BACKWARD VOLUME CONTRACTION FOR ENDOMORPHISMS WITH EVENTUAL VOLUME EXPANSION

JOSÉ F. ALVES, VILTON PINHEIRO, AND ARMANDO CASTRO

Abstract. We consider smooth maps on compact Riemannian manifolds. We prove that under some mild condition of eventual volume expansion Lebesgue almost everywhere we have uniform backward volume contraction on every pre-orbit for Lebesgue almost every point.

1. Statement of results

Let M be a compact Riemannian manifold and let Leb be a volume form on M that we call Lebesgue measure. We take $f\colon M \to M$ any smooth map. Let $0 < a_1 \le a_2 \le a_3 \le \dots$ be a sequence converging to infinity. We define

$$h(x) = \min\{n > 0\colon |\det Df^n(x)| \ge a_n\}, \tag{1}$$

if this minimum exists, and $h(x) = \infty$, otherwise. For $n \ge 1$, we take

$$\Gamma_n = \{x \in M\colon h(x) \ge n\}. \tag{2}$$

Theorem 1.1. *Assume that $h \in L^p(\mathrm{Leb})$, for some $p > 3$, and take $\gamma < (p-3)/(p-1)$. Choose any sequence $0 < b_1 \le b_2 \le b_3 \le \dots$ such that $b_k b_n \ge b_{k+n}$ for every $k, n \in \mathbb{N}$, and assume that there is $n_0 \in \mathbb{N}$ such that $b_n \le \min\{a_n, \mathrm{Leb}(\Gamma_n)^{-\gamma}\}$ for every $n \ge n_0$. Then, for* Leb *almost every $x \in M$, there exists $C_x > 0$ such that $|\det Df^n(y)| > C_x b_n$ for every $y \in f^{-n}(x)$.*

Date: March 15, 2004.

Work carried out at the Federal University of Bahia. Partially supported by FCT through CMUP and UFBA.

We say that $f\colon M \to M$ is *eventually volume expanding* if there exists $\lambda > 0$ such that for Lebesgue almost every $x \in M$

$$\sup_{n\geq 1} \frac{1}{n} \log |\det Df^n(x)| > \lambda. \tag{3}$$

Let h and Γ_n be defined as in (1) and (2), associated to the sequence $a_n = e^{\lambda n}$.

Corollary 1.2. *If f is eventually volume expanding, then for Lebesgue almost every point $x \in M$ there are $C_x > 0$ and $\sigma_n \to \infty$ such that $|\det Df^n(y)| > C_x \sigma_n$ for every $y \in f^{-n}(x)$. Moreover, given $\alpha > 0$ there is $\beta > 0$ such that*

1. *if $\mathrm{Leb}(\Gamma_n) \leq \mathcal{O}(e^{-\alpha n})$, then we may take $\sigma_n \geq e^{\beta n}$;*
2. *if $\mathrm{Leb}(\Gamma_n) \leq \mathcal{O}(e^{-\alpha n^\tau})$ for some $\tau > 0$, then we may take $\sigma_n \geq e^{\beta n^\tau}$;*
3. *if $\mathrm{Leb}(\Gamma_n) \leq \mathcal{O}(n^{-\alpha})$ and $\alpha > 2$, then we may take $\sigma_n \geq n^\beta$.*

Specific rates will be obtained in Section 4 for some eventually volume expanding endomorphisms. In particular, non-uniformly expanding maps such as quadratic maps and Viana maps will be considered.

2. Concatenated collections

Let $(U_n)_n$ be a collection of measurable subsets of M whose union covers a full Lebesgue measure subset of M. We say that $(U_n)_n$ is a *concatenated collection* if:

$$x \in U_n \quad \text{and} \quad f^n(x) \in U_m \quad \Rightarrow \quad x \in U_{n+m}.$$

Given $x \in \bigcup_{n\geq 1} U_n$, we define $u(x)$ as the minimum $n \in \mathbb{N}$ for which $x \in U_n$. Note that by definition we have $x \in U_{u(x)}$. We define the *chain generated by* $x \in \bigcup_{n\geq 1} U_n$ as $C(x) = \{x, f(x), \ldots, f^{u(x)-1}(x)\}$.

Lemma 2.1. *Let $(U_n)_n$ be a concatenated collection. If*

$$\sum_{n\geq 1} \sum_{j=0}^{n-1} \mathrm{Leb}(f^j(u^{-1}(n))) < \infty,$$

then we have $\sup\left\{ u(y) \;:\; y \in \bigcup_{n\geq 1} U_n \text{ and } x \in C(y) \right\} < \infty$ for Lebesgue almost every $x \in M$.

Proof. Assume that for a given $x \in M$ there exists an infinite number of chains $C_j = \{y_j, f(y_j), \dots, f^{s_j-1}(y_j)\}$, $j \geq 1$, containing x with $s_j \to \infty$. For each $j \geq 1$ let $1 \leq r_j < s_j$ be such that $x = f^{r_j}(y_j)$. First we verify that $\lim r_j = \infty$. If not, then replacing by a subsequence, we may assume that there is $N > 0$ such that $r_j < N$ for every $j \geq 1$. This implies that $y_j \in \bigcup_{i=1}^{N} f^{-i}(x)$ for every $j \geq 1$. Since $\#(\bigcup_{i=1}^{N} f^{-i}(x)) < \infty$ and the number of chains is infinite, we have a contradiction. Since $r_j \to \infty$ and $x = f^{r_j}(y_j) \in f^{r_j}(u^{-1}(s_j))$, then we have $x \in \bigcup_{n \geq k} \bigcup_{j=0}^{n-1} f^j(u^{-1}(n))$ for every $k \geq 1$. Since we are assuming $\sum_{n\geq 1} \sum_{j=0}^{n-1} \mathrm{Leb}(f^j(u^{-1}(n))) < \infty$, we have $\mathrm{Leb}\left(\bigcup_{n \geq k} \bigcup_{j=0}^{n-1} f^j(u^{-1}(n))\right) \to 0$, when $k \to \infty$. This completes the proof of Lemma 2.1. □

Lemma 2.2. *Let $(U_n)_n$ be a concatenated collection. If*

$$\sup\{\, u(y) \;:\; y \in \cup_{n\geq 1} U_n \text{ and } x \in C(y) \,\} \leq N,$$

then $f^{-n}(x) \subset U_n \cup \cdots \cup U_{n+N}$ for all $n \geq 1$.

Proof. Assume that $\sup\{\, u(y) \;:\; y \in \cup_{n\geq 1} U_n \text{ and } x \in C(y) \,\} \leq N$, and take $z \in f^{-n}(x)$. Let $z_j = f^j(z)$ for each $j \geq 0$. We distinguish the cases $x \in C(z)$ and $x \notin C(z)$. If $x \in C(z)$, then $n \leq u(z) \leq n + N$. Hence $z \in U_{u(z)} \subset U_n \cup \cdots \cup U_{n+N}$. If $x \notin C(z)$, then letting $u_0 = u(z)$ we must have $u_0 < n$. Let $u_1 = u(z_{u_0})$. If $u_0 + u_1 < n$ we take $u_2 = u(z_{u_0+u_1})$. We proceed in this way until we find the first $s \leq n$ such that $n \leq u_0 + \cdots + u_s$. Note that $u_s = u(z_{u_0+\cdots+u_{s-1}})$, and by the choice of s we must have $x \in C(z_{u_0+\cdots+u_{s-1}})$. Our assumption implies that $u(z_{u_0+\cdots+u_{s-1}}) \leq N$, and so $u_0 + \cdots + u_s \leq n + N$. By construction we have

$$\begin{aligned} z &\in U_{u_0} \\ f^{u_0}(z) = z_{u_0} &\in U_{u_1} \\ f^{u_0+u_1}(z) = z_{u_0+u_1} &\in U_{u_2} \\ &\vdots \\ f^{u_0+\cdots u_{s-1}}(z) = z_{u_0+\cdots u_{s-1}} &\in U_{u_s} \end{aligned}$$

By the definition of a concatenated collection we conclude that $z \in U_{u_0+u_1+\cdots+u_s}$. □

3. Proofs of main results

Let us now prove Theorem 1.2. Suppose that $h \in L^p(\mathrm{Leb})$, for some $p > 3$. This implies that $\sum_{n\geq 1} n^p \,\mathrm{Leb}(h^{-1}(n)) < \infty$, and so there exists some constant $K > 0$ such that

$$\mathrm{Leb}(h^{-1}(n)) \leq Kn^{-p}, \quad \text{for every } n \geq 1.$$

Now, taking $0 < \gamma < (p-3)/(p-1)$ we have for some $K' > 0$

$$\sum_{n=1}^{\infty} n \left(\sum_{k=n}^{\infty} \mathrm{Leb}(h^{-1}(k)) \right)^{1-\gamma} \leq \sum_{n=1}^{\infty} n (K'/n^{p-1})^{1-\gamma} < \infty.$$

Defining

$$U_n = \{x \in M \ : |\det Df^n(x)| \geq b_n\},$$

then we have that $(U_n)_n$ is a concatenated collection with respect to the Lebesgue measure. Moreover, setting

$$U_n^* = U_n \setminus (U_1 \cup ... \cup U_{n-1})$$

one has $U_n^* \subset \bigcup_{m\geq n} h^{-1}(m)$, for otherwise there would be $x \in U_n^* \cap h^{-1}(m)$ with $m < n$, and so $a_m \geq b_m > |\det Df^m(x)| \geq a_m$, which is not possible. As $|\det Df^j(x)| < b_j$ for every $x \in U_n^*$ and $j < n$, we get $\mathrm{Leb}(f^j(U_n^*)) \leq b_j \,\mathrm{Leb}(U_n^*)$ for each $j < n$. Hence

$$\begin{aligned}
\sum_{n=n_0+1}^{\infty} \sum_{j=0}^{n-1} \mathrm{Leb}(f^j(U_n^*)) &\leq \sum_{n=n_0+1}^{\infty} \sum_{j=0}^{n-1} b_j \,\mathrm{Leb}(U_n^*) \\
&\leq \sum_{n=n_0+1}^{\infty} \sum_{j=0}^{n_0-1} b_j \,\mathrm{Leb}(U_n^*) + \sum_{n=n_0+1}^{\infty} \sum_{j=n_0}^{n-1} b_j \,\mathrm{Leb}(U_n^*) \\
&\leq \sum_{j=0}^{n_0-1} b_j + \sum_{n=n_0+1}^{\infty} \sum_{j=n_0}^{n-1} b_j \,\mathrm{Leb}(U_n^*)
\end{aligned}$$

Now we just have to check that the last term in the sum above is finite. Indeed,

$$\begin{aligned}
\sum_{n=n_0+1}^{\infty}\sum_{j=n_0}^{n-1} b_j \operatorname{Leb}(U_n^*) &\le \sum_{n=n_0+1}^{\infty}\sum_{j=n_0}^{n-1} b_j \sum_{k=n}^{\infty}\operatorname{Leb}(h^{-1}(k)) \\
&\le \sum_{n=n_0+1}^{\infty} n b_n \sum_{k=n}^{\infty}\operatorname{Leb}(h^{-1}(k)) \\
&\le \sum_{n=n_0+1}^{\infty} n \left(\sum_{k=n}^{\infty}\operatorname{Leb}(h^{-1}(k)\right)^{-\gamma} \sum_{k=n}^{\infty}\operatorname{Leb}(h^{-1}(k)) \\
&= \sum_{n=n_0+1}^{\infty} n \left(\sum_{k=n}^{\infty}\operatorname{Leb}(h^{-1}(k))\right)^{1-\gamma} < \infty.
\end{aligned}$$

Applying Lemmas 2.1 and 2.2, we get for each generic point $x \in M$ a positive integer number N_x such that if $y \in f^{-n}(x)$ then $y \in U_{n+s}$ for some $0 \le s \le N_x$. Therefore, $|\det Df^{n+s}(y)| > b_{n+s} \ge b_n$. Then, taking $C_x = K^{-N_x}$, where $K = \sup\{|\det Df(z)|\colon z \in M\}$, we obtain the conclusion of Theorem 1.1:

$$|\det Df^n(y)| = \frac{|\det Df^{n+s}(y)|}{|\det Df^s(x)|} > C_x b_n.$$

Now we explain how we use Theorem 1.1 to prove Corollary 1.2. Recall that in Corollary 1.2 we have $a_n = e^{\lambda n}$ for each $n \in \mathbb{N}$. Assume first that $\operatorname{Leb}(\Gamma_n) \le \mathcal{O}(e^{-c'n})$ for some $c' > 0$. Then it is possible to choose $c > 0$ such that $b_n = e^{cn}$, for $n \ge n_0$. The other two cases are obtained under similar considerations.

4. Examples: non-uniformly expanding maps

An important class of dynamical systems where we can immediately apply our results are the non-uniformly expanding dynamical maps introduced in [2]. As particular examples of this kind of systems we present below quadratic maps and the higher dimensional Viana maps. Quadratic maps. Let $f_a\colon [-1,1] \to [-1,1]$ be given by $f_a(x) = 1 - ax^2$, for $0 < a \le 2$. Results in [3, 8] give that for a positive Lebesgue measure set of parameters f_a in non-uniformly expanding. Ongoing work [5] gives that for a positive Lebesgue measure set of parameters there are $C, c > 0$ such that $\text{Leb}(\Gamma_n) \leq Ce^{-cn}$ for every $n \geq 1$.

Thus, it follows from Corollary 1.2 that *we may find $\beta > 0$ such for Lebesgue almost every $x \in I$ there is $C_x > 0$ such that $|(f^n)'(y)| > C_x e^{\beta n}$ for every $y \in f^{-n}(x)$.*

Viana maps. Let $a_0 \in (1, 2)$ be such that the critical point $x = 0$ is pre-periodic for the quadratic map $Q(x) = a_0 - x^2$. Let $S^1 = \mathbb{R}/\mathbb{Z}$ and $b : S^1 \to \mathbb{R}$ given by $b(s) = \sin(2\pi s)$. For fixed small $\alpha > 0$, consider the map $\hat{f}$ from $S^1 \times \mathbb{R}$ into itself given by $\hat{f}(s, x) = \big(\hat{g}(s), \hat{q}(s, x)\big)$, where $\hat{q}(s, x) = a(s) - x^2$ with $a(s) = a_0 + \alpha b(s)$, and $\hat{g}$ is the uniformly expanding map of S^1 defined by $\hat{g}(s) = ds \pmod{\mathbb{Z}}$ for some integer $d \geq 2$. For $\alpha > 0$ small enough there is an interval $I \subset (-2, 2)$ for which $\hat{f}(S^1 \times I)$ is contained in the interior of $S^1 \times I$. Thus, any map f sufficiently close to $\hat{f}$ in the C^0 topology has $S^1 \times I$ as a forward invariant region. Moreover, there are $C, c > 0$ such that $\text{Leb}(\Gamma_n) \leq Ce^{-c\sqrt{n}}$ for every $n \geq 1$; see [1, 4, 9].

Thus, it follows from Corollary 1.2 that *we may find $\beta > 0$ such for Lebesgue almost every $X \in S^1 \times I$ there is a constant $C_X > 0$ such that $|\det Df^n(Y)| > C_X e^{\beta\sqrt{n}}$ for every $Y \in f^{-n}(X)$.*

References

[1] J. F. Alves, V. Araújo, *Random perturbations of nonuniformly expanding maps*, Astérisque **286** (2003), 25-62.

[2] J. F. Alves, C. Bonatti, M. Viana, *SRB measures for partially hyperbolic systems whose central direction is mostly expanding*, Invent. Math. **140** (2000), 351-398.

[3] M. Benedicks, L. Carleson, *On iterations of* $1 - ax^2$ *on* $(-1, 1)$, Ann. Math. **122** (1985), 1-25.

[4] J. Buzzi, O. Sester, M. Tsujii, *Weakly expanding skew-products of quadratic maps*, Ergodic Theory Dynam. Systems **23** (2003), no. 5, 1401–1414

[5] J. Freitas, in preparation.

[6] A. Castro, *Backward inducing and exponential decay of correlations for partially hyperbolic attractors*, Israel J. Math. **130** (2002), 29-75.

[7] A. Castro, *Fast mixing for attractors with mostly contracting central direction*, Ergodic Th. Dynam. & Syst., to appear.

[8] M. Jakobson, *Absolutely continuous invariant measures for one-parameter families of one-dimensional maps*, Comm. Math. Phys. **81** (1981), 39-88.

[9] M. Viana, *Multidimensional non-hyperbolic attractors*, Publ. Math. IHES **85** (1997), 63-96.

Departamento de Matemática Pura, Faculdade de Ciências do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal

E-mail address: jfalves@fc.up.pt

Departamento de Matemática, Universidade Federal da Bahia, Av. Ademar de Barros s/n, 40170-110 Salvador, Brazil.

E-mail address: viltonj@ufba.br

Departamento de Matemática, Universidade Federal da Bahia, Av. Ademar de Barros s/n, 40170-110 Salvador, Brazil.

E-mail address: armando@im.ufba.br