NON-ZERO LYAPUNOV EXPONENTS, NO SIGN CHANGES AND AXIOM A

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Abstract. Let $f : M \to M$ be a C^1 diffeomorphism of a compact manifold M admitting a dominated splitting $TM = E^{cs} \oplus E^{cu}$. We show that if the Lyapunov exponents of f are nonzero and have the same sign along the E^{cs} and E^{cu} directions on a total probability set (a set with probability one with respect to every f-invariant measure), then f is Axiom A. We also show that a f-ergodic measure whose Lyapunov exponents are all negative must be concentrated on the orbit of a sink (without using Hölder continuity on the derivative Df).

1. Introduction

Let $f : M \to M$ be a C^1 diffeomorphism of a compact finite dimensional manifold M endowed with a Riemannian metric which induces a norm $\|\cdot\|$ on the tangent bundle of M, a distance dist on M and a volume form m that we call Lebesgue measure.

We will assume that f admits a *dominated splitting* $TM = E^{cs} \oplus E^{cu}$, that is, there exists $\lambda \in (0,1)$ such that

$$\|Df \mid E^{cs}(x)\| \cdot \|(Df \mid E^{cu}(f(x)))^{-1}\| \le \lambda. \tag{1.1}$$

This will prevent expansion (respectively contraction) along the E^{cs} (resp. E^{cu}) direction to overcome expansion (resp. contraction) along the E^{cu} (resp. E^{cs}) direction. The names *center-stable* bundle for E^{cs} and *center-unstable* bundle for E^{cu} are commonly used [4, 6]. We let the dimensions of these subbundles to be written $u = \dim E^{cu}$ and $s = \dim E^{cs}$, which are constant on M because *dominated splittings are continuous*[5].

A large theory has been build around systems admitting invariant measures without zero Lyapunov exponents [19, 18, 21], also called *non-uniformly hyperbolic* measures [12, Addendum].

It is well known that for uniquely ergodic systems the behavior of time averages is rather rigid [24]. Some of this rigidity has been extended in

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similar settings for systems with non-uniform expansion and positive Lyapunov exponents or negative Lyapunov exponents along prescribed directions [23, 3, 9, 10] on *total probability sets*: subsets of M with measure one with respect to every f-invariant probability measure. Here we refine these results assuming only non-zero extreme (maximum and minimum) Lyapunov exponents along the two subbundles E^{cs} and E^{cu} on a total probability set, obtaining a characterization for the global dynamics of f.

1.1. **Statements of results.** The Multiplicative Ergodic Theorem of Oseledets [16] ensures that the following asymptotic growth rates exist on a total probability subset

$$\chi^*_\pm = \lim_{n\to+\infty} \frac{1}{n} \log \|(Df^n \mid E^*)^{\pm 1}\| \quad (* = cs \quad \text{or} \quad cu). \tag{1.2}$$

If for some x we have $\chi^*_-(x) \cdot \chi^*_+(x) < 0$, then this means that either *every Lyapunov exponent is positive* or *every Lyapunov exponent is negative* along the E^* direction at x. Another way of stating this is to say that *there are no zero Lyapunov exponents and there are no changes of sign for the Lyapunov exponents along a given subbundle.*

We recall that a f-invariant compact set Λ with a continuous Df-invariant splitting $T_\Lambda M = E^{cs} \oplus E^{cu}$ is said to be *(uniformly) hyperbolic* if there exist constants $C > 0$ and $\sigma > 1$ such that $\|Df^n \mid E^{cs}(x)\| \le C\sigma^{-n}$ and $\|(Df^n \mid E^{cu}(x))^{-1}\| \le C\sigma^{-n}$ for every $x \in \Lambda$ and every $n \ge 1$. Also a diffeomorphism f is *Axiom A* if the set $\mathrm{Per}(f)$ of periodic points of f is hyperbolic and $\Omega(f) = \mathrm{closure}(\mathrm{Per}(f))$ — see e.g. [22].

Axiom A systems have been thoroughly studied both in their geometric and ergodic aspects [7, 8, 17, 22]. We show that in our setting the diffeomorphism f must be Axiom A.

Theorem 1.1. *Let f be a C^1 diffeomorphism of a compact manifold M admitting a Df-invariant dominated splitting on M: $TM = E^{cs} \oplus E^{cu}$.*

If we have both $\chi^{cs}_+ \cdot \chi^{cs}_- < 0$ and $\chi^{cu}_+ \cdot \chi^{cu}_- < 0$ on a total probability set, then f is Axiom A.

Since the support $\mathrm{supp}\,\mu$ or every f-invariant probability measure μ is contained in the non-wandering set $\Omega(f)$ of f, we obtain the following characterization: *a diffeomorphism with a dominated splitting is Axiom A if, and only if, it has non-zero Lyapunov exponents on a total probability set and these exponents do not change sign along the subbundles of the splitting on the hyperbolic subsets.*

The arguments proving this statement provide also the following essential result for our characterization.

Theorem 1.2. *Let μ be an ergodic probability measure with respect to a C^1-diffeomorphism $f : M \to M$ having all its Lyapunov exponents negative. Then μ is concentrated on the orbit of a periodic attractor (sink).*

We stress that *we do not need Hölder continuity of the derivative* in the arguments proving Theorem 1.2 — compare with Corollary S.5.2 of [12, Supplement] where the usual Hölder condition on the derivative in *Pesin's Theory*, or non-uniform hyperbolic theory, is used to construct *hyperbolic blocks*. Obviously by exchanging f with f^{-1} we obtain that *every ergodic probability measure whose Lyapunov exponents are all positive must be concentrated on the orbit of a periodic repeller (source).*

First in the following subsection we present preliminary results already known needed for the arguments proving Theorem 1.2, in Section 2. Then we show how to obtain Theorem 1.1 in Section 3.

1.2. **Preliminary results.** This standard result, the Ergodic Decomposition Theorem, will be repeatedly used.

Proposition 1.3. *Let $f : X \to X$ be a measurable (Borelean) invertible transformation on the compact metric space X such that the set of f-invariant probability measures $\mathcal{M}(f, X)$ is non-empty. Then there exists a total probability subset Σ such that*

- *for every $x \in \Sigma$ the weak* limit of $|n|^{-1}\sum_{j=0}^{n-1}\delta_{f^j(x)}$ when $n \to \pm\infty$ exists and equals an f-ergodic probability measure μ_x;*
- *for every $\mu \in \mathcal{M}(f, X)$ and every μ-integrable $\varphi : X \to \mathbf{R}$, φ is μ_x-integrable for μ-almost every x and*

$$\int \varphi \, d\mu = \int\!\left(\int \varphi \, d\mu_x\right) \, d\mu(x).$$

Proof. See Chapter 2 of Mañé [15]. □

In what follows we use the notation and definitions of the previous section.

Lemma 1.4. *Letting $* = cs$ or cu, then $\chi_+^* + \chi_-^* \geq 0$ on a total probability set.*

Proof. Clearly $\|Df^n \mid E^*\| \cdot \|(Df^n \mid E^*)^{-1}\| \geq \|Id\| = 1$ for every $x \in M$ and all $n \in \mathbf{N}$. □

Lemma 1.5. *If the splitting $TM = E^{cs} \oplus E^{cu}$ is dominated, then $\chi_+^{cs} + \chi_-^{cu} < -\log\lambda < 0$ on a total probability set.*

Proof. Clearly $\|Df^n \mid E^{cs}\| \cdot \|(Df^n \mid E^{cu})^{-1}\| \leq \lambda^n$ for every $x \in M$ and all $n \in \mathbf{N}$ from (1.1). □

These inequalities will be very useful to study several cases in what follows.

2. Negative Lyapunov exponents and sinks

Here we prove Theorem 1.2. Let f be a C^1 diffeomorphism of a compact closed manifold M and μ an f-ergodic probability measure such that all Lyapunov exponents are negative μ-almost everywhere:

$$\lim_{n\to\infty} \frac{1}{n} \log \|Df^n(x)\| < 0 \quad \mu - \text{a.e.} x \in M.$$

Since f is invertible the Lyapunov exponents for $g = f^{-1}$ are all positive μ-almost everywhere (see e.g. [12, Supplement]). By Kingman's Subadditive Ergodic Theorem [14, 13] and a corollary obtained by Furstenberg-Kesten [11] (see [24] for statements in the setting of Differentiable Ergodic Theory) we know that

$$\inf_{n\ge1}\frac{1}{n}\int\log\|(Dg^n)^{-1}\|\,d\mu=\int\lim_{n\to\infty}\frac{1}{n}\log\|(Dg^n)^{-1}\|\,d\mu<0.$$

Hence for some $k\ge1$ we have $\int\log\|(Dg)^{-1}\|\,d\mu<0$ and μ is g^k-invariant. From now on we write g for g^k to simplify indexes in what follows. This ensures that

$$\int\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log\|Dg(g^j(x))^{-1}\|\,d\mu(x)=\int\log\|(Dg)^{-1}\|\,d\mu<0.$$

Thus there exists a positive μ-measure subset E of M such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log\|Dg(g^j(x))^{-1}\|<\frac{1}{2}\int\log\|(Dg)^{-1}\|\,d\mu<0\quad\text{for all } x\in E.$$

The following result from Pliss will be very useful together with the last inequality above.

Lemma 2.1. *Let $H\ge c_2>c_1>0$ and $\zeta=(c_2-c_1)/(H-c_1)$. Given real numbers $a_1,\dots,a_N$ satisfying*

$$\sum_{j=1}^{N}a_j\ge c_2N\quad\textit{and } a_j\le H\ \textit{ for all } 1\le j\le N,$$

there are $\ell>\zeta N$ and $1<n_1<\dots<n_\ell\le N$ such that

$$\sum_{j=n+1}^{n_i}a_j\ge c_1\cdot(n_i-n)\ \textit{ for each } 0\le n<n_i,\ i=1,\dots,\ell.$$

Proof. See [20] or [15, Chapter IV.11]. □

Now we set

$$H=-\log\inf_{x\in M}\|Dg(x)^{-1}\|,\quad c_2=-\int\log\|(Dg)^{-1}\|\,d\mu\quad\text{and } c_1=c_2/2,$$

and also for fixed $x\in E$ we let $a_j=-\log\|Dg(g^j(x))\|$ for all $j\ge1$. Then for $\zeta=c_2/(2H-c_2)>0$ and for big enough N Pliss' result above ensures that there are $\ell>\zeta N$ and $1<n_1<\dots<n_\ell\le N$ such that for each $0\le n<n_i$ and $i=1,\dots,\ell$

$$\prod_{j=n+1}^{n_i}\|Dg(g^j(x))^{-1}\|\le e^{-c_1(n_i-n)}. \tag{2.3}$$

We call these times n_i *hyperbolic times* for the g-orbit of x — see [4, 2, 1] for definitions, properties and examples of application of this notion. Polis' Lemma above ensures that for every $x\in E$ there are infinitely many hyperbolic times with positive density at infinity (bigger than ζ). It is worth noting that N above must be taken sufficiently large depending on the chosen $x \in E$ but ζ depends only on the minimum value of $\|Dg(z)^{-1}\|$ for $z \in M$ and on $\int \log \|(Dg)^{-1}\| \, d\mu$. Now (2.3) shows that

$$\|Dg^{n_i-n}(g^{n+1}(x))\| \le \lambda^{n_i-n} \quad \text{with} \quad \lambda = \exp\big(-\frac{1}{2}\int \log \|(Dg)^{-1}\| \, d\mu\big) \tag{2.4}$$

for all $i = 1, \ldots, \ell$ and $0 \le n < n_i$. This uniform contractive property together with the diffeomorphism character of g are enough for the following result.

Lemma 2.2. *There exist $\delta_1 > 0$ (depending only on g and c_1) and $\lambda_1 = \sqrt{\lambda} \in (0,1)$ such that if n is a hyperbolic time for $x \in M$, then for every $0 \le j < n$ there are neighborhoods V_{n-j} of $g^{n-j}(x)$ such that $g^{n-j}(V_{n-j}) = B(g^n(x), \delta_1)$ and for all $y, z \in B(g^n(x), \delta_1)$ we have*

$$\operatorname{dist}(g^{n-j}(y), g^{n-j}(z)) \le \lambda_1^j \operatorname{dist}(y, z).$$

Proof. See Lemma 5.2 and Corollary 5.3 of [4]. □

Now we use the compactness of M together with the fact that $x \in E$ has infinitely many hyperbolic times $n_1 < n_2 < \ldots$ to obtain two such times $m > n$ satisfying $\operatorname{dist}(g^m(x), g^n(x)) < (1-\lambda_1)\delta_1/2$. By Lemma 2.2 we have a neighborhood V_n of $g^n(x)$ such that

$$g^{m-n}(V_n) = B(g^m(x), \delta_1) \quad \text{and} \quad V_n \subset B(g^n(x), \lambda_1^{m-n}\delta_1) \subset B(g^m(x), \delta_1),$$

where the last inclusion comes from the choice of m, n and because $m-n \ge 1$.

This shows that g^{n-m} sends the ball $B(g^m(x), \delta_1)$ strictly inside itself. Thus every point in the ball belongs to the basin of attraction of a periodic sink for g^{-1}, which will be also a sink for f. In particular $g^n(x)$ and also x are in the basin of attraction of this periodic attracting orbit of f.

The point $x \in E$ may be taken μ-generic since $\mu(E) > 0$. Since the positive f-orbit of x tends to the orbit of a sink, $\mu = \lim_n n^{-1} \sum_{j=0}^{n-1} \delta_{f^j(x)}$ must equal the Dirac mass concentrated on the orbit of the sink. This concludes the proof of Theorem 1.2.

2.1. Negative upper Lyapunov exponents and sinks.

The following result, obtained as a consequence of Theorem 1.2, will be used in the proof of Theorem 1.1 in the next section.

Lemma 2.3. *Let f be a C^1 diffeomorphism of a compact manifold M admitting a Df-invariant dominated splitting on M: $TM = E^{cs} \oplus E^{cu}$. We assume also that we have $\chi_+^{cu} \cdot \chi_-^{cu} < 0$ on a total probability set.*

Let $x \in M$ be such that the upper Lyapunov exponents along the center-unstable direction are negative

$$\overline{\chi}_+^{cu}(x) = \limsup_{n\to+\infty} \frac{1}{n} \log \|Df^n \mid E^*(x)\| < 0. \tag{2.5}$$

Then there exists a sink $s \in M$ such that $x \in B(\{s\})$.

Here $B(\{s\}) = \{x \in M : \text{dist}(f^n(x), \mathcal{O}(s)) \to 0 \text{ when } n \to \infty\}$ is the topological basin of the sink $s \in S$.

Proof. Let f and x be as in the statement. Since $|\det Df^n \mid E^{cu}(x)| \leq u \cdot \|Df^n(x)\|$ we see that

$$\limsup_{n\to\infty} \frac{1}{n} \log|\det Df^n \mid E^{cu}(x)| = \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log|\det Df \mid E^{cu}(f^j(x))|$$

is also negative. Then for every given weak* accumulation point μ of $\mu_n = n^{-1}\sum_{j=0}^{n-1} \delta_{f^j(x)}$ we must have $\mu(\log|\det Df \mid E^{cu}(x)|) < 0$.

The Ergodic Decomposition Theorem ensures that there exists some μ-generic point y such that ergodic component μ_y of μ satisfies

$$\mu_y(\log|\det Df \mid E^{cu}(x)|) < 0.$$

Moreover we can assume without loss that y belongs to the support of μ since $\mu(\operatorname{supp}\mu) = 1$.

The Multiplicative Ergodic Theorem now assures that there exists some negative Lyapunov exponent along the E^{cu} direction μ_y-almost everywhere. The domination condition ensures (Lemma 1.5) that all Lyapunov exponents along the E^{cs} direction are negative μ_y-almost everywhere, and the total probability condition ($\chi_+^{cu} \cdot \chi_-^{cu} < 0$) guarantees that every Lyapunov exponent along the E^{cu} direction is negative as well.

According to Theorem 1.2 the measure μ_y must be a Dirac mass concentrated on the orbit of a sink s for f. Since $\mu(B(y,\varepsilon)) > 0$ for all $\varepsilon > 0$ and $\mu(\partial B(y,\varepsilon)) = 0$ and also $B(y,\varepsilon) \subset B(\{s\})$ for arbitrarily small values of ε, there must exist $n > 1$ such that $\mu_n(B(y,\varepsilon)) > 0$. Hence $x \in B(\{s\})$, finishing the proof of the lemma. □

3. Proof of Theorem 1.1

In the setting of the statement of Theorem 1.1, we have the following cases. Let x be a point in a total probability subset Σ of M with respect to f.

Case A: $\chi_+^{cs}(x) < 0$ and $\chi_+^{cu}(x) < 0$.

According to Lemma 2.3, this x must be in the topological basin of some periodic attracting orbit (sink).

Case B: $\chi_-^{cs}(x) < 0$ and $\chi_+^{cu}(x) < 0$.

Since $\chi_-^{cs}(x) < 0$, Lemma 1.4 ensures that $\chi_+^{cs}(x) > 0$ and thus $\chi_-^{cu}(x) < 0$ by Lemma 1.5. Hence $\chi_+^{cu}(x) < 0$ is impossible again by Lemma 1.4. This case is excluded in the present setting.

Case C: $\chi_-^{cs}(x) < 0$ and $\chi_-^{cu}(x) < 0$.

Now x has positive exponents in every direction and $x \in \Sigma$. Then there exists a f-ergodic probability measure μ such that $\mu = \mu_x$ as in Proposition 1.3. Since f is a diffeomorphism, the Lyapunov exponents also exist when $n \to -\infty$ and have the opposite sign of the exponents calculated when $n \to +\infty$. Hence we are in Case A for f^{-1}. Thus x

belongs to the basin of a sink s for f^{-1} and so μ is the Dirac mass concentrated on this sink. But the only way for $\mu = \mu_x$ is for x to belong to the same periodic orbit, thus x belongs to the orbit of a periodic repeller (source) for f.

The set A of points in Case A forms an open subset of M: it is the union of the topological basins of attraction of an at most denumerable family of sinks. The set C of points in Case C is a denumerable subset. Both these subsets are f-invariant, hence every f-ergodic probability measure gives weight zero or one to each of them.

In addition, the set C is *forward isolated*, that is, for every orbit $\mathcal{O}(r)$ of a repeller r in C there exists a neighborhood U of $\mathcal{O}(r)$ such that every point $z \in U \setminus \mathcal{O}(r)$ eventually leaves U: $f^k(z) \in M \setminus U$ for some $k \geq 1$. Hence there exists an open neighborhood $\mathcal{U}$ of C whose complement is closed, thus compact, and forward invariant: $f(M \setminus \mathcal{U}) \subset M \setminus \mathcal{U}$.

Case D: $\chi^{cs}_+(x) < 0$ and $\chi^{cu}_-(x) < 0$.

Now we have positive Lyapunov exponents in the E^{cu} direction and negative Lyapunov exponents in the E^{cs} direction.

Let D be the set of points satisfying the properties of Case D above. This set is also f-invariant and every f-ergodic measure μ supported in D is such that μ-almost every point x has positive Lyapunov exponents in the E^{cu} direction and negative Lyapunov exponents in the E^{cs} direction. Hence D does not intersect A nor C and so is contained in the closed forward invariant set $F = M \setminus (\mathcal{U} \cup A)$. Thus $D \subset \Lambda = \cap_{j\geq 1} f^j(F)$ where Λ is compact.

We claim that D is a total probability subset of Λ. For if μ is a f-invariant probability measure supported in Λ, then every Lyapunov exponent of μ must be as in Case D. Otherwise there would exist an ergodic component of μ having Lyapunov exponents as in Cases A or C, which is not possible since Λ is disjoint from $A \cup C$.

We have shown that *on a total probability subset of* Λ *the Lyapunov exponents along* E^{cu} *are positive and the Lyapunov exponents along* E^{cs} *are negative*, thus by the results in [23, 9] Λ must be a uniformly hyperbolic set for f.

3.1. **Finiteness of sinks and sources.**

Now we show that the number of sinks and sources must be finite.

We argue by contradiction, assuming that there are infinitely many orbits of sinks and letting μ_n be the normalized Dirac masses concentrated on these orbits, $n \geq 1$. These are f-ergodic probability measures and $\mu_n \neq \mu_n$ for all $m \neq n$. Now we let μ be any weak* accumulation point of the sequence $(\mu_n)_n$, which is a f-invariant probability measure.

We have that both $\mu_n(\log|\det Df \mid E^{cs}|)$ and $\mu_n(\log|\det Df \mid E^{cu}|)$ are strictly negative for every $n \geq 1$, since these integrals equal the integrated sum of the Lyapunov exponents along the respective directions (this follows from Oseledets' Multiplicative Ergodic Theorem [16]). Hence both $\mu(\log|\det Df \mid E^{cs}|)$ and $\mu(\log|\det Df \mid E^{cu}|)$ are non-positive. Therefore there exists $x \in \operatorname{supp}\mu$ such that the minimum of the Lyapunov exponents along E^{cu} over $\mathcal{O}(x)$ is non-positive, i.e., the Lyapunov exponents along $\mathcal{O}(x)$ are well defined and $\chi_-^{cu} \geq 0$.

The standing hypothesis about the Lyapunov exponents forces $\chi_+^{cu}(x) < 0$ and the dominated decomposition implies $\chi_+^{cs}(x) < 0$ also.

Now Lemma 2.3 shows that x is in the basin of some sink. Since $x \in \operatorname{supp}\mu$ we have $\mu(B(x,\varepsilon)) > 0$ and $\mu(\partial B(x,\varepsilon)) = 0$ for values of $\varepsilon > 0$ arbitrarily close to zero. Thus for a fixed small $\varepsilon > 0$ we can assume that $B(x,\varepsilon)$ is contained in the basin of a sink and $\mu_n(B(x,\varepsilon)) > 0$ for all n big enough. This shows that for big n the f-ergodic measures μ_n must coincide with the Dirac masses along the orbit of a fixed sink, contradicting the initial assumption that the μ_n where pairwise distinct.

We conclude that there can only be finitely many sinks for f in this setting. Since source are sinks for f^{-1}, the same arguments apply dually to conclude that there are finitely many sources also.

3.2. **Hyperbolicity of the chain recurrent set.** To finish the proof of Theorem 1.1 we study the chain recurrent set for f.

Let $x \in M$ be chain recurrent for f. Then either x is a sink or a source, or x belongs to Λ. Indeed, every chain recurrent point that does not belong either to the orbit of a sink or to the orbit of a source, must be in $M \setminus (\mathcal{U} \cup A)$ — recall that A is the union of the basins of the sinks of f and $\mathcal{U}$ is an isolating neighborhood of the set C of sources.

This shows that $\mathcal{R}(f) = S \cup C \cup \Lambda$ where S is the set of orbits of sinks. Hence $\mathcal{R}(f)$ is hyperbolic, thus $\mathcal{R}(f) = \text{closure}(\text{Per}(f))$ and consequently f is Axiom A, see [22].

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