SYMMETRIES OF PROJECTED WALLPAPER PATTERNS

I. S. LABOURIAU, E. M. PINHO

Abstract. In this paper we study periodic functions of one and two variables that are invariant under a subgroup of the Euclidean group. Starting with a function defined on the plane we obtain a function of one variable by two methods: we project the values of the function on a strip into its edge, by integrating along the width; and we restrict the function to a line. If the functions had been obtained by solving a partial differential equation equivariant under the Euclidean group, how do their symmetries compare to those of solutions of equations formulated directly in one dimension?

Some of the symmetries of projected and of restricted functions can be obtained knowing the symmetries of the original functions only. There are also some extra symmetries arising for special widths of the strip and for some special positions of the line used for restriction. We obtain a general description of the two types of symmetries and discuss how they arise in the wallpaper groups (crystalographic groups of the plane). We show that the projections and restrictions of solutions of p.d.e.s in the plane may have symmetry groups larger than those of solutions of problems formulated in one dimension.

1. Introduction

The symmetry of a problem is an essential tool in mathematical modelling. In particular, symmetry restricts the space of solutions. The domains of theoretical solutions and of real solutions may not be the same. For example, three-dimensional experiments on very thin layers may be modelled by the projection into $\mathbf{R}^2$ of a function defined in $\mathbf{R}^3$. The model may be simplified by considering a function defined in $\mathbf{R}^2$. Such simplifications may modify or hide the symmetry of the problem. For instance, some patterns observed on thin layers of gel are not expected to appear in a two-dimensional domain. Gomes [7] shows that some of these patterns may be obtained as a projection of a three-dimensional pattern.

Patterns arise on a great variety of problems, either from inner characteristics or because symmetries allow an easier mathematical approach. The last situation is usual in the treatment of bifurcation on $\mathbf{E}(2)$-equivariant problems. See chapter 5 of [5] for a complete review of the subject and [2], [3] and [4] for examples.

Suppose we have a problem with solutions forming tiling patterns that are periodic along two noncolinear directions, also called wallpaper patterns. What kind of symmetry should one expect to find in the solutions of one-dimensional simplifications of this problem?

In answer to this question, we describe the symmetries of the projection of functions defined in $\mathbf{R}^2$ onto a line and compare them to those of the restriction of the functions to a line. We work with functions $f : \mathbf{R}^2 \longrightarrow \mathbf{R}$ invariant under the action of of one of the 17 wallpaper groups. In particular, they are periodic in two noncolinear directions, the generators of the subgoup of translations. Note that the invariant functions do not have the same symmetries as the lattice of all their periods.

Symmetries of functions with a Fourier expansion (for instance in the space L^2) can be rewritten as relations on the Fourier coefficients. The periodicity of projected functions is studied in section 4 and their invariance under reflections in section 6. The main result on periodicity (section 4) is the following: suppose the projection of all functions with symmetry Γ have a common period $P > 0$. If $(P, 0)$ is not already a period of the original functions, then either the projection width y_0 is a period in Γ or Γ contains a glide reflection related to the projection width y_0.

In section 5 we show that if the projected functions are invariant for the reflection in a point and if Γ does not already contain a reflection then it contains a rotation of order two related to y_0.

A similar study for the restriction of $f(x, y)$ to the line $y = c$ is carried out in section 6.

Symmetries of projections and of restrictions are compared in section 7 for each wallpaper group. In section 8 we discuss the differences between one-dimensional models that are direct solutions of an equivariant equation, and the projections or restrictions of two-dimensional solutions of equivariant problems. We show that the symmetry of the projected problem is in many ways richer than that of simple one-dimensional problems.

2. Preliminary results and definitions: patterns and symmetries

For general results and definitions on symmetries, we refer the reader to [5] and [6] whose notation we use.

2.1. **Euclidean group.** The Euclidean group, $\mathbf{E}(n)$, of all the isometries of the space $\mathbf{R}^n$, may be writen as a semi-direct sum $\mathbf{E}(n) = \mathbf{R}^n \dot{+} \mathbf{O}(n)$ where $\mathbf{R}^n$ represents the group of all the translations on the space $\mathbf{R}^n$ with the usual vector addition, and $\mathbf{O}(n)$ is the orthogonal group.

For $v \in \mathbf{R}^n$ and $\delta \in \mathbf{O}(n)$, a generic element of $\mathbf{R}^n \dot{+} \mathbf{O}(n)$ may be written as (v, δ) acting on $\mathbf{R}^n$ by $(v, \delta) \cdot x = v + \delta x$, where $x \in \mathbf{R}^n$. The element $\delta \in \mathbf{O}(n)$ is called the orthogonal component of $(v, \delta) \in \mathbf{R}^n \dot{+} \mathbf{O}(n)$. The group operation is given by $(v_1, \delta_1) \cdot (v_2, \delta_2) = (v_1 + \delta_1 v_2, \delta_1 \delta_2)$. Thus $(v, \delta)^{-1} = (-\delta^{-1} v, \delta^{-1})$.

2.2. **$\mathbf{E}(n)$ acting on functions.** The action of $\mathbf{E}(n)$ on the space $\mathbf{R}^n$ induces an action on the functions $f : \mathbf{R}^n \longrightarrow \mathbf{R}$. The *scalar action*

$$(\gamma \cdot f)(x) = f(\gamma^{-1} \cdot x) \quad \forall \gamma \in \mathbf{E}(n), \forall x \in \mathbf{R}^n,$$

is a particular case of a *physical action* as defined on [8]. Although the action of $\mathbf{E}(n)$ on $\mathbf{R}^n$ is affine, it induces a linear action on the set of functions whose domain is $\mathbf{R}^n$, *i.e.*, a representation.

2.3. **spaces of invariant functions.** Let Γ be a subgroup of $\mathbf{E}(n)$. If for a given $\gamma \in \Gamma$ the function $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ satisfies $f(\gamma \cdot x) = f(x)$ for all $x \in \mathbf{R}^n$ we say that f is *γ-invariant* and a function f is *Γ-invariant* if it is γ-invariant for all $\gamma \in \Gamma$. Let X be a space of functions $f : \mathbf{R}^n \longrightarrow \mathbf{R}$. We write X_Γ for the subspace

$$X_\Gamma = Fix(\Gamma) = \{f \in X : f \text{ is } \Gamma\text{-invariant}\}.$$

2.4. **$\mathbf{E}(n)$-equivariant systems .** Let X be a space of functions $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ and let $\mathcal{P} : X \times \mathbf{R} \longrightarrow X$ be a $\mathbf{E}(n)$-equivariant operator. Suppose $\mathcal{P}(f_0, \lambda) = 0$ and f_0 is a Γ-invariant function with $\Gamma \leq \mathbf{E}(n)$. The group Γ is called the *isotropy subgroup of* f_0 and may be written as Γ_{f_0}. It is also called the group of symmetries of the pattern f_0.

The elements in the *orbit of* f_0,

$$\mathbf{E}(n) \cdot f_0 = \{\gamma \cdot f_0 : \gamma \in \mathbf{E}(n)\},$$

are also solutions of $\mathcal{P} = 0$ and have isotropy subgroups that are conjugate to Γ_{f_0}, *i.e.*, $\gamma \cdot f_0$ is a solution of $\mathcal{P}(f_0, \lambda) = 0$ and has isotropy subgroup

$$\Gamma_{\gamma \cdot f_0} = \gamma \cdot \Gamma_{f_0} \cdot \gamma^{-1} = \{\gamma \cdot (u, \xi) \cdot \gamma^{-1} : (u, \xi) \in \Gamma_{f_0}\}.$$

For a proof, see §1 on chapter XIII of [6].

For $\gamma = (v, \delta)$, the elements on $\Gamma_{\gamma \cdot f_0}$ have the form

$$(v, \delta) \cdot (u, \xi) \cdot (-\delta^{-1} v, \delta^{-1}) = (v + \delta u - \delta \xi \delta^{-1} v, \delta \xi \delta^{-1}). \tag{1}$$

2.5. **translation subgroups.** The *translation subgroup* of a group Γ is composed of all the elements whose orthogonal component is the identity.

Proposition 2.1. *Let* Γ *be a subgroup of* $\mathbf{E}(n)$*.*

If Γ *has a translation subgroup* $\mathbf{G} \cong \{(u, I) : u \in \mathbf{G}\}$ *then the conjugate subgroup* $\gamma \cdot \Gamma \cdot \gamma^{-1}$*, with* $\gamma = (v, \delta) \in \mathbf{E}(n)$*, has the translation subgroup* $\delta \cdot \mathbf{G} \cong \{(\delta u, I) : u \in \mathbf{G}\}$*.*

Proof: By expression (1), the elements in $\gamma \cdot \Gamma \cdot \gamma^{-1}$ have the form $(v + \delta u - \delta \xi \delta^{-1} v, \delta \xi \delta^{-1})$ where $(u, \xi) \in \Gamma$. The translations have trivial orthogonal component, *i.e.*, $\delta \xi \delta^{-1} = I$, which implies $\xi = I$ and $u \in \mathbf{G}$. Therefore the translation subgroup of $\gamma \cdot \Gamma \cdot \gamma^{-1}$ is

$$\gamma \cdot \mathbf{G} \cdot \gamma^{-1} \cong \{(v + \delta u - Iv, I) = (\delta u, I) : u \in \mathbf{G}\}.$$

□

2.6. **lattices and wallpaper groups.** Let Γ be a subgroup of $\mathbf{E}(2)$. If the translation subgroup of Γ can be identified to a plane lattice, $\mathcal{L} = \{m_1 l_1 + m_2 l_2 : m_1, m_2 \in \mathbf{Z}\}$ denoted as $\{l_1, l_2\}_{\mathbf{Z}}$ with $l_1, l_2 \in \mathbf{R}^2$ noncolinear, then Γ is called a *wallpaper group*. The wallpaper groups, also called *plane crystalographic groups*, are the symmetry groups of the *tiling patterns* which are periodic in two different directions.

The lattice $\mathcal{L}$ is a normal subgroup of Γ. We will use the symbol $\mathcal{L}$ indistinctly for the lattice $\mathcal{L} \subset \mathbf{R}^2$ and for the subgroup of the translations of Γ, $(\mathcal{L}, +)$. The lattice $\mathcal{L}$ has the structure of a module over the ring $\mathbf{Z}$.

The projection $(v, \delta) \longrightarrow \delta$ of a wallpaper group into $\mathbf{O}(2)$ is a homomorphism. Its kernel is the lattice $\mathcal{L}$ and its image is a subgroup $\mathbf{J} \subset \mathbf{O}(2)$ that leaves $\mathcal{L}$ invariant. Therefore $\Gamma / \mathcal{L} \cong \mathbf{J}$. The *holohedry* of a lattice is the largest subgroup of $\mathbf{O}(2)$ that leaves the lattice invariant and $\mathbf{J}$ is a subgroup of the holohedry. Chapters 25 and 26 of [1] have a good description of lattices and wallpaper patterns and prove the results above.

There is a dual lattice $\mathcal{L}^* = \{m_1 l_1^* + m_2 l_2^* : m_1, m_2 \in \mathbf{Z}\}$ whose generators, l_1^* and l_2^*, satisfy the duality relation $< l_i, l_j^* >= \delta_{ij}$, where $< k, x >$ is the usual inner product in $\mathbf{R}^2$.

Any two generators l_1 and l_2 of a lattice form a parallelogram called *fundamental cell.* Although the shape depends on the choice of the generators, its area is an invariant of the lattice given by $\rho = l_1 \wedge l_2$. Conversely, given any $l_1 \in \mathcal{L}$, not the origin, there is some integer n such that l_1/n is the smallest element of $\mathcal{L}$ along the direction of l_1 and the set $\{\frac{l_1}{n}, l_2\}$ generates $\mathcal{L}$ for any $l_2 \in \mathcal{L}$ such that $\frac{l_1}{n} \wedge l_2 = \rho$. Furthermore lattices are invariant under a rotation of π around the origin.

Let Γ be a wallpaper group with lattice $\mathcal{L}$ as translation subgroup. A Γ-invariant function is, in particular, a $\mathcal{L}$-invariant one. This property is also known as $\mathcal{L}$-periodicity.

2.7. **periodicity and parity.** Let Σ be a subgroup of $\mathbf{E}(1) = \mathbf{R}\dot{+}\mathbf{O}(1)$, where $\mathbf{O}(1) \cong \mathbf{Z}_2 = \{1, -1\}$, and let Y_Σ be the space of Σ-invariant functions $g : \mathbf{R} \longrightarrow \mathbf{R}$. If Σ has the elements $(P, 1)$ or $(2x_0, -1)$ then the functions in Y_Σ are, respectively, periodic with period P or invariant under a reflection on the point x_0. Notice that we call *parity* the $(2x_0, -1)$-invariance, even when $x_0 \neq 0$.

3. Preliminary results and definitions: Fourier expansions

The aim of this section is the characterization of Fourier expansions of functions invariant by a wallpaper group. If Γ is a wallpaper group with translation subgroup $\mathcal{L}$, then we have already seen that a Γ-invariant function is always $\mathcal{L}$-invariant. Thus, the Γ-invariant functions form a subspace of the space $X_\mathcal{L}$ of $\mathcal{L}$-invariant functions that we describe below.

3.1. **the spaces $X_\mathcal{L}$ and $\mathcal{F}_\mathcal{L}$.** Let $X_\mathcal{L}$ be the space of the $\mathcal{L}$-periodic functions $f : \mathbf{R}^2 \longrightarrow \mathbf{R}$ having a unique formal Fourier expansion. The $\mathcal{L}$-periodicity of functions $f \in X_\mathcal{L}$ restricts its Fourier expansion to a sum over waves that are themselves $\mathcal{L}$-periodic, *i.e.*, waves of the form $\omega_k(x) = \mathrm{e}^{2\pi i <k,x>}$, where $k \in \mathcal{L}^*$. Thus, the formal Fourier expansion of $f \in X_\mathcal{L}$ is given by

$$f(x) = \sum_{k \in \mathcal{L}^*} C(k)\omega_k(x)$$

with coefficients given by a function $C : \mathcal{L}^* \longrightarrow \mathbf{C}$.

We denote by $\mathcal{F}_\mathcal{L}$ the space of all possible Fourier coefficients. They satisfy $\overline{C(k)} = C(-k)$ since the elements of $X_\mathcal{L}$ are real functions. Thus the coefficients are $\mathbf{Z}_2$-equivariant functions with $\mathbf{Z}_2$ acting on $\mathcal{L}^*$ by $-I$ and on $\mathbf{C}$ by complex conjugation.

3.2. $X_{\mathcal{L}}$ **structure.** The last statement restricts the Fourier expansion of every function $f \in X_{\mathcal{L}}$ to

$$f(x) = \frac{1}{2} \sum_{k \in \mathcal{L}^*} C(k)\omega_k(x) + \overline{C(k)}\omega_{-k}(x).$$

Thus, the space $X_{\mathcal{L}}$ has, for each $k \in \mathcal{L}^*$, the following one-dimensional $\mathcal{L}$-irreducible subspaces:

$$V_k^e = \{r(\omega_k(x) + \omega_{-k}(x)) : r \in \mathbf{R}\} \text{ and}$$
$$V_k^o = \{ri(\omega_k(x) - \omega_{-k}(x)) : r \in \mathbf{R}\},$$

where the index e or o denotes even and odd functions.

Therefore $X_{\mathcal{L}} = \sum_{k \in \mathcal{L}^*} V_k^e \oplus V_k^o$. This sum, however, takes each term twice because $V_k^e = V_{-k}^e$ and $V_k^o = V_{-k}^o$. This ambiguity can be eliminated by changing the index to, say, $(k_1, k_2) \in \mathcal{L}^*$ such that either $k_1 > 0$ or both $k_1 = 0$ and $k_2 > 0$.

3.3. Γ **action on** $X_{\mathcal{L}}$**.** Let Γ be a wallpaper group with lattice $\mathcal{L}$ and let $f \in X_{\mathcal{L}}$, then $\gamma \cdot f$ is in $X_{\mathcal{L}}$. For the induced action of Γ on $X_{\mathcal{L}}$, every element of $X_{\mathcal{L}}$ is fixed by the subgroup $\mathcal{L} \subseteq \Gamma$. Thus the effective action of Γ on $X_{\mathcal{L}}$ is that of the group $\Gamma/\mathcal{L} \cong \mathbf{J}$. The group $\mathbf{J}$ is always a finite subgroup of $\mathbf{O}(2)$. It can be either $\mathbf{Z}_n$ or $\mathbf{D}_n$ with $n = 1, 2, 3, 4$ or 6. See [1] for a proof.

Given $\delta \in \mathbf{J}$, there are several choices of v such that $(v, \delta) \in \Gamma$, since for $l \in \mathcal{L}$, $(v + l, \delta)$ also lies in Γ. There is only one of choice of v, call it v_δ, in the fundamental cell of the lattice, unless v_δ belongs to the edges of the parallelogram. In that case we choose v_δ in the edge that contains the origin and if v_δ is one of the vertices we choose the origin. Thus, we have a good way of identifying the elements that effectively act on $X_{\mathcal{L}}$.

Let $\tilde{\Gamma} = \{(v_\delta, \delta) : \delta \in \mathbf{J}\}$ with operation $(v_\delta, \delta)(v_\xi, \xi) = (v_{\delta\xi}, \delta\xi)$. The action of Γ on $X_{\mathcal{L}}$ is identical to the action of $\tilde{\Gamma}$ and $\tilde{\Gamma} \cong \mathbf{J}$ is a compact Lie group.

3.4. Γ **action on Fourier coefficients.** The action of Γ on $X_{\mathcal{L}}$ induces an action on $\mathcal{F}_{\mathcal{L}}$, in the following way: for $\gamma = (v, \delta)$ we have

$$(\gamma \cdot f)(x) = f(\gamma^{-1} \cdot x) = \sum_{k \in \mathcal{L}^*} C(k)\omega_k(\gamma^{-1} \cdot x) =$$
$$\sum_{k \in \mathcal{L}^*} C(\delta^{-1}k)\omega_k(-v)\omega_k(x).$$

Thus the affine action of Γ on $\mathbf{R}^2$ induces the linear action of Γ on $\mathcal{F}_{\mathcal{L}}$

$$((v, \delta), C(k)) \longmapsto C(\delta^{-1}k)\omega_k(-v)$$

and a function in $X_{\mathcal{L}}$ is (v, δ)-invariant if and only if its Fourier coefficients satisfy $C(k) = C(\delta^{-1}k)\omega_k(-v)$ for all $k \in \mathcal{L}^*$.

The linear map $A : X_{\mathcal{L}} \longrightarrow \mathcal{F}_{\mathcal{L}}$ that assigns to each each function $f \in X_{\mathcal{L}}$ its Fourier coefficients C is Γ-equivariant, $A(\gamma \cdot f) = \gamma \cdot A(f)$.

Furthermore, the complex conjugation on $\mathcal{F}_{\mathcal{L}}$ is represented by the action of the element $(0, -I)$, that does not necessarily belong to Γ:

$$(0, -I) \cdot C(k) = C(-k) = \overline{C(k)}.$$

Consider the subset X_Γ of real Γ-invariant functions, $X_\Gamma = \mathrm{Fix}(\Gamma) \subset X_{\mathcal{L}}$, and let $\mathcal{F}_\Gamma = A(X_\Gamma) \subset \mathcal{F}_{\mathcal{L}}$, then $\mathcal{F}_\Gamma = \mathrm{Fix}(\Gamma)$ for the action of Γ on $\mathcal{F}_{\mathcal{L}}$.

3.5. X_Γ **structure.** The action of the element $(v_\delta, \delta) \in \Gamma$ on a wave is

$$(v_\delta, \delta) \cdot \omega_k(x) = \omega_{\delta k}(-v_\delta)\omega_{\delta k}(x).$$

Let $I_k(x)$ be the sum of all elements in the orbit $\Gamma \cdot \omega_k(x)$, given by

$$I_k(x) = \sum_{\delta \in \mathbf{J}} \omega_{\delta k}(-v_\delta)\omega_{\delta k}(x)$$

then, the sum of all elements in the Γ-*orbit* of the functions $\omega_k(x) + \omega_{-k}(x)$ and $i\omega_k(x) - i\omega_{-k}(x)$ is, respectively, $I_k(x) + I_{-k}(x)$ and $iI_k(x) - iI_{-k}(x)$. Thus, the space X_Γ can be written as the direct sum of the irreducible one-dimensional subspaces generated by $I_k(x) + I_{-k}(x)$ and by $iI_k(x) - iI_{-k}(x)$, for $k \in \mathcal{L}^*$.

4. Projection of invariant functions and symmetry - Periodicity on $\Pi_{y_0}(X_\Gamma)$

In this section we relate the periodicity of projected functions to their symmetry group before the projection. The main tools are the Fourier expansions of the functions, which relate periodicity to the dual lattice $\mathcal{L}^*$.

4.1. **projection.** From now on, $\mathcal{L} \subset \mathbf{R}^2$ is a lattice with two generators containing the origin and Γ is a wallpaper group on $\mathcal{L}$.

Given $y_0 > 0$, let Π_{y_0} be the projection operator $\Pi_{y_0}(f)(x) = \int_0^{y_0} f(x, y)dy$ and let $\Pi_{y_0}(X_\Gamma)$ be the set of possible results when the operator Π_{y_0} acts on all Γ-invariant functions. Some aspects of the Γ-invariance are preserved after projection. The main purpose of this section and the next one is to clarify this idea.

We denote the one-dimensional waves by $\omega_{k_j}(x) = \mathrm{e}^{2\pi i k_j x}$, with $k_j, x \in \mathbf{R}$. When the coordinates of the elements of $\mathcal{L}^*$ are $k = (k_1, k_2)$ we have $\omega_k(x, y) = \omega_{k_1}(x)\omega_{k_2}(y) = \mathrm{e}^{2\pi i <k,(x,y)>}$ for $(x, y) \in \mathbf{R}^2$.

If $f \in X_\Gamma$ then, formally,

$$\Pi_{y_0}(f)(x) = \int_0^{y_0} \sum_{k \in \mathcal{L}^*} C(k)\omega_k(x, y)dy =$$

$$= \sum_{k \in \mathcal{L}^*} C(k)\omega_{k_1}(x) \int_0^{y_0} \omega_{k_2}(y)dy =$$

$$= \sum_{k_1 \in \mathcal{L}_1^*} \omega_{k_1}(x) \sum_{k_2 : (k_1, k_2) \in \mathcal{L}^*} C(k_1, k_2) \int_0^{y_0} \omega_{k_2}(y)dy,$$

where $\mathcal{L}_1^* = \{k_1 : (k_1, k_2) \in \mathcal{L}^*\}$.

4.2. **periodicity of $\Pi_{y_0}(X_\Gamma)$ related to $\mathcal{L}^*$.** We will use the symbol σ for the reflection $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ until the end of the article.

Proposition 4.1. *All functions in $\Pi_{y_0}(X_\Gamma)$ have a common period $P \in \mathbf{R} - \{0\}$ if and only if, for each $k = (k_1, k_2) \in \mathcal{L}^*$, one of the following conditions holds:*

a) $k_1 P \in \mathbf{Z}$,
b) $k_2 y_0 \in \mathbf{Z} - \{0\}$,
c) $\sigma \in \mathbf{J}$ and $< k, \sigma v_\sigma + (0, y_0) > +\frac{1}{2} \in \mathbf{Z}$.

Conditions a), b) and c) define subsets of dual lattice $\mathcal{L}^*$, respectively,

$$\mathcal{M}_P^* = \{k \in \mathcal{L}^* : < k, (P, 0) > \in \mathbf{Z}\},$$
$$\mathcal{M}_{y_0}^* = \{k \in \mathcal{L}^* : < k, (0, y_0) > \in \mathbf{Z} - \{0\}\} \quad \text{and}$$
$$\mathcal{N}^* = \left\{k \in \mathcal{L}^* : < k, \sigma v_\sigma + (0, y_0) > + \frac{1}{2} \in \mathbf{Z}\right\} \quad \text{if } \sigma \in \mathbf{J},$$
$$\mathcal{N}^* = \emptyset \quad \text{if } \sigma \notin \mathbf{J}.$$

We will refer to condition a) either as $k_1 P \in \mathbf{Z}$ or as $k \in \mathcal{M}_P^*$, according to the tools we intend to use. The same will happen with conditions b) and c).

The sets $\mathcal{M}_P^*$, $\mathcal{M}_{y_0}^*$ and $\mathcal{N}^*$ play a central role on the proof of Proposition 4.1. Note that $\mathcal{M}_P^*$ is a module. Let $\mathcal{M}_0^*$ be the module

$$\mathcal{M}_0^* = \{k \in \mathcal{L}^* : < k, (0, y_0) > = 0\}.$$

Therefore, $\mathcal{M}_0^* \cap \mathcal{M}_{y_0}^* = \emptyset$ and $\mathcal{M}_0^* \cup \mathcal{M}_{y_0}^*$, represented by $\overline{\mathcal{M}_{y_0}^*}$, is the smallest module containing $\mathcal{M}_{y_0}^*$. Similarly, if $\mathcal{N}_0^*$ is the module

$$\mathcal{N}_0^* = \{k \in \mathcal{L}^* : < k, \sigma v_\sigma + (0, y_0) > \in \mathbf{Z}\},$$

the disjoint union $\mathcal{N}_0^* \cup \mathcal{N}^* = \overline{\mathcal{N}^*}$ is the smallest module containing $\mathcal{N}^*$. Moreover $\mathcal{N}^*$ has the property:

$$(2) \qquad \begin{array}{c} v_1, v_2 \in \mathcal{N}^* \\ n_1, n_2 \in \mathbf{Z} \end{array} \Rightarrow n_1 v_1 + n_2 v_2 \in \left\{ \begin{array}{ll} \mathcal{N}_0^* & \text{if } n_1 + n_2 \text{ even} \\ \mathcal{N}^* & \text{if } n_1 + n_2 \text{ odd} \end{array} \right. .$$

With this notation we can restate Proposition 4.1 as:

Proposition. 4.1 *All functions in $\Pi_{y_0}(X_\Gamma)$ have a common period $P \in \mathbf{R} - \{0\}$ if and only if $\mathcal{L}^* = \mathcal{M}_P^* \cup \mathcal{M}_{y_0}^* \cup \mathcal{N}^*$.*

Proof of Proposition 4.1: A function with Fourier expansion $\sum_{k_1 \in \mathcal{L}_1^*} D(k_1) \omega_{k_1}(x)$ has period P if and only if, for each $k_1 \in \mathcal{L}_1^*$, either

a') ω_{k_1} has period P or b') $D(k_1) = 0$.

Given $k_1 \in \mathcal{L}_1^*$, conditions a) and a') are equivalent.

The Fourier coefficients for the projection of $f \in X_\Gamma$ are given by

$$D(k_1) = \sum_{k_2:(k_1,k_2)\in\mathcal{L}^*} C(k_1,k_2)\int_0^{y_0} \omega_{k_2}(y)dy. \tag{3}$$

Condition b) is equivalent to $\int_0^{y_0}\omega_{k_2}(y)dy = 0$.

Suppose either a), b) or c) holds for each $k \in \mathcal{L}^*$. If $k_1 P \in \mathbf{Z}$ then ω_{k_1} has period P. Suppose $k_1 P \notin \mathbf{Z}$ for some $k \in \mathcal{L}^*$, and that $k_2 y_0 \in \mathbf{Z} - \{0\}$. Then the summation in $D(k_1)$ is taken over indexes k_2 such that $k_2 y_0 \in \mathbf{Z} - \{0\}$ and therefore $\int_0^{y_0}\omega_{k_2}(y)dy = 0$ (condition b').

In order to show that the conditions are sufficient it remains to study the case where $k_1 P \notin \mathbf{Z}$ and $k_2 y_0 \notin \mathbf{Z} - \{0\}$, when c) holds and therefore:

$$\omega_{\sigma k}(-v) = \mathrm{e}^{2\pi i<\sigma k,-v>} = \mathrm{e}^{2\pi i\left(<k,(0,y_0)>+\frac{1}{2}\right)} = \mathrm{e}^{2\pi i<k,(0,y_0)>}\mathrm{e}^{\pi i} = -\omega_{k_2}(y_0).$$

From the invariance of Fourier coefficients it follows that $C(\sigma k') = C(k')\omega_{\sigma k'}(-v)$ for all $k' \in \mathcal{L}^*$. Therefore

(4)

$$D(k_1) = \frac{1}{2}\sum_{k_2:(k_1,k_2)\in\mathcal{L}^*}\left(C(k_1,k_2)\int_0^{y_0}\omega_{k_2}(y)dy + C(k_1,-k_2)\int_0^{y_0}\omega_{-k_2}(y)dy\right)$$

because $(k_1,k_2) \in \mathcal{L}^*$ implies $\sigma(k_1,k_2) = (k_1,-k_2) \in \mathcal{L}^*$. Thus

$$D(k_1) = \frac{1}{2}\sum_{k_2:(k_1,k_2)\in\mathcal{L}^*} C(k_1,k_2)\left(\int_0^{y_0}\omega_{k_2}(y)dy - \omega_{k_2}(y_0)\int_0^{y_0}\omega_{-k_2}(y)dy\right)$$

and the last expression is always zero because

$$\int_0^{y_0}\omega_{k_2}(y)dy - \omega_{k_2}(y_0)\int_0^{y_0}\omega_{-k_2}(y)dy = 0. \tag{5}$$

Therefore conditions a), b) and c) are sufficient.

Now we prove that the conditions are necessary. Suppose $\Pi_{y_0}(f)$ has period P for all functions $f \in X_\Gamma$. It follows that, for each $k_1 \in \mathcal{L}_1^*$, either ω_{k_1} has period P, and therefore condition a) holds, or $D(k_1) = 0$. We will show that in the second case either b) or c) holds.

The hypothesis holds, in particular, for the simplest Γ-invariant functions, the real and imaginary parts of $I_k(x,y) = \sum_{\delta\in\mathbf{J}}\omega_{\delta k}(-v_\delta)\omega_{\delta k}(x,y)$, $k \in \mathcal{L}^*$. Denoting the j$^{\underline{\text{th}}}$ coordinate of δk by $\delta k|_j$, the projection of I_k is:

$$\Pi_{y_0}(I_k)(x) = \sum_{\delta\in\mathbf{J}}\omega_{\delta k|_1}(x)D'(\delta,k)$$

where

$$D'(\delta,k) = \omega_{\delta k}(-v_\delta)\int_0^{y_0}\omega_{\delta k|_2}(y)dy.$$

For any $k=(k_1,k_2)\in\mathcal{L}^*$ the coefficient $D(k_1)$ in $\Pi_{y_0}(I_k)$ is

$$S(k)=\sum_{\delta\in\mathbf{J}^+(k)} D'(\delta,k) \qquad \text{where} \qquad \mathbf{J}^+(k)=\{\delta\in\mathbf{J}\ :\ \delta k|_1=k_1\}\,.$$

The remainder of the proof is divided in three lemmas. In order to treat the condition $S(k)=0$ we start by describing the possibilities for $\mathbf{J}^+(k)$ in Lemma 4.1. In Lemma 4.2 we show that $S(k)=0$ implies $k\in\mathcal{M}^*_{y_0}\cup\mathcal{N}^*\cup\mathcal{O}^*$, for a certain subset $\mathcal{O}^*$ of $\mathcal{L}^*$, therefore $\mathcal{L}^*=\mathcal{M}^*_P\cup\mathcal{M}^*_{y_0}\cup\mathcal{N}^*\cup\mathcal{O}^*$. In Lemma 4.3 we show that this implies $\mathcal{L}^*=\mathcal{M}^*_P\cup\mathcal{M}^*_{y_0}\cup\mathcal{N}^*$, and thus the result follows. □

Lemma 4.1. *Let $\mathbf{J}^+=\mathbf{J}\cap\{I,\sigma\}$. The set $\mathbf{J}^+(k)$ satisfies:*

1. $\mathbf{J}^+(k)=\{\delta\in\mathbf{J}\ :\ \delta k=k\vee\delta k=\sigma k\}$.
2. *For any $k\in\mathcal{L}^*$, $\mathbf{J}^+\subset\mathbf{J}^+(k)$ and $\mathbf{J}^+(0,0)=\mathbf{J}$.*
3. *For any $k\in\mathcal{L}^*$, $k=(k_1,k_2)\neq(0,0)$, if $\delta\in\mathbf{J}^+(k)-\mathbf{J}^+$ then $\delta k=(k_1,-|\delta|k_2)$ where $|.|$ is the determinant.*
4. *If $\mathbf{J}^+=\{I\}$ and $k\neq(0,0)$ then, for any $k\in\mathcal{L}^*$, $\mathbf{J}^+(k)$ contains at most one element $\delta\neq I$.*
5. *If $\mathbf{J}^+=\{I,\sigma\}$ then for any $k\in\mathcal{L}^*$, $k\neq(0,0)$, either $\mathbf{J}^+(k)=\mathbf{J}^+$ or $\mathbf{J}^+(k)-\mathbf{J}^+$ contains exactly two elements, δ and $\sigma\delta$.*

Proof: Assertion *1.* follows by orthogonality of $\mathbf{J}$, since any element of the orbit $\mathbf{J}(k_1,k_2)$ with first component k_1 is of the form $(k_1,\pm k_2)$ and *2.* follows directly from *1.* and the definition of $\mathbf{J}^+(k)$. To prove *3.* let $\delta\in\mathbf{J}^+(k)-\mathbf{J}^+$ with $k\neq(0,0)$. If $\delta k=k$ then $|\delta|=-1$, since the only element of $\mathbf{O}(2)$, with determinant 1, that fixes points beyond the origin is the identity. Similarly if $\delta k=\sigma k$ then $|\sigma\delta|=-1$ and $|\delta|=1$.

Suppose now there are two elements, δ and $\xi\neq\delta$, in $\mathbf{J}^+(k)-\mathbf{J}^+$. Then either $\xi k=\delta k$ or $\xi k=\sigma\delta k$. The first case happens either when both $\xi k=k$ and $\delta k=k$ or when $\xi k=\sigma k$ and $\delta k=\sigma k$. Therefore, by *3.*, $|\xi|=|\delta|$. Since $k=\xi^{-1}\delta k$ then by the proof of *3.* it follows that $\xi^{-1}\delta=I$, contradicting our hypothesis.

Thus $\xi k=\sigma\delta k$ and, by *3.*, $|\xi|=-|\delta|$. This implies $\xi^{-1}\sigma\delta=I$ and both $\xi=\sigma\delta$ and σ are in $\mathbf{J}$. □

Let $\mathcal{M}^*_+=\{k:\mathbf{J}^+(k)=\mathbf{J}^+\}$ and, if $\delta\in\mathbf{J}-\mathbf{J}^+\neq\emptyset$, for $\delta\in\mathbf{J}-\mathbf{J}^+$, let

$$\mathcal{M}^*_\delta=\{k\in\mathcal{L}^*\ :\ \delta k=(k_1,-|\delta|k_2)\}\,.$$

Note that δ in cases *4.* and *5.* may be either a rotation such that $\delta k=(k_1,-k_2)$, or a reflection that keeps k fixed. Therefore $\mathcal{M}^*_\delta$ is the intersection of $\mathcal{L}^*$ with the line fixed either by $\sigma\delta$ or by δ and may be only the origin. If $\sigma\in\mathbf{J}$ and if $\delta\in\mathbf{J}-\mathbf{J}^+$, then $\sigma\delta\in\mathbf{J}-\mathbf{J}^+$ and

$\mathcal{M}^*_\delta = \mathcal{M}^*_{\sigma\delta}$. Then we can write

$$\mathcal{L}^* = \mathcal{M}^*_+ \cup \bigcup_{\delta \in \mathbf{J}-\mathbf{J}^+} \mathcal{M}^*_\delta$$

where any two sets in the right hand side either are disjoint or coincide, with the convention $\bigcup_{\delta \in \mathbf{J}-\mathbf{J}^+} \mathcal{M}^*_\delta = \emptyset$ if $\delta \in \mathbf{J} - \mathbf{J}^+ = \emptyset$.

Lemma 4.2. *Suppose $S(k) = 0$ for some $k \in \mathcal{L}^*$. Then*

$$k \in \mathcal{M}^*_{y_0} \cup \mathcal{N}^* \cup \mathcal{O}^*$$

where

$$\mathcal{O}^* = \bigcup_{\delta \in \mathbf{J}-\mathbf{J}^+} \mathcal{N}^*_\delta \qquad \textit{for} \qquad \mathcal{N}^*_\delta = \{k \in \mathcal{M}^*_\delta \ : \ S(k) = 0\}.$$

Proof: The condition $S(k) = \sum_{\delta \in \mathbf{J}^+(k)} D'(\delta, k) = 0$, with $k \in \mathcal{L}^*$, depends on the set $\mathbf{J}^+(k)$ and will have different expressions for each of the subsets of $\mathcal{L}^*$ defined above, after the proof of Lemma 4.1.

Suppose $k \in \mathcal{M}^*_+$. If $\mathbf{J}^+(k) = \mathbf{J}^+ = \{I\}$ then

$$S(k) = D'(I, k) = \int_0^{y_0} \omega_{k_2}(y)dy = 0$$

and therefore $k_2 y_0 \in \mathbf{Z} - \{0\}$ or, which is equivalent, $k \in \mathcal{M}^*_{y_0}$.

If $\mathbf{J}^+(k) = \mathbf{J}^+ = \{I, \sigma\}$ then

$$S(k) = D'(I,k) + D'(\sigma,k) = \int_0^{y_0} \omega_{k_2}(y)dy + \omega_{(k_1,-k_2)}(-v_1,-v_2)\int_0^{y_0} \omega_{-k_2}(y)dy = 0.$$

If

$$\int_0^{y_0} \omega_{k_2}(y)dy = \int_0^{y_0} \omega_{-k_2}(y)dy = 0$$

then $k_2 y_0 \in \mathbf{Z} - \{0\}$. Otherwise, using (5) we get $\omega_{(k_1,-k_2)}(-v_1,-v_2) = -\omega_{k_2}(y_0)$, which is equivalent to $\langle (k_1,k_2),(v_1,-v_2)+(0,y_0)\rangle + \frac{1}{2} \in \mathbf{Z}$, *i.e.*, $k \in \mathcal{N}^*$. □

Lemma 4.3. *If $\mathcal{L}^* = \mathcal{M}^*_P \cup \mathcal{M}^*_{y_0} \cup \mathcal{N}^* \cup \mathcal{O}^*$ then $\mathcal{L}^* = \mathcal{M}^*_P \cup \mathcal{M}^*_{y_0} \cup \mathcal{N}^*$.*

Proof: At first, we prove that $(\mathcal{M}^*_0 \cap \mathcal{O}^*) - \{(0,0)\} = \emptyset$ and so $\mathcal{M}^*_0 \subset (\mathcal{M}^*_P \cup \mathcal{N}^*)$. If $k \in \mathcal{M}^*_0$ then $k = (k_1, 0)$ for some $k_1 \in \mathbf{R}$. If, moreover, $k \in \mathcal{O}^*$ then either $\delta k = k$ or $\delta k = \sigma k$ for some $\delta \in \mathbf{J} - \mathbf{J}^+$. By orthogonality, for $k_1 \neq 0$, this implies either $\delta = I$ or $\delta = \sigma$, *i.e.*, $\delta \in \mathbf{J}^+$.

Let k be any element of $\mathcal{L}^*$. If $k = (0,0)$ then $k \in \mathcal{M}^*_P$. For $k \neq (0,0)$, let g be the smallest element of $\mathcal{L}^*$ in the direction of k and $h \in \mathcal{L}^*$ such that $\mathcal{L}^* = \{g, h\}_{\mathbf{Z}}$. Now we prove that the affine submodule $A_k = \{k + nh : n \in \mathbf{Z}\}$ is a subset of $\mathcal{M}^*_P \cup \mathcal{M}^*_{y_0} \cup \mathcal{N}^*$ and so, in particular, that k belongs to that set.

The intersection $A_k \cap \mathcal{M}_P^*$ is either the empty set or a set with only a point or an infinite set of equally spaced points. This happens because $\mathcal{M}_P^*$ is a module and the existence of any two distinct elements of $A_k \cap \mathcal{M}_P^*$, $k + n_1 h$ and $k + n_2 h$, implies $(n_2 - n_1)h \in \mathcal{M}_P^*$ and $\{k + n_1 h + n(n_2 - n_1)h : n \in \mathbf{Z}\} \subset \mathcal{M}_P^*$. A characteristic period, τ_1, is given by the smallest difference between two elements of $A_k \cap \mathcal{M}_P^*$.

Analogously, the intersection $A_k \cap \mathcal{M}_{y_0}^*$ is either the empty set or a set with only a point or an infinite set of points. In the last case we can define a period, τ_2, given by the smallest difference between two of those points. This period belongs to the module $\overline{\mathcal{M}_{y_0}^*}$ and the set $(A_k \cap \overline{\mathcal{M}_{y_0}^*}) \subset (\mathcal{M}_P^* \cup \mathcal{M}_{y_0}^* \cup \mathcal{N}^*)$ has infinite equally spaced points.

For the set $A_k \cap \mathcal{N}^*$ there are also the three possible results. Although $\mathcal{N}^*$ is not a module, the smallest difference between two elements of $A_k \cap \mathcal{N}^*$ defines a period $\tau_3 \in \mathcal{N}_0^*$, by 2. Thus, whenever $A_k \cap \mathcal{N}^*$ has more than one element, if $k + n_1 h \in \mathcal{N}^*$ then $\{k + n_1 h + n\tau_3 : n \in \mathbf{Z}\} = A_k \cap \mathcal{N}^*$.

The set $A_k - \mathcal{O}^*$ is infinite and so, at least, one of the periods τ_1, τ_2 or τ_3 must exist. The least common multiple of the existing periods is a period, τ, of $A_k \cap (\mathcal{M}_P^* \cup \mathcal{M}_{y_0}^* \cup \mathcal{N}^*)$. Therefore $A_k - (\mathcal{M}_P^* \cup \mathcal{M}_{y_0}^* \cup \mathcal{N}^*)$ is either the empty set or an infinite set with period τ. However that is a subset of the finite set $A_k \cap \mathcal{O}^*$ and so $A_k - (\mathcal{M}_P^* \cup \mathcal{M}_{y_0}^* \cup \mathcal{N}^*) = \emptyset$, implying $A_k \subset (\mathcal{M}_P^* \cup \mathcal{M}_{y_0}^* \cup \mathcal{N}^*)$. $\square$

4.3. **periodicity of $\Pi_{y_0}(X_\Gamma)$ related to $\mathcal{L}$.** The set $\Pi_{y_0}(X_\Gamma)$ inherits periodicity, through the invariant functions $I_k(x, y)$, from the lattice structure of $\mathcal{L}$. The duality relation between $\mathcal{L}$ and $\mathcal{L}^*$ allows us to establish the conditions over $\mathcal{L}$ for periodicity of the projected functions. The main result of this section is the next theorem. Recall that $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Theorem 4.1. *All functions in $\Pi_{y_0}(X_\Gamma)$ have a common period $P \neq 0$ if and only if one of the following conditions holds:*

a) $(P, 0) \in \mathcal{L}$,
b) $(0, y_0)$ *and* $(P, y_1) \in \mathcal{L}$ *for some* $y_1 \in \mathbf{R}$,
c) $((P, y_0), \sigma) \in \Gamma$,
d) $(0, y_0) \in \mathcal{L}$ *and* $((P, y_1), \sigma) \in \Gamma$ *for some* $y_1 \in \mathbf{R}$.

Proof: A direct calculation of the integrals in Lemma 4.4 shows that the conditions are sufficient.

For the necessity, note that if all functions in $\Pi_{y_0}(X_\Gamma)$ have some common period $P \neq 0$ then, by Proposition 4.1, $\mathcal{L}^* = \mathcal{M}_P^* \cup \mathcal{M}_{y_0}^* \cup \mathcal{N}^*$.

The case $\sigma \notin \mathbf{J}$, where $\mathcal{N}^* = \emptyset$ is treated in Lemma 4.5 below, where we show that either a) or b) holds.

Suppose now that $\sigma \in \mathbf{J}$. This imposes restrictions on the position and the generators of $\mathcal{L}$, that we describe in Lemma 4.6 below. In particular, we obtain the classical result that the oblique lattice cannot have a reflection in its holohedry. There are two possibilities for $\mathcal{L}$. The first is that $\mathcal{L}$ is either the square or the rectangular lattice in standard position. The second possibility is that $\mathcal{L}$ is either the square or the centred rectangular or the hexagonal lattice, in what we call *diamond position*.

In Lemma 4.7 below we relate the coordinates of the translation component of (v_σ, σ) to the generators of $\mathcal{L}$. For the square or rectangular lattices in standard position, (v_σ, σ) is either a reflection or a glide reflection. For the lattices in diamond position, (v_σ, σ) must be a reflection. Then in Lemma 4.8 we use the information that $\mathcal{L}^* = \mathcal{M}_P^* \cup \mathcal{M}_{y_0}^* \cup \mathcal{N}^*$ from Proposition 4.1 to relate the period P to the coordinates of one of the generators of $\mathcal{L}$.

If $\sigma \in \mathbf{J}$ then it will follow from Lemma 4.6 that $\mathcal{M}_0^* \neq \{(0,0)\}$ and so there exists some generator l_1^* of $\mathcal{L}^*$ such that $l_1^* \in \mathcal{M}_0^*$. In Lemma 4.9 we show that this, together with $\mathcal{L}^* = \mathcal{M}_P^* \cup \mathcal{M}_{y_0}^* \cup \mathcal{N}^*$, imposes restrictions on the other generator l_2^* of $\mathcal{L}^*$. This is then used to show that conditions a), b), c), d) hold, for the square and rectangular lattices in standard position in Lemma 4.10 and for the lattices in diamond position in Lemma 4.11. $\square$

Lemma 4.4. *If one of the conditions a), b), c), or d) of Theorem 4.1 holds then $\Pi_{y_0}(X_\Gamma)$ is a set of periodic functions with period $P \neq 0$.*

Proof: If a) holds then for all $f \in X_\Gamma$, $f(x,y) = f(x+P,y)$ and $\Pi_{y_0}(f)(x) = \int_0^{y_0} f(x,y)dy = \int_0^{y_0} f(x+P,y)dy = \Pi_{y_0}(f)(x+P)$.

If condition b) holds then for all $f \in X_\Gamma$, $\Pi_{y_0}(f)(x) = \int_0^{y_0} f(x,y)dy = \int_0^{y_0} f(x+P, y_0-y)dy$ which, for $z = y_0 - y$, equals $\int_0^{y_0} f(x+P,z)dz = \Pi_{y_0}(f)(x+P)$.

If c) holds then $\Pi_{y_0}(f)(x) = \int_0^{y_0} f(x+P, y_1 \pm y)dy$ which either equals $\int_{y_1}^{y_1+y_0} f(x+P,z)dz$ or $\int_{y_1-y_0}^{y_1} f(x+P,z)dz$. Every function $f \in X_\Gamma$ has period $(0,y_0)$ and so both integrals equal $\int_0^{y_0} f(x+P,z)dz = \Pi_{y_0}(f)(x+P)$. $\square$

Lemma 4.5. *If $\mathcal{L}^* = \mathcal{M}_P^* \cup \mathcal{M}_{y_0}^*$, then one of the following conditions holds:*

a) $(P,0) \in \mathcal{L}$,

b) $(0,y_0) \in \mathcal{L}$ *and* $(P,y_1) \in \mathcal{L}$ *for some* $y_1 \in \mathbf{R}$.

Proof:

Since $\mathcal{L}^*$ is the union of the two modules $\mathcal{L}^* = \mathcal{M}_P^* \cup \overline{\mathcal{M}_{y_0}^*}$, then either $\mathcal{L}^* = \mathcal{M}_P^*$, implying condition a), or $\mathcal{L}^* = \overline{\mathcal{M}_{y_0}^*}$.

In the second case, we have $(0, y_0) \in \mathcal{L}$ and $\mathcal{M}_0^* \subset \mathcal{M}_P^*$. Let n be the largest integer such that $l_1 = \left(0, \frac{y_0}{n}\right) \in \mathcal{L}$, then $\mathcal{L}$ is generated by l_1 and $l_2 = \left(\frac{\rho n}{y_0}, y_2\right)$ for some $y_2 \in \mathbf{R}$. Since

$$l_2^* = \left(\frac{y_0}{\rho n}, 0\right) \in \mathcal{M}_0^* \subset \mathcal{M}_P^*$$

it follows that $\frac{\rho n}{y_0} = \frac{P}{m}$ for some $m \in \mathbf{Z}$. Therefore $l_2 = \left(\frac{P}{m}, y_2\right) \in \mathcal{L}$, implying condition b). □

Lemma 4.6. *If $\sigma \in \mathbf{J}$ then there are generators for $\mathcal{L}$ of one of the forms below, with $\alpha \neq 0 \neq \beta$,*

$$\mathcal{L} = \{(\alpha, 0), (0, \beta)\}_{\mathbf{Z}} \quad or \quad \mathcal{L} = \left\{\left(\frac{\alpha}{2}, \frac{\beta}{2}\right), (0, \beta)\right\}_{\mathbf{Z}}.$$

Proof: Let (a, b) be any element of $\mathcal{L}$. As $\sigma \in \mathbf{J}$ then $\sigma(\mathcal{L}) = \mathcal{L}$ and so $(a, -b) \in \mathcal{L}$. Thus, $(2a, 0)$ and $(0, 2b)$ belong to $\mathcal{L}$.

Let $(\alpha, 0)$ and $(0, \beta)$ be the smallest elements of $\mathcal{L}$ in the coordinate axes and let $\mathcal{M} \subset \mathcal{L}$ be the submodule generated by them. If $\mathcal{L} \neq \mathcal{M}$, then it follows that $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$ lies in $\mathcal{L}$, since $(a, b) \in \mathcal{L} - \mathcal{M}$ implies $(2a, 0)$ and $(0, 2b)$ lie in $\mathcal{L}$. □

From now on, we refer to $\mathcal{L} = \{(\alpha, 0), (0, \beta)\}_{\mathbf{Z}}$ as the lattice in *standard position* and to $\mathcal{L} = \left\{\left(\frac{\alpha}{2}, \frac{\beta}{2}\right), (0, \beta)\right\}_{\mathbf{Z}}$ as the lattice in *diamond position*. In section 7 below we show that in both cases, the presence of a reflection $\sigma \in \mathbf{J}$ on a horizontal line implies that Γ contains glide reflections along the same direction.

Lemma 4.7. *If $\sigma \in \mathbf{J}$ then either $v_\sigma = (0, v_2)$ or $v_\sigma = \left(\frac{\alpha}{2}, v_2\right)$ for some v_2, $0 < v_2 < \beta$ for the α and β in Lemma 4.6. For the lattice in diamond position, only the first possibility occurs.*

Proof: As defined in Section 3, either $v_\sigma = (v_1, v_2)$ belongs to the interior of the fundamental cell of $\mathcal{L} = \{l_1, l_2\}_{\mathbf{Z}}$ or v_σ lies in one of the segments tl_i, with $t \in [0, 1)$ and $i = 1$ or 2. For the generators in Lemma 4.6, this implies $v_1 \in [0, \alpha)$, $v_2 \in [0, \beta)$. Moreover, $(v_\sigma, \sigma)^2 = (v_\sigma + \sigma v_\sigma, I) = ((2v_1, 0), I) \in \Gamma$, and thus $(2v_1, 0) \in \mathcal{L}$ or, equivalently, $2v_1 = n\alpha$ for some $n \in \mathbf{Z}$. For the lattice in diamond position, this implies $v_1 = 0$. For the lattice in standard position there is also the possibility $v_1 = \frac{\alpha}{2}$. □

The possible structures of $\mathcal{L}$ due to the presence of (v_σ, σ) in Γ are already defined. Lemma 4.8, below, scales those lattices according to the relation $\mathcal{L}^* = \mathcal{M}_P^* \cup \mathcal{M}_{y_0}^* \cup \mathcal{N}^*$.

Lemma 4.8. *If $\mathcal{L}^* = \mathcal{M}_P^* \cup \mathcal{M}_{y_0}^* \cup \mathcal{N}^*$ and if $\sigma \in \mathbf{J}$ then α, defined in Lemma 4.6, satisfies $\alpha = \frac{2P}{n}$ for some integer n.*

Proof: From Lemma 4.6 it follows that either $\mathcal{L}^* = \left\{\left(\frac{1}{\alpha}, 0\right), \left(0, \frac{1}{\beta}\right)\right\}_{\mathbf{Z}}$ or $\mathcal{L}^* = \left\{\left(\frac{2}{\alpha}, 0\right), \left(-\frac{1}{\alpha}, \frac{1}{\beta}\right)\right\}_{\mathbf{Z}}$. In both cases, $k = \left(\frac{2}{\alpha}, 0\right) \in \mathcal{L}^*$. Since $k \in \mathcal{M}_0^*$, then either $k \in \mathcal{M}_P^*$ or $k \in \mathcal{N}^*$. By Lemma 4.7, we have $< k, \sigma v_\sigma + (0, y_0) >$ is either 0 or 1, and therefore $k \notin \mathcal{N}^*$ and the result follows. $\square$

In the next Lemma we obtain more information about the generators of $\mathcal{L}^*$ in a form that will also be suitable for use in section 5.

Lemma 4.9. *Let $\mathcal{L}^* = \mathcal{M}^* \cup \mathcal{M}_{y_0}^* \cup \mathcal{N}^*$, where $\mathcal{M}^*$ is a module, and suppose $\mathcal{L}^* =< l_1^*, l_2^* >_{\mathbf{Z}}$ and $l_1^* \in \mathcal{M}_0^*$. Then either $l_1^* \in \mathcal{M}^*$ or $l_1^* \in \mathcal{N}^*$.*
Moreover, the two cases above can be subdivided in the following 6 cases:

1. $l_1^* \in \mathcal{M}^*$ *and either*
   i) $l_2^* \in \mathcal{M}^*$, *or*
   ii) $l_2^*, l_1^* + l_2^* \in \mathcal{N}^*$, *or*
   iii) $(0, y_0) \in \mathcal{L}$
2. $l_1^* \in \mathcal{N}^*$ *and either*
   (i) $l_2^* \in \mathcal{N}^*$ *and* $l_1^* + l_2^* \in \mathcal{M}^*$, *or*
   (ii) $l_2^* \in \mathcal{M}^*$ *and* $l_1^* + l_2^* \in \mathcal{N}^*$, *or*
   (iii) $(0, y_0) \in \mathcal{L}$

Moreover, $\mathcal{L}^* = \mathcal{M}^*$ in case 1i), $\mathcal{L}^* = \overline{\mathcal{M}_{y_0}^*}$ in cases 1iii) and 2iii) and $\mathcal{L}^* = \overline{\mathcal{N}^*}$ in the remaining cases.

Proof: Any two of the three elements l_1^*, l_2^* and $l_1^* + l_2^*$ generate $\mathcal{L}^*$ and, by hypothesis, $l_1^* \in \mathcal{M}^* \cup \mathcal{N}^*$.

If either l_2^* or $l_1^* + l_2^*$ belongs to $\mathcal{M}_{y_0}^*$ then $\mathcal{L}^* = \overline{\mathcal{M}_{y_0}^*}$ and $(0, y_0) \in \mathcal{L}$ and we have cases 1iii) and 2iii).

Now suppose neither l_2^* nor $l_1^* + l_2^*$ belong to $\mathcal{M}_{y_0}^*$. Then $l_1^*, l_2^*, l_1^* + l_2^* \in (\mathcal{M}^* \cup \mathcal{N}^*)$ and so at least two of them belong to the same set, either $\mathcal{M}^*$ or $\mathcal{N}^*$. Therefore, $\mathcal{L}^* = \mathcal{M}^* \cup \mathcal{N}^* = \mathcal{M}^* \cup \overline{\mathcal{N}^*}$. Since both $\mathcal{M}^*$ and $\overline{\mathcal{N}^*}$ are modules, it follows that either $\mathcal{L}^* = \mathcal{M}^*$ or $\mathcal{L}^* = \overline{\mathcal{N}^*}$. Case 1i) is equivalent to $\mathcal{L}^* = \mathcal{M}^*$. If $\mathcal{L}^* = \overline{\mathcal{N}^*}$, we recall that by the properties (2) of $\mathcal{N}^*$, only two of the elements l_1^*, l_2^* and $l_1^* + l_2^*$ can belong to $\mathcal{N}^*$ and the three possible cases are 1ii), 2i) and 2ii). $\square$

Lemma 4.10. *Suppose $(v_\sigma, \sigma) \in \Gamma$ and $\mathcal{L}^* = \mathcal{M}_P^* \cup \mathcal{M}_{y_0}^* \cup \mathcal{N}^*$. Then, for the rectangular lattices $\mathcal{L} = \{(\alpha, 0), (0, \beta)\}_{\mathbf{Z}}$ one of the conditions a), c) or d) of Theorem 4.1 holds.*

Proof: By Lemma 4.8 we have $\alpha = \frac{2P}{n}$ and $\mathcal{L}^* = \{l_1^*, l_2^*\}_{\mathbf{Z}}$ where $l_1^* = \left(\frac{n}{2P}, 0\right)$ and $l_2^* = \left(0, \frac{1}{\beta}\right)$. Since $l_1^* \in \mathcal{M}_0^*$, we can use Lemma 4.9 with $\mathcal{M}^* = \mathcal{M}_P^*$.

We have $l_1^* \in \mathcal{M}_P^*$, if and only if n is even. This, together with $\left(\frac{2P}{n}, 0\right) \in \mathcal{L}$, implies $(P, 0) \in \mathcal{L}$. Therefore, in case 1) of Lemma 4.9 we obtain condition a).

Suppose now that $l_1^* \notin \mathcal{M}_P^*$ and therefore that $l_1^* + l_2^* \notin \mathcal{M}_P^*$, *i.e.*, one of the two cases 2ii) or 2iii) of Lemma 4.9 holds, and moreover n is odd.

Let $v_\sigma = (v_1, v_2)$, by Lemma 4.7, either $v_1 = 0$ or $v_1 = \frac{P}{n}$. Since $l_1^* \in \mathcal{N}^*$ then $\frac{n}{2P}v_1 + \frac{1}{2} \in \mathbf{Z}$ and $v_1 = \frac{P}{n}$. Thus $\left(\left(\frac{P}{n}, v_2\right), \sigma\right) \in \Gamma$ and $\left(\left(\frac{P}{n}, v_2\right), \sigma\right)^n = ((P, v_2), \sigma) \in \Gamma$. Therefore case 2iii) of Lemma 4.9 implies condition d).

In case 2iii) we have $l_1^* + l_2^* \in \mathcal{N}^* \Leftrightarrow \frac{1}{\beta}(y_0 - v_2) \in \mathbf{Z}$, then $(0, y_0 - v_2) \in \mathcal{L}$ and

$$((0, y_0 - v_2), I) \cdot ((P, v_2), \sigma) = ((P, y_0), \sigma) \in \Gamma,$$

and condition c) holds. □

Lemma 4.11. *Suppose* $(v_\sigma, \sigma) \in \Gamma$ *and* $\mathcal{L}^* = \mathcal{M}_P^* \cup \mathcal{M}_{y_0}^* \cup \mathcal{N}^*$. *Then, for the lattices* $\mathcal{L} = \left\{\left(\frac{\alpha}{2}, \frac{\beta}{2}\right), (0, \beta)\right\}_{\mathbf{Z}}$, *one of the conditions a), c) or d) of Theorem 4.1 holds.*

Proof: The dual lattice is of the form $\mathcal{L}^* = \left\{\left(\frac{n}{P}, 0\right), \left(-\frac{n}{2P}, \frac{1}{\beta}\right)\right\}_{\mathbf{Z}}$. For $l_1^* = \left(\frac{n}{P}, 0\right) \in \mathcal{M}_0^*$, $l_2^* = \left(-\frac{n}{2P}, \frac{1}{\beta}\right)$ and $\mathcal{M}^* = \mathcal{M}_P^*$ Lemma 4.9 holds.

First notice that $l_1^* \in \mathcal{M}_P^*$ and $l_2^* \notin \mathcal{N}^*$, as $v_\sigma = (0, v_2)$ by Lemma 4.7. Therefore case 2 of Lemma 4.9 never happens and case 1iii) implies condition d) because $\left(P, n\frac{\beta}{2}\right) \in \mathcal{L}$.

In case 1i0, l_2^* belongs to $\mathcal{M}_P^*$ and so $\frac{n}{2P}P \in \mathbf{Z} \Leftrightarrow n$ is even. Thus $(P, 0) \in \mathcal{L}$ and condition a) is verified.

For case 1ii) n is odd and $\frac{1}{\beta}(y_0 - v_2) + \frac{1}{2} \in \mathbf{Z} \Leftrightarrow y_0 - v_2 = m\beta + \frac{\beta}{2}$ for some $m \in \mathbf{Z}$. Thus $\left(\frac{P}{n}, y_0 - v_2\right) \in \mathcal{L}$ and condition c) follows because

$$\left(\left(\frac{P}{n}, y_0 - v_2\right), I\right) \cdot ((0, v_2), \sigma) = \left(\left(\frac{P}{n}, y_0\right), \sigma\right) \in \Gamma$$

and $\left(\left(\frac{P}{n}, y_0\right), \sigma\right)^n = ((P, y_0), \sigma) \in \Gamma$. □

5. Projection of invariant functions and symmetry - Parity on $\Pi_{y_0}(X_\Gamma)$

The translations in the group of symmetries of a projected pattern have been described in the previous section. In order to describe its full symmetry it remains to obtain the orthogonal part. As we have remarked in section 2, for one spatial dimension this takes the form of parity.

5.1. parity of $\Pi_{y_0}(X_\Gamma)$ related to $\mathcal{L}^*$.

Recall that $-\sigma = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $-I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

Proposition 5.1. *All functions in $\Pi_{y_0}(X_\Gamma)$ are invariant for the reflection in the point x_0 if and only if one of the conditions holds:*
a) $-\sigma \in \mathbf{J}$ and for each $k \in \mathcal{L}^*$ either $< k, (0, y_0) > \in \mathbf{Z} - \{0\}$ or $< k, -\sigma v_{-\sigma} + (2x_0, 0) > \in \mathbf{Z}$,
b) $-I \in \mathbf{J}$ and for each $k \in \mathcal{L}^*$ either $< k, (0, y_0) > \in \mathbf{Z} - \{0\}$ or $< k, -v_{-I} + (2x_0, y_0) > \in \mathbf{Z}$,
c) $\pm\sigma \in \mathbf{J}$ and for each $k \in \mathcal{L}^*$ either $< k, (0, y_0) > \in \mathbf{Z} - \{0\}$ or $< k, -\sigma v_{-\sigma} + (2x_0, 0) > \in \mathbf{Z}$ or $< k, \sigma v_\sigma + (0, y_0) > + \frac{1}{2} \in \mathbf{Z}$.

Proposition 5.1 can be rephrased in terms of the subsets of $\mathcal{L}^*$ of section 4 plus the following submodules of $\mathcal{L}^*$:

$$\mathcal{M}^*_{-\sigma} = \{k \in \mathcal{L}^* : < k, -\sigma v_{-\sigma} + (2x_0, 0) > \in \mathbf{Z}\} \text{ and}$$

$$\mathcal{M}^*_{-I} = \{k \in \mathcal{L}^* : < k, -v_{-I} + (2x_0, y_0) > \in \mathbf{Z}\},$$

with the conventions $\mathcal{M}^*_{-\sigma} = \emptyset$ if $-\sigma \notin \mathbf{J}$, $\mathcal{M}^*_{-I} = \emptyset$ if $-I \notin \mathbf{J}$.

Proposition. 5.1 *All functions in $\Pi_{y_0}(X_\Gamma)$ are invariant for the reflection in the point x_0 if and only if one of the conditions holds:*
a) $-\sigma \in \mathbf{J}$ and $\mathcal{L}^* = \mathcal{M}^*_{y_0} \cup \mathcal{M}^*_{-\sigma}$,
b) $-I \in \mathbf{J}$ and $\mathcal{L}^* = \mathcal{M}^*_{y_0} \cup \mathcal{M}^*_{-I}$,
c) $\pm\sigma \in \mathbf{J}$ and $\mathcal{L}^* = \mathcal{M}^*_{y_0} \cup \mathcal{M}^*_{-\sigma} \cup \mathcal{N}^*$.

Proof: A function $g : \mathbf{R} \longrightarrow \mathbf{R}$ is invariant under reflections in a point x_0 if and only if $g(x) = g(2x_0 - x)$ for all $x \in \mathbf{R}$. Therefore, $\Pi_{y_0}(f)(x) = \sum_{k_1 \in \mathcal{L}_1^*} D(k_1)\omega_{k_1}(x)$ is invariant under reflection in a point x_0 if and only if, for each $k_1 \in \mathcal{L}_1^*$,

$$D(k_1) = \omega_{k_1}(-2x_0)D(-k_1), \tag{6}$$

since $-\mathcal{L}_1^* = \mathcal{L}_1^*$.

First we prove that each one of the conditions a), b) and c) is sufficient. If a) is valid then

$$D(k_1) = \sum_{k_2 : (k_1, k_2) \in \mathcal{L}^*} C(k_1, k_2) \int_0^{y_0} \omega_{k_2}(y) dy$$

and

$$\omega_{k_1}(-2x_0)D(-k_1) = \omega_{k_1}(-2x_0) \sum_{k_2:(-k_1,k_2)\in\mathcal{L}^*} C(-k_1,k_2)\int_0^{y_0}\omega_{k_2}(y)dy =$$

$$= \omega_{k_1}(-2x_0) \sum_{k_2:-\sigma(k_1,k_2)\in\mathcal{L}^*} C(k_1,k_2)\omega_k(\sigma v_{-\sigma})\int_0^{y_0}\omega_{k_2}(y)dy,$$

where $k=(k_1,k_2)$. As $-\sigma(\mathcal{L}^*)=\mathcal{L}^*$, this equals

$$\omega_{k_1}(-2x_0) \sum_{k_2:(k_1,k_2)\in\mathcal{L}^*} C(k_1,k_2)\omega_k(\sigma v_{-\sigma})\int_0^{y_0}\omega_{k_2}(y)dy$$

and therefore $D(k_1)-\omega_{k_1}(-2x_0)D(-k_1)$ is given by

$$\sum_{k_2:(k_1,k_2)\in\mathcal{L}^*} C(k_1,k_2)\int_0^{y_0}\omega_{k_2}(y)dy\left(1-\omega_{k_1}(-2x_0)\omega_k(\sigma v_{-\sigma})\right).$$

The last expression is allways zero because, by a), either $\int_0^{y_0}\omega_{k_2}(y)dy = 0$ or $\omega_{k_1}(-2x_0)\omega_{(k_1,k_2)}(\sigma v_{-\sigma})=1$ for each term of the sum. Therefore condition (6) is verified for all $k\in\mathcal{L}^*$.

Suppose now that b) happens. The expression $\omega_{k_1}(-2x_0)D(-k_1)$ becomes

$$\omega_{k_1}(-2x_0) \sum_{k_2:(-k_1,k_2)\in\mathcal{L}^*} C(-k_1,k_2)\int_0^{y_0}\omega_{k_2}(y)dy$$

$$= \omega_{k_1}(-2x_0) \sum_{-k_2:(-k_1,-k_2)\in\mathcal{L}^*} C(-k_1,-k_2)\int_0^{y_0}\omega_{-k_2}(y)dy$$

$$= \omega_{k_1}(-2x_0) \sum_{-k_2:(k_1,k_2)\in\mathcal{L}^*} C(k_1,k_2)\omega_k(v_{-I})\omega_{k_2}(-y_0)\int_0^{y_0}\omega_{k_2}(y)dy.$$

Therefore, $D(k_1)-\omega_{k_1}(-2x_0)D(-k_1)$ is

$$\sum_{k_2:(k_1,k_2)\in\mathcal{L}^*} C(k_1,k_2)\int_0^{y_0}\omega_{k_2}(y)dy\left(1-\omega_{k_1}(-2x_0)\omega_k(v_{-I})\omega_{k_2}(-y_0)\right).$$

But, by b), either $\int_0^{y_0}\omega_{k_2}(y)dy=0$ or $\omega_{k_1}(-2x_0)\omega_k(v_{-I})\omega_{k_2}(-y_0)=1$ and so expression (6) is verified for all $k\in\mathcal{L}^*$.

If c) happens then

$$D(k_1) = \frac{1}{2}\sum_{k_2:(k_1,k_2)\in\mathcal{L}^*}\left(C(k_1,k_2)\int_0^{y_0}\omega_{k_2}(y)dy + C(k_1,-k_2)\int_0^{y_0}\omega_{-k_2}(y)dy\right)$$

because $(v_\sigma,\sigma)\in\Gamma$. Thus,

$$D(k_1) = \frac{1}{2}\sum_{k_2:(k_1,k_2)\in\mathcal{L}^*} C(k_1,k_2)\left(\int_0^{y_0}\omega_{k_2}(y)dy + \omega_{\sigma k}(-v_\sigma)\int_0^{y_0}\omega_{-k_2}(y)dy\right)$$

$$= \frac{1}{2}\sum_{k_2:(k_1,k_2)\in\mathcal{L}^*} C(k_1,k_2)\int_0^{y_0}\omega_{k_2}(y)dy\left(1+\omega_k(-\sigma v_\sigma)\omega_{k_2}(-y_0)\right).$$

Similarly,

$$D(-k_1) = \frac{1}{2}\sum_{k_2:(-k_1,k_2)\in\mathcal{L}^*}\left(C(-k_1,k_2)\int_0^{y_0}\omega_{k_2}(y)dy + C(-k_1,-k_2)\int_0^{y_0}\omega_{-k_2}(y)dy\right)$$

$$= \frac{1}{2}\sum_{k_2:-\sigma(k_1,k_2)\in\mathcal{L}^*} C(k_1,k_2)\left(\omega_{-\sigma k}(-v_{-\sigma})\int_0^{y_0}\omega_{k_2}(y)dy + \omega_{-k}(-v_{-I})\int_0^{y_0}\omega_{-k_2}(y)dy\right)$$

because the elements $(v_{-\sigma},-\sigma)$ and (v_σ,σ) of Γ ensure that $(v_{-I},-I)\in\Gamma$ for some v_{-I}. Therefore,

$$D(-k_1) = \frac{1}{2}\sum_{k_2:(k_1,k_2)\in\mathcal{L}^*} C(k_1,k_2)\int_0^{y_0}\omega_{k_2}(y)dy\left(\omega_k(\sigma v_{-\sigma}) + \omega_k(v_{-I})\omega_{k_2}(-y_0)\right).$$

Thus, expression $D(k_1) - \omega_{k_1}(-2x_0)D(-k_1)$ becomes

$$\frac{1}{2}\sum_{k_2:(k_1,k_2)\in\mathcal{L}^*} C(k_1,k_2)\int_0^{y_0}\omega_{k_2}(y)dy\, G(k_1,k_2)$$

where

$$\begin{aligned} G(k_1,k_2) &= 1+\omega_k(-\sigma v_\sigma)\omega_{k_2}(-y_0) - \omega_{k_1}(-2x_0)\left(\omega_k(\sigma v_{-\sigma}) + \omega_k(v_{-I})\omega_{k_2}(-y_0)\right) \\ &= \left(1-\omega_{k_1}(-2x_0)\omega_k(\sigma v_{-\sigma})\right)\left(1+\omega_k(-\sigma v_\sigma)\omega_{k_2}(-y_0)\right) \end{aligned}$$

because $(v_{-\sigma},-\sigma)(v_\sigma,\sigma) = (v_{-\sigma}-\sigma v_\sigma,-I)$ and $(v_{-\sigma},-\sigma)(v_{-\sigma},-\sigma) = (v_{-\sigma}-\sigma v_{-\sigma}, I)$ imply, respectivelly, $\omega_k(v_{-I}) = \omega_k(v_{-\sigma})\omega_k(-\sigma v_\sigma)$ and $\omega_k(\sigma v_{-\sigma}) = \omega_k(v_{-\sigma})$.

Moreover, by c), for each $k\in\mathcal{L}^*$, one of the following conditions holds: $\int_0^{y_0}\omega_{k_2}(y)dy = 0$ or $1-\omega_{k_1}(-2x_0)\omega_k(v_{-\sigma}) = 0$ or $1+\omega_k(-\sigma v_\sigma)\omega_{k_2}(-y_0) = 0$. Therefore, expression (6) is valid for all $k\in\mathcal{L}^*$.

Conversely, suppose that $\Pi_{y_0}(f)$ is invariant under a reflection in a point x_0 for all $f\in X_\Gamma$. In particular, this is true for the projection of the Γ-invariant functions I_k+I_{-k} and iI_k-iI_{-k}, the real and imaginary components of I_k. For

$$\Pi_{y_0}(I_k)(x) = \sum_{\delta\in\mathbf{J}}\omega_{\delta k|_1}(x)D'(\delta,k),$$

expression (6) is equivalent to

$$S'(k) = \sum_{\delta\in\mathbf{J}^+(k)} D'(\delta,k) - \omega_{k_1}(-2x_0)\sum_{\delta\in\mathbf{J}^-(k)} D'(\delta,k) = 0, \tag{7}$$

where

$$\mathbf{J}^+(k) = \{\delta\in\mathbf{J}\ :\ \delta k|_1 = k_1\} \quad\text{and}\quad \mathbf{J}^-(k) = \{\delta\in\mathbf{J}\ :\ \delta k|_1 = -k_1\},$$

and must be verified for all $k=(k_1,k_2)\in\mathcal{L}^*$. The result follows by Lemma 5.3 below. □

Lemma 5.1. *Let $\mathbf{J}^- = \mathbf{J}\cap\{-I,-\sigma\}$. The set $\mathbf{J}^-(k)$ satisfies:*

1. $\mathbf{J}^-(k) = \{\delta \in \mathbf{J} \; : \; \delta k = -k \vee \delta k = -\sigma k\}$.
2. *For any* $k \in \mathcal{L}^*$, $\mathbf{J}^- \subset \mathbf{J}^-(k)$ *and* $\mathbf{J}^-(0,0) = \mathbf{J}$.
3. *For any* $k \in \mathcal{L}^*$, $k = (k_1, k_2) \neq (0,0)$, *if* $\delta \in \mathbf{J}^-(k) - \mathbf{J}^-$ *then* $\delta k = -(k_1, -|\delta| k_2)$ *where* $|.|$ *is the determinant.*
4. *If* $\mathbf{J}^+ = \{I\}$ *and* $k \neq (0,0)$ *then, for any* $k \in \mathcal{L}^*$, $\mathbf{J}^-(k) - \mathbf{J}^-$ *contains at most one element.*
5. *If* $\mathbf{J}^+ = \{I, \sigma\}$ *then for any* $k \in \mathcal{L}^*$, $k \neq (0,0)$, *either* $\mathbf{J}^-(k) = \mathbf{J}^-$ *or* $\mathbf{J}^-(k) - \mathbf{J}^-$ *contains exactly two elements,* δ *and* $\sigma\delta$.

We omit the proof of this Lemma because it is similar, step by step, to the proof of Lemma 4.1.

Let $\mathcal{M}^*_\pm = \{k : \mathbf{J}^+(k) = \mathbf{J}^+ \text{ and } \mathbf{J}^-(k) = \mathbf{J}^-\}$ and, if $\delta \in \mathbf{J} - (\mathbf{J}^+ \cup \mathbf{J}^-) \neq \emptyset$, for $\delta \in \mathbf{J} - (\mathbf{J}^+ \cup \mathbf{J}^-)$, let

$$\mathcal{M}^*_\delta = \{k \in \mathcal{L}^* \; : \; \delta k = \pm(k_1, -|\delta| k_2)\}.$$

If δ is a rotation and $k \in \mathcal{M}^*_\delta$ then either $\delta k = (k_1, -k_2)$ or $\delta k = -(k_1, -k_2)$, *i.e.*, k belongs to the line fixed either by $\sigma\delta$ or $-\sigma\delta$. Therefore $\mathcal{M}^*_\delta$ is the intersection of those lines with $\mathcal{L}^*$. Similarly, if δ is a reflection then $\mathcal{M}^*_\delta$ is the intersection of $\mathcal{L}^*$ with the lines fixed either by δ or by $-\delta$.

We can write

$$\mathcal{L}^* = \mathcal{M}^*_\pm \cup \bigcup_{\delta \in \mathbf{J} - (\mathbf{J}^+ \cup \mathbf{J}^-)} \mathcal{M}^*_\delta \tag{8}$$

where any two sets in the right hand side either are disjoint or coincide, with the convention $\bigcup_{\delta \in \mathbf{J} - (\mathbf{J}^+ \cup \mathbf{J}^-)} \mathcal{M}^*_\delta = \emptyset$ if $\delta \in \mathbf{J} - (\mathbf{J}^+ \cup \mathbf{J}^-) = \emptyset$.

Lemmas 4.1 and 5.1 characterize the subsets of $\mathcal{L}^*$ with, respectivelly, common $\mathbf{J}^+(k)$ and common $\mathbf{J}^-(k)$ and Lemma 5.2 restricts $k \in \mathcal{M}^*_\pm$ when $S'(k) = 0$.

Lemma 5.2. *Suppose* $S'(k) = 0$ *for some* $k \in \mathcal{M}^*_\pm$.
i) If $\mathbf{J}^- = \emptyset$ *then* $k \in \mathcal{M}^*_{y_0} \cup \mathcal{N}^*$.
ii) If $-\sigma \in \mathbf{J}^-$ *then* $k \in \mathcal{M}^*_{y_0} \cup \mathcal{M}^*_{-\sigma} \cup \mathcal{N}^*$.
iii) If $\mathbf{J}^- = \{-I\}$ *then* $\mathbf{J}^+ = \{I\}$ *and* $k \in \mathcal{M}^*_{y_0} \cup \mathcal{M}^*_{-I}$.

Proof: If $\mathbf{J}^- = \emptyset$ then $S'(k) = \sum_{\delta \in \mathbf{J}^+} D'(\delta, k) = S(k) = 0$ which, by Lemma 4.2, implies either $k \in \mathcal{M}^*_{y_0}$, if $\mathbf{J}^+ = \{I\}$, or $k \in \mathcal{M}^*_{y_0} \cup \mathcal{N}^*$ if $\mathbf{J}^+ = \{I, \sigma\}$. It follows i) by the definition of set $\mathcal{N}^*$.

Now suppose $-\sigma \in \mathbf{J}^-$. If $\mathbf{J}^- = \{-\sigma\}$ then $\mathbf{J}^+ = \{I\}$ and

$$S'(k) = D'(I, k) - \omega_{k_1}(-2x_0) D'(-\sigma, k) = 0.$$

Therefore

$$\int_0^{y_0} \omega_{k_2}(y) dy - \omega_{k_1}(-2x_0)\omega_{-\sigma k}(-v_{-\sigma}) \int_0^{y_0} \omega_{k_2}(y) dy = 0$$

$$\Leftrightarrow \int_0^{y_0} \omega_{k_2}(y) dy = 0 \text{ or } \omega_{k_1}(-2x_0)\omega_k(\sigma v_{-\sigma}) = 1,$$

i.e., $k \in \mathcal{M}^*_{y_0} \cup \mathcal{M}^*_{-\sigma}$. The expression in ii) follows because $\mathcal{N}^*$ is the empty set. If $\mathbf{J}^- = \{-I, -\sigma\}$ then

$$S'(k) = D'(I,k) + D'(\sigma,k) - \omega_{k_1}(-2x_0)\left(D'(-I,k) + D'(-\sigma,k)\right) = 0.$$

Therefore,

$$\int_0^{y_0} \omega_{k_2}(y)dy\,(1 - \omega_{k_1}(-2x_0)\omega_{-\sigma k}(-v_{-\sigma})) + \int_0^{y_0} \omega_{-k_2}(y)dy\,(\omega_{\sigma k}(-v_\sigma) - \omega_{k_1}(-2x_0)\omega_{-k}(-v_{-I}))$$

$$= \int_0^{y_0} \omega_{k_2}(y)dy\,(1 - \omega_{k_1}(-2x_0)\omega_{-\sigma k}(-v_{-\sigma}) + \omega_{k_2}(-y_0)\,(\omega_{\sigma k}(-v_\sigma) - \omega_{k_1}(-2x_0)\omega_{-k}(-v_{-I})))$$

$$= \int_0^{y_0} \omega_{k_2}(y)dy G(k_1,k_2) = 0,$$

with $G(k_1,k_2)$ as defined above, and so either $k_2 y_0 \in \mathbf{Z} - \{0\}$ or

$$G(k_1,k_2) = (1 - \omega_{k_1}(-2x_0)\omega_k(\sigma v_{-\sigma}))\,(1 + \omega_k(-\sigma v_\sigma)\omega_{k_2}(-y_0)) = 0$$

which implies $k \in \mathcal{M}^*_{y_0} \cup \mathcal{M}^*_{-\sigma} \cup \mathcal{N}^*$.

Finally, if $\mathbf{J}^- = \{-I\}$. Thus, $\mathbf{J}^+ = \{I\}$ and

$$S'(k) = D'(I,k) - \omega_{k_1}(-2x_0)D'(-I,k) = 0$$

implying

$$\int_0^{y_0} \omega_{k_2}(y)dy - \omega_{k_1}(-2x_0)\omega_{-k}(-v_{-I}) \int_0^{y_0} \omega_{-k_2}(y)dy = 0$$

$$\Leftrightarrow \int_0^{y_0} \omega_{k_2}(y)dy = 0 \text{ or } \omega_{k_1}(-2x_0)\omega_k(v_{-I}) = \omega_{k_2}(y_0)$$

or, equivalently, $k \in \mathcal{M}^*_{y_0} \cup \mathcal{M}^*_{-I}$. □

Lemma 5.3. *If $S'(k) = 0$ for all $k \in \mathcal{L}^*$ then one of the following conditions holds.*

a) $\mathbf{J}^- = \{-\sigma\}$ and $\mathcal{L}^* = \mathcal{M}^*_{y_0} \cup \mathcal{M}^*_{-\sigma}$.
b) $\mathbf{J}^- = \{-I\}$ and $\mathcal{L}^* = \mathcal{M}^*_{y_0} \cup \mathcal{M}^*_{-I}$.
c) $\mathbf{J}^- = \{-I, -\sigma\}$ and $\mathcal{L}^* = \mathcal{M}^*_{y_0} \cup \mathcal{M}^*_{-\sigma} \cup \mathcal{N}^*$.

Proof: We prove the result using Lemmas 5.2 and 4.3. In order to use Lemma 4.3 we redefine $\mathcal{O}^* = \bigcup_{\delta \in \mathbf{J}-(\mathbf{J}^+ \cup \mathbf{J}^-)} \mathcal{M}^*_\delta$ and show that $\mathcal{M}^*_0 \cap \mathcal{O}^* = \emptyset$ if $\mathbf{J} - (\mathbf{J}^+ \cup \mathbf{J}^-) = \emptyset$ or else $\mathcal{M}^*_0 \cap \mathcal{O}^* = \{(0,0)\}$, for the sets $\mathcal{M}^*_\delta$ defined in this section.

If $k \in \mathcal{M}^*_0$ then $k = (k_1, 0)$ for some $k_1 \in \mathbf{R}$. If, moreover, $k_1 \neq 0$ and $k \in \mathcal{M}^*_\delta$, for some $\delta \in \mathbf{J} - (\mathbf{J}^+ \cup \mathbf{J}^-)$, then $\delta k = \pm(k_1, 0)$ and, by orthogonality, $\delta = \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}$, *i.e.*, $\delta \in \mathbf{J}^+ \cup \mathbf{J}^-$.

If $S'(k) = 0$ for all $k \in \mathcal{L}^*$ then, by relation (8) and Lemma 5.3, i follows that

i) If $\mathbf{J}^- = \emptyset$ then $\mathcal{L}^* = \mathcal{M}^*_{y_0} \cup \mathcal{N}^* \cup \mathcal{O}^*$.
ii) If $-\sigma \in \mathbf{J}^-$ then $\mathcal{L}^* = \mathcal{M}^*_{y_0} \cup \mathcal{M}^*_{-\sigma} \cup \mathcal{N}^* \cup \mathcal{O}^*$.

iii) If $\mathbf{J}^- = \{-I\}$ then $\mathbf{J}^+ = \{I\}$ and $\mathcal{L}^* = \mathcal{M}^*_{y_0} \cup \mathcal{M}^*_{-I} \cup \mathcal{O}^*$.

Case iii) implies condition b) by Lemma 4.3, with module $\mathcal{M}^*_{-I}$ in the place of $\mathcal{M}^*_P$ and with $\mathcal{N}^* = \emptyset$. Analogously, Lemma 4.3, with module $\mathcal{M}^*_{-\sigma}$ instead of $\mathcal{M}^*_P$, takes case ii) to either condition a) or c).

If $\mathcal{L}^* = \mathcal{M}^*_{y_0} \cup \mathcal{N}^* \cup \mathcal{O}^*$ then $\mathcal{M}^*_0 \subset (\mathcal{N}^* \cup \mathcal{O}^*)$. For $\mathcal{O}^* = \emptyset$ it follows that $\mathcal{M}^*_0 \subset \mathcal{N}^*$. Otherwise, $\mathcal{M}^*_0 \subset (\mathcal{N}^* \cup \{(0,0)\})$. In the first case we obtain an impossible condition because $\mathcal{M}^*_0$ is a module but $\{(0,0)\} \notin \mathcal{N}^*$.

Now suppose $\mathcal{M}^*_0 \subset (\mathcal{N}^* \cup \{(0,0)\})$ and let $\mathcal{N}^* \neq \emptyset$ and $k \neq (0,0)$. If $k \in \mathcal{M}^*_0$ then $2k \in \mathcal{M}^*_0$ but, by the properties (2) of $\mathcal{N}^*$, $2k \notin \mathcal{N}^*$ which contradicts our assumption. If $\mathcal{N}^* = \emptyset$ then $\mathcal{M}^*_0 = \{(0,0)\}$. Moreover, $\mathcal{L}^* = \mathcal{M}^*_{y_0} \cup \mathcal{O}^*$ which, by Lemma 4.3, would imply $\mathcal{L}^* = \overline{\mathcal{M}^*_{y_0}}$. However this imposes $\mathcal{M}^*_0 \neq \{(0,0)\}$ (see proof of Lemma 4.5) and so we may conclude that $\mathbf{J}^- \neq \emptyset$. □

5.2. parity of $\Pi_{y_0}(X_\Gamma)$ related to $\mathcal{L}$.

Theorem 5.1. *All functions in $\Pi_{y_0}(X_\Gamma)$ are invariant for the reflection in the point x_0 if and only if one of the following conditions holds:*

a) $((2x_0, 0), -\sigma) \in \Gamma$,

b) $((2x_0, y_0), -I) \in \Gamma$,

c) $(0, y_0) \in \mathcal{L}$ *and for some* $y_1 \in \mathbf{R}$ $((2x_0, y_1), -\sigma) \in \Gamma$.

d) $(0, y_0) \in \mathcal{L}$ *and for some* $y_1 \in \mathbf{R}$ $((2x_0, y_1), -I) \in \Gamma$.

Proof: A direct calculation of the integrals shows in Lemma 5.4 that the conditions are sufficient

For the necessity, suppose $\Pi_{y_0}(X_\Gamma)$ is a set of functions invariant under a reflection in a point x_0, and therefore one of the three conditions of Proposition 5.1 must hold. In Lemma 5.5 below, we show that conditions a) and b) of Proposition 5.1 imply the conditions of Theorem 5.1.

Condition c) of Proposition 5.1 yelds a dual lattice with a structure similar to that in the proof of Theorem 4.1. Since $\sigma \in \mathbf{J}$ then Lemma 4.6 shows that $\mathcal{L}$ is either a lattice in standard position (square or rectangular) or a lattice in diamond position (square, centered rectangular or hexagonal). Lemma 4.7 restricts the form of v_σ for each type of lattice. In Lemma 5.6 below we obtain analogous restrictions to the form of $v_{-\sigma}$ and Lemma 5.7 scales the possible lattices according to the value of $v_{-\sigma}$. Thus, one of the next conditions must hold, with $\alpha = \frac{2(2x_0 - u_1)}{n}$:

i) $\mathcal{L} = \{(\alpha, 0), (0, \beta), \}_{\mathbf{Z}}$, either $v_\sigma = (0, v_2)$ or $v_\sigma = \left(\frac{\alpha}{2}, v_2\right)$ and either $v_{-\sigma} = (u_1, 0)$ or $v_{-\sigma} = \left(u_1, \frac{\beta}{2}\right)$;

ii) $\mathcal{L} = \left\{\left(\frac{\alpha}{2}, \frac{\beta}{2}\right), (0, \beta)\right\}_{\mathbf{Z}}$, $v_\sigma = (0, v_2)$ and either $v_{-\sigma} = (0, 0)$ or $v_{-\sigma} = \left(u_1, \frac{\beta}{2}\right)$ or $v_{-\sigma} = (u_1, \beta)$.

Finally, Lemmas 5.8 and 5.9, below, prove the result for, respectivelly, case i) and ii). □

Lemma 5.4. *If one of the conditions a), b), c), or d) of Theorem 5.1 holds then $\Pi_{y_0}(X_\Gamma)$ is a set of functions invariant under reflection in the point x_0.*

Proof: A function $g : \mathbf{R} \longrightarrow \mathbf{R}$ is invariant under reflections in a point x_0 if $g(x) = g(2x_0 - x)$ for all $x \in \mathbf{R}$. If a) holds then for all $f \in X_\Gamma$, $f(x,y) = f(2x_0 - x, y)$ and $\Pi_{y_0}(f)(x) = \int_0^{y_0} f(x,y)dy = \int_0^{y_0} f(2x_0 - x, y)dy = \Pi_{y_0}(f)(2x_0 - x)$.

Similarly, if b) holds then $\Pi_{y_0}(f)(x) = \int_0^{y_0} f(2x_0 - x, y_0 - y)dy$ which, for $z = y_0 - y$, equals $\int_0^{y_0} f(2x_0 - x, z)dz = \Pi_{y_0}(f)(2x_0 - x)$.

In case c) the projection satisfies either $\Pi_{y_0}(f)(x) = \int_{y_1}^{y_1+y_0} f(2x_0 - x, z)dz$ or $\Pi_{y_0}(f)(x) = \int_{y_1-y_0}^{y_1} f(2x_0 - x, z)dz$. In both cases every function $f \in X_\Gamma$ has period $(0, y_0)$ and thus $\Pi_{y_0}(f)(x) = \int_0^{y_0} f(2x_0 - x, z)dz = \Pi_{y_0}(f)(2x_0 - x)$. □

Lemma 5.5.

1) If $-\sigma \in \mathbf{J}$ and $\mathcal{L}^* = \mathcal{M}^*_{y_0} \cup \mathcal{M}^*_{-\sigma}$ then either a) or c) of Theorem 5.1 holds.

2) If $-I \in \mathbf{J}$ and $\mathcal{L}^* = \mathcal{M}^*_{y_0} \cup \mathcal{M}^*_{-I}$ then either b) or d) of Theorem 5.1 holds.

Proof: Let $\xi = -\sigma$ in case *1)* and $\xi = -I$ in case *2)*. Let $\mathcal{M}^* = \mathcal{M}^*_\xi$, and $(v_1, v_2) = v_\xi$.

Since both $\mathcal{M}^*$ and $\overline{\mathcal{M}^*_{y_0}}$ are modules, then from $\mathcal{L}^* = \overline{\mathcal{M}^*_{y_0}} \cup \mathcal{M}^*$ it follows that either $\mathcal{L}^* = \overline{\mathcal{M}^*_{y_0}}$ or $\mathcal{L}^* = \mathcal{M}^*$.

In case *1)* if $\mathcal{L}^* = \mathcal{M}^*_{-\sigma}$ then $-\sigma v_{-\sigma} + (2x_0, 0) \in \mathcal{L}$ and

$$(v_{-\sigma}, -\sigma) \cdot (\sigma v_{-\sigma} - (2x_0, 0), I) = ((2x_0, 0), -\sigma) \in \Gamma,$$

i.e., condition a) is satisfied.

In case *2)* if $\mathcal{L}^* = \mathcal{M}^*_{-I}$ then $-v_{-I} + (2x_0, y_0) \in \mathcal{L}$ and

$$((2x_0, y_0) - v_{-I}, I) \cdot (v_{-I}, -I) = ((2x_0, y_0), -I) \in \Gamma,$$

i.e., condition b) holds.

Now suppose $\mathcal{L}^* = \overline{\mathcal{M}^*_{y_0}}$ and thus $(0, y_0) \in \mathcal{L}$. For an integer $n \neq 0$ and some $y_2 \in \mathbf{R}$, we choose generators for $\mathcal{L}$ and $\mathcal{L}^*$ as follows:

$$\begin{aligned} \mathcal{L} &= \{l_1 = (0, y_0/n), l_2 = (n\rho/y_0, y_2)\}_{\mathbf{Z}} \\ \mathcal{L}^* &= \{l_1^* = (-y_2/\rho, n/y_0), l_2^* = (y_0/n\rho, 0)\}_{\mathbf{Z}}. \end{aligned}$$

Since $l_2^* \in \mathcal{M}^*_0$ then $l_2^* \in \mathcal{M}^*$.

In case *1)* we have $\langle l_2^*, -\sigma v_{-\sigma} + (2x_0, 0)\rangle = y_0/n\rho(2x_0 - v_1) = m \in \mathbf{Z}$ and in case *2)* we have $\langle l_2^*, -av_{-I} + (2x_0, y_0)\rangle = y_0/n\rho(2x_0 - v_1) = m \in \mathbf{Z}$.

In both cases it follows that $ml_2 = (2x_0 - v_1, my_0) \in \mathcal{L}$ and thus,

$$((2x_0 - v_1, my_2), I) \cdot ((v_1, v_2), \xi) = ((2x_0, y_1), \xi) \in \Gamma,$$

where $y_1 = my_2 + v_2$, implying condition c) in case *1)* and condition d) in case *2)*. $\square$

Now we obtain restrictions on $v_{-\sigma}$ like those on v_σ (Lemma 4.7).

Lemma 5.6. *If* $\pm\sigma \in \mathbf{J}$, *then:*
If $\mathcal{L} = \{(\alpha, 0), (0, \beta)\}_{\mathbf{Z}}$ *then either* $v_{-\sigma} = (u_1, 0)$ *or* $v_{-\sigma} = \left(u_1, \frac{\beta}{2}\right)$,
If $\mathcal{L} = \left\{\left(\frac{\alpha}{2}, \frac{\beta}{2}\right), (0, \beta)\right\}_{\mathbf{Z}}$ *then either* $v_{-\sigma} = (0, 0)$ *or* $v_{-\sigma} = \left(u_1, \frac{\beta}{2}\right)$ *or* $v_{-\sigma} = (u_1, \beta)$. *Moreover, if* $v_{-\sigma} = (u_1, \beta)$ *then* $((0, \beta), -\sigma) \in \Gamma$.

Proof: Let $v_{-\sigma} = (u_1, u_2)$, with $(v_{-\sigma}, -\sigma)^2 = ((0, 2u_2), I) \in \Gamma$. For the lattice in standard position, we have $u_2 \in [0, \beta)$ with $(0, 2u_2) \in \mathcal{L}$. Therefore, either $u_2 = 0$ or $u_2 = \frac{\beta}{2}$.

For the lattice in diamond position, $u_2 \in \left[0, \frac{3\beta}{2}\right)$ with $(0, 2u_2) \in \mathcal{L}$ and thus u_2 is either 0 or $\frac{\beta}{\alpha}$ or β . Moreover, $(0, 0)$ is the only element of the fundamental cell of this lattice having null second coordinate. If $u_2 = \beta$ then $((0, -\beta), I) \cdot ((u_1, \beta), -\sigma) = ((u_1, 0), -\sigma) \in \Gamma$. $\square$

Lemma 5.7. *If* $\pm\sigma \in \mathbf{J}$ *and* $\mathcal{L}^* = \mathcal{M}^*_{-\sigma} \cup \mathcal{M}^*_{y_0} \cup \mathcal{N}^*$, *then for* $v_{-\sigma} = (u_1, u_2)$ *and for* α *defined in Lemma 4.6 we have* $\alpha = \frac{2(2x_0 - u_1)}{n}$ *for some integer* n.

Proof: As in the proof of Lemma 4.8, using $\mathcal{M}^*_{-\sigma}$ instead of $\mathcal{M}^*_P$, we obtain $k = \left(\frac{2}{\alpha}, 0\right) \in \mathcal{L}^*$ with $k \notin \mathcal{M}^*_{y_0} \cup \mathcal{N}^*$.

Therefore $k \in \mathcal{M}^*_{-\sigma}$, *i.e.*, $\left\langle \left(\frac{2}{\alpha}, 0\right), -\sigma v_{-\sigma} + (2x_0, 0)\right\rangle \in \mathbf{Z}$ or, equivalently, $\frac{2}{\alpha}(2x_0 - u_1) = n \in \mathbf{Z}$ and the result follows. $\square$

Lemma 5.8. *Suppose* $\pm\sigma \in \mathbf{J}$ *and* $\mathcal{L}^* = \mathcal{M}^*_{y_0} \cup \mathcal{M}^*_{-\sigma} \cup \mathcal{N}^*$. *If* $\mathcal{L} = \{(\alpha, 0), (0, \beta)\}_{\mathbf{Z}}$, *then one of the conditions of Theorem 5.1 holds.*

Proof: Writing $\mathcal{L}^* = \left\{ l_1^* = \left(\frac{1}{\alpha}, 0\right), l_2^* = \left(0, \frac{1}{\beta}\right)\right\}_{\mathbf{Z}}$ then $l_1^* \in \mathcal{M}^*_0$ and by Lemma 4.9, with $\mathcal{M}^* = \mathcal{M}^*_{-\sigma}$, we have either $l_1^* \in \mathcal{M}^*_{-\sigma}$ or $l_1^* \in \mathcal{N}^*$. Let $v_\sigma = (v_1, v_2)$, $v_{-\sigma} = (u_1, u_2)$, with $\alpha = 2(2x_0 - u_1)/n$ for some $n \in \mathbf{Z}$, as in Lemma 5.7.

If $l_1^* \in \mathcal{M}^*_{-\sigma}$, then n is even and so $(2x_0 - u_1, 0) \in \mathcal{L}$ implying $((2x_0 - u_1, 0), I) \cdot ((u_1, u_2), -\sigma) = ((2x_0, u_2), -\sigma) \in \Gamma$. By Lemma 4.9 there are three possibilities:

1iii) $(0, y_0) \in \mathcal{L}$ and condition c) follows.

1i) $l_2^* \in \mathcal{M}_{-\sigma}^* \Leftrightarrow \frac{1}{\beta}u_2 \in \mathbf{Z}$. Then $u_2 = 0$, by Lemma 5.6, and thus condition a) holds.

1ii) Since $l_2^* \in \mathcal{N}^*$, we have $\frac{1}{\beta}(y_0 - v_2) + \frac{1}{2} \in \mathbf{Z}$ and therefore, $y_0 - v_2 + \frac{\beta}{2} = m\beta$, for some $m \in \mathbf{Z}$. From $l_2^*, l_1^* + l_2^* \in \mathcal{N}^*$ it follows that $\frac{v_1}{\alpha} \in \mathbf{Z}$ and by Lemma 4.7 either $v_1 = \alpha/2$, a contradiction, or else $v_1 = 0$.

Since $ml_2 = (0, y_0 - v_2 + u_2) \in \mathcal{L}$, then condition b) follows because

$$((0, y_0 - v_2 + u_2), I) \cdot ((0, v_2), \sigma) \cdot ((2x_0, u_2), -\sigma) = ((2x_0, y_0), -I) \in \Gamma.$$

If $l_1^* \in \mathcal{N}^*$ then n is odd, $\frac{v_1}{\alpha} + \frac{1}{2} \in \mathbf{Z}$ and, by Lemma 4.7, $v_1 = \frac{\alpha}{2}$. Lemma 4.9 divides this into three cases:

2iii) $(0, y_0) \in \mathcal{L}$ and condition d) holds, because

$$((2x_0 - u_1, v_2), \sigma) \cdot ((u_1, u_2), -\sigma) = ((2x_0, v_2 - u_2), -I) \in \Gamma.$$

2i) $l_1^* + l_2^* \in \mathcal{M}_{-\sigma}^* \Leftrightarrow \frac{n}{2} + \frac{1}{\beta}u_2 \in \mathbf{Z}$ and so $u_2 = \frac{\beta}{2}$. Since $l_2^* \in \mathcal{N}^*$, then $\frac{1}{\beta}(y_0 - v_2) + \frac{1}{2} \in \mathbf{Z}$. Condition b) follows from

$$((0, y_0 - v_2 + u_2), I) \cdot ((2x_0 - u_1, v_2), \sigma) \cdot ((u_1, u_2), -\sigma) = ((2x_0, y_0), -I) \in \Gamma.$$

2ii) If $l_2^* \in \mathcal{M}_{-\sigma}^*$ and $l_1^* + l_2^* \in \mathcal{N}^*$, then both $\frac{1}{\beta}u_2$ and $\frac{1}{\beta}(y_0 - v_2)$ are integers. Condition b) follows since $u_2 = 0$, $(0, y_0 - v_2) \in \mathcal{L}$ and

$$((0, y_0 - v_2), I) \cdot ((2x_0 - u_1, v_2), \sigma) \cdot ((u_1, 0), -\sigma) = ((2x_0, y_0), -I) \in \Gamma.$$

$\square$

Lemma 5.9. *Suppose $\pm\sigma \in \mathbf{J}$ and $\mathcal{L}^* = \mathcal{M}_{y_0}^* \cup \mathcal{M}_{-\sigma}^* \cup \mathcal{N}^*$. If $\mathcal{L} = \left\{ \left(\frac{\alpha}{2}, \frac{\beta}{2}\right), (0, \beta) \right\}_{\mathbf{Z}}$, then one of the conditions of Theorem 5.1 holds.*

Proof: We use Lemma 4.7 to have $v_\sigma = (0, v_2)$. Let $v_{-\sigma} = (u_1, u_2)$, then either $v_{-\sigma} = (0, 0)$ or u_2 is either $\frac{\beta}{2}$ or β, by Lemma 5.6. We also have $\alpha = \frac{2}{n}(2x_0 - u_1)$ by Lemma 5.7.

The dual lattice is $\mathcal{L}^* = \left\{ l_1^* = \left(\frac{2}{\alpha}, 0\right), l_2^* = \left(-\frac{1}{\alpha}, \frac{1}{\beta}\right) \right\}_{\mathbf{Z}}$. Since $l_1^* \in \mathcal{M}_0^*$, and $l_1^* \notin \mathcal{N}^*$ then, by Lemma 4.9 with $\mathcal{M}^* = \mathcal{M}_{-\sigma}^*$, we have $l_1^* \in \mathcal{M}_{-\sigma}^*$ and there are three possible cases:

1iii) If $(0, y_0) \in \mathcal{L}$ then condition c) follows because $n\left(\frac{\alpha}{2}, \frac{\beta}{2}\right) = \left(2x_0 - u_1, n\frac{\beta}{2}\right) \in \mathcal{L}$ and thus

$$((2x_0 - u_1, n\beta/2), I) \cdot ((u_1, u_2), -\sigma) = ((2x_0, u_2 + n\beta/2), -\sigma) \in \Gamma.$$

1i) $l_2^* \in \mathcal{M}_{-\sigma}^* \Leftrightarrow \frac{1}{\beta}u_2 - \frac{n}{2} \in \mathbf{Z}$.

If n is even then $\frac{n}{2}(\alpha,0) = (2x_0 - u_1, 0) \in \mathcal{L}$ and, by Lemma 5.6, we have either $u_2 = 0$ or $u_2 = \beta$ and $((u_1,0),-\sigma) \in \Gamma$. Thus

$$((2x_0 - u_1, 0), I) \cdot ((u_1,0),-\sigma) = ((2x_0,0),-\sigma) \in \Gamma$$

and condition a) follows.

If n is odd then $u_2 = \frac{\beta}{2}$, by Lemma 5.6. Since $n\left(\frac{\alpha}{2},\frac{\beta}{2}\right) = ((2x_0 - u_1, nu_2) \in \mathcal{L}$ and $((u_1,nu_2),-\sigma) \in \Gamma$. Condition a) holds because

$$((2x_0 - u_1, nu_2), I) \cdot ((u_1,nu_2),-\sigma) = ((2x_0,0),-\sigma) \in \Gamma.$$

1ii) Since $l_2^* \in \mathcal{N}^*$, then $\frac{1}{\beta}(y_0 - v_2) \pm \frac{1}{2} \in \mathbf{Z}$. By Lemma 5.6, either $((u_1,0),-\sigma) \in \Gamma$ or $((u_1,\beta/2),-\sigma) \in \Gamma$.

If $((u_1,0),-\sigma) \in \Gamma$ and n is even then $u_2/\beta \in \mathbf{Z}$ and $l_2^* \in \mathcal{M}^*_{-\sigma}$. We are in case 1i) and condition a) follows. For n odd, let $m = \frac{y_0}{\beta} - \frac{v_2}{\beta} - \frac{1}{2} \in \mathbf{Z}$. Then $m(0,\beta) + (\alpha/2,\beta/2) = (\alpha/2, y_0 - v_2) \in \mathcal{L}$ and

$$\left(\left(\alpha/2, y_0 - v_2\right), I\right) \cdot ((0,v_2),\sigma) = \left(\left(\alpha/2, y_0\right), \sigma\right) \in \Gamma$$

and so

$$\left(\left(\alpha/2, y_0\right), \sigma\right)^n \cdot ((u_1,0),-\sigma) = ((2x_0,y_0),-I) \in \Gamma,$$

implying condition b).

If $u_2 = \beta/2$ and n is odd, then $l_2^* \in \mathcal{M}^*_{-\sigma}$ (case 1i) and condition a) follows. If n is even, let $m = \frac{y_0}{\beta} - \frac{v_2}{\beta} + \frac{1}{2} \in \mathbf{Z}$. Then $m(0,\beta) = (0, y_0 - v_2 + u_2) \in \mathcal{L}$ and $n(\frac{\alpha}{2},\frac{\beta}{2}) - \frac{n}{2}(0,\beta) = (2x_0 - u_1, 0) \in \mathcal{L}$ and so

$$((2x_0 - u_1, y_0 - v_2 + u_2), I) \cdot ((0,v_2),\sigma) \cdot ((u_1,u_2),-\sigma) = ((2x_0,y_0),-I) \in \Gamma,$$

implying condition b).

$\square$

6. Restriction of invariant functions and symmetry

6.1. **restriction.** We want to compare the projection of a function to its restriction to a line. In this section we obtain for the restriction results analogous to those obtained in sections 4 and 5 for the projection operator.

For each $c \in \mathbf{R}$, let Φ_c be the operator that maps $f(x,y)$ to its restriction to the horizontal line $\{(x,c) : x \in \mathbf{R}\}$ given by $\Phi_c(f)(x) = f(x,c)$. Let $\Phi_c(X_\Gamma)$ be the set of possible results when the operator Φ_c acts on all Γ-invariant functions.

If $f \in X_{\mathcal{L}}$ then, formally,

$$\Phi_c(f)(x) = \sum_{k \in \mathcal{L}^*} C(k)\omega_k(x,c) =$$

$$= \sum_{k\in\mathcal{L}^*} C(k)\omega_{k_1}(x)\omega_{k_2}(c) =$$

$$= \sum_{k_1\in\mathcal{L}_1^*} \omega_{k_1}(x) \sum_{k_2:(k_1,k_2)\in\mathcal{L}^*} C(k_1,k_2)\omega_{k_2}(c),$$

where $\mathcal{L}_1^* = \{k_1 : (k_1,k_2) \in \mathcal{L}^*\}$, as before.

Notice that for any function $f \in X_\Gamma$ its restriction, $\Phi_c(f)$, and its projection, $\Pi_{y_0}(f)$, have analogous formal Fourier series. The difference lies on the term $\omega_{k_2}(c)$ of the restriction, that corresponds in the projection to the integral $\int_0^{y_0} \omega_{k_2}(y)dy$. In this section we will present results concerning the restriction Φ_c that are similar to the ones proved throughout the previous sections for the projection Π_{y_0}. For the Propositions 6.1 and 6.2 we do not present the proof because it is analogous to the ones of, respectively, Propositions 4.1 and 5.1, with $\omega_{k_2}(c)$ instead of $\int_0^{y_0} \omega_{k_2}(y)dy$. The condition $\omega_{k_2}(c) = 0$ is never verified and so we don't have an analogous to the item $k_2y_0 \in \mathbf{Z} - \{0\}$ of the previous Propositions. Moreover, the expression

$$\int_0^{y_0} \omega_{k_2}(y)dy - \omega_{k_2}(y_0)\int_0^{y_0} \omega_{-k_2}(y)dy = 0$$

has the analogous

$$\omega_{k_2}(c) - \omega_{k_2}(2c)\omega_{-k_2}(c) = 0.$$

It follows that $2c$ appears where originally we had the variable y_0. As a consequence, the theorems of this section will be very similar to the ones already presented. The items with the condition $(0, y_0) \in \mathcal{L}$ disappear and y_0 is changed to $2c$ on other conditions.

6.2. periodicity of $\Phi_c(X_\Gamma)$.

Proposition 6.1. *All functions in $\Phi_c(X_\Gamma)$ have a common period $P \in \mathbf{R} - \{0\}$ if and only if, for each $k = (k_1, k_2) \in \mathcal{L}^*$, one of the following conditions holds:*

a) $k_1P \in \mathbf{Z}$,

b) $(v_\sigma, \sigma) \in \Gamma$ and $< k, \sigma v_\sigma + (0, 2c) > +\frac{1}{2} \in \mathbf{Z}$ with $\sigma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Lemma 6.1. *Let $P \neq 0$. If either $(P, 0) \in \mathcal{L}$ or $((P, 2c), \sigma) \in \Gamma$ then $\Phi_c(X_\Gamma)$ is a set of functions with period P.*

Proof: If $(P, 0) \in \mathcal{L}$ then for all $f \in X_\mathcal{L}$, $f(x, c) = f(x + P, c)$ and $\Phi_c(f)(x) = f(x, c) = f(x + P, c) = \Phi_c(f)(x + P)$.

If the second condition holds then, for all $f \in X_\Gamma$, $\Phi_c(f)(x) = f(x, c) = f(x + P, 2c - c) = f(x + P, c) = \Phi_c(f)(x + P)$. □

We will use the modules

$$\mathcal{M}_P^* = \{k \in \mathcal{L}^* : < k, (P, 0) > \in \mathbf{Z}\},$$

$$\mathcal{R}^* = \{k \in \mathcal{L}^* : < k, \sigma v_\sigma + (0,2c) > \in \mathbf{Z}\} \text{ and}$$
$$\overline{\mathcal{N}^*} = \mathcal{R}^* \cup \mathcal{N}^*,$$
where $\mathcal{N}^* = \left\{k \in \mathcal{L}^* : < k, \sigma v_\sigma + (0,2c) > +\frac{1}{2} \in \mathbf{Z}\right\}$.

Lemma 6.2. *If* $\mathcal{L}^* = \mathcal{M}_P^* \cup \mathcal{N}^*$ *then* $(2P, 0) \in \mathcal{L}$.

Proof: If $\mathcal{L}^* = \mathcal{M}_P^* \cup \mathcal{N}^*$ then $\mathcal{R}^* \subset \mathcal{M}_P^*$ and so $(P, 0) \in \mathcal{R}$. Besides, $2\overline{\mathcal{N}^*} \subset \mathcal{R}^*$ and, consequently, $(2P, 0) \in \overline{\mathcal{N}}$. Moreover, if $\mathcal{L}^* = \mathcal{M}_P^* \cup \mathcal{N}^*$ then either $\mathcal{L}^* = \mathcal{M}_P^*$ or $\mathcal{L}^* = \overline{\mathcal{N}^*}$. Therefore, either $\mathcal{L} = \mathcal{M}_P$ or $\mathcal{L} = \overline{\mathcal{N}}$ and the result follows. □

Theorem 6.1. *All functions in* $\Phi_c(X_\Gamma)$ *have a common period* $P \neq 0$ *if and only if one of the following conditions holds:*
a) $(P, 0) \in \mathcal{L}$,
b) $((P, 2c), \sigma) \in \Gamma$.

Proof: By Lemma 6.1, conditions a) and b) are sufficient. Conversely, if $\Phi_c(X_\Gamma)$ is a set of functions with some common period $P \neq 0$ then, by Proposition 6.1, one of the following conditions holds:
i) $\mathcal{L}^* = \mathcal{M}_P^*$,
ii) $(v_\sigma, \sigma) \in \Gamma$ and $\mathcal{L}^* = \mathcal{M}_P^* \cup \mathcal{N}^*$.
Case i) is equivalent to $\mathcal{L} = \mathcal{M}_P = \mathcal{L} + \{(P, 0)\}_{\mathbf{Z}}$. Therefore $(P, 0) \in \mathcal{L}$, *i.e.*, condition a).
If ii) happens then, by Lemmas 4.6, 4.7, **??** and 6.2, the lattice $\mathcal{L}^*$ has one of the forms below:

$$\mathcal{L}^* = \left\{\left(\frac{n}{2P}, 0\right), \left(0, \frac{1}{\beta}\right)\right\}_{\mathbf{Z}},$$
$$\mathcal{L}^* = \left\{\left(\frac{n}{P}, 0\right), \left(-\frac{n}{2P}, \frac{1}{\beta}\right)\right\}_{\mathbf{Z}},$$

where n is the greatest integer such that $\left(\frac{2P}{n}, 0\right) \in \mathcal{L}$. Now we use the proof of Theorem 4.1 where $\mathcal{M}_P^*$ is calculated and the set $\mathcal{L}^* - \mathcal{M}_P^*$ is constrained to be a subset of $\mathcal{M}_{y_0}^* \cup \mathcal{N}^*$ in order to restrict v_σ. The only changes are the use of $2c$ instead of y_0 and the elimination of the conditions related to $\mathcal{M}_{y_0}^*$. Therefore the result is proved. □

6.3. **parity of $\Phi_c(X_\Gamma)$.**

Proposition 6.2. *All functions in* $\Phi_c(X_\Gamma)$ *are invariant for the reflection in the point* x_0 *if and only if one of the conditions holds:*
a) $(v_{-\sigma}, -\sigma) \in \Gamma$ *and* $< k, -\sigma v_{-\sigma} + (2x_0, 0) > \in \mathbf{Z}$ *for all* $k \in \mathcal{L}^*$,
b) $(v_{-I}, -I) \in \Gamma$ *such that* $< k, -v_{-I} + (2x_0, 2c) > \in \mathbf{Z}$ *for all* $k \in \mathcal{L}^*$,
c) both $(v_{-\sigma}, -\sigma)$ *and* (v_σ, σ) *belong to* Γ *and for each* $k \in \mathcal{L}^*$ *either* $< k, -\sigma v_{-\sigma} + (2x_0, 0) > \in \mathbf{Z}$ *or* $< k, \sigma v_\sigma + (0, 2c) > +\frac{1}{2} \in \mathbf{Z}$.

Lemma 6.3. *If one of the following conditions holds then $\Phi_c(X_\Gamma)$ is a set of invariant functions under reflection in the point x_0,*
a) $((2x_0, 0), -\sigma) \in \Gamma$,
b) $((2x_0, 2c), -I) \in \Gamma$.

Proof: A function $f : \mathbf{R} \longrightarrow \mathbf{R}$ is invariant under reflections around x_0 if $f(x) = f(2x_0 - x)$ for all $x \in \mathbf{R}$. If a) holds then for all $f \in X_\Gamma$, $f(x, c) = f(2x_0 - x, c)$ and $\Phi_c(f)(x) = f(x, c) = f(2x_0 - x, c) = \Phi_c(f)(2x_0 - x)$.
Similarly, if b) holds then $\Phi_c(f)(x) = f(2x_0 - x, 2c - c) = f(2x_0 - x, c) = \Phi_c(f)(2x_0 - x)$. □

Theorem 6.2. *All functions in $\Phi_c(X_\Gamma)$ are invariant for the reflection in the point x_0 if and only if one of the following conditions holds:*
a) $((2x_0, 0), -\sigma) \in \Gamma$,
b) $((2x_0, 2c), -I) \in \Gamma$.

Proof: Conditions a) and b) are sufficient by Lemma 6.3. Conversely, if $\Phi_c(X_\Gamma)$ is a set of invariant functions under a reflection in the point x_0 then, by Proposition 6.2, we have one of the following:
i) $(v_{-\sigma}, -\sigma) \in \Gamma$ and $\mathcal{L} = \mathcal{L} + \{-\sigma v_{-\sigma} + (2x_0, 0)\}_{\mathbf{Z}}$, where $v_{-\sigma} = (v_1, v_2)$ and $((0, -2v_2), I) \in \Gamma$. Therefore

$$((2x_0, 0), -\sigma) = ((0, -2v_2), I) \cdot ((2x_0 - v_1, v_2), I) \cdot ((v_1, v_2), -\sigma) \in \Gamma.$$

ii) $(v_{-I}, -I) \in \Gamma$ and $\mathcal{L} = \mathcal{L} + \{-v_{-I} + (2x_0, 2c)\}_{\mathbf{Z}}$, where $v_{-I} = (v_1, v_2)$. Therefore,

$$((2x_0, 2c), -I) = ((2x_0 - v_1, 2c - v_2), I) \cdot ((v_1, v_2), -I) \in \Gamma.$$

iii) both $(v_{-\sigma}, -\sigma)$ and (v_σ, σ) belong to Γ and

$$\mathcal{L}^* = \mathcal{M}^*_{-\sigma} \cup \mathcal{N}^*$$

where $\mathcal{M}^*_{-\sigma} = \{k \in \mathcal{L}^* : < k, -\sigma v_{-\sigma} + (2x_0, 0) >\in \mathbf{Z}\}$. We prove this case with Lemmas 5.8 and 5.9 considering that $(0, y_0) \notin \mathcal{L}$ and that $\mathcal{L}^*$ can be written as the union of the two sets $\mathcal{M}^*_{-\sigma}$ and $\mathcal{N}^*$. Thus, the result follows using $2c$ instead of y_0 in the definition of $\mathcal{N}^*$. □

7. Characterization of wallpaper groups

In this section we summarize the results proved in the previous sections with a different point of view. Our aim is to emphasize the relation between the wallpaper group fixing the space of functions X_Γ and the group of symmetries of the projected or restricted functions, *i.e.* the group that fixes $\Pi_{y_0}(X_\Gamma)$ or $\Phi_c(X_\Gamma)$. The results of sections 4 - 6 are easier to use if one knows the symmetry of $\Pi_{y_0}(X_\Gamma)$ or $\Phi_c(X_\Gamma)$

and wants to find the original symmetry group, Γ. Here we want to predict the symmetry, given Γ.

Let Γ be a wallpaper group with lattice $\mathcal{L}$ and let X_Γ be the space of Γ-invariant functions $f : \mathbf{R}^2 \longrightarrow \mathbf{R}$. Suppose we apply to these functions one of the operators Π_{y_0} or Φ_c. We call $\Sigma(y_0)$ and $\Sigma(c)$ the subgroups of $\mathbf{E}(1)$ that fix, respectivelly, the space $\Pi_{y_0}(X_\Gamma)$ and $\Phi_c(X_\Gamma)$. These groups depend on the group Γ, in particular on its lattice $\mathcal{L}$, and on the variables y_0 and c.

The next tables are built to be read from left to right, meaning that we begin to state the existence of some element in Γ and then use the tables to find information on $\Sigma(y_0)$ or $\Sigma(c)$. Recall from section 2 that the elements of Σ are either translactions, $(a, 1)$, or reflections, $(a, -1)$, with $a \in \mathbf{R}$.

Conditions that do not apply to the restriction operator are indicated by "n.a." (not applicable) in the tables. In many cases the hypothesis restricts both the lattice and the wallpaper groups. We use the standard notation for these groups - a complete description of this notation can be obtained in [1] and on pages 156 and 157 there is a helpful illustration of the seventeen wallpaper groups.

Notice that if $((a,b),I) \in \Gamma$ is a generator then (a,b) is the smallest element of $\mathcal{L}$ in its direction. When we say that $((a,b),I)$ and $((c,d),I)$ are generators we mean $\mathcal{L} = \{(a,b),(c,d)\}_{\mathbf{Z}}$.

We will use the notation $\sigma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ along this section. Let $\det \delta = -1$. If $\delta v = -v$ then the element (v, δ) is a *reflection*, otherwise (v,δ) is a *glide reflection.* Notice that, since $\delta \in \mathbf{O}(2)$, when $\det \delta = -1$ then $\delta^2 = I$.

7.1. **all wallpaper groups.**

original group Γ contains	validity set y_0	validity set c	image group Σ contains
$((P,0),I)$	$\mathbf{R}^+$	$\mathbf{R}$	$(P,1)$
$((P,a),I)$ and $((0,\beta),I)$ generator	$\{n\beta : n \in \mathbf{N}\}$	*n.a.*	$(P,1)$

These results are given by the theorems of periodicity in sections 4 and 6.

By definition, a wallpaper group has a lattice as translation subgroup. Therefore conditions $((a,0),I) \in \Gamma$ or $((0,a),I) \in \Gamma$ are only restrictions on the position of the lattice. We discuss this in section 8, below.

7.2. **wallpaper groups with a glide reflection.**

<table>
<tr><th rowspan="2">original group Γ contains</th><th colspan="2">validity set</th><th rowspan="2">image group Σ contains</th></tr>
<tr><th>y_0</th><th>c</th></tr>
<tr><td rowspan="2">$((P,a),\sigma)$,
$((2P,0),I)$ and
$((0,\beta),I)$ generators</td><td>$\{a+n\beta : n\in \mathbf{N_0}\}$</td><td>$\left\{\frac{a}{2}+n\frac{\beta}{2} : n\in \mathbf{Z}\right\}$</td><td>$(P,1)$</td></tr>
<tr><td>$\mathbf{R}^+$</td><td>$\mathbf{R}$</td><td>$(2P,1)$</td></tr>
<tr><td rowspan="3">$((P,a),\sigma)$,
$((2P,0),I)$ and
$\left(\left(P,\frac{\beta}{2}\right),I\right)$ generators</td><td>$\{a+n\beta : n\in \mathbf{N_0}\}$</td><td>$\left\{\frac{a}{2}+n\frac{\beta}{2} : n\in \mathbf{Z}\right\}$</td><td>$(P,1)$</td></tr>
<tr><td>$\mathbf{R}^+$</td><td>$\mathbf{R}$</td><td>$(2P,1)$</td></tr>
<tr><td>$\{n\beta : n\in \mathbf{N}\}$</td><td>n.a.</td><td>$(P,1)$</td></tr>
<tr><td rowspan="2">$\left(\left(2x_0,\frac{\beta}{2}\right),-\sigma\right)$,
$((P,0),I)$ and
$((0,\beta),I)$ generators</td><td>$\{n\beta : n\in \mathbf{N}\}$</td><td>n.a.</td><td>$(2x_0,-1)$</td></tr>
<tr><td>$\mathbf{R}^+$</td><td>$\mathbf{R}$</td><td>$(P,1)$</td></tr>
<tr><td rowspan="4">$\left(\left(2x_0,\frac{\beta}{2}\right),-\sigma\right)$,
$((0,\beta),I)$ and
$\left(\left(P,\frac{\beta}{2}\right),I\right)$ generators,
$((2x_0+P,0),-\sigma)$</td><td>$\{n\beta : n\in \mathbf{N}\}$</td><td>n.a.</td><td>$(2x_0,-1)$</td></tr>
<tr><td>$\mathbf{R}^+$</td><td>$\mathbf{R}$</td><td>$(2P,1)$</td></tr>
<tr><td>$\{n\beta : n\in \mathbf{N}\}$</td><td>n.a.</td><td>$(P,1)$</td></tr>
<tr><td>$\mathbf{R}^+$</td><td>$\mathbf{R}$</td><td>$(2x_0+P,-1)$</td></tr>
</table>

The results in this table are given by the theorems about periodicity and parity of the sections 4 to 6, together with some considerations about the way a glide reflection restricts the lattice and the corresponding wallpaper groups.

First we explain the two cases where $((P,a),\sigma)\in\Gamma$. Suppose $((Q,a),\sigma)\in\Gamma$, thus $((Q,a),\sigma)^2=((2Q,0),I)\in\Gamma$ and there is some $m\in\mathbf{N}$ such that $\left(\left(\frac{2Q}{m},0\right),I\right)$ is a generator. If m is even, $m=2n$, then $\left(\left(\frac{Q}{n},0\right),I\right)^n=((Q,0),I)\in\Gamma$ and, up to elements of $\mathcal{L}$, the element $((Q,a),\sigma)$ is a reflection and not a glide. We treat this case separately in the next subsection. Now suppose m is odd. Therefore

$$\left(\left(\frac{Q}{m},a\right),\sigma\right)=\left(\left(-\frac{2Q}{m},0\right),I\right)^{\frac{m-1}{2}}\cdot((Q,a),\sigma)\in\Gamma$$

and so, for $P=\frac{Q}{m}$, there exists an element of the form $((P,a),\sigma)$ in Γ and $((2P,0),I)$ is a generator.

By Lemma 4.6 we know that the missing generator is either $((0,\beta),I)$ or $\left(\left(P,\frac{\beta}{2}\right),I\right)$. The first case corresponds to wallpaper groups with rectangular or square lattices and, at least, a glide. These are the groups pg, p2mg, p2gg and p4gm. Notice that the group p4mm also has a glide but not on this particular relative position to the lattice.

In the second case we have either a centred rectangular or an hexagonal or a square lattice and, at least, one glide reflection. The wallpaper groups with these characteristics are cm, c2mm, p4mm, p4gm, p3m1, p31m and p6mm.

Now we explain the cases where $\left(\left(2x_0,\frac{\beta}{2}\right),-\sigma\right)\in\Gamma$. By a method analogous to the previous one, we can show that $((0,\beta),I)$ generates the lattice together with either $((P,0),I)$ or $\left(\left(P,\frac{\beta}{2}\right),I\right)$. In the first case the result presented on the table follows by the theorems already mentioned. If the elements $(2x_0,-1)$ and $(P,1)$ belong to Σ then $(2x_0+mP,-1)\in\Sigma$ for all $m\in\mathbf{Z}$.

The last case of the table is somewhat more complex. In the wallpaper groups with centred rectangular or hexagonal lattices, reflections and glide reflections arise together on parallel lines. This is also valid for wallpaper groups with square lattices, but only for those glide reflections and reflections on lines parallel to the diagonal of the square. When these lines are horizontal, *i.e.* if σ is the orthogonal component, only the glide is relevant for our study. For $-\sigma$, as can be see on Theorem 5.1, both the glide reflections and the reflections play an important role. Formally, if $\left(\left(2x_0,\frac{\beta}{2}\right),-\sigma\right)\in\Gamma$ and $\mathcal{L}=\left\{(0,\beta),\left(P,\frac{\beta}{2}\right)\right\}_{\mathbf{Z}}$ then

$$((2x_0+P,0),-\sigma)=\left(\left(2x_0,\frac{\beta}{2}\right),-\sigma\right)\cdot\left(\left(P,\frac{\beta}{2}\right),I\right)^{-1}$$

also belongs to Γ.

As in the previous case, by the existence of the period $(2P,1)$, the element $(2x_0+(2m+1)P,-1)$ belongs to Σ for all $m\in\mathbf{Z}$.

7.3. wallpaper groups with a reflection.

original group Γ contains	validity set y_0	validity set c	image group Σ contains
$((2x_0,0),-\sigma)$, $((P,0),I)$ and $((0,\beta),I)$ generators	$\mathbf{R}^+$	$\mathbf{R}$	$(2x_0,-1)$
	$\mathbf{R}^+$	$\mathbf{R}$	$(P,1)$

As explained above, a reflection restricts the lattice associated to the wallpaper group. This table presents the case where the lattice is rectangular or square and is relevant for the wallpaper groups having one of these lattices and a reflection: pm, p2mm, p2mg and p4mm. The group p4gm has a glide but with a different direction. The existence of the element $((2x_0',0),-\sigma)$ associated to centred rectangular, hexagonal or rotated square lattices are presented on the previous table with $2x_0'=2x_0+P$.

7.4. wallpaper groups with a rotation of order two.

original group Γ contains	validity set y_0	validity set c	image group Σ contains
$((2x_0,a),-I)$	a	$\frac{a}{2}$	$(2x_0,-1)$
$((2x_0,a),-I)$ and $((0,\beta),I)$ generator	a	$\frac{a}{2}$	$(2x_0,-1)$
	$\{n\beta:n\in\mathbf{N}\}$	n.a.	$(2x_0,-1)$
	$\{n\beta:n\in\mathbf{N}\}$	n.a.	$(P,1)$

These cases are relevant whenever the wallpaper group has a rotation of order two. The ones with this property are p2, p2mm, p2mg, p2gg, c2mm, p4, p4mm, p4gm, p6 and p6mm.

Notice that in the first case of the table the projected function is not periodic. This is the only result where the projection of a Γ-invariant function, where Γ is a wallpaper group, exhibits some symmetry without being invariant under a group with a lattice in $\mathbf{R}$. All the other results consist of Γ-invariant functions whose projection has the symmetry of a group that is an analogue, in $\mathbf{R}$, to the crystalographic groups.

8. Equivariant approach

Suppose we have a model whose solutions have domain $\mathbf{R}$ but are a projection of Γ-invariant solutions of a $\mathbf{E}(2)$-equivariant system, where Γ is a wallpaper group. We study some properties of this model using equivariant theory.

8.1. **$\mathbf{E}(2)$-equivariance.** Let X be a space of functions $f : \mathbf{R}^2 \longrightarrow \mathbf{R}$ and let $\mathcal{P} : X \times \mathbf{R} \longrightarrow X$ be a $\mathbf{E}(2)$-equivariant operator.

Suppose $\mathcal{P}(f_0, \lambda) = 0$ and f_0 is a Γ-invariant function where Γ is a wallpaper group with lattice $\mathcal{L}$. Thus, (see section 2.4), for all $\gamma \in \mathbf{E}(2)$, the functions $\gamma \cdot f_0$ are also solutions of $\mathcal{P} = 0$. By Proposition 2.1, the solution $\gamma \cdot f_0 = (v, \delta) \cdot f_0$ is a $\delta \cdot \mathcal{L}$-periodic function. Therefore the orbit of solutions $\mathbf{E}(2) \cdot f_0 = \{\gamma \cdot f_0 : \gamma \in \mathbf{E}(2)\}$ induces the orbit $\mathbf{O}(2) \cdot \mathcal{L} = \{\delta \cdot \mathcal{L} : \delta \in \mathbf{O}(2)\}$ in the space of all the possible lattices for wallpaper groups.

The results presented in this article are about spaces of functions $X_{\mathcal{L}}$ and their subspaces X_Γ. If the given functions are solutions of a $\mathbf{E}(2)$-equivariant problem then we should look at the orbits $\mathbf{E}(2) \cdot X_{\mathcal{L}} = \{X_{\delta \cdot \mathcal{L}} : \delta \in \mathbf{O}(2)\}$, instead of a single space $X_{\mathcal{L}}$.

8.2. **projecting orbits of solutions.** Now suppose we project the solutions of $\mathcal{P} = 0$. Instead of working with $\Pi_{y_0}(X_{\mathcal{L}})$ we will study the set

$$\{\Pi_{y_0}(X_{\delta \cdot \mathcal{L}}) : \delta \in \mathbf{O}(2)\}$$

and the results concerning the lattice $\mathcal{L}$ must be reformulated for the orbit $\mathbf{O}(2) \cdot \mathcal{L}$. This is a set whose elements are all the orthogonal rotations and reflections of the lattice $\mathcal{L}$.

Let l be any element of $\mathcal{L}$. We can rotate $\mathcal{L}$ in order to make l an horizontal vector or, formally,

$$\forall l \in \mathcal{L}\ \exists \delta \in \mathbf{O}(2) : (\|l\|, 0) \in \delta \cdot \mathcal{L},$$

where $\|l\|$ is the usual norm of $\mathbf{R}^2$. Therefore, by Theorem 4.1, for all $l \in \mathcal{L}$ there is some $\delta \in \mathbf{O}(2)$ such that $\Pi_{y_0}(X_{\delta \cdot \mathcal{L}})$ is a set of functions with period $\|l\|$. It follows the result:

Corollary 8.1. *A $\mathcal{L}$-periodic solution f_0 of $\mathcal{P} = 0$ belongs, after a projection Π_{y_0}, to an orbit of solutions*

$$\{\Pi_{y_0}(\gamma \cdot f_0) : \gamma \in \mathbf{E}(2)\}$$

where we can find a function with period $\|l\|$ for all $l \in \mathcal{L}$.

This result does not depend on the variable y_0 and is also valid for the set $\{\Phi_c(\gamma \cdot f_0) : \gamma \in \mathbf{E}(2)\}$ and for all $c \in \mathbf{R}$.

8.3. **$\mathbf{E}(1)$-equivariance.** When looking for solutions of a $\mathbf{E}(1)$-equivariant system $\tilde{\mathcal{P}} = 0$, with $\tilde{\mathcal{P}} : Y \times \mathbf{R} \longrightarrow Y$ and Y a space of functions $g : \mathbf{R} \longrightarrow \mathbf{R}$, we do not expect a result such as Corollary 8.1.

If we have a solution g_0 with smallest period P, this means $\{P\}_{\mathbf{Z}}$ is the translation subgroup of the group Σ that leaves g_0 invariant. Thus, by Proposition 2.1, the solutions on the orbit $\mathbf{E}(1) \cdot g_0$ have translation subgroups $\delta \cdot \{P\}_{\mathbf{Z}} = \{P\}_{\mathbf{Z}}$, because $\delta \in \{1, -1\} = \mathbf{O}(1)$. Therefore, on a $\mathbf{E}(1)$-equivariant approach all the solutions on an orbit have the same periods.

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E. M. Pinho — Departamento de Matemática Aplicada da Universidade do Porto, Rua do Campo Alegre, 687, 4169-007 Porto, Portugal
E-mail address: eopinho@fc.up.pt

Isabel S. Labouriau — Centro de Matemática da Universidade do Porto, Rua do Campo Alegre, 687, 4169-007 Porto, Portugal
E-mail address: islabour@fc.up.pt