STOCHASTIC STABILITY AT THE BOUNDARY OF EXPANDING MAPS

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Dedicated to C. Gutierrez on the occasion of his 60th birthday.

ABSTRACT. We consider endomorphisms of a compact manifold which are expanding except for a finite number of points and prove the existence and uniqueness of a physical measure and its stochastical stability. We also characterize the zero-noise limit measures for a model of the intermittent map and obtain stochastic stability for some values of the parameter. The physical measures are obtained as zero-noise limits which are shown to satisfy Pesin´s Entropy Formula.

CONTENTS

1. INTRODUCTION

After the long and deep developments in the last decades on the structural stability theory of dynamical systems, we know that this form of stability is too strong to be a generic property. Recently there has been some emphasis on the study of stochastic stability of dynamical systems, among other forms of stability.

On the one hand, one of the challenging problems of smooth Ergodic Theory is to prove the existence of "nice" invariant measures called physical measures or sometimes SRB (Sinai-Ruelle-Bowen) measures. On the other hand, a natural formulation of stochastic stability of dynamical systems assumes the existence of physical measures. However, the characterization of zero-noise limit measures involved in the study of stochastic stability may provide ways to

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construct physical measures. In this work the study of zero-noise limit measures for endomorphisms which are expanding except at a finite number of points yields a construction of physical measures and also their stochastic stability.

Let M be a compact and connected Riemannian manifold and $\mathcal{T} := C^{1+\alpha}(M,M)$ be the space of $C^{1+\alpha}$ maps of M where $\alpha > 0$. We write m for some fixed measure induced by a normalized volume form on M that we call *Lebesgue measure*, dist for the Riemannian distance on M and $\|\cdot\|$ for the induced Riemannian norm on TM.

We recall that an invariant probability measure μ for a transformation $T : M \to M$ on a manifold M is *physical* if the *ergodic basin*

$$B(\mu) = \left\{ x \in M : \frac{1}{n}\sum_{j=0}^{n-1} \varphi(T^j(x)) \to \int \varphi \, d\mu \text{ for all continuous } \varphi : M \to \mathbb{R} \right\}$$

has positive Lebesgue measure.

Let $(\theta_\varepsilon)_{\varepsilon>0}$ be a family of Borel probability measures on $(\mathcal{T}, \mathcal{B}(\mathcal{T}))$, where we write $\mathcal{B}(X)$ the Borel $\sigma-$algebra of a topological space X. We are dealing with random dynamical systems generated by independent and identically distributed maps of $\mathcal{T}$ and θ_ε will be the common probability distribution when choosing the maps to generate random dynamics.

We say that a probability measure μ^ε on M is *stationary for the random system* $(\hat{T}, \theta_\varepsilon)$ if the following holds

$$\int\int \varphi(T(x)) \, d\mu^\varepsilon(x) d\theta_\varepsilon(T) = \int \varphi \, d\mu^\varepsilon \quad \text{for all continuous } \varphi : M \to \mathbb{R}. \tag{1.1}$$

We assume that the support of θ_ε shrinks to T when $\varepsilon \to 0$ in a suitable topology. A classical result in random dynamical systems (see [14] or [4]) implies that *every weak* accumulation point of the stationary measures* $(\mu^\varepsilon)_{\varepsilon>0}$ *when* $\varepsilon \to 0$ *is a* T*-invariant probability measure,* which is called a *zero-noise limit measure.* This naturally leads to the study of the kind of zero noise limits that can arise and to the notion of stochastic stability.

Definition 1. *A map T is stochastically stable (under the random perturbation $(\hat{T}, \theta_\varepsilon)_{\varepsilon>0}$) if every accumulation point μ of the family of stationary measures $(\mu^\varepsilon)_{\varepsilon>0}$, when $\varepsilon \to 0$, is a linear convex combination of the physical measures of T.*

Uniformly expanding maps and uniformly hyperbolic systems are known to be stochastically stable [13, 14, 28, 29]. Some non-uniformly hyperbolic systems, like quadratic maps, Hénon maps and Viana maps, were shown to be stochastically stable much more recently [3, 6, 7]. These systems either exhibit expansion/contraction everywhere or are expanding/contracting away from a critical region with slow recurrence rate to it. This allows for a probabilistic argument which shows that the visits to a neighborhood of the critical region are negligible on the average, and also that this behavior persists under small random perturbations.

It is not obvious how to apply the standard techniques to systems whose typical orbits do not have a slow recurrence rate of visits to the non-hyperbolic regions. This is the case of *intermittent maps* [20]. These applications are expanding, except at a neutral fixed point. The local behavior near this neutral point is responsible for various phenomena.

Consider $\alpha > 0$ and the map $T : [0,1] \to [0,1]$ defined as follows

$$T(x) = \begin{cases} x + 2^{\alpha} x^{1+\alpha} & x \in [0, \frac{1}{2}) \\ x - 2^{\alpha}(1-x)^{1+\alpha} & x \in [\frac{1}{2}, 1] \end{cases} \tag{1.2}$$

This map defines a $C^{1+\alpha}$ map of the unit circle $\mathbb{S}$ into itself. The unique fixed point is 0 and $DT(0) = 1$. The above family of maps provides many interesting results in Ergodic Theory. If $\alpha \geq 1$, i.e if the order of tangency at zero is high enough, then the Dirac mass at zero δ_0 is the unique physical probability measure and so the Lyapunov exponent of Lebesgue almost all points vanishes [27]. The situation is completely different for $0 < \alpha < 1$: in this case there exists a unique absolutely continuous invariant probability measure μ_{SRB}, which is therefore a physical measure and whose basin has full Lebesgue measure [26].

Another point of interest is that these maps provide examples of dynamical systems with polynomial decay of correlations. M. Holland has obtained even sub-polynomial rate of mixing modifying these intermittent maps [11]. In particular μ_{SRB} is always mixing, when it exists.

1.1. **Statement of the results.** We consider additive noise applied to a map T of $\mathbb{S}$ with an indifferent fixed point at 0 and expanding everywhere else, as in example (1.2). Let $\alpha > 0$ be fixed and consider $T_t := T + t$ for $|t| \leq \varepsilon$. Then $\hat{T} : [-1/2, 1/2] \to C^{1+\alpha}(\mathbb{S}, \mathbb{S}), t \mapsto T_t$ is a (smooth) family of $C^{1+\alpha}$ maps of $\mathbb{S}$.

Let θ_ε be an absolutely continuous probability measure, with respect to the Lebesgue measure m on $\mathbb{S}$, whose support is contained in $[-\varepsilon, \varepsilon]$ (e.g. $\theta_\varepsilon = (2\varepsilon)^{-1} m \mid [-\varepsilon, \varepsilon], \varepsilon > 0$). This naturally induces a probability measure on $\mathcal{T} = \{T_t, t \in [-1/2, 1/2]\}$ which we denote by the same symbol θ_ε (the meaning being clear from the context).

In this setting it is well known that there always exist a stationary probability measure μ^ε for all $\varepsilon > 0$. Moreover this measure is ergodic and is the unique absolutely continuous stationary measure for $(\hat{T}, \theta_\varepsilon)$ (see Subsection 2.2).

Let us fix now $\alpha \in (0,1)$ and let

$$\mathbb{E} = \{t\delta_0 + (1-t)\mu_{SRB} : 0 \leq t \leq 1\}$$

be the set of linear convex combinations of the Dirac mass at 0 with the unique absolutely continuous invariant probability measure for these maps.

Theorem A. *Let μ_0 be any accumulation point of the stationary measures $(\mu^\varepsilon)_{\varepsilon>0}$ when $\varepsilon \to 0$ for the random perturbation $(\hat{T}, \theta_\varepsilon)_{\varepsilon>0}$ with $\alpha \in (0,1)$. Then $\mu_0 \in \mathbb{E}$.*

In the case $\alpha \geq 1$ there does not exist any absolutely continuous invariant probability measure. However the Dirac measure δ_0 is the unique physical measure. In this case we are able to obtain stochastic stability.

Theorem B. *Let $\alpha \geq 1$ in (1.2) and let $(\mu^\varepsilon)_{\varepsilon>0}$ be the family of stationary measures for the random perturbation $(\hat{T}, \theta_\varepsilon)_{\varepsilon>0}$. Then $\mu^\varepsilon \to \delta_0$ when $\varepsilon \to 0$ in the weak* topology.*

However, taking a different family f_t unfolding the saddle-node at 0, e.g.

$$f_t(x) = \begin{cases} tx + 2^{\alpha}(2-t)x^{1+\alpha} & x \in [0, \frac{1}{2}) \\ 1 - t(1-x) - 2^{\alpha}(2-t)(1-x)^{1+\alpha} & x \in [\frac{1}{2}, 1] \end{cases} \tag{1.3}$$

with $\alpha \in (0,1)$ we obtain an example of *non-stochastic stability*. In fact, since $f_t'(0) = t$ then for $t < 1$ the fixed point 0 is a sink for f_t (see Figure 1) and we prove that the physical measure for the random system is always δ_0 for restricted choices of the probability measures θ_ε.

Theorem C. *For every small enough $\varepsilon > 0$ there are $a(\varepsilon) < b(\varepsilon) < 1$ such that $a(\varepsilon) \to 1$ when $\varepsilon \to 0$ and, for any given probability measure θ_ε supported in $[a(\varepsilon), b(\varepsilon)]$, the unique stationary measure μ^ε for the random system equals δ_0.*

Since $f_1 = T$ admits an absolutely continuous invariant measure μ_{SRB} and clearly δ_0 cannot converge to this physical measure, we have an example of a stochastically unstable system (under this kind of perturbations).

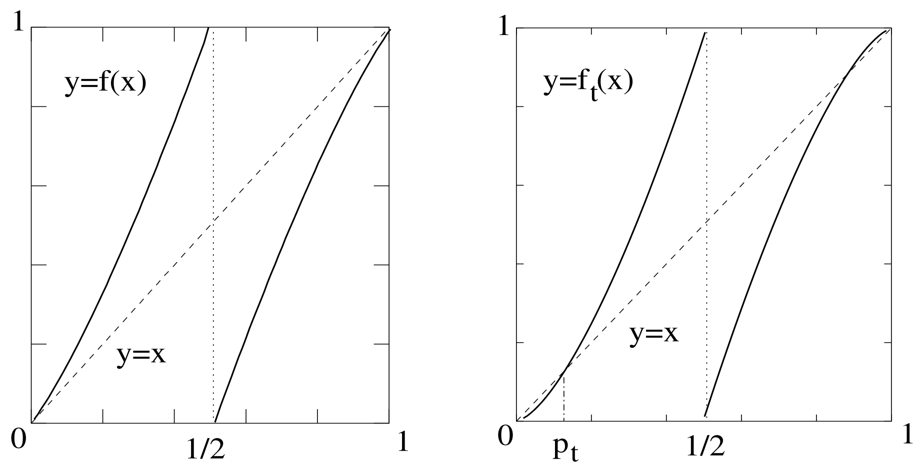

FIGURE 1. The map $f = T$ (left) and the map f_t for $0 < t < 1$ (right).

Using the same kind of additive perturbations considered in Theorems A and B, our methods provide the following results for maps in higher dimensions.

Theorem D. *Let $f : M \to M$ be a $C^{1+\alpha}$ local diffeomorphism such that*

1. $\|Df(x)^{-1}\| \leq 1$ *for all $x \in M$;*
2. $K = \{x \in M : \|Df(x)^{-1}\| = 1\}$ *is finite and $|\det Df(x)| > 1$ for every $x \in K$.*

Then, for any non-degenerate random perturbation $(\hat{f}, \theta_\varepsilon)_{\varepsilon>0}$, there exists a unique ergodic stationary probability measure μ^ε for all $\varepsilon > 0$. Moreover μ^ε converges, in the weak* topology when $\varepsilon \to 0$, to a unique absolutely continuous f-invariant probability measure μ_0 whose basin has full Lebesgue measure, and f is stochastically stable.

Here we will assume that M is a n-dimensional torus since the maps f satisfying the conditions on Theorem D are at the boundary of expanding maps, which can only exist on special manifolds [25, 10], the best known example being the tori. Since these manifolds are parallelizable, we can define additive perturbations just as we did on the circle. If $\mathbb{T}^n$ is a n-dimensional torus, then $T\mathbb{T}^n \simeq \mathbb{R}^n$ and $\hat{f} : B \subset \mathbb{R}^n \to C^{1+\alpha}(M,M), v \to f + v$, where B is a ball around the origin of $\mathbb{R}^n$ (together with a family $(\theta_\varepsilon)_{\varepsilon>0}$ of absolutely continuous probability measures on B, see

Subsection 2.3 for the definition of non-degenerate random perturbation) will be a the kind of additive perturbation we will consider.

These results will be derived from the following more technical one, but also interesting in itself.

Theorem E. *Let $f : M \to M$ be a $C^{1+\alpha}$ local diffeomorphism such that*

1. $\|Df(x)^{-1}\| \le 1$ *for all* $x \in M$*;*
2. $K = \{x \in M : \|Df(x)^{-1}\| = 1\}$ *is finite.*

Then, for any non-degenerate random perturbation $(\hat{f}, \theta_\varepsilon)_{\varepsilon>0}$, every weak* accumulation point μ of the sequence $(\mu^\varepsilon)_{\varepsilon>0}$, when $\varepsilon \to 0$, is an equilibrium state for the potential $-\log|\det Df(x)|$, i.e.

$$h_\mu(f) = \int \log|\det Df(x)|\, d\mu(x). \tag{1.4}$$

Moreover every equilibrium state μ as above is a convex linear combination of an absolutely continuous invariant probability measure with finitely many Dirac measures concentrated on periodic orbits whose Jacobian equals 1.

Cowieson and Young have presented results similar to ours for C^2 or C^∞ diffeomorphisms. However their assumptions are on the convergence of the sum of the positive Lyapunov exponents for the random maps to the same sum for the original map, and they obtain *SRB measures*, not necessarily physical ones, see [9] for more details. We make much stronger assumptions on both the kind of maps being perturbed (expanding except at finitely many points) and the kind of perturbations used (additive besides being absolutely continuous), and we obtain physical measures for $C^{1+\alpha}$ endomorphisms.

In what follows, we first present some examples of applications and then general results about random dynamical systems (Section 2) to be used to prove Theorem E (Section 3). At this point we are ready to obtain Theorem D (Section 4). Finally we apply the ideas to the specific case of the intermittent maps (Section 5), completing the proof of Theorems A, B and C.

1.2. **Examples.** In what follows we write $\mathbb{T}$ for $\mathbb{S} \times \mathbb{S}$ and consider $\mathbb{S} = [0,1]/\{0 \sim 1\}$. We always assume that these spaces are endowed with the metrics induced by the standard Euclidean metric through the identifications. The Lebesgue measure on these spaces will be denoted by m (area) on $\mathbb{T}$ and m_1 (length) on $\mathbb{S}$.

An extra example is the intermittent map itself, dealt with in Section 5.

1.2.1. *Direct product "intermittent×expanding".* Let $f : \mathbb{T} \to \mathbb{T}, (x,y) \mapsto (T_\alpha(x), g(y))$, where T_α is defined at the Introduction with $\alpha > 0$, and $g : \mathbb{S} \to \mathbb{S}$ is $C^{1+\alpha}$, admits a fixed point $g(0) = 0$ and $g' = Dg > 1$.

Since f is a direct product, if $\alpha \in (0,1)$, then f admits an invariant probability measure $\nu = \mu_\alpha \times \lambda$, where μ_α is the unique absolutely continuous invariant measure for T_α and λ is the unique absolutely continuous invariant measure for g, i.e., $\mu_\alpha \ll m_1$ and $\lambda \ll m_1$. Hence the product measure is absolutely continuous: $\nu \ll m = m_1 \times m_1$. These measures are ergodic and also mixing, and the basins of μ_α and λ equal $\mathbb{S}, m_1$ mod 0. Thus their direct product ν is ergodic and so $B(\nu) = \mathbb{T}, m$ mod 0.

If $\alpha \geq 1$, then $\nu = \delta_0 \times \lambda$ is again an ergodic invariant probability measure for f with $B(\nu) = \mathbb{T}, m$ mod 0, since λ is the same as before and so is mixing for g, and δ_0 is T_α-ergodic, with the basin of both measures equal to $\mathbb{S}$.

Here $K = \{0\} \times \mathbb{S}$ (the definition of K is given at the statement of Theorem D) is not finite, and the conclusion of Theorem D does not hold when $\alpha \geq 1$: we have a physical measure which is not absolutely continuous with respect to m. Note that clearly $\|(Df)^{-1}\| \leq 1$ everywhere and since K contains fixed (and periodic) points, f is not uniformly expanding.

1.2.2. *Direct product "intermittent×intermittent".* Let $f : \mathbb{T} \to \mathbb{T}, (x,y) \mapsto (T_\alpha(x), T_\beta(y))$ where $\alpha, \beta > 0$. Now $K = \{0\} \times \mathbb{S} \cup \mathbb{S} \times \{0\}$ and, by the same reasoning of the previous example, the probability measure $\nu = \mu_\alpha \times \mu_\beta$ is the unique physical measure for f. Moreover $B(\nu) = \mathbb{T}$ as before. However ν is absolutely continuous with respect to m if, and only if, $\alpha, \beta \in (0,1)$.

1.2.3. *Skew-product "intermittent⋊expanding".* Let $f : \mathbb{T} \to \mathbb{T}, (x,y) \mapsto (T_\alpha(x) + \eta y, g(y))$, for $\alpha > 0, \eta \in (0,1)$, and $g : \mathbb{S} \to \mathbb{S}$ as in example 1.2.1.

In this case we easily calculate $Df = \begin{pmatrix} DT_\alpha & \eta \\ 0 & Dg \end{pmatrix}$ and so $K = \{(0,0)\}$.

Clearly $\|(Df)^{-1}\| \leq 1$ everywhere and since K is a fixed point the map f is not uniformly expanding. Applying Theorem D we get an absolutely continuous invariant probability measure μ for f with $\int \log \|(Df)^{-1}\| \, d\mu < 0$. Hence the Lyapunov exponents for Lebesgue almost every point on the basin of μ are all positive, so f is a non-uniformly expanding transformation.

This map is stochastically stable, since every weak* accumulation point of $(\mu^\varepsilon)_{\varepsilon>0}$ when $\varepsilon \to 0$ equals μ by the uniqueness part of Theorem D. We stress that since the value of α played no role in the arguments, these conclusions hold for any $\alpha > 0$.

2. Preliminary results

Throughout this section we outline some general results about random dynamical systems to be used in what follows.

Having a parameterized family of maps $\hat{T} : X \to \mathcal{T}, t \mapsto T_t$, where X is some connected compact metric space, enables us to identify a sequence $T_1, T_2, \dots$ of maps from $\mathcal{T}$ with a sequence $\omega_1, \omega_2, \dots$ of parameters in X. The probability measure θ_ε can then be assumed to be supported on X.

We set $\Omega = X^{\mathbb{N}}$, the space of sequences $\omega = (\omega_i)_{i \geq 1}$ with elements in X. Then we endow Ω with the standard infinite product topology, which makes Ω a compact metrizable space, with distance given by (for example) $d(\omega, \omega') = \sum_{j \geq 1} 2^{-1} d_X(\omega_j, \omega'_j)$ where d_X is the distance on X. We also take the standard product probability measure $\theta^\varepsilon = \theta_\varepsilon^{\mathbb{N}}$, which makes $(\Omega, \mathcal{B}, \theta^\varepsilon)$ a probability space. Here $\mathcal{B} = \mathcal{B}(X)$ is the σ-algebra generated by cylinder sets, that is, the minimal $\sigma-$algebra of subsets of Ω containing all sets of the form $\{\omega \in \Omega : \omega_1 \in A_1, \omega_2 \in A_2, \cdots, \omega_l \in A_l\}$ for any sequence of Borel subsets $A_i \subset X, i = 1, \cdots, l$ and $l \geq 1$.

The following skew-product map is the natural setting for many definitions connecting random with standard dynamical systems

$$\begin{aligned} S \quad : \quad & \Omega \times M \to \Omega \times M \\ & (\omega, x) \mapsto (\sigma(\omega), T_{\omega_1}(x)) \end{aligned}$$

where σ is the left shift on Ω, defined as $(\sigma(\omega))_n = \omega_{n+1}$ for all $n \geq 1$. It is an exercise to check that μ^ε is a stationary measure for the random system $(\hat{T}, \theta_\varepsilon)$ (i.e. satisfying (1.1)) if, and only if, $\theta^\varepsilon \times \mu^\varepsilon$ on $\Omega \times M$ is invariant by S. Ergodicity of stationary measures is defined in a natural way. A Borel set $A \in \mathcal{B}(M)$ is called invariant if for μ^ε-almost every point $x \in M$

$$\begin{aligned} x \in A \quad &\Rightarrow \quad T_t(x) \in A \quad \text{ for } \theta_\varepsilon - \text{almost every } t \in X; \text{ and} \\ x \in A^c \quad &\Rightarrow \quad T_t(x) \in A^c \quad \text{ for } \theta_\varepsilon - \text{almost every } t \in X. \end{aligned}$$

Definition 2. *A stationary measure μ^ε is said to be ergodic if every Borel invariant set has either μ^ε-measure zero or one.*

It is not difficult to prove that μ^ε is ergodic if and only if $\theta^\varepsilon \times \mu^\varepsilon$ is an ergodic measure for S (see for example [18]).

2.1. **Metric entropy of Random Dynamical Systems.** The notion of metric entropy can be defined for random dynamical systems in different ways. We point out two definitions which will be used in this paper and relate them. The following results can be found in the book of Kifer [12, Section II].

Let μ be a stationary measure for the random system $(\hat{T}, \theta_\varepsilon)$ as defined in the beginning of this section.

Theorem 2.1. [12, Thm. 1.3] *For any finite measurable partition ξ of M the limit*

$$h_{\mu^\varepsilon}((\hat{T}, \theta_\varepsilon), \xi) = \lim_{n \to \infty} \frac{1}{n} \int H_{\mu^\varepsilon}\Big(\bigvee_{k=0}^{n-1} (T_\omega^k)^{-1} \xi\Big) d\theta^\varepsilon(\omega)$$

exists. This limit is called the entropy of the random dynamical system with respect to ξ and to μ^ε.

Remark 2.2. As in the deterministic case the above limit can be replaced by the infimum.

Definition 3. *The metric entropy of the random dynamical system $(\hat{T}, \theta_\varepsilon)$ is given by $h_{\mu^\varepsilon}(\hat{T}, \theta_\varepsilon) = \sup h_{\mu^\varepsilon}((\hat{T}, \theta_\varepsilon), \xi)$, where the supremum is taken over all measurable partitions.*

It seems natural to define the entropy of a random system by $h_{\theta^\varepsilon \times \mu^\varepsilon}(S)$ where S is the corresponding skew-product map. Kifer [12, Thm. 1.2] shows that this definition is not very convenient: under some mild conditions the entropy of S is infinite. However considering an appropriate $\sigma-$algebra, the conditional entropy of $\theta^\varepsilon \times \mu^\varepsilon$ coincides with the entropy as defined in Definition 3.

Let $\mathcal{B} \times M$ denote the minimal $\sigma-$algebra containing all products of the form $A \times M$ with $A \in \mathcal{B}$. In what follows we denote by $h^{\mathcal{B} \times M}_{\theta^\varepsilon \times \mu^\varepsilon}(S)$ the conditional metric entropy of S with respect to the σ-algebra $\mathcal{B} \times M$. (See e.g. [8] for a definition and properties of conditional entropy.)

Theorem 2.3. [12, Thm. 1.4] *Let μ^ε be a stationary probability measure for the random system $(\hat{T},\theta_\varepsilon)$. Then $h_{\mu^\varepsilon}(\hat{T},\theta_\varepsilon) = h^{\mathcal{B}\times M}_{\theta^\varepsilon\times\mu^\varepsilon}(S)$.*

The useful Kolmogorov-Sinai result about generating partitions is also available in a random version. We denote $\mathcal{A} = \mathcal{B}(M)$ the Borel σ-algebra of M.

Theorem 2.4. [12, Cor. 1.2] *If ξ is a random generating partition for $\mathcal{A}$, that is ξ is a finite partition of M such that*

$$\bigvee_{k=0}^{+\infty} (T^k_\omega)^{-1}\xi = \mathcal{A} \quad \textit{for} \quad \theta^\varepsilon - \textit{almost all}\ \omega \in \Omega,$$

then $h_{\theta^\varepsilon\times\mu^\varepsilon}(\hat{T},\theta_\varepsilon) = h_{\mu^\varepsilon}((\hat{T},\theta_\varepsilon),\xi)$.

2.2. **Topological mixing.** Here we show that in the setting of Theorems D and E we always have topological mixing for the transformation T. *This ensures uniqueness of stationary measures under non-degenerate random perturbations*, as we shall see.

Since $T : M \to M$ is a local diffeomorphism on a compact Riemannian manifold, there is a positive number ρ such that $T \mid B(x,\rho)$ is a diffeomorphism onto its image and $B(x,\rho)$ is a convex neighborhood for every $x \in M$, i.e., for every pair of points y,z in $B(x,\rho)$ there exists a smooth geodesic $\gamma : [0,1] \to M$ connecting them whose length equals $\operatorname{dist}(y,z)$ and, moreover, $\gamma \mid [s,t]$ is the curve of minimal length between any pair of points $\gamma(s),\gamma(t)$ with $s < t, s,t \in [0,1]$.

Lemma 2.5. *Let $T : M \to M$ satisfy the conditions of Theorem E. Then for every open subset U there exists an iterate $n \geq 1$ such that $T^n(U) = M$.*

Proof. Arguing by contradiction, let us suppose that there exists a ball $B(x,r)$, for some $x \in M$ and small $r > 0$, such that $T^n(B(x,r)) \neq M$ for every $n > 1$.

In what follows we fix $n > 1$ such that $T^k(B(x,r)) \neq M$ for all $k = 1,\dots,n$. Then there is $y \in M \setminus T^n(B(x,r))$ and a smooth curve $\gamma : [0,1] \to M$ such that $\gamma(0) = T^n(x), \gamma(1) = y$ and whose length is bounded by $\kappa = \operatorname{diam}(M) + 1$ (which is finite, because M is compact).

Now we fix $\delta \in (0, r/(10k))$ and $\delta' \in (0,\delta)$ small enough such that

- the k connected components of $B(K,\delta')$ are convex neighborhoods;
- the connected components of $T(B(K,\delta'))$ (there are at most k of them) are also convex neighborhoods with diameter smaller than 2δ.

Moreover we choose $\lambda_1 \in (0,1)$ such that $r > 2k\delta/\lambda_1$ and set $\lambda = \max\{\|DT(x)^{-1}\| : x \in M \setminus B(K,\delta')\}$ and $\lambda_0 = \lambda + \lambda_1(1-\lambda) < 1$.

We write γ_0 to denote a smooth curve such that $T \circ \gamma_0 = \gamma$ in what follows.

Let $\overline{\gamma}_0 = \gamma_0 \mid \gamma_0^{-1} B(K,\delta')$ be the portion of γ_0 inside $B(K,\delta')$. Since every connected component of $\overline{\gamma} = T \circ \overline{\gamma}_0$ is inside a convex neighborhood of diameter at most 2δ, we may assume that the length $\ell(\overline{\gamma})$ of $\overline{\gamma}$ is at most 2δ. For otherwise we may replace $\overline{\gamma}$ by portions of minimizing geodesics connecting the endpoints of each connected component, with smaller total length. Now we obtain, by the non-contracting character of T and by the definitions of the constants above

$$\begin{aligned}\ell(\gamma_0) &= \ell(\gamma_0\setminus\bar\gamma_0)+\ell(\bar\gamma_0)\le\lambda\cdot\ell(\gamma\setminus\bar\gamma)+\ell(\bar\gamma)\\ &= \ell(\gamma)\left(\frac{\lambda(\ell(\gamma)-\ell(\bar\gamma))+\ell(\bar\gamma)}{\ell(\gamma)}\right)=\ell(\gamma)\left(\lambda+(1-\lambda)\frac{\ell(\bar\gamma)}{\ell(\gamma)}\right)\\ &\le \ell(\gamma)\left(\lambda+(1-\lambda)\frac{2k\delta}{r}\right)\le\lambda_0\cdot\ell(\gamma)\le\lambda_0\kappa,\end{aligned}$$

as long as $\ell(\gamma)\ge r$. This is true by the definition of γ, the assumption on $B(x,r)$ and the non-contracting derivative of the local diffeomorphisms T. Indeed, if $\ell(\gamma)<r$, then letting $\gamma_1:[0,1]\to M$ be the only piecewise smooth curve satisfying $\gamma_1(0)=x$ and $T^n\circ\gamma_1=\gamma$, we must have $\ell(\gamma)\ge\ell(\gamma_1)$ and thus $\gamma_1(1)\in B(x,r)$ and $T^n(\gamma_1(1))=y$. This contradicts our assumption that $y\notin T^n(B(x,r))$.

Now if we choose γ_0 such that $\gamma_0=T^{n-1}\circ\gamma_1$ then, as above, we have both

$$\ell(\gamma_0)=\ell(T^{n-1}\circ\gamma_1)\ge r \quad\text{and}\quad \ell(T^{n-1}\circ\gamma_1)\le\lambda_0\cdot\ell(T^n\circ\gamma_1)=\lambda_0\cdot\ell(\gamma).$$

Hence, by induction on k, we get for every $k=1,\dots,n$ that

$$r\le\ell(T^{n-k}\circ\gamma_1)\le\lambda_0^k\cdot\ell(\gamma)\le\lambda_0^k\kappa.$$

However, this cannot be true for arbitrarily big values of n, since $r,\kappa>0$ are fixed and $\lambda_0\in(0,1)$. This shows that for every $x\in M$ and $r>0$ there is n such that $T^n(B(x,r))=M$, ending the proof of the lemma. □

Proposition 2.6. *Let $\hat T: B\subset\mathbb{R}^n\to C^{1+\alpha}(M,M), v\to T+v$, with B a ball around the origin of $\mathbb{R}^n$, as defined at the Introduction, where $T:M\to M$ satisfies the conditions of Theorem E. Then for every $v\in B$, all $x\in M$ and every given $\varepsilon>0$, there exists $n\in\mathbb{N}$ such that $T_v^n(B(x,\varepsilon))=M$.*

Proof. We just have to note that the subset K does not depend on v for the maps T_v since $DT_v=DT$. Hence if T satisfies the conditions of Theorem E, then every T_v does also. Thus we can use Lemma 2.5 with T_v in the place of T. □

2.3. **Non-degenerate random perturbations.** Here we recall the setting of *non-degenerate random perturbations* as defined in [4]. For a complete list of propositions and proofs see [2].

We assume that the family $(\theta_\varepsilon)_{\varepsilon>0}$ of probability measures on X is such that their supports have non-empty interior and

$$\operatorname{supp}(\theta_\varepsilon)\to\{t_0\}\quad\text{when}\quad\varepsilon\to 0,\quad\text{such that } T_{t_0}=T.$$

For $\omega=(\omega_1,\omega_2,\cdots)\in\Omega$ and for $n\ge1$ we set

$$T_\omega^n=T_{\omega_n}\circ\cdots\circ T_{\omega_1}.$$

Given $x\in M$ and $\omega\in\Omega$ we call the sequence $(T_\omega^n(x))_{n\ge1}$ a *random orbit* of x.

In what follows we need the map $\tau_x:X\to M,\tau_x(t)=T_t(x)$.

Definition 4. *We say that $(\hat T,\theta_\varepsilon)_{\varepsilon>0}$ is a non-degenerate random perturbation of T if, for every small enough ε and fixed t_* in the interior of* $\operatorname{supp}(\theta_\varepsilon)$, *there is $\delta_1=\delta_1(\varepsilon)>0$ such that for all $x\in M$:*

1. $\{T_t(x) : t \in \operatorname{supp}(\theta_\varepsilon)\}$ *contains a ball of radius* δ_1 *around* $T_{t_*}(x)$;
2. $(\tau_x)_*\theta_\varepsilon$ *is absolutely continuous with respect to* m.

Remark 2.7. We note that θ_ε cannot have atoms because of the non-degeneracy condition (2) above.

We outline some interesting consequences of the non-degeneracy conditions — for a proof see [4].

- Any stationary measure μ^ε is absolutely continuous with respect to m.
- $\operatorname{supp}(\mu^\varepsilon)$ has non-empty interior and $T_t(\operatorname{supp}(\mu^\varepsilon)) \subset \operatorname{supp}(\mu^\varepsilon)$ for any $t \in \operatorname{supp}(\theta_\varepsilon)$.
- $\operatorname{supp}(\mu^\varepsilon) \subseteq B(\mu^\varepsilon)$.

Here $B(\mu^\varepsilon)$ is the *ergodic basin* of μ^ε

$$B(\mu^\varepsilon) = \left\{ x \in M : \frac{1}{n}\sum_{j=0}^{n-1} \varphi(T^j_\omega(x)) \to \int \varphi \, d\mu \text{ for all } \varphi \in C(M,\mathbb{R}) \text{ and } \theta^\varepsilon\text{-a.e. } \omega \in \Omega \right\}$$

which by the above properties has positive Lebesgue measure in M.

If T is in the setting of Theorem E, then after Proposition 2.6 we deduce that, since the support of a stationary measure μ^ε has non-empty interior and is forward invariant by T_t for any $t \in \operatorname{supp}\theta_\varepsilon$, the support must contain M, and so *there exists only one physical measure* μ^ε for all $\varepsilon > 0$, because the support is contained in the basin, m mod 0.

These non-degeneracy conditions are not too restrictive since there always exists a non-degenerate random perturbation of any differentiable map of a compact manifold of finite dimension with X the closed ball of radius 1 around the origin of a Euclidean space, see [4]. In the setting $M = \mathbb{T}^n$ with additive noise, as explained at the Introduction, we have also that Df_t may be identified with Df, so that $\|(Df_t)^{-1}\| = \|(Df)^{-1}\|, t \in X$, which is very important in our arguments.

3. Zero-noise limits are equilibrium measures

In what follows we present a proof of Theorem E. Let $f : M \to M$ be a local diffeomorphism on a manifold M satisfying the conditions stated in the above mentioned theorem. Let also $\hat{f} : X \to C^{1+\alpha}(M,M), t \mapsto f_t$ be a continuous family of maps, where X is a metric space with $f_{t_0} \equiv f$ for some fixed $t_0 \in X$, and $(\theta_\varepsilon)_{\varepsilon>0}$ be a family of probability measures on X such that $(\hat{f}, (\theta_\varepsilon)_{\varepsilon>0})$ is non-degenerate random perturbation of f.

The strategy is to find a *fixed random generating partition* for the system $(\hat{f}, \theta_\varepsilon)$ for *every small* $\varepsilon > 0$ and use the absolute continuity of the stationary measure μ^ε, together with the "non-contractive" conditions on f to obtain (using the same notations and definitions from Section 2) a semicontinuity property for entropy on zero-noise limits.

Theorem 3.1. *Let us assume that there exists a finite partition* ξ *of* M *(Lebesgue modulo zero) which is generating for random orbits, for every small enough* $\varepsilon > 0$.

Let μ^0 be a weak* accumulation point of $(\mu^\varepsilon)_{\varepsilon>0}$ when $\varepsilon \to 0$. If $\mu^{\varepsilon_k} \to \mu^0$ for some $\varepsilon_k \to 0$ when $k \to \infty$, then [1]

$$\limsup_{k\to\infty} h_{\mu^{\varepsilon_k}}((\hat{f},\theta_{\varepsilon_k}),\xi) \leq h_{\mu^0}(f,\xi).$$

The absolute continuity of μ^ε and the quasi-expansion enable us to use a random version of the Entropy Formula for endomorphisms (for a more general setting see [16, 5]).

Theorem 3.2. *If an ergodic stationary measure μ^ε for a $C^{1+\alpha}$ random perturbation $(\hat{f},\theta_\varepsilon)$ is absolutely continuous and $\iint \log\|Df_t(x)^{-1}\|\,d\theta_\varepsilon(t)d\mu^\varepsilon(x) < 0$ for a given $\varepsilon > 0$, then*

$$h_{\mu^\varepsilon}(\hat{f},\theta_\varepsilon) = \iint \log|\det Df_t(x)|\,d\mu^\varepsilon(x)d\theta_\varepsilon(t).$$

Putting Theorems 3.1 and 3.2 together shows that $h_{\mu^0}(f) \geq \int \log|\det Df(x)|\,d\mu^0(x)$, since $\theta_\varepsilon \to \delta_{t_0}$ in the weak* topology when $\varepsilon \to 0$, by the assumptions on the support of θ_ε in subsection 2.3. The reverse inequality holds in general (Ruelle's inequality [24]) proving Theorem E.

3.1. **Random Entropy Formula.**

Now we explain the meaning of Theorem 3.2.

Let $\varepsilon > 0$ be fixed in what follows. The Lyapunov exponents $\lim_{n\to\infty} n^{-1}\log\|Df^n_\omega(x)\cdot v\|$ exist for $\theta^\varepsilon\times\mu^\varepsilon$-almost every (ω,x) and every $v \in T_xM\setminus\{0\}$, and are always positive in this setting. In fact, since the random perturbations are additive we have $Df_t = Df$ and, moreover, μ^ε is ergodic, absolutely continuous and $\mu^\varepsilon(K) = 0$, so for $\theta^\varepsilon\times\mu^\varepsilon$-almost every (ω,x)

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log\|Df_{\omega(j+1)}(f^j_\omega(x))^{-1}\| = \int \log\|Df_t(x)^{-1}\|\,d\mu^\varepsilon(x)d\theta_\varepsilon(t) < 0.$$

(Setting $\varphi(t,x) = \log\|Df_t(x)^{-1}\|$ then this is just the Ergodic Theorem applied to $S : \Omega\times M \to \Omega\times M$ with $\psi = \varphi\circ\pi$, where $\pi : \Omega\times M \to X\times M, (\omega,x) \mapsto (\omega(1),x)$.) This readily ensures the positivity of the growth exponent in every direction under random perturbations because

$$\log\|(Df^n_\omega(x))^{-1}\| \leq \sum_{j=0}^{n-1}\log\|(Df_{\omega(j+1)}(f^j_\omega(x)))^{-1}\|.$$

According to the Multiplicative Ergodic Theorem (Oseledets [22]) the sum of the Lyapunov exponents (with multiplicities) equals the following limit $\theta^\varepsilon\times\mu^\varepsilon$-almost everywhere

$$\lim_{n\to\infty}\frac{1}{n}\log|\det Df^n_\omega(x)| = \iint \log|\det Df_t(x)|\,d\mu^\varepsilon(x)d\theta_\varepsilon(t) > 0,$$

and the identity above follows from the Ergodic Theorem, since the value of the limit is S-invariant, thus constant.

Now Pesin's Entropy Formula states that for $C^{1+\alpha}$ maps, $\alpha > 0$, with positive Lyapunov exponents everywhere, as in our setting, the metric entropy with respect to an invariant measure μ^ε satisfies the relation in Theorem 3.2 if, and only if, μ^ε is absolutely continuous. In general we integrate the sum of the positive Lyapunov exponents, see Liu [19] for a proof in the C^2 setting.

[1] Cowieson-Young [9] obtain the same result for random diffeomorphisms without assuming the existence of a uniform generating partition but need either a local entropy condition or that the maps $\hat{f}$ involved be of class C^∞.

In our setting of the proof that $\mu^\varepsilon \ll m$ implies the Entropy Formula is an exercise using the bounded distortion provided by the Hölder condition on the derivative.

3.2. **Random generating partition.** Here we construct the uniform random generating partition assumed in the statement of Theorem 3.1. In what follows we fix a weak* accumulation point μ^0 of μ^ε when $\varepsilon \to 0$: there exists $\varepsilon_k \to 0$ when $k \to \infty$ such that $\mu = \lim_k \mu^{\varepsilon_k}$.

To understand how to obtain a generating partition, we need a preliminary result.

For the following lemma, recall that $K = \{x \in M : \|Df(x)^{-1}\| = 1\}$. In what follows $B(K,\delta_0) = \cup_{z\in K} B(z,\delta_0)$ is the δ_0-neighborhood of K and $\rho > 0$ is such that $f \mid B(x,\rho)$ is a diffeomorphism onto its image and $B(x,\rho)$ is a convex neighborhood for every $x \in M$, as in Subsection 2.2. Using uniform continuity, we let $\rho_0 > 0$ be such that for every $x, y \in M$ and $t \in X$, if $\operatorname{dist}(x,y) < \rho_0$, then $\operatorname{dist}(f_t(x), f_t(y)) < \rho$.

Lemma 3.3. *Let $(f_t)_{t\in X}$ be a family of maps from a non-degenerate (additive) random perturbation. For any given $\delta_0 \in (0,\rho_0)$ there exists $\beta > 0$ such that if $x \in M$ and $y \in M \setminus B(K,\delta_0)$ are such that $\delta_0 \le \operatorname{dist}(x,y) \le \rho_0$, then $\operatorname{dist}(f_t(x), f_t(y)) \ge \operatorname{dist}(x,y) + \beta$ for every $t \in X$.*

Proof. Let us assume that $x \in K$, let $y \in M \setminus B(K,\delta_0)$ be such that $\operatorname{dist}(x,y) \in [\delta_0,\rho_0]$ and let $t \in X$ be fixed. By the choice of ρ there is a smooth geodesic $\gamma : [0,1] \to M$ with $\gamma(0) = f_t(x)$ and $\gamma(1) = f_t(y)$ and $\operatorname{dist}(f_t(x), f_t(y)) = \int_0^1 \|\gamma'(s))\|\,ds < \rho$. In addition, there exists a unique smooth curve $\gamma_0 : [0,1] \to M$ such that $f \circ \gamma_0 = \gamma$, $\gamma_0(0) = x$ and $\gamma_0(1) = y$.

Let us set $b = \|Df_t(y)^{-1}\| = \|Df(y)^{-1}\| < 1$ and $K(a) = \{z \in M : \|Df_t(z)^{-1}\| \ge a\}$ for $a \in (0,1)$. Then there must be $b_1, b_2 \in (b,1)$ with $b_1 < b_2$ such that $K(b_1)$ (a compact set) is in the interior of $K(b_2)$ (recall that $z \mapsto \|Df_t(z)^{-1}\|$ is continuous and we are assuming that $x \in K$, that is, $\|(Df_t)^{-1}\|$ assumes the value 1).

We notice that $\|Df_t(z)^{-1}\| < b_1$ for all $z \in K(b_2) \setminus K(b_1)$ and, moreover, that $\Gamma = \gamma^{-1}(K(b_2) \setminus K(b_1))$ has nonempty interior, thus positive Lebesgue measure on $[0,1]$. Then

$$
\begin{aligned}
\operatorname{dist}(f_t(x), f_t(y)) &= \int_0^1 \|\dot\gamma(s))\|\,ds = \int_0^1 \|Df_t(\gamma(s)) \cdot Df_t(\gamma(s))^{-1} \cdot \dot\gamma(s))\|\,ds \\
&\ge \frac{1}{b_1}\int_\Gamma \|Df_t(\gamma(s))^{-1} \cdot \dot\gamma(s))\|\,ds + \int_{[0,1]\setminus\Gamma} \|Df_t(\gamma(s))^{-1} \cdot \dot\gamma(s))\|\,ds \\
&> \int_0^1 \|\dot\gamma_0(s))\|\,ds \ge \operatorname{dist}(x,y).
\end{aligned}
$$

If $x \in M \setminus K$, then there exists $b \in (0,1)$ such that both $x, y \in M \setminus K(b)$ and thus we may take $\Gamma = [0,1]$ in the calculations above, arriving at the same sharp inequality.

Hence if $\Delta(\delta_0) = \{(x,y) \in M \times M : \operatorname{dist}(x,y) \ge \delta_0 \text{ and } y \in M \setminus B(K,\delta_0)\}$, then

$$h : X \times M \times M \to \mathbb{R}, \quad (t,x,y) \mapsto \operatorname{dist}(f_t(x), f_t(y)) - \operatorname{dist}(x,y)$$

is positive on the compact set $X \times \Delta(\delta_0)$, since in the calculations above $t \in X$ was arbitrary. We just have to take $\beta = \min h \mid X \times \Delta(\delta_0)$. $\square$

We now are able to construct a random generating partition under the conditions of Theorem E.

Let us take a finite cover $\{B(x_i,\rho_0/2), i = 1,\dots,\ell\}$ of M by $\rho_0/2$-balls, where $\rho_0 > 0$ was already defined. Since μ^0 is a probability measure, we may assume that $\mu^0(\partial\xi) = 0$, for otherwise we can replace each ball by $B(x_i,\gamma\rho_0/2)$, for some $\gamma\in(1,3/2)$ and for all $i=1,\dots,k$. Now let ξ be the finest partition of M obtained through all possible intersections of these balls: $\xi = B(x_1,\gamma\rho_0/2)\vee\cdots\vee B(x_\ell,\gamma\rho_0/2)$. In the following lemma we let ρ stand for this new radius.

Remark 3.4. The partition ξ is such that all atoms of $\vee_{j=0}^{n-1}(f_\omega^j)^{-1}\xi$ have boundary (which is a union of pieces of boundaries of open balls) with zero Lebesgue measure, for all $n\ge 1$ and every $\omega\in\Omega$. Moreover, since μ^0 is f-invariant and $\mu^0(\partial\xi)=0$, then $\mu^0(\vee_{j=0}^{n-1}f^{-j}\xi)=0$ also, for all $n\ge 1$.

Lemma 3.5. $\bigvee_{k=0}^{+\infty}(f_\omega^k)^{-1}\xi=\mathcal{A}$ *when $n\to+\infty$ for each $\omega\in\Omega$.*

Proof. Let us argue by contradiction assuming that there are two points x,y such that for some fixed $\omega\in\Omega$: $\operatorname{dist}(f_\omega^j(x),f_\omega^j(y))\in[\delta_0,\rho]$ for some $\delta_0>0$ and $y\in(\bigvee_{i=0}^n f_\omega^{-i}\xi)(x)$ for every $n\ge 1$.

Let $\delta_1=\min\{\operatorname{dist}(z_1,z_2): z_1,z_2\in K, z_1\ne z_2\}$ be the minimum separation between points in K (we recall that K is finite) and take $V=B(K,\min\{\delta_1,\delta_0\}/4)$. Then it is not possible that both $f_\omega^j(x),f_\omega^j(y)$ are in the same connected component of V. Using the fact that every ρ-neighborhood is a convex neighborhood and expressing $\operatorname{dist}(f_\omega^j(x),f_\omega^j(y))$ through the length of a geodesic, we get a point $z\in M\setminus V$ and $\beta=\beta(\delta_0,\delta_1)>0$ such that

$$\begin{aligned}\operatorname{dist}(f_\omega^j(x),f_\omega^j(y)) &= \operatorname{dist}(f_\omega^j(x),f_\omega^j(z))+\operatorname{dist}(f_\omega^j(z),f_\omega^j(y))\\ &\ge \operatorname{dist}(f_\omega^{j-1}(x),f_\omega^{j-1}(z))+\operatorname{dist}(f_\omega^{j-1}(z),f_\omega^{j-1}(y))+2\beta\\ &\ge \operatorname{dist}(f_\omega^{j-1}(x),f_\omega^{j-1}(y))+2\beta\end{aligned}$$

for every $j>0$, applying Lemma 3.3 twice. But then the upper bound ρ for the distance between iterates of x and y cannot hold for all $j\ge 1$. This shows that δ_0 cannot be positive, hence the diameter of the atoms of the refined partitions tends to zero. This is enough to conclude the statement of the lemma. $\square$

This last lemma implies that ξ is a random generating partition as in the statement of the Random Kolmogorov-Sinai Theorem 2.4. Hence we conclude that $h_{\mu^{\varepsilon_k}}((\hat f,\theta_{\varepsilon_k}),\xi)=h_{\mu^{\varepsilon_k}}(\hat f,\theta_{\varepsilon_k})$ for all $k\ge 1$.

3.3. **Semicontinuity of entropy on zero-noise.** Now we start the proof of Theorem 3.1. We need to construct a sequence of partitions of $\Omega\times M$ according to the following result. For a partition $\mathcal{P}$ of a given space Y and $y\in Y$ we denote by $\mathcal{P}(y)$ the element (atom) of $\mathcal{P}$ containing y. We set $\omega_0=(t_0,t_0,t_0,\dots)\in\Omega$ in what follows.

Lemma 3.6. *There exists an increasing sequence of measurable partitions $(\mathcal{B}_n)_{n\ge1}$ of Ω such that*

1. $\omega_0\in\operatorname{int}\mathcal{B}_n(\omega_0)$ *for all $n\ge 1$;*
2. $\mathcal{B}_n\nearrow\mathcal{B}$, θ^{ε_k} mod 0 *for all $k\ge1$ when $n\to\infty$;*
3. $\lim_{n\to\infty}H_\rho(\xi\mid\mathcal{B}_n)=H_\rho(\xi\mid\mathcal{B})$ *for every measurable finite partition ξ and any S-invariant probability measure ρ.*

Proof. For the first two items we let $\mathcal{C}_n$ be a finite θ_{ε_k} mod 0 partition of X such that $t_0 \in \operatorname{int} \mathcal{C}_n(t_0)$ with $\operatorname{diam} \mathcal{C}_n \to 0$ when $n \to \infty$. Example: take a cover $(B(t,1/n))_{t\in X}$ of X by $1/n$-balls and take a subcover $U_1,\dots,U_k$ of $X \setminus B(t_0,2/n)$ together with $U_0 = B(t_0,3/n)$; then let $\mathcal{C}_n = U_0 \vee \cdots \vee U_k$.

We observe that we may assume that the boundary of these balls has null θ_{ε_k}-measure for all $k \geq 1$, since $(\theta_{\varepsilon_k})_{k\geq 1}$ is a denumerable family of non-atomic probability measures on X (see Remark 2.7). Now we set

$$\mathcal{B}_n = \mathcal{C}_n \times \overset{n}{\cdots} \times \mathcal{C}_n \times \Omega \quad \text{for all } n \geq 1.$$

Then since $\operatorname{diam} \mathcal{C}_n \leq 2/n$ for all $n \geq 1$ we have that $\operatorname{diam} \mathcal{B}_n \leq 2/n$ also and so tends to zero when $n \to \infty$. Clearly $\mathcal{B}_n$ is an increasing sequence of partitions. Hence $\vee_{n\geq 1} \mathcal{B}_n$ generates the σ-algebra $\mathcal{B}$, θ^{ε_k} mod 0 (see e.g. [8, Lemma 3, Chpt. 2]) for all $k \geq 1$. This proves items (1) and (2).

Item (3) of the statement of the lemma is Theorem 12.1 of Billingsley [8]. $\square$

Now we use some properties of conditional entropy to obtain the right inequalities. We start with

$$\begin{aligned} h_{\mu^{\varepsilon_k}}(\hat{f},\theta_{\varepsilon_k}) &= h_{\mu^{\varepsilon_k}}((\hat{f},\theta_{\varepsilon_k}),\xi) = h^{\mathcal{B}\times M}_{\theta^{\varepsilon_k}\times\mu^{\varepsilon_k}}(S,\Omega\times\xi) \\ &= \inf \frac{1}{n} H_{\theta^{\varepsilon_k}\times\mu^{\varepsilon_k}}\left(\bigvee_{j=0}^{n-1}(S^j)^{-1}(\Omega\times\xi) \mid \mathcal{B}\times M\right) \end{aligned}$$

where the first equality comes from subsection 3.2 and the second one can be found in Kifer [12, Thm. 1.4, Chpt. II], with $\Omega\times\xi = \{\Omega\times A : A\in\xi\}$. Hence for arbitrary fixed $N \geq 1$ and for any $m \geq 1$

$$\begin{aligned} h_{\mu^{\varepsilon_k}}(\hat{f},\theta_{\varepsilon_k}) &\leq \frac{1}{N} H_{\theta^{\varepsilon_k}\times\mu^{\varepsilon_k}}\left(\bigvee_{j=0}^{N-1}(S^j)^{-1}(\Omega\times\xi) \mid \mathcal{B}\times M\right) \\ &\leq \frac{1}{N} H_{\theta^{\varepsilon_k}\times\mu^{\varepsilon_k}}\left(\bigvee_{j=0}^{N-1}(S^j)^{-1}(\Omega\times\xi) \mid \mathcal{B}_m\times M\right) \end{aligned}$$

because $\mathcal{B}_m \times M \subset \mathcal{B}\times M$. Now we fix N and m, let $k\to\infty$ and note that since $\mu^0(\partial\xi) = 0 = \delta_{\omega_0}(\partial\mathcal{B}_m)$ it must be that

$$(\delta_{\omega_0}\times\mu^0)(\partial(B_i\times\xi_j)) = 0 \quad \text{for all } B_i\in\mathcal{B}_m \text{ and } \xi_j\in\xi,$$

where δ_{ω_0} is the Dirac mass concentrated at $\omega_0\in\Omega$. Thus we get by weak* convergence of $\theta^{\varepsilon_k}\times\mu^{\varepsilon_k}$ to $\delta_{\omega_0}\times\mu^0$ when $k\to\infty$

$$\limsup_{k\to\infty} h_{\mu^{\varepsilon_k}}(\hat{f},\theta_{\varepsilon_k}) \leq \frac{1}{N} H_{\delta_{\omega_0}\times\mu^0}\left(\bigvee_{j=0}^{N-1}(S^j)^{-1}(\Omega\times\xi) \mid \mathcal{B}_m\times M\right) = \frac{1}{N} H_{\mu^0}\big(\bigvee_{j=0}^{N-1} f^{-j}\xi\big). \tag{3.1}$$

Here it is easy to see that the middle conditional entropy of (3.1) (involving only finite partitions) equals $N^{-1}\sum_i \mu^0(P_i)\log\mu^0(P_i)$, with $P_i = \xi_{i_0}\cap f^{-1}\xi_{i_1}\cap\cdots\cap f^{-(N-1)}\xi_{i_{N-1}}$ ranging over every possible sequence of $\xi_{i_0},\dots,\xi_{i_{N-1}}\in\xi$.

Finally, since N was an arbitrary integer, Theorem 3.1 follows from (3.1). We have completed the proof of the first part of Theorem E.

4. Existence of a.c.i.m. and stochastic stability

Let $f : M \to M$ be as in the statement of Theorem E. The assumptions on f ensure that for every $x \in M$ and all $v \in T_xM \setminus \{0\}$ we have

$$\liminf_{n\to\infty} \frac{1}{n} \log \|Df^n(x)\cdot v\| \geq 0.$$

Thus the Lyapunov exponents for any given f-invariant probability measure μ are non-negative. Hence the sum $\chi(x)$ of the positive Lyapunov exponents of a μ-generic point x is such that

$$\chi(x) = \lim_{n\to\infty} \frac{1}{n} \log|\det Df^n(x)| \quad \text{and} \quad \int \chi\, d\mu = \int \log|\det Df|\, d\mu \tag{4.1}$$

by the Multiplicative Ergodic Theorem and the standard Ergodic Theorem.

Using Theorem E we know that there is only one stationary measure for every $\varepsilon > 0$ (see the beginning of Section 3), and every weak* accumulation point μ of the stationary measures $(\mu^\varepsilon)_{\varepsilon>0}$, when $\varepsilon \to 0$, is an equilibrium state for $-\log|\det Df|$, that is (1.4) holds. We may and will assume that μ is ergodic due to the following

Lemma 4.1. *Almost every ergodic component of an equilibrium state for $-\log|\det Df|$ is itself an equilibrium state for this same function.*

Proof. Let μ be an f-invariant measure satisfying $h_\mu(f) = \int \log|\det Df|\, d\mu$. On the one hand, the Ergodic Decomposition Theorem (see e.g Mañé [21]) ensures that

$$\int \log|\det Df|\, d\mu = \int\int \log|\det Df|\, d\mu_z\, d\mu(z) \quad \text{and} \quad h_\mu(f) = \int h_{\mu_z}(f)\, d\mu(z). \tag{4.2}$$

On the other hand, Ruelle's inequality guarantees for a μ-generic z that (recall (4.1))

$$h_{\mu_z}(f) \leq \int \log|\det Df|\, d\mu_z. \tag{4.3}$$

By (4.2) and (4.3), and because μ is an equilibrium state (1.4), we conclude that we have equality in (4.3) for μ-almost every z. □

Now we note that since K is finite, if $\mu(K) > 0$, then μ (which is ergodic) is concentrated on a periodic orbit. Hence $h_\mu(f) = 0$ and so by the entropy formula these orbits are non-volume-expanding (the Jacobian equals 1).

Finally, for an ergodic equilibrium state μ with $\mu(K) = 0$, we must have

$$\lim_{n\to\infty} \frac{1}{n} \log\|(Df^n(x))^{-1}\| \leq \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log\|Df(f^j(x))^{-1}\| = \int \log\|(Df)^{-1}\|\, d\mu < 0,$$

μ-almost everywhere. This means that the Lyapunov exponents of μ are strictly positive, where f is a $C^{1+\alpha}$ endomorphism, $\alpha > 0$.

Now the extension of the Entropy Formula for endomorphisms obtained by Liu [17] (in the C^2 setting, but the distortion estimates need only a Hölder condition on the derivative) ensures that an equilibrium state whose Lyapunov exponents are all positive must be absolutely continuous with respect to Lebesgue measure. Hence $\mu \ll m$.

The previous discussion shows that f is non-uniformly expanding in the sense of Alves-Bonatti-Viana [1]. We obtain that

$$\limsup_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| < 0, \quad \mu-\text{almost every } x. \tag{4.4}$$

These authors show that any absolutely continuous invariant measure μ in this setting has basin $B(\mu)$ containing an open subset U Lebesgue modulo 0. By the topological mixing property of f (Lemma 2.5) and because f is a regular map, we deduce that $B(\mu)$ must contain all of M, Lebesgue modulo 0, and is thus unique.

These arguments hold true for every ergodic component of any weak* accumulation point of stationary measures when $\varepsilon \to 0$, thus every such accumulation point is a linear convex combination of an absolutely continuous invariant measure with finitely many Dirac masses concentrated on non-volume-expanding orbits. This ends the proof of Theorem E.

4.1. **Stochastic stability.** Now we assume that $f : M \to M$ satisfies all the conditions of the statement of Theorem D. We arrive at the same conclusion of Theorem E but since we assume that $|\det Df| > 1$ on K, we would arrive at a contradiction if μ is an equilibrium state with $\mu(K) = 0$, that is, the expanding volume condition avoids Dirac masses on periodic orbits as ergodic components of zero-noise limit measures. Thus in this setting we must have $\mu(K) = 0$ for all ergodic equilibrium states. Hence there is only one equilibrium state: the unique absolutely continuous invariant probability measure μ of f.

Therefore every weak* accumulation point of the stationary measures, when $\varepsilon \to 0$, must be equal to μ, showing the stochastic stability of μ and concluding the proof of Theorem D.

5. Random perturbations of the intermittent map

Let T be defined as in Introduction and consider a non-degenerate random perturbation of T. For any small $\varepsilon > 0$, we know that there exists a single stationary measure μ^ε, see Subsection 2.2.

5.1. **Characterization of zero-noise measures.** Let T be as above for $\alpha \in (0,1)$. Here we prove Theorem A in two steps. Let μ be a weak* accumulation point of μ^ε, when $\varepsilon \to 0$. We show that $\mu \in \mathbb{E} = \{t\delta_0 + (1-t)\mu_{\text{SRB}} : 0 \le t \le 1\}$.

Firstly we prove that $h_\mu(T) = \int \log DT\, d\mu$, and in the sequel we deduce that any T-invariant measure satisfying the entropy formula as above should belong to $\mathbb{E}$. The latter is proved also in [23] by different methods.

As we are considering additive random perturbation of T, we can apply Theorem E and conclude that μ is an equilibrium state of $-\log DT$. As in the previous section, we may and will assume that μ is ergodic, by Lemma 4.1.

Now we consider two cases: either $\mu(\{0\}) > 0$, or $\mu(\{0\}) = 0$.

In the first case, the ergodicity of μ ensures that $\mu = \delta_0$, since 0 is a fixed point for T.

For the second case, as $\log DT > 0$ for all points in $\mathbb{S}$ except 0, we have that

$$h_\mu(T) = \int \log DT \, d\mu > 0$$

and the Ergodic Theorem together with the fact that $\mathbb{S}$ is one-dimensional guarantees that the Lyapunov exponent of μ is positive. Thus μ is an ergodic probability measure with positive Lyapunov exponent and positive entropy which satisfies the Entropy Formula. Now we apply a version Pesin's Entropy Formula obtained by Ledrappier [15], which holds for $C^{1+\alpha}$ endomorphisms of $\mathbb{S}^1$, to conclude that μ must be absolutely continuous with respect to Lebesgue measure.

The above arguments show that any typical ergodic component of every zero-noise limit measure μ equals either δ_0 or μ_{SRB}. Hence a straightforward application the Ergodic Decomposition Theorem to μ concludes the proof of Theorem A.

5.2. **Stochastic stability without a.c.i.m.** After Theorem E, any zero-noise limit measure μ for the additive random perturbation of the intermittent map T is an equilibrium state for $-\log DT$. For $\alpha \geq 1$ this is enough to deduce stochastic stability of $T = T_\alpha$.

Indeed, let μ be a weak* accumulation point of μ^ε when $\varepsilon \to 0$. As in the proof of Theorem A (in the previous subsection), we consider the ergodic decomposition of μ.

We claim that almost all ergodic components of μ equal the Dirac measure δ_0 concentrated on 0. Arguing by contradiction, we suppose that for some ergodic component η of μ we have $\eta(\{0\}) = 0$. Thus by the same reasoning of the previous subsection (using positive Lyapunov exponents and Entropy Formula for one-dimensional maps), this implies that η is an absolutely continuous invariant probability measure for T.

However, because for $\alpha \geq 1$ the intermittent map T is C^2, it is well known that T does not admit any absolutely continuous invariant probability measure in this setting — see e.g. []Vi97b for a proof of this fact.

Hence if some ergodic component η of μ is such that $\eta(\{0\}) = 0$ we arrive at contradiction. Thus $\eta(\{0\}) > 0$ and $\eta = \delta_0$ by ergodicity, for every ergodic component of μ. Therefore $\mu = \delta_0$.

This proves *stochastic stability of the intermittent map when it does not admit absolutely continuous invariant probability measures* and ends the proof of Theorem B.

5.3. **Stochastically unstable random perturbation.** Here we prove Theorem C. Let f_t be the family (1.3) and let $0 < s < 1$. Then there exists a unique fixed source

$$p_s = \frac{1}{2}\left(\frac{s-1}{s-2}\right)^{1/\alpha} \in (0, 1/2) \quad \text{such that} \quad f_s'(p_s) = 1 + \alpha(1-s) > 1.$$

Now we choose $u \in (s, 1)$ such that $f_t' \mid [p_u, p_s] > 1$ for all $t \in [s, u]$. For this we just have to take u close enough to s.

Clearly $f_u^n(x) \to 0$ for all $x \in (1-p_u, p_u)$ when $n \to \infty$, see Figure 1 and recall that the maps f_t are symmetric ($f_t(x) = 1 - f_t(1-x)$) on $\mathbb{S} = [0,1]/0 \sim 1$.

Lemma 5.1. *For every $x \in (1-p_u, p_u)$ and every sequence $\underline{t} \in [s,u]^{\mathbb{N}}$ we have that $f_{\underline{t}}^n(x) \to 0$ when $n \to \infty$.*

Proof. It is straightforward to check that the graph of $f_t \mid [0,1/2]$ is below the graph of f_u and above the graph of f_s for every $t \in (s,u)$. Hence $f_{\underline{t}}^n(x) \leq f_u^n(x) \to 0$ when $n \to \infty$ for every $x \in (0, p_u)$. Using the symmetry we arrive at $f_{\underline{t}}^n(x) \to 0$ when $n \to \infty$ for all $x \in (1-p_u, 0)$. □

Now we let θ be any probability measure with support contained in $[s,u]$ and set $\theta_0 = \theta^{\mathbb{N}}$.

Proposition 5.2. *For $\theta_0 \times m$-almost every $(\underline{t},x) \in [s,u]^{\mathbb{N}} \times [p_u, 1-p_u]$ there exists $n \geq 1$ such that $f_{\underline{t}}^n(x) \notin [p_u, 1-p_u]$.*

Combining the two results above we conclude that for $\theta_0 \times m$-almost every $(\underline{t},x) \in [s,u]^{\mathbb{N}} \times \mathbb{S}$ we have that

$$\frac{1}{n}\sum_{j=0}^{n-1} \delta_{f_{\underline{t}}^j(x)} \to \delta_0 \text{ in the weak}^* \text{ topology when } n \to \infty,$$

finishing the proof of Theorem C.

To prove Proposition 5.2 we need the following result whose proof follows standard steps, using the uniform expansion and the $C^{1+\alpha}$ condition on every f_t. Let us fix $\underline{t} \in [s,u]^{\mathbb{N}}$ and a point $0 < r < p_u$ such that $f_t'(s) > 1$ for all $t \in [s,u]$. Let also $\beta_1 = \min\{f_t'(x) : x \in [r, 1-r], t \in [s,u]\} > 1$ and $\beta_2 = \max\{f_t'(x) : x \in \mathbb{S}, t \in [s,u]\} > 1$.

Lemma 5.3. *There exists $C > 1$ such that for any interval $I \subset [r, 1-r]$, all $\underline{t} \in [s,u]^{\mathbb{N}}$ and $k \geq 1$ such that $f_{\underline{t}}^j(I) \subset [r, 1-r]$ for every $j = 0, \dots, k-1$, it holds*

$$\frac{1}{C} \leq \frac{(f_{\underline{t}}^k)'(x)}{(f_{\underline{t}}^k)'(y)} \leq C$$

for all $x, y \in I$.

Proof of the Proposition. For $\underline{t} \in [s,u]^{\mathbb{N}}$ we define $E_k(\underline{t}) = (f_{\underline{t}}^k)^{-1}[p_u, 1-p_u]$ for $k \geq 1$. We will show that $\cap_{k \geq 1} E_k(\underline{t})$ has zero Lebesgue measure for any $\underline{t}$, which is enough to conclude the statement of the lemma. In fact, this means that $n(\underline{t},x) = \min\{k \geq 1 : f_{\underline{t}}^k(x) \notin [p_u, 1-p_u]\}$ is finite for every x in a set $X(\underline{t})$ with $m(X(\underline{t})) = 1$, for every given $\underline{t} \in [s,u]^{\mathbb{N}}$. Thus $\Delta = \cup_{\underline{t} \in [s,u]^{\mathbb{N}}} \{\underline{t}\} \times X(\underline{t})$ is measurable and $(\theta_0 \times m)(\Delta) = 1$.

Let us fix $\underline{t} \in [s,u]^{\mathbb{N}}$, take a nonempty interval $I \subset [p_u, 1-p_u]$ and show that $I \cap \cap_{k \geq 1} E_k(\underline{t})$ has zero Lebesgue measure.

Let $k > 1$ be the first time such that $f_{\underline{t}}^j(I) \not\subset [r, 1-r]$. There exists such k since by uniform expansion $m(f_{\underline{t}}^k(I)) \geq \beta_1^k m(I)$ whenever $f_{\underline{t}}^j(I) \subset [r, 1-r]$ for $j = 0, \dots, k-1$. Now there are two possibilities: either $f_{\underline{t}}^k(I) \subset (1-p_u, p_u)$ or we have $f_{\underline{t}}^k(I) \cap [p_u, 1-p_u] \neq \emptyset \neq [1-r, r] \cap f_{\underline{t}}^k(I)$.

In the former case we conclude that $I \cap \cap_{k \geq 1} E_k(\underline{t}) = \emptyset$, and the argument ends.

In the latter case, we let $F = f_{\underline{t}}^k(I) \cap (1-p_u, p_u)$ and observe that either $F \supset [s, p_u]$ or $F \supset [1-p_u, 1-s]$, so $m(F) \geq p_u - s$. Since $f_{\underline{t}}^{k-1}(I) \subset [r, 1-r]$ we have $m(f_{\underline{t}}^k(I)) \leq \beta_2(1-2r)$ and hence

$$\frac{m(G)}{m(I)} \geq C \frac{m(F)}{m(f_{\underline{t}}^k(I))} \geq C \frac{p_u - r}{\beta_2(1-2r)}$$

where $G = (f_{\underline{t}}^k \mid I)^{-1}(\text{closure}(F))$ and $C > 0$ is a bounded distortion constant from Lemma 5.3.

This shows that $m(I \setminus G) \leq (1 - C(p_u - r)/(\beta_2(1-2r)))m(I)$. We may take r so close to p_u that

$$0 < \gamma = 1 - C\frac{p_u - r}{\beta_2(1-2r)} < 1.$$

If $m(I \setminus G) = 0$, we are done. Otherwise we apply the same argument to each connected component of $I_1 = I \setminus G$ inductively, as follows.

Let $I_k \subset I$ be a compact set formed by finitely many pairwise disjoint closed intervals $I_k = I_{k,1} \cup \cdots \cup I_{k,i_k}$ such that for each $j \in \{1, \dots, i_k\}$ there is a maximal iterate n_j so that $f_{\underline{t}}^n(I_{k,j}) \subset [r, 1-r]$ for all $n = 1, \dots, n_j - 1$.

We observe that γ does not depend on the number of iterates of the first exit. Then either $f_{\underline{t}}^{n_j}(I_{k,j}) \subset (1 - p_u, p_u)$, or there exists $G_{k,j} \subset I_{k,j}$ maximal such that $f_{\underline{t}}^{n_j}(G_{k,j}) \subset (1 - p_u, p_u)$ and $m(I_{k,j} \setminus G_{k,j}) \leq \gamma \cdot m(I_{k,j})$, as before.

In the former case we delete $I_{k,j}$ from I_{k+1}. In the latter case, we add the connected components I' of $I_{k,j} \setminus G_{k,j}$ to I_{k+1} and associate to each of them the maximal number of iterates n' such that $f_{\underline{t}}^j(I') \subset [r, 1-r]$ for $j = 1, \dots, n' - 1$. Then $m(I_{k+1}) \leq \gamma \cdot m(I_k)$. This shows that $m(I_k) \leq \gamma^k m(I) \to 0$ when $k \to \infty$. Since by construction $\cap_{k \geq 1} I_k$ contains $I \cap \cap_{k \geq 1} E_k(\underline{t})$, the proof is complete. □

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