, by proposition 12, $a_{j0} = 0$ in U_+. So the induction hypothesis is verified for $|Q| = 0$.

Suppose that $\forall Q : |Q| \leq q$ we have $a_{iQ} = 0$, $\forall i : a_{iQ} \in \Lambda_{Z_\alpha}(U_+)$. Then g_+ is of the type

$$(c_2, \dots, c_n) \mapsto (c_2 + \sum_{j=1}^{k_2} a_{2Q_{2j}} c^{Q_{2j}}, \dots, c_n + \sum_{j=1}^{k_n} a_{nQ_{nj}} c^{Q_{nj}})$$

where $|Q_{ij}| \geq q + 1$, $\forall i = 2, \dots, n$ and $1 \leq j \leq k_j$.

With this supposition, the composition $g_- \circ g_+$ is given by

$$c \mapsto (c_2 + \sum_{j=1}^{k_2} a_{2Q_{2j}} c^{Q_{2j}} + \sum_{\substack{Q \neq e_1 \\ Q:(Q,\gamma)-\gamma_2 \notin R}} a_{2Q} \prod_{i=2}^{n} (c_i + \sum_{j=1}^{k_i} a_{iQ_{ij}} c^{Q_{ij}})^{q_i},$$

$$, \dots, c_n + \sum_{j=1}^{k_n} a_{nQ_{nj}} c^{Q_{nj}} + \sum_{\substack{Q \neq e_{n-1} \\ Q:(Q,\gamma)-\gamma_n \notin R}} a_{nQ} \prod_{i=2}^{n} (c_i + \sum_{j=1}^{k_i} a_{iQ_{ij}} c^{Q_{ij}})^{q_i})$$

As it must be the identity we have:

(10)

$$\begin{cases} \sum_{j=1}^{k_2} a_{2Q_{2j}} c^{Q_{2j}} + \sum_{\substack{Q \neq e_1 \\ Q:(Q,\gamma)-\gamma_2 \notin R}} a_{2Q} \prod_{i=2}^{n} (c_i + \sum_{j=1}^{k_i} a_{iQ_{ij}} c^{Q_{ij}})^{q_i} = 0 \\ \vdots \\ \sum_{j=1}^{k_n} a_{nQ_{nj}} c^{Q_{nj}} + \sum_{\substack{Q \neq e_{n-1} \\ Q:(Q,\gamma)-\gamma_n \notin R}} a_{nQ} \prod_{i=2}^{n} (c_i + \sum_{j=1}^{k_i} a_{iQ_{ij}} c^{Q_{ij}})^{q_i} = 0 \end{cases}$$

Let Q_0 be such that $|Q_0| = q + 1$ and $(Q_0, \gamma) - \gamma_i \in R$ for some $i = 2, \dots, n$. We look for the coefficient of c^{Q_0} in the $(i-1)^{th}$ equation of the system (10).

The term c^{Q_0} appears in $\prod_{p=2}^{n} (c_p + \sum_{j=1}^{k_p} a_{pQ_{pj}} c^{Q_{pj}})^{q_p}$ if $(q_2, \dots, q_n) = e_{i-1}$ or $(q_2, \dots, q_n) = Q_0$. However, as $(Q_0, \gamma) - \gamma_i \in R$, both $(n-1)$-tuples are forbidden to take in the second sum.

We can ask if there are other hypothesis to obtain a constant times c^{Q_0} in $\prod_{p=2}^{n} (c_p + \sum_{j=1}^{k_p} a_{pQ_{pj}} c^{Q_{pj}})^{q_p}$. As the terms of $\sum_{j=1}^{k_p} a_{pQ_{pj}} c^{Q_{pj}}$ involves only monomials of order greater or equal to $q + 1$, the only chance is the existence of $j \neq i$ such that $(Q_0, \gamma) - \gamma_j \in R$: in this case we should take $Q = e_{j-1}$.

Suppose that $(Q_0, \gamma) - \gamma_k = 0$ only for $k = i$ and $k = j$.

If we look to the $(j-1)^{th}$ equation we can obtain c^{Q_0} in $\Pi_{p=2}^{n} (c_p + \sum_{j=1}^{k_p} a_{pQ_{pj}} c^{Q_{pj}})^{q_p}$ if we take $Q = e_{i-1}$.

However $(e_{j-1},\gamma)-\gamma_i=\gamma_j-\gamma_i$ and $(e_{i-1},\gamma)-\gamma_j=\gamma_i-\gamma_j$, i.e., one is symmetrical of the other. As the complement of $U\cup(U+\pi)$ is the union of two sectors of amplitude smaller than π and neither U nor $U+\pi$ contain singular directions, we have that one and exactly one of the two numbers belongs to R.

Suppose that $(e_{j-1},\gamma)-\gamma_i\in R$. Then $Q=e_{j-1}$ is forbidden in the second sum of the $(i-1)^{th}$ equation, and so the coefficient of c^{Q_0} is a_{iQ_0}. Thus a_{iQ_0} must be 0.

Suppose that $(e_{i-1},\gamma)-\gamma_j\in R$. By the argument described above, the coefficient of c^{Q_0} in the $(j-1)^{th}$ equation is a_{jQ_0} and consequently a_{jQ_0} is zero. In this way the term $a_{jQ_0}c^{Q_0}$ does not appear in the second sum of the $(i-1)^{th}$ equation. So a_{iQ_0} is also zero.

As $\sharp\{(Q,\gamma)-\gamma_i:(Q,\gamma)-\gamma_i\in R \text{ for some } i=2,\dots,n\}$ is finite, this process stops in a finite number of steps. So we proved that $g_+=id$.

To prove that g_- is also the identity function it is sufficient to see that the last system reduces to

$$\begin{cases}\sum_{\substack{Q\neq e_1\\ Q:(Q,\gamma)-\gamma_2\notin R}} a_{2Q}c_2^{q_2}\dots c_n^{q_n}=0\\ \vdots\\ \sum_{\substack{Q\neq e_{n-1}\\ Q:(Q,\gamma)-\gamma_n\notin R}} a_{nQ}c_2^{q_2}\dots c_n^{q_n}=0\end{cases}$$

and consequently $a_{iQ}=0$, $\forall i=2,\dots,n$ and $\forall Q\in\mathbb{N}_0$.

We have supposed that $\sharp\{i:(Q_0,\gamma)-\gamma_i=0\}=2$, which is not necessarily true, specialy for great dimensions. Let us see how to solve the problem when $\sharp\{i:(Q_0,\gamma)-\gamma_i=0\}>2$.

Let us consider the sector U, defined before, as close to the real axis as possible.

Suppose that $\{i:(Q_0,\gamma)-\gamma_i=0\}=\{i_1,i_2,\dots,i_k\}$ with $i_1<i_2<\dots<i_k$. We want to prove that $a_{iQ_0}=0$, $\forall i=i_1,\dots,i_k$.

The coefficient of c^{Q_0} on the $(i_j-1)^{th}$ equation of the system (10) is given by

$$a_{i_jQ_0}+\sum_{\substack{l\neq j\\ l:(e_{i_l-1},\gamma)-\gamma_{i_j}\notin R}} a_{i_je_{i_l-1}}a_{i_lQ_0}$$

and is equal to zero. So we have a system of k equations in k unknowns: $a_{i_jQ_0}$, $j=1,\dots,k$. Denote by $B=(b_{ij})$ the matrix associated to this new system.

We can assume that

$$\Im(\gamma_{i_1})\le\Im(\gamma_{i_2})\le\dots\le\Im(\gamma_{i_k})$$

and that if $\Im(\gamma_{i_j})=\Im(\gamma_{i_{j+1}})$ then $\Re(\gamma_{i_j})<\Re(\gamma_{i_{j+1}})$. If this is not true we can reorder the variables and the lines of the system in order to have the inequality given above.

We have already seen that $(e_{i_j-1},\gamma)-\gamma_{i_l}=\gamma_{i_j}-\gamma_{i_l}$, $(e_{i_l-1},\gamma)-\gamma_{i_j}=\gamma_{i_l}-\gamma_{i_j}$ and only one of them belongs to R. As $\gamma_{i_j}-\gamma_{i_l}\in R$ means that e_{i_j-1} does not belong to the second sum of the $(i_l-1)^{th}$ equation of system (10), if $\gamma_{i_j}-\gamma_{i_l}\in R$ then $b_{lj}=0$.

By hypothesis

$$i_l > i_j \quad \text{and} \quad \Im(\gamma_{i_l}) > \Im(\gamma_{i_j}) \quad \Rightarrow \quad \Im(\gamma_{i_l}-\gamma_{i_j}) > 0$$

As we can choose U so close to the real axis as possible and $\sharp\{i : (Q_0,\gamma)-\gamma_i=0\}$ is finite, we can say that

$$i_l > i_j \quad \Rightarrow \quad \gamma_{i_l}-\gamma_{i_j}\notin R$$

i.e.,

$$i_l > i_j \quad \Rightarrow \quad \gamma_{i_j}-\gamma_{i_l}\in R \quad \Rightarrow \quad b_{lj}=0$$

If $\Im(\gamma_{i_l})=\Im(\gamma_{i_j})$ for $i_l>i_j$ then $\Im(\gamma_{i_l}-\gamma_{i_j})=0$. However $\Re(\gamma_{i_l})>\Re(\gamma_{i_j})$. Thus $\Re(\gamma_{i_l}-\gamma_{i_j})>0$ and $\Re(\gamma_{i_j}-\gamma_{i_l})<0$, which means that $\gamma_{i_j}-\gamma_{i_l}\in R$. So $b_{lj}=0$.

We have just proved that the matrix B is of the form

$$B=\begin{pmatrix} 1 & a_{i_i e_{i_2-1}} & a_{i_1 e_{i_3-1}} & \dots & a_{i_2 e_{i_k-1}} \\ 0 & 1 & a_{i_2 e_{i_3-1}} & \dots & a_{i_2 e_{i_k-1}} \\ 0 & 0 & 1 & \dots & a_{i_3 e_{i_k-1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

and so we can conclude that $a_{i_j Q_o}=0$, $\forall j=1,\dots,k$.

The induction over $|Q|$ stops in a finite number of steps. So $g_+=id$.

We prove that g_- is also the identity in the same way: we see that system (10) reduces to

$$\begin{cases} \sum_{\substack{Q\neq e_1 \\ Q:(Q,\gamma)-\gamma_2\notin R}} a_{2Q}c_2^{q_2}\dots c_n^{q_n}=0 \\ \vdots \\ \sum_{\substack{Q\neq e_{n-1} \\ Q:(Q,\gamma)-\gamma_n\notin R}} a_{nQ}c_2^{q_2}\dots c_n^{q_n}=0 \end{cases}$$

and consequently $a_{iQ}=0$, $\forall i=2,\dots,n$ and $\forall Q\in\mathbb{N}_0$.

□

Corollary 1. *Let X be a vector field of type $Y_{1,\alpha}$. Then X is semi-complete iff X is analytically conjugated to its formal normal form.*

Proof. Suppose that X, of type $Y_{1,\alpha}$ is semi-complete. By lemma 2 and proposition 12 there is an invariant manifold tangent to the x_1-axis. Lemma 3 guarantees that the holonomy relative to that invariant manifold is the identity. So the result follow, by proposition 15.

The other implication results by the fact that semi-completude is preserved by analytical conjugacy.

□

It remains to analyse which formal normal form are semi-complete. This is a much simpler problem.

Proposition 16. *Let X be a vector field of type*

$$X : \begin{cases} \dot{x_1} = x_1^2 \\ \dot{x_2} = x_2(1+\alpha_2 x_1) \\ \vdots \\ \dot{x_n} = x_n(\gamma_n + \alpha_n x_1) \end{cases}$$

Then X is semi-complete iff $\alpha_i \in \mathbb{Z}$, $\forall i = 2, \dots, n$.

Proof. Consider the vector field X given above and suppose that X is semi-complete. Remark that the x_1-axis is an invariant manifold of the differential equation associated to the vector field X. Then, as we proved before, the holonomy relative to the x_1-axis is the identity.

Consider the ordinary differential equation

$$\begin{cases} \dfrac{dx_2}{dx_1} = \dfrac{x_2(1+\alpha_2 x_1)}{x_1^2} \\ \vdots \\ \dfrac{dx_n}{dx_1} = \dfrac{x_n(\gamma_n + \alpha_n x_1)}{x_1^2} \end{cases} \tag{11}$$

equivalent to the differential equation associated to the vector field X.

Let Σ be the transversal section to the first axis, through the point $(r, 0, \dots, 0)$, given by $\Sigma = \{(r, x_2, \dots, x_n) : 0 \leq \sum_{i=2}^{n} |x_i|^2 < \varepsilon < r\}$. Taking $x_1 = re^{2\pi i t}$, $t \in [0,1]$, and substituting in (11) we obtain

$$\begin{cases} \dfrac{dx_2}{dt} = \dfrac{dx_2}{dx_1}\dfrac{dx_1}{dt} = 2\pi i(\alpha_2 + \frac{1}{r}e^{-2\pi i t})x_2 \\ \vdots \\ \dfrac{dx_n}{dt} = \dfrac{dx_n}{dx_1}\dfrac{dx_1}{dt} = 2\pi i(\alpha_n + \frac{1}{r}\gamma_n e^{-2\pi i t})x_n \end{cases}$$

Integrating for t between 0 and 1 we obtain

$$\begin{aligned} x_i(1) &= x_i(0)e^{\int_0^1 2\pi i(\alpha_i + \frac{1}{r}\gamma_i e^{-2\pi i t})dt} = x_i(0)e^{[2\pi i \alpha_i t - \frac{1}{r}\gamma_i e^{-2\pi i t}]_0^1} \\ &= x_i(0)e^{2\pi i \alpha_i} \end{aligned}$$

for all $i = 2, \dots, n$.

So the holonomy is given by

$$h(x_2, \dots, x_n) = (x_2 e^{2\pi i \alpha_2}, \dots, x_n e^{2\pi i \alpha_n})$$

As the holonomy is the identity then $\alpha_i \in \mathbb{Z}$ for all $i = 2, \dots, n$

$$\begin{cases} e^{2\pi i \alpha_2} = 1 \\ \vdots \\ e^{2\pi i \alpha_n} = 1 \end{cases} \quad \Leftrightarrow \quad (\alpha_2, \dots, \alpha_n) \in \mathbb{Z}^{n-1}$$

The other implication is immediate. We can easily solve the differencial equation associated to the vector field X:

$$\frac{dx_1}{dT} = x_1^2 \Leftrightarrow \frac{dx_1}{x_1^2} = dT \Leftrightarrow x_1(T) = \frac{x_1(0)}{1 - x_1(0)T}$$

Substituting $\frac{x_1(0)}{1 - x_1(0)T}$ for $x(T)$ in the other equations we obtain the linear system

$$\begin{cases} \dfrac{dx_2}{dT} = x_2(1 + \alpha_2 \dfrac{x_1(0)}{1 - x_1(0)T}) \\ \vdots \\ \dfrac{dx_n}{dT} = x_n(\gamma_n + \alpha_n \dfrac{x_1(0)}{1 - x_1(0)T}) \end{cases}$$

whose solution is given by

$$\begin{cases} x_2(T) = x_2(0) \dfrac{e^T}{(1 - x_1(0)T)^{\alpha_2}} \\ \vdots \\ x_n(T) = x_n(0) \dfrac{e^{\gamma_n T}}{(1 - x_1(0)T)^{\alpha_n}} \end{cases} \tag{12}$$

As $\alpha_i \in \mathbb{Z}$, $\forall i = 2, \ldots, n$, $(1 - x_1(0)T)^{\alpha_i}$ is well defined for all $T \in \mathbb{C} \setminus \{\frac{1}{x_1(0)}\}$.

So, the application $\Phi : \Omega = \{(T, x_1, \ldots, x_n) : T \neq \frac{1}{x_1}\} \subseteq \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ given by

$$(T, x_1, x_2, \ldots, x_n) \mapsto (\frac{x_1}{1 - x_1 T}, x_2 \frac{e^T}{(1 - x_1 T)^{\alpha_2}}, \ldots, x_n \frac{e^{\gamma_n T}}{(1 - x_1 T)^{\alpha_n}})$$

is obviously a semi-complete flow associated to X: for each $(x_1, \ldots, x_n)$ fixed, $(T_i, x_1, \ldots, x_n) \to \partial\Omega$ iff $T_i \to \frac{1}{x_1}$. We have that

$$\lim_{T \to \frac{1}{x_1}} e^{\gamma_i T} = e^{\frac{\gamma_i}{x_1}} \neq 0 \quad \text{and} \quad \lim_{T \to \frac{1}{x_1}} (1 - x_1 T)^{\alpha_i} = 0$$

So, $(T_i, x_1, \ldots, x_n) \to \partial\Omega$ implies that $\|\Phi(T_i, x_1, \ldots, x_n)\| \to \infty$, i.e., $\Phi(T_i, x_1, \ldots, x_n)$ tends to the boundary of $\mathbb{C}^n$.

In particular X is semi-complete relatively to $\mathbb{C}^n$.

$\square$

Our objective is to prove theorem 3:

A saddle-node foliation is associated to a semi-complete vector field iff it admits

$$\begin{cases} \dot{x_1} = x_1^2 \\ \dot{x_2} = x_2(\lambda_2 + \alpha_2 x_1) \\ \vdots \\ \dot{x_n} = x_n(\lambda_n + \alpha_n x_1) \end{cases} \quad , \quad (\alpha_2, \dots, \alpha_n) \in \mathbb{Z}^{n-1}$$

as normal form.

If $\mathcal{F}$ is the foliation associated to $Y_{1,\alpha}$, with $\alpha \in \mathbb{Z}^{n-1}$ and $Y_{1,\alpha}$ is analytically conjugated to its formal normal form Z_α, then $\mathcal{F}$ is associated to a semi-complete vector field in a neighbourhood of the origin, as we have just proved.

We are going to prove that there are no more foliations of saddle-node type associated to semi-complete vector fields.

First we will prove the next result:

Proposition 17. *Let f be a holomorphic function ($f : \mathbb{C}^n \to \mathbb{C}$) such that $f(0) \neq 0$ and consider the vector field $Y = fY_{1,\alpha}$. Suppose that Y is semi-complete in $B(0, \varepsilon)$ with f non zero in this open set. Then there is a holomorphic invariant manifold tangent to the x_1-axis at the origin, i.e., a holomorphic central manifold.*

Proof. Suppose that Y does not have a holomorphic invariant manifold tangent to the x_1-axis at the origin. As Y and $Y_{1,\alpha}$ have exactly the same foliation in $B(0, \varepsilon)$ then $Y_{1,\alpha}$ does not also have a holomorphic invariant manifold tangent to the x_1-axis at the origin.

Consider the curve $c(t) = (re^{2\pi i t}, 0, \dots, 0)$, for r sufficiently small (in particular $|r| < \varepsilon$).

We know that in a neighbourhood of the origin Π_1 is transverse to the leaves of $Y_{1,\alpha}$ not contained in $\{x_1 = 0\}$. Let c_L be the lift of c to the leaf, L, containing $H_1^{-1}(U_1 \setminus \{0\} \times \{(0, \dots, 0)\})$. We have $H_1(L \cap (U_1 \times (\mathbb{C}^{n-1}, 0))) \subseteq \{x_i = 0, i = 2, \dots, n\}$ and so L is parametrized by $(0, \dots, 0)$.

We choose r in such a manner that $\Pi_1(c_L(0)) \in U_+$. As $U_- = U_+ + \pi$, $\Pi_1(c_L(\frac{1}{2})) \in U_-$. The application g_- is tangent to the identity so, $(0, \dots, 0)$ is transformed into $(0, \dots, 0)$ by g_-. This means that $H_2(L \cap (U_2 \times (\mathbb{C}^{n-1}, 0))) \subseteq \{x_i = 0, i = 2, \dots, n\}$. We also have that $\Pi_1(c_L(1)) \in U_+$, so

$$(0, \dots, 0) \mapsto (a_{20\dots0}, \dots, a_{n0\dots0})$$

As there is no holomorphic central manifold $(a_{20\dots0}, \dots, a_{n0\dots0}) \neq (0, \dots, 0)$, by proposition 12. This means that $c_L(0) \neq c_L(1)$, i.e., the curve c_L is not closed.

Consider the conjugacy

$$Y_\lambda = (DH_\lambda)^{-1} Y \circ H_\lambda$$

where $H_\lambda : \mathbb{C}^n \to \mathbb{C}^n$ is the homothety $H_\lambda(x) = \lambda x$. Thus Y_λ and Y are analytically conjugated.

If Y is semi-complete in $B(0,\varepsilon)$, then Y_λ is semi-complete in $B(0, \dfrac{\varepsilon}{|\lambda|})$. In the particular case that $\lambda = \varepsilon$, Y_λ is semi-complete in $B(0,1)$. We will allways take $\lambda = \varepsilon$.

As $f(0) \neq 0$, Π_1 is transverse to the leaves of Y except to the invariant mainifold $\{x_1 = 0\}$, in a neighbourhood of the origin. We have that

$$dT^Y = \frac{dx_1}{f(x)x_1^2}$$

so

$$dT^{Y_\lambda} = H_\lambda^*(dT^Y) = \frac{d(\lambda x_1)}{f(\lambda x)\lambda^2 x_1^2} = \frac{dx_1}{\lambda f(\lambda x)x_1^2}$$

However, for a given curve c

$$\int_c \frac{dx_1}{\lambda f(\lambda x)x_1^2} = 0 \quad \Leftrightarrow \quad \int_c \frac{dx_1}{f(\lambda x)x_1^2} = 0$$

Consider the curve

$$c^\lambda(t) = (\frac{\lambda}{2}e^{2\pi i t}, 0, \dots, 0) \quad , \quad t \in [0,1]$$

For each λ consider $(x_2^\lambda, \dots, x_n^\lambda)$ sufficiently close to $(0,\dots,0) \in \mathbb{C}^{n-1}$ in such a manner that $(\frac{\lambda}{2}, x_2^\lambda, \dots, x_n^\lambda)$ belongs to L and that the lift of c to L, denoted by c_L^λ, is contained in $B(0,\varepsilon)$.

Let

$$c_\lambda = H_\lambda^{-1}(c_L^\lambda)$$

Remark 14. We can choose $(x_2^\lambda, \dots, x_n^\lambda)$ in such a manner that

$$\lambda \mapsto (x_2^\lambda, \dots, x_n^\lambda)$$

is a continuous function of λ. Consequently $\lambda \mapsto c_L^\lambda$ is also a continuous function of λ, and so is $\lambda \mapsto c_\lambda$ because H_λ is a holomorphic function of λ.

As $c_L^\lambda \subseteq B(0,\varepsilon)$, then $c_\lambda \subseteq B(0,1)$. But c_λ has another important property: as c_L^λ is not closed and H_λ is a homothety, then c_λ is also not closed.

On the other hand

$$\Pi_1(c_\lambda) = \Pi_1(H_\lambda^{-1}(c_L)) = H_\lambda^{-1}(\Pi_1(c_L)) = H_\lambda^{-1}(\frac{\lambda}{2}e^{2\pi i t}) = \frac{1}{2}e^{2\pi i t}$$

for all λ, because H_λ is a homothety. In this way

$$\lim_{\lambda \to 0} \int_{c_\lambda} \frac{dx_1}{f(\lambda x)x_1^2} = \int_{\frac{1}{2}e^{2\pi i t}} \frac{dx_1}{x_1^2} = 0$$

Remark 15. As $\lambda \mapsto c_\lambda$ is a continuous function of λ and $c_\lambda \subseteq D(0,1)$, which is a compact set, there exists $\lim_{\lambda\to 0} c_\lambda$. We only know that $\lim_{\lambda\to 0} \Pi_1(c_L) = \frac{1}{2}e^{2\pi i t}$, which is the only property necessary to the proof.

Let $W \subseteq \mathbb{C}_{x_1}$ be a simply connected neighbourhood of $\Pi_1(c_\lambda(1)) = \frac{1}{2}$, not containing the origin. In this neighbourhood we can write x_i as function of x_1 for all $i = 2, \dots, n$ (remember that Π_1 is transverse to all leaves except to those in the invariant manifold $\{x_1 = 0\}$ and we are excluding $x_1 = 0$ from W).

Define

$$\begin{aligned} I_\lambda : W \to \quad & \mathbb{C} \\ p \mapsto & \int_{c_p} \frac{dx_1}{f(\lambda x)x_1^2} \end{aligned}$$

where $c_p \subseteq W$ is a curve joining $\frac{1}{2}$ to p.

The function f is non zero at the origin, so it can be written in the form

$$f(x) = k + g(x)$$

where $g(0) = 0$ and $k = f(0)$, i.e., $g(x) = x_1 g_1(x) + \ldots + x_n g_n(x)$.

In this way we can rewrite the application I_λ in the following way:

$$\begin{aligned} I_\lambda(p) &= \int_{c_p} \frac{dx_1}{(k + g(\lambda x_1, \lambda x_2(x_1), \dots, \lambda x_n(x_1))x_1^2} \\ &= \int_{c_p} \frac{dx_1}{kx_1^2 + \lambda h(\lambda, x_1)} \end{aligned}$$

Let $m(x_1, \lambda) = \lambda h(\lambda, x_1)$. As m is holomorphic in W and $m(x_1, 0) = 0$, lemma 1 guarantees the existence of real and positive numbers λ_0 and θ such that

$$B(0,\theta) \subseteq I_\lambda(W) \quad , \quad \forall \lambda : |\lambda| \leq \lambda_0$$

As

$$\lim_{\lambda\to 0} \int_{c_\lambda} \frac{dx_1}{f(\lambda x)x_1^2} = 0$$

there exists λ_1 such that $|\lambda_1| < \lambda_0$ and

$$\int_{c_{\lambda_1}} \frac{dx_1}{f(\lambda x)x_1^2} = \alpha$$

with $|\alpha| < \theta$. However $B(0,\theta) \subseteq I_{\lambda_1}(W)$. Thus there exists $p \in W$ such that

$$I_{\lambda_1}(p) = -\alpha$$

If $p \notin c_{\lambda_1}([0,1])$ let $\tilde{c}$ be the curve joining $c_{\lambda_1}(0)$ to p obtained by concatenating c_{λ_1} to c_p. If $p \in c_{\lambda_1}([0,1])$, i.e., $p = c_{\lambda_1}(t_0)$ for some

$0 < t_0 < 1$, let $\tilde{c} = c_{|[0,t_0]}$. Thus

$$\int_{\tilde{c}} \frac{dx_1}{f(\lambda x) x_1^2} = 0$$

But, in both cases, $\tilde{c}$ is an one-to-one embedded curve. This result contradicts the semi-completude of the vector field Y_{λ_1} and, consequently, the semi-completude of the vector field Y.

So, if Y is semi-complete then there exists a holomorphic invariant manifold tangent to the x_1-axis at the origin.

□

Next we will prove:

Proposition 18. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function such that $f(0) \neq 0$ and consider the vector field $Y = fY_{1,\alpha}$. Suppose that Y is semi-complete in $B(0, \varepsilon)$ where ε is such that f is non zero in this open set. Then the holonomy relative to the holomorphic invariant manifold tangent to the x_1-axis (whose existence is guaranteed in Proposition 17) is the identity.*

Proof. Suppose that Y is semi-complete in a neighbourhood of the origin. Then there is a holomorphic invariant manifold tangent to the x_1-axis. By a holomorphic change of coordinates we can suppose that this holomorphic invariant manifold is the x_1-axis itself. Thus the vector field can be written in the form

$$\begin{cases} \dot{x_1} = x_1^2 h(x) \\ \dot{x_2} = \lambda_2 x_2 + x_1 \sum_{j=2}^n x_j f_{2j}(x) \\ \vdots \\ \dot{x_n} = \lambda_n x_n + x_1 \sum_{j=2}^n x_j f_{nj}(x) \end{cases}$$

where $h(0) \neq 0$.

As the vector field is semi-complete in a neighbourhood of the origin, its restriction to the x_1-axis is also semi-complete in a neighbourhood of the origin. But this restriction is the vector field

$$X = x^2 h(x, 0, \dots, 0) \frac{\partial}{\partial x}$$

which is equivalent to the 1-dimensional vector field

$$\dot{x} = x^2 h(x, 0, \dots, 0)$$

But we know that a 1-dimensional meromorphic semi-complete vector field in a neighbourhood of the origin (of $\mathbb{C}$) is analytically conjugated to $X(x) = (\lambda x + \dots)\dfrac{\partial}{\partial x}$, $X(x) = x^2 \dfrac{\partial}{\partial x}$ or X is constant, [10]. Thus the vector field $\dot{x} = x^2 h(x, 0, \dots, 0)$ must be analytically conjugated to $\dot{x} = x^2$.

Consider the curve $c(t) = (re^{2\pi it}, 0, \dots, 0)$, $t \in [0,1]$, for r sufficiently close to 0 and let $c' = (H(\Pi_1(c)), 0, \dots, 0)$, where H is the diffeomorphism that conjugates the vector fields $x^2 h(x, 0, \dots, 0)\dfrac{\partial}{\partial x}$ and $x^2\dfrac{\partial}{\partial x}$. Then

$$\int_c dT_{\{x_i=0, i=2,\dots,n\}} = \int_{c'} \frac{dx_1}{x_1^2} = 0$$

Suppose that the holonomy is not the identity: there exists a neighbourhood of $0 \in \mathbb{C}^{n-1}$ such that, for every $\bar{x}_0 = (x_2^0, \dots, x_n^0)$ sufficiently close to $(0, \dots, 0)$, the lift c_L of c to the leaf L through $(r, x_2^0, \dots, x_n^0)$ is not closed. Thus $c_L(0) \neq c_L(1)$ although $\Pi_1(c_L(0)) = \Pi_1(c_L(1))$.

As $h(0) \neq 0$, there exists a neighbourhood $B(0, \varepsilon)$ of the origin such that $h(x) \neq 0$, $\forall x \in B(0, \varepsilon)$. In particular, we choose r and $\bar{x}_0$, with $\|\bar{x}_0\| \leq \varepsilon_1$, such that $c_L \subseteq B(0, \varepsilon)$.

As before, the projection Π_1 is transverse to all leaves, except to those contained in the invariant manifold $\{x_1 = 0\}$ in a neighbourhood of the origin. As $\Pi_1(c_L(1)) = r \neq 0$ there is a simply connected neighbourhood of r in $\mathbb{C}_{x_1} \setminus \{0\}$, W, such that we can write x_i as function of x_1 for all $i = 2, \dots, n$

$$\begin{cases} x_2 = x_2(x_1; \bar{x}_0) \\ \vdots \\ x_n = x_n(x_1; \bar{x}_0) \end{cases} \tag{13}$$

in each leaf of $Y|_{W \times \mathbb{C}^{n-1}}$.

Substituting (13) in the first equation of the differential system associted to Y, we obtain the differential equation

$$\dot{x_1} = x_1^2 h(x_1, x_2(x_1; \bar{x}_0), \dots, x_n(x_1; \bar{x}_0))$$

where $\bar{x}_0$ is considered as a pararameter.

Consider the application

$$\begin{aligned} I_{\bar{x}_0} : W &\to \mathbb{C} \\ p &\mapsto \int_{c_p} \frac{dx_1}{x_1^2 h(x_1, x_2(x_1; \bar{x}_0), \dots, x_n(x_1, \bar{x}_0))} \end{aligned}$$

where $c_p \subseteq W$ represents a curve joining r to p.

We have $I_{\bar{x}_0}(r) = 0$, $\forall \bar{x}_0 : \|\bar{x}_0\| \leq \varepsilon_1$. In particular $I_0(r) = 0$. On the other hand, as $I_0'(r) = \frac{1}{r^2 h(r, 0, \dots, 0)} \neq 0$ and $I_{\bar{x}_0}'$ is a continuous function of $\bar{x}_0$ there exists $0 < \varepsilon_2 < \varepsilon_1$ such that

$$I_{\bar{x}_0}'(r) \in B\left(\frac{1}{r^2 h(r, 0, \dots, 0)}, \frac{1}{2|r^2 h(r, 0, \dots, 0)|}\right), \quad \forall \bar{x}_0 : \|\bar{x}_0\| \leq \varepsilon 2$$

By the same argument used in the proof of lemma 1, there exist real and positive numbers θ and ε_0 such that

$$\forall \bar{x}_0 : \|\bar{x}_0\| \leq \varepsilon_0, \quad B(0, \theta) \subseteq I_{\bar{x}_0}(W).$$

However

$$\int_{c_L} dT_L \stackrel{\bar{x}_0 \to 0}{\longrightarrow} 0$$

Thus, there exists $\bar{x}_0$ with $\|\bar{x}_0\| \leq \varepsilon_0$ such that

$$\int_{c_L} dT_L = \alpha$$

where $|\alpha| < \theta$. By lemma 1, there exists $p \in W$ such that

$$\int_{c_p} \frac{dx_1}{x_1^2 h(x_1, x_2(x_1; \bar{x}_0), \ldots , x_n(x_1, \bar{x}_0))} = -\alpha$$

The curve c_p can allways be choosen in such a manner that its lift to L does not intersect c_L except when $P = (p, x_2(p), \ldots , x_n(p))$ belongs to c_L.

If $P \notin c_L([0, 1])$ we denote by $\tilde{c}$ the curve obtained by the concatenation of c_L to the lift of c_p to L. If $P = c_L(t_0)$ for some $0 < t_0 < 1$, we denote by $\tilde{c}$ the curve $c_L|_{[0,t_0]}$.

In both cases

$$\int_{\tilde{c}} dT_L = 0$$

and $\tilde{c}$ is an one-to-one embedded curve, contradicting the fact that Y is semi-complete.

Thus the holonomy is the identity.

□

Finally we are going to prove Theorem 3.

Proof of Theorem 3. Let $\mathcal{F}$ be a foliation associated o a vector field in $\mathfrak{x}$ (with an isolated singularity at the origin). For each vector field X, whose foliation coincides with $\mathcal{F}$ (in a small neighbourhood of the origin), there exist p and a holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$, verifying $f(0) \neq 0$, such that X can be written in the form fY_p.

We proved that if $\mathcal{F}$ is associated to a semi-complete vector field then $p = 1$.

Consider now the vector field of the type $Y = fY_{1,\alpha}$, with $f(0) \neq 0$, and suppose that Y is semi-complete. Proposition 17 tells us that Y has a holomorphic invariant manifold tangent to the x_1-axis. Proposition 18 says that the holonomy relative to that invariant manifold is the identity.

As Y and $Y_{1,\alpha}$ have the same foliation (in a neighbourhood of the origin), then $Y_{1,\alpha}$ has a holomorphic invariant manifold tangent to the x_1-axis and the holonomy relative to that invariant manifold is the identity.

By proposition 15, $Y_{1,\alpha}$ is analytically conjugated to its formal normal form Z_α.

So, if $Y_{1,\alpha}$ is not analytically conjugated to its formal normal form, $fY_{1,\alpha}$ is not semi-complete in any neighbourhood of the origin, for any holomorphic function f such that $f(0) \neq 0$.

On the other hand, the holonomy of the vector field Z_α

$$Z_\alpha : \begin{cases} \dot{x_1} = x_1^2 \\ \dot{x_2} = x_2(1 + \alpha_2 x_1) \\ \vdots \\ \dot{x_n} = x_n(\gamma_n + \alpha_n x_2) \end{cases}$$

is the identity iff $\alpha \in \mathbb{Z}^{n-1}$. So, even if $Y_{1,\alpha}$ is analyticaly conjugated to its formal normal form Z_α, if $\alpha \notin \mathbb{Z}^{n-1}$ Y canot be semi-complete in any neighbourhood of the origin, because the holonomy relative to the x_1-axis is not the identity.

It remains to analyse the foliations associated to vector fields X whose linear part is not diagonal, but is diagonalizable. Suppose that $\mathcal{F}$ is a foliation in that condition. We know that there exists a linear change of coordinates, H, that linearizes its linear part.

Consider the vector field $Y = (DH)^{-1}.X \circ H$. Suppose that X is semi-complete, then Y is also semi-complete and, consequently, of the form $fY_{1,\alpha}$ for some function f such that $f(0) \neq 0$. Then $Y_{1,\alpha}$ is anayticaly conjugated to Z_α with $\alpha \in \mathbb{Z}^{n-1}$, as we have just proved.

However, to prove that $\mathcal{F}$ addmits Z_α as normal form it remains to prove that $\mathcal{F}$ has a representant analytically conjugated to Z_α, or equivalently, to $Y_{1,\alpha}$ (as Z_α and $Y_{1,\alpha}$ are analytically conjugated).

Consider the vector field $\dfrac{1}{(f \circ H^{-1})}X$. Then

$$\begin{aligned} (DH)^{-1}.((\frac{1}{f \circ H^{-1}})X) \circ H &= (DH)^{-1}.\frac{1}{f}(X \circ H) \\ &= \frac{1}{f}(DH)^{-1}.X \circ H = \frac{1}{f} fY_{1,\alpha} \\ &= Y_{1,\alpha} \end{aligned}$$

and $\dfrac{1}{(f \circ H^{-1})}X$ is also a representant of $\mathcal{F}$.

Suppose now that $\mathcal{F}$ addmits Z_α, with $\alpha \in \mathbb{Z}^{n-1}$ as normal form. So there exists a vector field X, analytically conjugated to Z_α, whose foliation is given by $\mathcal{F}$. Thus X is a semi-complete vector field. So $\mathcal{F}$ is associated to a semi-complete vector field.

So, $\mathcal{F}$ is associated to a semi-complete vector field in a neighbourhood of the origin iff $\mathcal{F}$ addmits Z_α as normal form, with $\alpha \in \mathbb{Z}^{n-1}$.

□

6. Saddle-node with a non isolated singularity

In this section we are going to classify the semi-complete vector fields with diagonal linear part associated to a saddle-node foliation, but whose set of singularities coincides with the holomorphic invariant manifold transverse to the x_1-axis, whose existence is guaranteed in [1].

Proposition 19. [1] *Consider a vector field $X \in \mathfrak{X}$. The vector field X has a holomorphic invariant manifold transverse to the x_1-axis.*

By proposition 5 we can assume that the holomorphic invariant manifold transverse to the x_1-axis is the hyperplane $\{x_1 = 0\}$. In this way it is sufficient to study the vector fields of type $fx_1^{-k}Y_p$, where $f(0) \neq 0$ and $k \in \mathbb{Z} \backslash \{0\}$. The case studied before (where the origin is an isolated singularity) corresponds to $k = 0$.

We should remark that the foliation associated to Y_p coincides with the foliation associated to $fx_1^{-k}Y_p$ outside $\{x_1 = 0\}$. The foliation restricted to $\{x_1 = 0\}$ in the first case is of Poincaré type, while in the second case is a set of singular points, if $k < 0$, or does not exists, if $k > 0$. In this last case $\{x_1 = 0\}$ is a singular set in the sense that it does not belong to the domain of the vector field.

As the foliation of $fx_1^{-k}Y_p$ coincides with the foliation of fY_p, outside the invariant hypersurface that is transformed to a set of singularities, abusing notation we still call $fx_1^{-k}Y_p$ of saddle-node type, but remarking that it has no more an isolated singularity.

Proposition 20. *Let X be a vector field of type $fx_1^{-k}Y_p$, where f is a holomorphic function such that $f(0) \neq 0$, $k \in \mathbb{Z}$ and X defined in an open neighbourhood $U \subseteq \mathbb{C}^n$ of the origin. Suppose that X is semi-complete in a neighbourhood of the origin, then $k \in \{p-1, p, p+1\}$.*

Proof. Consider the vector field

$$X = fx_1^{-k}Y_p : \begin{cases} \dot{x_1} = f(x)x_1^{-k+p+1} \\ \dot{x_2} = f(x)x_1^{-k}(\lambda_2 x_2 + x_1 a_1(x)) \\ \vdots \\ \dot{x_n} = f(x)x_1^{-k}(\lambda_n x_n + x_1 b_n(x)) \end{cases}$$

We are going to prove that if $k \notin \{p-1, p, p+1\}$, then X is not semi-complete in any neighbourhood of the origin.

Suppose that $k \notin \{p-1, p, p+1\}$. If $k \leq p-2$ then $-k+p+1 \geq 3$ and if $k \geq p+2$ then $-k+p+1 \leq -1$. Those values will be important in the sequence (remember proposition 4).

The proof of this proposition is totally identical to the proof of the proposition 6 which says that if fY_p is semi-complete, with $f(0) \neq 0$, then $p = 1$.

We can easily verify that the fibres of Π_1 are transverse to the leaves of the foliation $\mathcal{F}$ associated to the vector field $fx_1^{-k}Y_p$ in a neighbourhood of the origin, except to those contained in the manifold $\{x_1 = 0\}$:

$$D\Pi_1(x).X(x) = f(x)x_1^{-k+p+1}$$

with $f(0) \neq 0$, and, consequently, non zero in a sufficiently small neighbourhood of the origin.

Fix a disc $B(0, \varepsilon) \subseteq \mathbb{C}^n$ of center at the origin of $\mathbb{C}^n$ and radius $\varepsilon > 0$ relatively to which X is semi-complete.

Suppose $f \equiv k \in \mathbb{C}$. Consider the curve $c(t) = (re^{2\pi it/(-k+p)}, 0, \dots, 0)$, $t \in [0, 1]$, which is an one-to-one embedded curve, because $|-k+p| \geq 2$.

As $\Pi_1(c(t)) \neq 0$, $\forall t \in [0, 1]$, for each $(r, x_2, \dots, x_n)$ sufficiently close to $(0, \dots, 0)$, we can lift the curve c to a curve c_L contained in $L \cap B(0, \varepsilon)$, where L is the leaf (of the foliation $\mathcal{F}$) through $(r, x_2, \dots, x_n)$. As we are assuming $f \equiv k$ we have

$$\int_{c_L} dT_L = \int_c \frac{dx_1}{kx_1^{-k+p+1}} = 0$$

As c is an one-to-one embedded curve, so is c_L. This contradicts the fact that X is semi-complete.

We are going to treat now the case $f \not\equiv k$. In this case, we obtain the differential 1-form

$$dT_L^X = \frac{dx_1}{x_1^{-k+p+1}f(x)}$$

where $x = (x_1, \dots, x_n)$ and $f(0) \neq 0$.

Consider $S \subseteq \mathbb{C}$, an angular sector with vertex at the origin and angle greater than $\frac{2\pi}{|-k+p|}$ and less than 2π. As Π_1 is transverse to the leaves, except to the the manifold $\{x_1 = 0\}$, for each leaf in $S \setminus \{0\} \times (\mathbb{C}^{n-1}, 0)$, we can write

$$\begin{cases} x_2 = x_2^L(x) \\ \vdots \\ x_n = x_n^L(x) \end{cases}$$

univocaly.

Let c_L be an one-to-one embedded curve in L. We have that

$$\int_{c_L} dT_L = \int_{c_L} \frac{dx_1}{x_1^{-k+p+1}f(x)} = \int_{\Pi_1(c_L)} \frac{dx_1}{x_1^{-k+p+1}f(x_1, x_2^L(x_1), \dots, x_n^L(x_1))}$$

By transversality, as c_L is an one-to-one embedded curve, so is $\Pi_1(c_L)$. So we reduced the study of the semi-completude of a vector field in $S \times U$, where U is a neighbourhood of the origin of $\mathbb{C}^{n-1}$, to the study of the semi-completude of a vector field in S.

Remark that if $fx_1^{-k}Y_p$ is not semi-complete in any neighbourhood of the origin of the type $(S \times U) \cap B(0, \varepsilon)$, then it canot be semi-complete in any neighbourhood of the origin $B(0, \varepsilon)$ (remark 5).

We have already proved that any unidimensional vector field of type

$$\dot{x} = x^k f(x)$$

with $f(0) \neq 0$ and $k \geq 3$ or $k \leq -1$ is not semi-complete relatively to any sector of amplitude greater than $\dfrac{2\pi}{|k-1|}$. In our case we are analysing the vector field

$$Y: \quad \dot{x} = x^{-k+p+1} f(x, x_2^L(x), \ldots, x_n^L(x))$$

with $f(0) \neq 0$ and $-k+p+1 \geq 3$ or $-k+p+1 \leq -1$. As the sector S has amplitude greater than $\dfrac{2\pi}{|-k+p|}$, Y is not semi-complete relatively to any neighbourhood of the origin of the type $S \cap B(0, \varepsilon)$. Thus X is not semi-complete relatively to any neighbourhood of the origin.

□

Immediatly, we can conclude:

Corollary 2. *There are no holomorphic semi-complete vector fields of saddle-node type with a diagonal linear part such that the invariant hypersurface transverse to the weak direction is contained in its set of singularities.*

We can also say that:

Corollary 3. *Let X be a holomorphic vector field of saddle-node type, with an isolated singularity at the origin, and M the invariant hypersurface transverse to the weak direction of X. If F is a holomorphic function such that $F(x) = 0 \Leftrightarrow x \in M$, then the holomorphic vector field FX is not semi-complete in any neighbourhood of the origin.*

Proof. It is sufficient to prove for vector fields X whose linear part is diagonalizable, but not diagonal. The diagonal case is expressed in the last corollary.

Let H be the linear transformation that linearizes the linear part of $DX(0)$. Consider the vector field:

$$Y = (DH)^{-1}(FX) \circ H = (F \circ H)(DH)^{-1} X \circ H$$

Thus Y is of type $(F \circ H) Y_p$ for some $p \geq 1$.

The hypersurface M is the invariant hypersurface transverse to the weak direction of X. So, $H^{-1}(M)$ is the hypersurface transverse to the weak direction of Y_p. M is given by $F = 0$, so $H^{-1}(M)$ is given by $F \circ H = 0$. So, the set of singularities of Y is given by $H^{-1}(M)$, i.e., is given by the hypersurface transverse to the weak direction of Y_p.

As Y_p has a diagonal linear part, Y is a holomorphic vector field of saddle-node type with a diagonal linear part whose hypersurface transverse to the weak direction is contained in its set of singularities. By Corollary 2, Y is not semi-complete.

As X is analytically conjugated to Y, X is not semi-complete.

$\square$

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