ON A NEW INDEX TRANSFORMATION RELATED TO THE PRODUCT OF MACDONALD FUNCTIONS

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Abstract

We deal with an integral transformation, which is associated with the product of the Macdonald functions $K_{i\tau}\left(\sqrt{x^2+y^2}-y\right)K_{i\tau}\left(\sqrt{x^2+y^2}+y\right)$, where $(x,y)\in\mathbf{R}_+\times\mathbf{R}_+$ and $i\tau,\ \tau\in\mathbf{R}_+$ is a pure imaginary index. An integration process is realized with respect to τ. In the limit case when $y=0$ it gives the Lebedev transformation with the square of Macdonald functions. The Bochner type representation theorem, the Plancherel theorem and Parsevals equality are proved by using a relationship with the Mellin and the Kontorovich-Lebedev transforms. An application of the introduced transformation is given to find a solution of the Neumann weighted problem for a second order partial differential equation.

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1 Introduction and preliminary results

Let $\mathbf{R}_+=(0,\infty)$ and $(x,y)\in\mathbf{R}_+\times\mathbf{R}_+$. Let $f:\mathbf{R}_+\to\mathbf{C}$ be a measurable function. We consider the following transformation

$$[\mathcal{G}f](x,y)\equiv[\mathcal{G};f(\tau)](x,y)=$$

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$$= \frac{2}{\sqrt{\pi}} \int_0^\infty K_{i\tau}\left(\sqrt{x^2+y^2}-y\right) K_{i\tau}\left(\sqrt{x^2+y^2}+y\right) f(\tau)d\tau, \tag{1.1}$$

where the integral is convergent in a definite sense, which we will clarify below. Here $K_\nu(z)$ is the modified Bessel function or the Macdonald function (cf. [1, Vol. II]) of the pure imaginary index $\nu = i\tau$. The function $K_\nu(z)$ satisfies the differential equation

$$z^2 \frac{d^2u}{dz^2} + z\frac{du}{dz} - (z^2+\nu^2)u = 0, \tag{1.2}$$

for which it is the solution that remains bounded as z tends to infinity on the real line. The Macdonald function has the asymptotic behaviour [1, Vol. II]

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}[1+O(1/z)], \qquad z\to\infty, \tag{1.3}$$

and near the origin

$$z^\nu K_\nu(z) = 2^{\nu-1}\Gamma(\nu) + o(1),\ \ z\to 0, \tag{1.4}$$

$$K_0(z) = -\log z + O(1),\ \ z\to 0. \tag{1.5}$$

It can be given by the following integral [1, Vol.II]

$$K_\nu(z) = \int_0^\infty e^{-z\cosh u}\cosh \nu u du,\ \ \mathrm{Re} z > 0. \tag{1.6}$$

It has another representation in terms of the inverse Mellin transform of the product of the Euler gamma-functions [3], [6]. Precisely we find

$$K_\nu(2z) = \frac{1}{8\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right) z^{-s}ds,\ \ \gamma>0. \tag{1.7}$$

The product of functions $K_{i\tau}(x)$ of different arguments can be represented in turn by the Macdonald formula (cf. [1, Vol.II], [6])

$$K_{i\tau}(x)K_{i\tau}(y) = \frac{1}{2}\int_0^\infty e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy}+\frac{yu}{x}\right)} K_{i\tau}(u)\frac{du}{u}. \tag{1.8}$$

In particular, for the kernel of the transformation (1.1) we obtain the following representation

$$K_{i\tau}\left(\sqrt{x^2+y^2}-y\right) K_{i\tau}\left(\sqrt{x^2+y^2}+y\right) = \frac{1}{2}\int_0^\infty e^{-2\frac{y^2}{x^2}u-\frac{x^2}{2u}} e^{-u} K_{i\tau}(u)\frac{du}{u}. \tag{1.9}$$

Meanwhile, the Mellin direct transform [1, Vol.I], [3]

$$f^{\mathcal{M}}(s) = \int_0^\infty f(x)x^{s-1}dx,\ \ s=\gamma+it, \tag{1.10}$$

is defined for the Lebesgue space $L_1(\mathbf{R}_+; x^{\gamma-1}dx)$ with the norm

$$||f||_1 = \int_0^\infty |f(x)|x^{\gamma-1}dx < +\infty$$

and maps it into the space of bounded continuous functions on the vertical line $\gamma + it,\ t \in \mathbf{R}$ vanishing at infinity $|t| \to +\infty$. However, if $f \in L_2(\mathbf{R}_+; x^{2\gamma-1}dx)$, which is normed by

$$||f||_2 = \left(\int_0^\infty |f(x)|^2 x^{2\gamma-1}dx\right)^{1/2} < +\infty, \tag{1.11}$$

then it forms a one-to-one correspondence and isomorphically maps on the space $L_2((\gamma - i\infty, \gamma + i\infty); dt)$. Integral (1.10) in this case converges in mean with respect to the norm of the latter L_2-space. The inverse operator is given by

$$f(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f^{\mathcal{M}}(s)x^{-s}ds,\ s = \gamma + it,\ x > 0, \tag{1.12}$$

where integral (1.12) is convergent in mean with respect to the norm (1.11). Moreover, the operator $f^{\mathcal{M}}$ is an isometric isomorphism between two mentioned Hilbert spaces and the following Parseval equality

$$\int_0^\infty |f(x)|^2 x^{2\gamma-1}dx = \frac{1}{2\pi}\int_{-\infty}^\infty |f^{\mathcal{M}}(\gamma + it)|^2 dt \tag{1.13}$$

holds true.

We define now operators of the Kontorovich-Lebedev [6] and the Lebedev [2], [7] transformations by the formulas

$$(KLf)(x) = \int_0^\infty K_{i\tau}(x)f(\tau)d\tau, \tag{1.14}$$

$$(Lef)(x) = \frac{2}{\sqrt{\pi}}\int_0^\infty K_{i\tau}^2(x)f(\tau)d\tau, \tag{1.15}$$

respectively. As it is easily seen, operator (1.15) is a limit case of the transformation (1.1) when we put $y = 0$. A process of integration in formulas (1.1), (1.14), (1.15) is realized with respect to the index (a parameter) of the Macdonald function. Therefore we call such a class of integral transforms as index transformations [6].

In order to prove the representation theorem for transformation (1.1) in the next section we introduce here (cf. [8], [9]) the following Banach space of functions $f \in L^*(\mathbf{R}_+)$ or space of A-type, whose Fourier cosine transforms

$$(F_c f)(x) \equiv (F_c; f(t))(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos tx dt \tag{1.16}$$

lie in $L_1(\mathbf{R}_+)$. For the norm in L^* we set

$$||f||_{L^*(\mathbf{R}_+)} = \int_0^\infty |(F_c f)(t)|dt. \tag{1.17}$$

As it follows from the theory of the Fourier transform elements of the space L^* are bounded, continuous functions, which vanish at infinity. As it is proved in [8] the Kontorovich-Lebedev transform (1.14) is a bounded operator in $L^*(\mathbf{R}_+)$ and we have

$$||KLf||_{L^*(\mathbf{R}_+)} \le \frac{\pi}{2}||f||_{L^*(\mathbf{R}_+)}.$$

Moreover the following Bochner's type representation theorem holds true.

Theorem 1. [8]. *Any $f \in L^*(\mathbf{R}_+)$ for all $\tau \in \mathbf{R}_+$ is represented as*

$$f(\tau) = \lim_{\varepsilon \to 0+} \frac{2}{\pi^2}\tau \sinh \pi\tau \int_0^\infty x^{\varepsilon-1} K_{i\tau}(x)dx \int_0^\infty K_{i\mu}(x) f(\mu) d\mu, \tag{1.18}$$

where the convergence is pointwise.

Finally in this section we mention useful formula (2.16.33.10) in [4] for the Mellin transform (1.10) with respect to x of the kernel (1.1)

$$\int_0^\infty x^{s-1} K_{i\tau}\left(\sqrt{x^2+y^2}-y\right) K_{i\tau}\left(\sqrt{x^2+y^2}+y\right) dx$$

$$= \frac{\sqrt{\pi}}{2} y^{s/2} K_{s/2}(2y) \frac{\Gamma\left(\frac{s}{2}+i\tau\right)\Gamma\left(\frac{s}{2}-i\tau\right)}{\Gamma((1+s)/2)}, \tag{1.19}$$

which is true for all $y, \tau > 0$ and $\gamma = \text{Re}s > 0$. Multiplying both sides of (1.18) by $y^{\omega-1}$, $\text{Re}\omega > 0$ we integrate with respect to $y > 0$. Then by using formula (2.16.2.2) in [4] we arrive at the value of the double Mellin transform [3] for the kernel (1.1) as

$$\int_0^\infty \int_0^\infty x^{s-1} y^{\omega-1} K_{i\tau}\left(\sqrt{x^2+y^2}-y\right) K_{i\tau}\left(\sqrt{x^2+y^2}+y\right) dxdy$$

$$= \frac{\sqrt{\pi}}{8}\Gamma(\omega)\Gamma\left(\frac{s+\omega}{2}\right)\frac{\Gamma\left(\frac{s}{2}+i\tau\right)\Gamma\left(\frac{s}{2}-i\tau\right)}{\Gamma((1+s)/2)}.$$

2 The representation theorem

In this section we prove an analog of Theorem 1 for the transformation (1.1). Indeed, we have

Theorem 2. *Let $f \in L^*(\mathbf{R}_+)$. Then for all $\tau \in \mathbf{R}_+$ the following expansion holds true*

$$f(\tau) = \lim_{\varepsilon \to 0+} \frac{16}{\pi^4}\tau \sinh 2\pi\tau \int_0^\infty \int_0^\infty K_{i\tau}\left(\sqrt{x^2+y^2}-y\right) K_{i\tau}\left(\sqrt{x^2+y^2}+y\right) x^{\varepsilon-1} dxdy$$

$$\times \int_0^\infty K_{i\mu}\left(\sqrt{x^2+y^2}-y\right) K_{i\mu}\left(\sqrt{x^2+y^2}+y\right) f(\mu)d\mu, \tag{2.1}$$

where the convergence with respect to $\varepsilon > 0$ is pointwise.

Proof. By taking integral (1.19) and applying the inversion formula (1.12) for the Mellin transform with respect to x we immediately obtain that for all positive x, y, μ the product of Macdonald functions is represented as follows

$$K_{i\mu}\left(\sqrt{x^2+y^2}-y\right) K_{i\mu}\left(\sqrt{x^2+y^2}+y\right) = \frac{1}{2\pi i}\frac{\sqrt{\pi}}{2}\int_{\gamma-i\infty}^{\gamma+i\infty} y^{s/2}K_{s/2}(2y)$$

$$\times\frac{\Gamma\left(\frac{s}{2}+i\mu\right)\Gamma\left(\frac{s}{2}-i\mu\right)}{\Gamma((1+s)/2)}x^{-s}ds, \tag{2.2}$$

where we may choose $\gamma \in (0, \varepsilon)$. We substitute the integral with respect to s from the right-hand side of (2.2) into the inner integral with respect to μ in (2.1) and we change the order of integration. Hence we find

$$\int_0^\infty K_{i\mu}\left(\sqrt{x^2+y^2}-y\right) K_{i\mu}\left(\sqrt{x^2+y^2}+y\right) f(\mu)d\mu = \frac{1}{2\pi i}\frac{\sqrt{\pi}}{2}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{y^{s/2}K_{s/2}(2y)}{\Gamma((1+s)/2)}x^{-s}ds$$

$$\times\int_0^\infty \Gamma\left(\frac{s}{2}+i\mu\right)\Gamma\left(\frac{s}{2}-i\mu\right) f(\mu)d\mu. \tag{2.3}$$

The change of the order of integration in (2.3) we motivate by the use of the Fubini theorem. In fact, since $f \in L^*(\mathbf{R}_+)$, then it belongs to the space of bounded continuous functions on $\mathbf{R}_+$. Consequently, the iterated integral in the right-hand side of (2.3) is majorized by

$$\left|\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{y^{s/2}K_{s/2}(2y)}{\Gamma((1+s)/2)}x^{-s}ds\int_0^\infty \Gamma\left(\frac{s}{2}+i\mu\right)\Gamma\left(\frac{s}{2}-i\mu\right) f(\mu)d\mu\right|$$

$$\leq \text{const.}\, y^{\gamma/2}x^{-\gamma}\int_{\gamma-i\infty}^{\gamma+i\infty}\left|\frac{K_{s/2}(2y)}{\Gamma((1+s)/2)}ds\right|\int_0^\infty \left|\Gamma\left(\frac{s}{2}+i\mu\right)\Gamma\left(\frac{s}{2}-i\mu\right)\right| d\mu. \tag{2.4}$$

It is easily seen, that the product of gamma-functions in the latter integral (2.4) is continuous with respect to $\mu \in \mathbf{R}_+$. Furthermore, making use the representation [10]

$$\Gamma\left(\frac{s}{2}+i\mu\right)\Gamma\left(\frac{s}{2}-i\mu\right) = \frac{\Gamma(s)}{2^{s-2}}\int_0^\infty \frac{\cos(2\mu y)dy}{\cosh^s y}$$

$$= \frac{\Gamma(s+1)}{\mu^2 2^{s+1}}\int_{-\infty}^\infty \frac{e^{2i\mu y}(1-s\sinh^2 y)}{\cosh^{s+2} y}dy, \tag{2.5}$$

we deduce

$$\left|\Gamma\left(\frac{s}{2}+i\mu\right)\Gamma\left(\frac{s}{2}-i\mu\right)\right| \leq \frac{|\Gamma(s+1)|}{\mu^2 2^\gamma}\int_0^\infty \frac{1+|s|\sinh^2 y}{\cosh^{\gamma+2} y}dy$$

$$\leq \frac{|\Gamma(s+1)|}{\mu^2 2^\gamma}\left[\int_0^\infty \frac{dy}{\cosh^2 y} + |s|\int_0^\infty \frac{dy}{\cosh^\gamma y}\right], \mu, \gamma > 0.$$

Now taking into account an inequality (cf. (1.6)) $|K_{s/2}(2y)| \leq K_{\gamma/2}(2y)$, the Stirling asymptotic formula for gamma-functions [1, Vol.I], equalities (2.5) and the latter estimate we majorize for each $y > 0, \gamma \in (0,\varepsilon)$ the iterated integral in the right-hand side of (2.4). This gives the chain of inequalities

$$\int_{\gamma-i\infty}^{\gamma+i\infty}\left|\frac{K_{s/2}(2y)}{\Gamma((1+s)/2)}ds\right|\int_0^\infty \left|\Gamma\left(\frac{s}{2}+i\mu\right)\Gamma\left(\frac{s}{2}-i\mu\right)\right| d\mu \leq K_{\gamma/2}(2y)\int_{\gamma-i\infty}^{\gamma+i\infty}\left|\frac{ds}{\Gamma((1+s)/2)}\right|$$

$$\times\left[\int_0^1 + \int_1^\infty\right]\left|\Gamma\left(\frac{s}{2}+i\mu\right)\Gamma\left(\frac{s}{2}-i\mu\right)\right| d\mu \leq \frac{K_{\gamma/2}(2y)}{2^{\gamma-2}}\int_{\gamma-i\infty}^{\gamma+i\infty}\left|\frac{\Gamma(s)ds}{\Gamma((1+s)/2)}\right|$$

$$\times\int_0^\infty \frac{du}{\cosh^\gamma u} + \frac{K_{\gamma/2}(2y)}{2^\gamma}\left[\int_{\gamma-i\infty}^{\gamma+i\infty}\left|\frac{\Gamma(s+1)ds}{\Gamma((1+s)/2)}\right|\int_0^\infty \frac{du}{\cosh^2 u}\right.$$

$$\left.+\int_{\gamma-i\infty}^{\gamma+i\infty}\left|\frac{s\Gamma(s+1)ds}{\Gamma((1+s)/2)}\right|\int_0^\infty \frac{du}{\cosh^\gamma u}\right] < \infty.$$

Consequently via the Fubini theorem it guarantees the change of the order of integration and equality (2.3). Substituting the right-hand side of (2.3) in (2.1) and denoting by $I_\varepsilon(\tau)$ the corresponding iterated integral under the limit sign we write it in the form

$$I_\varepsilon(\tau) = \frac{1}{2\pi i}\frac{8\tau\sinh 2\pi\tau}{\pi^3\sqrt{\pi}}\int_0^\infty\int_0^\infty K_{i\tau}\left(\sqrt{x^2+y^2}-y\right)K_{i\tau}\left(\sqrt{x^2+y^2}+y\right)x^{\varepsilon-1}dxdy$$

$$\times\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{y^{s/2}K_{s/2}(2y)}{\Gamma((1+s)/2)}x^{-s}ds\int_0^\infty \Gamma\left(\frac{s}{2}+i\mu\right)\Gamma\left(\frac{s}{2}-i\mu\right)f(\mu)d\mu. \tag{2.6}$$

We note here, that in a similar manner invoking the previous estimates and formulas of the asymptotic behaviour (1.3), (1.4), (1.5) for the Macdonald function it is not difficult to realize via Fubini's theorem an integration in (2.6) in any order. Therefore calculating the inner integral with respect to x by formula (1.19) it becomes

$$I_\varepsilon(\tau) = \frac{4}{\pi^3}\frac{\tau\sinh 2\pi\tau}{2\pi i}\int_0^\infty y^{\varepsilon/2}K_{s/2}(2y)K_{(\varepsilon-s)/2}(2y)dy$$

$$\times\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma\left(\frac{\varepsilon-s}{2}+i\tau\right)\Gamma\left(\frac{\varepsilon-s}{2}-i\tau\right)}{\Gamma((1+\varepsilon-s)/2)\Gamma((1+s)/2)}ds\int_0^\infty \Gamma\left(\frac{s}{2}+i\mu\right)\Gamma\left(\frac{s}{2}-i\mu\right)f(\mu)d\mu. \tag{2.7}$$

Hence we calculate the integral with respect to y by using relation 2.16.33.2 from [4], which gives

$$\int_0^\infty y^{\varepsilon/2}K_{s/2}(2y)K_{(\varepsilon-s)/2}(2y)dy = \frac{\sqrt{\pi}}{8\Gamma(1+\varepsilon/2)}\Gamma\left(\frac{1+\varepsilon}{2}\right)\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{1+\varepsilon-s}{2}\right).$$

Substituting this value into (2.7) we obtain

$$I_\varepsilon(\tau) = \frac{\Gamma\left(\frac{1+\varepsilon}{2}\right)}{2\pi^2\sqrt{\pi}\Gamma(1+\varepsilon/2)} \frac{\tau\sinh 2\pi\tau}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma\left(\frac{\varepsilon-s}{2}+i\tau\right)\Gamma\left(\frac{\varepsilon-s}{2}-i\tau\right) ds$$

$$\times \int_0^\infty \Gamma\left(\frac{s}{2}+i\mu\right)\Gamma\left(\frac{s}{2}-i\mu\right) f(\mu) d\mu. \tag{2.8}$$

The integral with respect to s in (2.8) can be treated by employing the Mellin formula (1.7) and the generalized Parseval equality for the Mellin transform (cf. [3], [5]). Thus we deduce

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma\left(\frac{\varepsilon-s}{2}+i\tau\right)\Gamma\left(\frac{\varepsilon-s}{2}-i\tau\right)\Gamma\left(\frac{s}{2}+i\mu\right)\Gamma\left(\frac{s}{2}-i\mu\right) ds$$

$$= 16\int_0^\infty K_{2i\tau}(2x)K_{2i\mu}(2x)x^{\varepsilon-1}dx.$$

Consequently, after substitution this result in (2.8) we appeal to Theorem 1 to write (2.8) in the form

$$I_\varepsilon(\tau) = \frac{8\Gamma\left(\frac{1+\varepsilon}{2}\right)}{\pi^2\sqrt{\pi}\Gamma(1+\varepsilon/2)}\tau\sinh 2\pi\tau \int_0^\infty K_{2i\tau}(2x)x^{\varepsilon-1}dx \int_0^\infty K_{2i\mu}(2x) f(\mu) d\mu$$

$$= \frac{2^{2-\varepsilon}\Gamma\left(\frac{1+\varepsilon}{2}\right)}{\pi^2\sqrt{\pi}\Gamma(1+\varepsilon/2)}\tau\sinh 2\pi\tau \int_0^\infty K_{2i\tau}(x)x^{\varepsilon-1}dx \int_0^\infty K_{i\mu}(x) f\left(\frac{\mu}{2}\right) d\mu. \tag{2.9}$$

Finally, invoking (1.18) we pass to the limit through equality (2.9) and we easily establish the pointwise convergence of $I_\varepsilon(\tau)$ to $f(\tau)$ when $\varepsilon \to 0+$. Theorem 2 is proved.

3 The Plancherel-type theorem

In this section we apply the theory of the modified Kontorovich-Lebedev transformation [6] in order to prove an analog of the Plancherel theorem for the transform $[\mathcal{G};\tau f(\tau)](x,y)$ (cf. (1.1)). Namely, we consider the Kontorovich-Lebedev transform of the form

$$[KLf](x) = 4\int_0^\infty \tau K_{2i\tau}(2x) f(\tau) d\tau. \tag{3.1}$$

As it is known (cf. [6, Chapter 2]), the Kontorovich-Lebedev operator (3.1) is the isometric isomorphism between Hilbert spaces

$$[KLf] : L_2\left(\mathbf{R}_+; \frac{\tau d\tau}{\sinh 2\pi\tau}\right) \leftrightarrow L_2\left(\mathbf{R}_+; x^{-1}dx\right), \tag{3.2}$$

where the integral (3.1) converges in mean with respect to the norm

$$||f||_{L_2(\mathbf{R}_+;\tau[\sinh 2\pi\tau]^{-1}d\tau)} = \left(\int_0^\infty \frac{\tau}{\sinh 2\pi\tau}|f(\tau)|^2 d\tau\right)^{1/2}. \tag{3.3}$$

Furthermore, we have the Parseval identity

$$\int_0^\infty |[KLf](x)|^2 \frac{dx}{x} = 2\pi^2 \int_0^\infty \frac{\tau}{\sinh 2\pi\tau}|f(\tau)|^2 d\tau. \tag{3.4}$$

By using a relationship with the Mellin and Kontorovich-Lebedev transforms we will prove that the operator $[\mathcal{G};\tau f(\tau)](x,y)$ is the one-to-one isometric and isomorphic map:

$$\mathcal{G} : L_2\left(\mathbf{R}_+; \frac{\tau d\tau}{\sinh 2\pi\tau}\right) \leftrightarrow L_2\left(\mathbf{R}_+ \times \mathbf{R}_+; \frac{dxdy}{x}\right), \tag{3.5}$$

where the norm in the space $L_2\left(\mathbf{R}_+ \times \mathbf{R}_+; x^{-1}dxdy\right)$ is defined by

$$||f||_{L_2(\mathbf{R}_+\times\mathbf{R}_+;x^{-1}dxdy)} = \left(\int_0^\infty \int_0^\infty |f(x,y)|^2 \frac{dxdy}{x}\right)^{1/2}. \tag{3.6}$$

Thus we arrive at

Theorem 3. *Let $f \in L_2(\mathbf{R}_+;\tau[\sinh 2\pi\tau]^{-1}d\tau)$. Then as $N \to \infty$ the integral*

$$\frac{2}{\sqrt{\pi}} \int_{1/N}^N \tau K_{i\tau}\left(\sqrt{x^2+y^2}-y\right) K_{i\tau}\left(\sqrt{x^2+y^2}+y\right) f(\tau)d\tau \tag{3.7}$$

converges in mean to $[\mathcal{G};\tau f(\tau)](x,y)$ with respect to the norm (3.6) and a reciprocal integral

$$\frac{8\sinh 2\pi\tau}{\pi^3\sqrt{\pi}} \int_{1/N}^N \int_{1/N}^N K_{i\tau}\left(\sqrt{x^2+y^2}-y\right) K_{i\tau}\left(\sqrt{x^2+y^2}+y\right) [\mathcal{G};\tau f(\tau)](x,y)\frac{dxdy}{x} \tag{3.8}$$

converges in mean to $f(\tau)$ with respect to the norm (3.3). Moreover, almost for all $\tau \in \mathbf{R}_+$ and $(x,y) \in \mathbf{R}_+ \times \mathbf{R}_+$, respectively, the following reciprocal formulas take place

$$f(\tau) = \frac{8\sinh 2\pi\tau}{\pi^3\sqrt{\pi}\tau}\frac{d}{d\tau}\int_0^\infty \int_0^\infty \int_0^\tau \xi K_{i\xi}\left(\sqrt{x^2+y^2}-y\right) K_{i\xi}\left(\sqrt{x^2+y^2}+y\right)$$
$$\times[\mathcal{G};\tau f(\tau)](x,y)\frac{d\xi dxdy}{x}, \tag{3.9}$$

$$[\mathcal{G};\tau f(\tau)](x,y) = \frac{2}{\sqrt{\pi}}\frac{\partial}{\partial x\partial y}\int_0^\infty \int_0^x \int_0^y \tau K_{i\tau}\left(\sqrt{u^2+v^2}-v\right) K_{i\tau}\left(\sqrt{u^2+v^2}+v\right)$$
$$\times f(\tau)dudvd\tau \tag{3.10}$$

with the Parseval identity

$$\int_0^\infty \int_0^\infty |[\mathcal{G};\tau f(\tau)](x,y)|^2 \frac{dxdy}{x} = \frac{\pi^3}{4} \int_0^\infty \frac{\tau}{\sinh 2\pi\tau} |f(\tau)|^2 d\tau. \tag{3.11}$$

Proof. We set $f(\tau) = f_N(\tau) \in L_2\left(\mathbf{R}_+;\tau[\sinh 2\pi\tau]^{-1}d\tau\right),\ N = 1, 2,\ \dots,$ and vanishes outside of the interval $(1/N, N)$. Then in view of the uniform estimate [6]

$$|K_{i\tau}(u)| \le e^{-\delta|\tau|} K_0(u\cos\delta),\ \ u > 0, \delta \in \left[0, \frac{\pi}{2}\right), \tag{3.12}$$

where $K_0(z)$ is the Macdonald function of the index zero, it follows that integral (3.7), which we call as $[\mathcal{G};\tau f_N(\tau)](x,y)$ exists in the Lebesgue sense. Hence we can calculate its Mellin transform (1.10) with respect to x. Changing the order of integration via Fubini's theorem and applying formula (1.19) we obtain

$$[\mathcal{G};\tau f_N(\tau)]^{\mathcal{M}}(s,y) = \frac{y^{s/2}K_{s/2}(2y)}{\Gamma((1+s)/2)} \int_0^\infty \tau\Gamma\left(\frac{s}{2}+i\tau\right)\Gamma\left(\frac{s}{2}-i\tau\right) f_N(\tau)d\tau, \tag{3.13}$$

where $y > 0, s = \gamma+it, \gamma > 0$. However, by using (1.7) we see that equality (3.13) represents the composition of the Kontorovich-Lebedev and Mellin transforms (3.1) and (1.10), respectively. In fact, it can be verified by substituting instead of the product of Gamma functions the value of the Mellin transform of the Macdonald function (cf. (2.16.2.2) in [4]). Then appealing to (3.12) and the Fubini theorem we invert the order of integration in the obtained iterated integral and arrive at the following composition representation

$$[\mathcal{G};\tau f_N(\tau)]^{\mathcal{M}}(s,y) = \frac{y^{s/2}K_{s/2}(2y)}{\Gamma((1+s)/2)} [KLf_N]^{\mathcal{M}}(s) \tag{3.14}$$

with

$$[KLf_N]^{\mathcal{M}}(s) = \int_{1/N}^N \tau\Gamma\left(\frac{s}{2}+i\tau\right)\Gamma\left(\frac{s}{2}-i\tau\right) f(\tau)d\tau,\ \ s = \gamma+it, \gamma > 0. \tag{3.15}$$

Further, we estimate the following double integral

$$I_N(\gamma) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_0^\infty \left|[\mathcal{G};\tau f_N(\tau)]^{\mathcal{M}}(\gamma+it,y)\right|^2 dydt. \tag{3.16}$$

Making use the representation (3.14) we substitute it in (3.16). This leads to the iterated integral in the form

$$I_N(\gamma) = \frac{1}{2\pi}\int_{-\infty}^\infty \left|\frac{[KLf_N]^{\mathcal{M}}(\gamma+it)}{\Gamma((1+\gamma+it)/2)}\right|^2 dt \int_0^\infty y^\gamma K_{(\gamma+it)/2}(2y)K_{(\gamma-it)/2}(2y)dy. \tag{3.17}$$

But the integral with respect to y in (3.17) can be calculated in view of the formula (2.16.33.2) in [4] (cf. Section 1). Thus, we insert in (3.17) the corresponding result and employ the Mellin-Parseval formula (1.13). Consequently, the integral I_N can be written as

$$I_N(\gamma) = \frac{\sqrt{\pi}\Gamma(\gamma+1/2)}{8\Gamma(\gamma+1)} \int_0^\infty |[KLf_N](x)|^2 x^{2\gamma-1} dx. \tag{3.18}$$

However on the other hand, I_N can be represented in terms of the square of norm of $[\mathcal{G};\tau f_N(\tau)](x,y)$ in the space $L_2(\mathbf{R}_+ \times \mathbf{R}_+; x^{2\gamma-1}dxdy)$. Indeed, by use of (1.13) we have

$$I_N(\gamma) = \int_0^\infty \int_0^\infty |[\mathcal{G};\tau f_N(\tau)](x,y)|^2 x^{2\gamma-1} dxdy. \tag{3.19}$$

Combining (3.18) and (3.19) we arrive at the equality

$$\int_0^\infty \int_0^\infty |[\mathcal{G};\tau f_N(\tau)](x,y)|^2 x^{2\gamma-1} dxdy = \frac{\sqrt{\pi}\Gamma(\gamma+1/2)}{8\Gamma(\gamma+1)}$$

$$\times \int_0^\infty |[KLf_N](x)|^2 x^{2\gamma-1} dx. \tag{3.20}$$

So the Parseval identity (3.11) for L_2-sequence f_N will follow from (3.20) by formal substitution $\gamma = 0$ and using the equality (3.4). We show that this is indeed possible. Appealing to inequality (3.12) we find that $[KLf_N](x) \in L_2(\mathbf{R}_+)$. Hence for all $\gamma,\ 0 \le \gamma \le 1/2$

$$\int_0^\infty x^{2\gamma-1}|[KLf_N](x)|^2 dx \le \int_0^\infty \left[\chi_{[0,1]}(x) + x\chi_{[1,\infty)}(x)\right] |[KLf_N](x)|^2 \frac{dx}{x} < \infty,$$

where $\chi_{(a,b)}(t)$ is the characteristic function of the respective interval. Therefore, $I_N(\gamma) < \infty,\ 0 \le \gamma \le 1/2$ and the limit $\gamma \to 0$ on the right-hand side of (3.20) can be taken under the integral sign because of the Lebesgue dominated convergence theorem.

Analogously we treat the left-hand side of (3.20). We write

$$\int_0^\infty \int_0^\infty |[\mathcal{G};\tau f_N(\tau)](x,y)|^2 x^{2\gamma-1} dxdy = \int_0^1 \int_0^\infty |[\mathcal{G};\tau f_N(\tau)](x,y)|^2 x^{2\gamma-1} dxdy$$

$$+ \int_1^\infty \int_0^\infty |[\mathcal{G};\tau f_N(\tau)](x,y)|^2 x^{2\gamma-1} dxdy. \tag{3.21}$$

Hence we can pass to the limit $\gamma \to 0$ in the first integral on the right-hand side of (3.21) by virtue of Levi's theorem. Then we invoke inequality (3.12) with $\delta = 0$, formulas (1.3, (1.4), (1.5) of the asymptotic behaviour of the Macdonald function near zero and infinity and majorize the second integral as follows

$$\int_1^\infty \int_0^\infty |[\mathcal{G};\tau f_N(\tau)](x,y)|^2 x^{2\gamma-1} dxdy \le \frac{4}{\pi} \left(\int_{1/N}^N \tau |f(\tau)| d\tau \right)^2$$

$$\times \int_1^\infty K_0(x)\,dx \int_0^\infty K_0^2\left(\sqrt{1+y^2}-y\right) K_0(y)\,dy < \infty.$$

Therefore the passage to the limit $\gamma \to 0$ is possible in the second integral due to the Lebesgue dominated convergence theorem. Finally we apply equality (3.4) and establish (3.11) from (3.20) for f_N. It remains true for the whole space (3.3) due to the continuity of norms.

Further, we have

$$\int_0^\infty \int_0^\infty |[\mathcal{G};\tau f_N(\tau)](x,y) - [\mathcal{G};\tau f_N(\tau)](x,y)|^2 \frac{dxdy}{x}$$

$$= \frac{\pi^3}{4}\left(\int_{1/M}^{1/N} + \int_N^M\right) \frac{\tau}{\sinh 2\pi\tau} |f_N(\tau) - f_M(\tau)|^2 d\tau. \tag{3.22}$$

Since the right-hand side of (3.22) tends to zero as $M \to \infty, N \to \infty$, so does the left-hand side. That is, $[\mathcal{G};\tau f_N(\tau)](x,y)$ converges in mean to a function, $[\mathcal{G};\tau f(\tau)](x,y)$ say, of the class $L_2\left(\mathbf{R}_+ \times \mathbf{R}_+; x^{-1}dxdy\right)$.

Now we show that apart from sets of measure zero, there is a one-to-one correspondence between $[\mathcal{G};\tau f(\tau)](x,y)$ and $f(\tau)$. Indeed, for $f_N(\tau)$ integral (3.7) has a finite range of integration and converges absolutely and uniformly by $(x,y) \in \mathbf{R}_+ \times \mathbf{R}_+, 0 < r \le \sqrt{x^2+y^2} \le R$. Therefore, we integrate with respect to x and y in (3.7) and making the interchange of the order of integration we arrive at the equality

$$\int_0^x \int_0^y [\mathcal{G};\tau f_N(\tau)](u,v)dudv = \frac{2}{\sqrt{\pi}} \int_0^\infty \tau \int_0^x \int_0^y K_{i\tau}\left(\sqrt{u^2+v^2}-v\right)$$

$$\times K_{i\tau}\left(\sqrt{u^2+v^2}+v\right) f_N(\tau) dudvd\tau. \tag{3.23}$$

If $N \to \infty$, then for each fixed $x > 0, y > 0$ the left-hand side of (3.23) tends to the expression $\int_0^x \int_0^y [\mathcal{G}f](u,v)dudv$ as a bounded linear functional. Thus if we prove that

$$\int_0^x \int_0^y K_{i\tau}\left(\sqrt{u^2+v^2}-v\right) K_{i\tau}\left(\sqrt{u^2+v^2}+v\right) dudv$$

$$\in L_2(\mathbf{R}_+; \tau \sinh 2\pi\tau d\tau), \tag{3.24}$$

then as it is easily seen via the Schwarz inequality the integral on the right-hand side of (3.23) converges absolutely. Moreover, making $N \to \infty$ for almost all positive x and y after differentiation of both sides in (3.23) we obtain formula (3.10). So this correspondence is unique in L_2-sense.

Similarly, if we prove that for each $\tau > 0$

$$\mathcal{K}_\tau(x,y) = \frac{2}{\sqrt{\pi}} \int_0^\tau \xi K_{i\xi}\left(\sqrt{x^2+y^2}-y\right) K_{i\xi}\left(\sqrt{x^2+y^2}+y\right) d\xi$$

$$\in L_2\left(\mathbf{R}_+ \times \mathbf{R}_+; \frac{dxdy}{x}\right) \tag{3.25}$$

then in the same manner we observe that integral (3.9) converges absolutely. Furthermore, by virtue of (3.24) and the Schwarz inequality we prove that the sequence f_N, which is given by the integral (3.8) has finite norms $||f_N||$ in the space $L_2(\mathbf{R}_+; \tau[\sinh 2\pi\tau]^{-1}d\tau)$ and therefore, converges to the limit function $\psi(\tau)$. We have to prove that $\psi(\tau) = f(\tau)$ almost for all $\tau \in \mathbf{R}_+$, and for this it suffices to show that

$$\int_0^\tau \frac{\xi\psi(\xi)}{\sinh 2\pi\xi} d\xi = \int_0^\tau \frac{\xi f(\xi)}{\sinh 2\pi\xi} d\xi, \tag{3.26}$$

where both integrals are absolutely convergent. In fact, invoking (3.8) we deduce

$$\int_0^\tau \frac{\xi}{\sinh 2\pi\xi}\psi(\xi)d\xi = \lim_{N\to\infty}\int_0^\tau \frac{\xi}{\sinh 2\pi\xi} f_N(\xi)d\xi$$

$$= \lim_{N\to\infty}\frac{8}{\pi^3\sqrt{\pi}}\int_0^\tau \xi d\xi \int_{1/N}^N\int_{1/N}^N K_{i\tau}\left(\sqrt{x^2+y^2}-y\right)K_{i\tau}\left(\sqrt{x^2+y^2}+y\right)$$

$$\times[\mathcal{G}f](x,y)\frac{dxdy}{x} = \lim_{N\to\infty}\frac{8}{\pi^3\sqrt{\pi}}\int_{1/N}^N\int_{1/N}^N\int_0^\tau \xi K_{i\tau}\left(\sqrt{x^2+y^2}-y\right)K_{i\tau}\left(\sqrt{x^2+y^2}+y\right)$$

$$\times[\mathcal{G}f](x,y)\frac{d\xi dxdy}{x} = \frac{8}{\pi^3\sqrt{\pi}}\int_0^\infty\int_0^\infty\int_0^\tau \xi K_{i\tau}\left(\sqrt{x^2+y^2}-y\right)K_{i\tau}\left(\sqrt{x^2+y^2}+y\right)$$

$$\times[\mathcal{G}f](x,y)\frac{d\xi dxdy}{x}. \tag{3.27}$$

On the hand, for two functions f, θ we have, as a consequence of (3.11) and the parallelogram identity,

$$\int_0^\infty\int_0^\infty [\mathcal{G}f](x,y)\overline{[\mathcal{G}\theta](x,y)}\frac{dxdy}{x} = \frac{\pi^3}{4}\int_0^\infty \frac{\xi}{\sinh 2\pi\xi} f(\xi)\overline{\theta(\xi)}d\xi. \tag{3.28}$$

We put

$$\theta(\xi) \equiv \theta_\tau(\xi) = \begin{cases} 1, & 0 \le \xi \le \tau, \\ 0, & \xi > \tau \end{cases}. \tag{3.29}$$

Then from relations (3.28), (3.7) and comparing with (3.27) we obtain (3.26). We substitute (3.29) in (3.28) and we differentiate with respect to τ in both sides of (3.28). This leads us to the formula (3.9), which is valid for almost all $\tau > 0$.

In order to complete the proof of Theorem 3 it only remains to prove propositions (3.24), (3.25). Indeed, since the integrand in (3.24) is a continuous function with respect to τ, then it suffices to show that for large $\Delta > 0$ the integral

$$\int_\Delta^\infty \tau\sinh 2\pi\tau\left|\int_0^x\int_0^y K_{i\tau}\left(\sqrt{u^2+v^2}-v\right)K_{i\tau}\left(\sqrt{u^2+v^2}+v\right)dudv\right|^2 d\tau < \infty. \tag{3.30}$$

To do this we apply the asymptotic formula for the Macdonald function with respect to the index $\tau \to +\infty$ and finite range of the argument (see, for example [6, p. 20])

$$K_{i\tau}(x) = \sqrt{\frac{2\pi}{\tau}} e^{-\pi\tau/2} \sin\left(\tau \log \frac{2\tau}{x} - \tau + \frac{\pi}{4} + \frac{x^2}{4\tau}\right) [1 + O(1/\tau)].$$

Then we substitute it in (3.30) and by employing simple trigonometric relations and Minkowski's inequality for the norms we obtain

$$\left(\int_{\Delta}^{\infty} \tau \sinh 2\pi\tau \left| \int_0^x \int_0^y K_{i\tau}\left(\sqrt{u^2+v^2} - v\right) K_{i\tau}\left(\sqrt{u^2+v^2} + v\right) dudv \right|^2 d\tau\right)^{1/2}$$

$$= O\left[\left(\int_{\Delta}^{\infty} \frac{d\tau}{\tau} \left| \int_0^x \int_0^y e^{i\left(\tau \log \frac{\sqrt{u^2+v^2}-v}{\sqrt{u^2+v^2}+v} + \frac{v\sqrt{u^2+v^2}}{2\tau}\right)} dudv \right|^2\right)^{1/2}\right.$$

$$\left. + \left(\int_{\Delta}^{\infty} \frac{d\tau}{\tau} \left| \int_0^x \int_0^y e^{i\left(2\tau\left(\log \frac{2\tau}{u} - 1\right) + \frac{u^2+2v^2}{2\tau}\right)} dudv \right|^2\right)^{1/2}\right]. \tag{3.31}$$

Further we make the substitution $u = tv$ in the inner integral of the first double integral on the right-hand side of (3.31). Hence we find

$$\int_0^x \int_0^y e^{i\left(\tau \log \frac{\sqrt{u^2+v^2}-v}{\sqrt{u^2+v^2}+v} + \frac{v\sqrt{u^2+v^2}}{2\tau}\right)} dudv = O\left(\frac{1}{2\tau} \int_0^y vdv \int_0^{x/v} \sqrt{t^2+1} e^{i\left(\tau \log \frac{\sqrt{t^2+1}-t}{\sqrt{t^2+1}+t}\right)}\right.$$

$$\left. \times\, d\left(\tau \log \frac{\sqrt{t^2+1}-t}{\sqrt{t^2+1}+t}\right)\right), \tau \to +\infty. \tag{3.32}$$

Then on integration by parts, we see that the latter iterated integral is $O(1/\tau)$, $\tau \to +\infty$. Similarly, we treat the second double integral on the right-hand side of (3.31). We have,

$$\left| \int_0^x \int_0^y e^{i\left(2\tau\left(\log \frac{2\tau}{u} - 1\right) + \frac{u^2+2v^2}{2\tau}\right)} dudv \right| = \left| \int_0^y e^{\frac{iv^2}{\tau}} dv \int_0^x e^{i\left(u^2/2\tau - 2\tau \log u\right)}\right.$$

$$\left. \times \frac{u\tau}{u^2 - 2\tau^2} d\left(-2\tau \log u + \frac{u^2}{2\tau}\right) \right| = O\left(\frac{1}{\tau}\right), \quad \tau \to +\infty. \tag{3.33}$$

Thus, combining (3.32) and (3.33) we deduce that

$$\left(\int_{\Delta}^{\infty} \tau \sinh 2\pi\tau \left| \int_0^x \int_0^y K_{i\tau}\left(\sqrt{u^2+v^2} - v\right) K_{i\tau}\left(\sqrt{u^2+v^2} + v\right) dudv \right|^2 d\tau\right)^{1/2}$$

$$= O\left[\left(\int_{\Delta}^{\infty}\frac{d\tau}{\tau^3}\right)^{1/2}\right],$$

which guarantees (3.30).

In the case of (3.25) we derive (see (1.13), (3.1), (1.19), (3.18))

$$\int_0^\infty\int_0^\infty |\mathcal{K}_\tau(x,y)|^2\frac{dxdy}{x} = \lim_{\gamma\to 0+}\int_0^\infty\int_0^\infty |\mathcal{K}_\tau(x,y)|^2 x^{2\gamma-1}dxdy$$

$$= \lim_{\gamma\to 0+}\frac{1}{2\pi}\int_{-\infty}^\infty\int_0^\infty \left|\mathcal{K}_\tau^{\mathcal{M}}(\gamma+it,y)\right|^2 dydt = \frac{\pi}{8}\int_0^\infty |\varphi(x,\tau)|^2\frac{dx}{x} < \infty, \tag{3.34}$$

where

$$\varphi(x,\tau) = 4\int_0^\tau \xi K_{2i\xi}(2x)d\xi. \tag{3.35}$$

To verify that the latter integral in (3.34) is finite we split it on two integrals over $0 < x < a, a < 1$ and $x \geq a$. Then we use the inequality $|K_{2i\xi}(2x)| \leq K_0(2x)$ for $a \leq x < \infty$, which gives the convegrence of the corresponding integral. Indeed, the Macdonald function $K_0(2x)$ is continuous there and according to (1.3) behaves as $O(e^{-2x}/\sqrt{2x}), x \to +\infty$. For $0 < x \leq a$ we represent the Macdonald function by its definition (cf. [1, Vol.II]) in terms of the combination of the modified Bessel functions

$$K_{2i\xi}(2x) = \frac{\pi}{2}\frac{I_{-2i\xi}(2x) - I_{2i\xi}(2x)}{i\sinh 2\pi\xi}.$$

Then we employ the series representation for the modified Bessel functions $I_{\pm 2i\xi}(2x)$. By substituting it in (3.35) and changing the order of integration and summation we easily obtain that for each $\tau > 0$ $\varphi(x,\tau) = O([\log x]^{-1}),\ x \to 0+$. This fact guarantees the convergence of the integral $\int_0^a |\varphi(x,\tau)|^2\frac{dx}{x}$. Theorem 3 is completely proved.

4 A solution of the Neumann problem

In this final section we consider an application of the transformation (1.1) to the Neumann problem for a second order partial differential equation. Let us deduce this equation and show that the kernel (1.9) is a fundamental solution. First by straightforward calculations it is not difficult to observe that for each $\tau \geq 0$ integral (1.9) and its partial derivatives with respect to $x \in [x_0, X_0] \subset \mathbf{R}_+$ and $y \in [y_0, Y_0] \subset \mathbf{R}_+$ converge absolutely and uniformly in a region $[x_0, X_0] \times [y_0, Y_0]$. Consequently we can differentiate through the sign of the integral in (1.9). Thus denoting by $S_{i\tau}(x,y)$ the product of Macdonald functions in (1.9) we easily find

$$\frac{\partial S_{i\tau}}{\partial y} == -\frac{2y}{x^2}\int_0^\infty e^{-2\frac{y^2}{x^2}u-\frac{x^2}{2u}}e^{-u}K_{i\tau}(u)du.$$

Hence

$$\frac{\partial}{\partial x}\left(-\frac{x^2}{2y}\frac{\partial S_{i\tau}}{\partial y}\right) = \int_0^\infty e^{-2\frac{y^2}{x^2}u-\frac{x^2}{2u}}e^{-u}K_{i\tau}(u)\left(-\frac{x}{u}+\frac{4y^2u}{x^3}\right)du$$

$$= -2xS_{i\tau}(x,y) + \frac{4y^2}{x^3}\int_0^\infty e^{-2\frac{y^2}{x^2}u-\frac{x^2}{2u}}e^{-u}K_{i\tau}(u)udu. \tag{4.1}$$

However,

$$\frac{\partial}{\partial y}\left(-\frac{x^2}{2y}\frac{\partial S_{i\tau}}{\partial y}\right) = -\frac{4y}{x^2}\int_0^\infty e^{-2\frac{y^2}{x^2}u-\frac{x^2}{2u}}e^{-u}K_{i\tau}(u)udu. \tag{4.2}$$

Therefore, combining (4.1) and (4.2) we arrive at the following second order partial differential equation with rational coefficients

$$\left(\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)\left(\frac{x^2}{2y}\frac{\partial S_{i\tau}}{\partial y}\right) = 2xS_{i\tau}. \tag{4.3}$$

After simple manipulations it can be written in the form

$$xy\frac{\partial^2 S_{i\tau}}{\partial y^2} + x\frac{\partial^2 S_{i\tau}}{\partial x\partial y} + (2-x)\frac{\partial S_{i\tau}}{\partial y} - 4yS_{i\tau} = 0. \tag{4.4}$$

So we finally get that the function

$$S_{i\tau}(x,y) = K_{i\tau}\left(\sqrt{x^2+y^2}-y\right)K_{i\tau}\left(\sqrt{x^2+y^2}+y\right)$$

is a fundamental solution of the equation (4.3)-(4.4) in the domain $\mathbf{R}_+\times\mathbf{R}_+$.

Now we show that for any $h(\tau)=\tau f(\tau)\in L^*(\mathbf{R}_+)$ the function

$$U(x,y) = \frac{2}{\sqrt{\pi}}\int_0^\infty \tau S_{i\tau}(x,y)f(\tau)d\tau \tag{4.5}$$

is a solution of the equation (4.4). Since $h(\tau)$ is a bounded continuous function on $\mathbf{R}_+$ then invoking inequality (3.12) and integral (1.9) due to Fubini's theorem we establish the following representation

$$U(x,y) = \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-2\frac{y^2}{x^2}u-\frac{x^2}{2u}}e^{-u}(KLh)(u)\frac{du}{u}, \tag{4.6}$$

where $(KLh)(u)$ is the Kontorovich-Lebedev operator (1.14). Meanwhile, for any fixed $\delta\in(0,\pi/2)$ we have

$$|(KLh)(u)| \le K_0(u\cos\delta)\int_0^\infty \tau e^{-\delta\tau}|f(\tau)|d\tau < C_\delta K_0(u\cos\delta), \tag{4.7}$$

where $C_\delta>0$ is a constant, which depends on δ. Hence we use asymptotic formulas for the Macdonald function $K_0(u)$ (1.3), (1.5) and uniform convergence of the corresponding integrals to verify by straightforward calculations that it admits a differentiation through the integral sign in (4.6) for all partial derivatives of the differential operator on the left-hand side of the equation (4.4). Moreover this fact allows us to conclude that

$$A_{x,y}U = \frac{2}{\sqrt{\pi}} \int_0^\infty \tau A_{x,y} S_{i\tau} f(\tau) d\tau = 0, \tag{4.8}$$

where we denote by

$$A_{x,y} \equiv xy\frac{\partial^2}{\partial y^2} + x\frac{\partial^2}{\partial x \partial y} + (2-x)\frac{\partial}{\partial y} - 4y. \tag{4.9}$$

Thus (4.5) is a solution of the equation (4.4).

Consider the following weighted Neumann problem for the differential operator (4.9)

$$A_{x,y}U = 0, \qquad x\frac{\partial}{\partial x}U(x,0) = g(x), \ x \in \mathbf{R}_+, \tag{4.10}$$

where $g(x)$ is a given bounded continuous function. Our goal now is to demonstrate that integral (4.5) is a solution of this problem, where $f(\tau)$ is expressed in terms of the Lebedev type transformation of the function g (cf. [7]). Indeed, in order to satisfy the Neumann boundary condition (4.10) we return to solution (4.5). Then taking into account representation (4.6) and the above discussions we deduce

$$x\frac{\partial}{\partial x}U(x,y) = \frac{2x}{\sqrt{\pi}} \int_0^\infty \tau \frac{\partial}{\partial x} S_{i\tau}(x,y) f(\tau) d\tau = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-2\frac{y^2}{x^2}u - \frac{x^2}{2u}} e^{-u}$$

$$\times (KLh)(u)\left(-\frac{x^2}{u} + \frac{4y^2 u}{x^2}\right)\frac{du}{u} = -\frac{x^2}{\sqrt{\pi}} \int_0^\infty e^{-2\frac{y^2}{x^2}u - \frac{x^2}{2u}} e^{-u} (KLh)(u)\frac{du}{u^2}$$

$$+\frac{4y^2}{x^2\sqrt{\pi}} \int_0^\infty e^{-2\frac{y^2}{x^2}u - \frac{x^2}{2u}} e^{-u} (KLh)(u) du. \tag{4.11}$$

Appealing to inequality (4.7) it is easily seen that for all $x > 0$

$$\lim_{y\to 0+} \frac{4y^2}{x^2\sqrt{\pi}} \int_0^\infty e^{-2\frac{y^2}{x^2}u - \frac{x^2}{2u}} e^{-u} (KLh)(u) du = 0. \tag{4.12}$$

Meantime, since (see (4.7))

$$\left| \int_0^\infty e^{-2\frac{y^2}{x^2}u - \frac{x^2}{2u}} e^{-u} (KLh)(u)\frac{du}{u^2} \right| \le \int_0^\infty e^{-u - \frac{x^2}{2u}} |(KLh)(u)| \frac{du}{u^2}$$

$$\le C_\delta \int_0^\infty e^{-u - \frac{x^2}{2u}} K_0(u\cos\delta)\frac{du}{u^2} < \infty, x > 0,$$

then by virtue of Lebesgue's dominated convergence theorem we can pass to a limit in (4.11) under the sign of the integral when $y \to 0+$. Combining with (4.12) we obtain

$$x\frac{\partial}{\partial x}U(x,0) = -\frac{x^2}{\sqrt{\pi}}\int_0^\infty e^{-u-\frac{x^2}{2u}}(KLh)(u)\frac{du}{u^2} = \frac{2x}{\sqrt{\pi}}\int_0^\infty \tau\frac{\partial}{\partial x}S_{i\tau}(x,0)f(\tau)d\tau$$

$$= \frac{2x}{\sqrt{\pi}}\int_0^\infty \tau\frac{\partial}{\partial x}K_{i\tau}^2(x)f(\tau)d\tau = g(x), \quad x > 0. \tag{4.13}$$

The latter integral in (4.13) represents a modification of the Lebedev transformation (1.15). We will use now results from [7] to find an inversion formula of the Lebedev transformation

$$g(x) = \frac{2x}{\sqrt{\pi}}\int_0^\infty \tau\frac{\partial}{\partial x}K_{i\tau}^2(x)f(\tau)d\tau \tag{4.14}$$

in order to express f in terms of a given bounded continuous function g. Indeed, starting from the Mellin-Barnes representation [7]

$$\frac{2x}{\sqrt{\pi}}\frac{\partial}{\partial x}K_{i\tau}^2(x) = -\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma\left(\frac{s}{2}+i\tau\right)\Gamma\left(\frac{s}{2}-i\tau\right)\frac{\Gamma(1+s/2)}{\Gamma((1+s)/2)}x^{-s}ds, \gamma > 0,$$

we substitute it in (4.14) and we change the order of integration by Fubini's theorem. This can be motivated in a similar manner as in the proof of Theorem 2 since the corresponding iterated integral is absolutely and uniformly convergent. Invoking (3.15) we get

$$g(x) = -\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(1+s/2)}{\Gamma((1+s)/2)}[KLf]^{\mathcal{M}}(s)x^{-s}ds. \tag{4.15}$$

Now we employ an auxiliary Lebedev's operator [2] with a combination of the modified Bessel functions

$$[KI\psi](x) = 2\sqrt{\pi}\int_0^\infty K_{i\xi}(x)\left[I_{i\xi}(x)+I_{-i\xi}(x)\right]\psi(\xi)\frac{\xi d\xi}{\cosh\xi}, \ x > 0. \tag{4.16}$$

According to [7], the kernel in (4.16) can be represented as

$$\frac{2\sqrt{\pi}}{\cosh\xi}K_{i\xi}(x)\left[I_{i\xi}(x)+I_{-i\xi}(x)\right] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma\left(\frac{s}{2}+i\xi\right)\Gamma\left(\frac{s}{2}-i\xi\right)\frac{\Gamma((1-s)/2)}{\Gamma(1-s/2)}x^{-s}ds.$$

Hence we substitute the latter Mellin-Barnes integral in (4.16) and we invert the order of integration assuming that $\xi\psi(\xi) \in L^*(\mathbf{R}_+)$. Taking into account (3.15) we arrive at the formula

$$[KI\psi](x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma((1-s)/2)}{\Gamma(1-s/2)}[KL\psi]^{\mathcal{M}}(s)x^{-s}ds. \tag{4.17}$$

As it is easily seen by virtue of the Stirling asymptotic formula for gamma-functions [1, Vol.I] we have the estimates

$$\left|\frac{\Gamma((1-s)/2)}{\Gamma(1-s/2)}\right|^2 = O(|t|^{-1}),\ s=\gamma+it, |t|\to\infty,$$

$$\left|\frac{\Gamma(1+s/2)}{\Gamma((1+s)/2)}\right|^2 = O(|t|),\ s=\gamma+it, |t|\to\infty.$$

Moreover, since $\xi\psi(\xi)\in L^*(\mathbf{R}_+)$, then it implies that $\psi(\xi)\in L_2(\mathbf{R}_+)\subset L_2(\mathbf{R}_+;\xi[\sinh 2\pi\xi]^{-1}d\xi)$. Therefore due to Parseval equalities (1.13), (3.4) we find from (3.20), (4.17) for $\gamma\in[0,1/2]$ that

$$\int_0^\infty |[KI\psi](x)|^2 x^{2\gamma-1}dx = \frac{1}{2\pi}\int_{-\infty}^\infty \left|\frac{\Gamma((1-\gamma-it)/2)}{\Gamma(1-(\gamma+it)/2)}[KL\psi]^{\mathcal{M}}(\gamma+it)\right|^2 dt$$

$$\leq \text{const.}\int_{-\infty}^\infty \left|[KL\psi]^{\mathcal{M}}(\gamma+it)\right|^2 dt = \text{const.}\int_0^\infty [KL\psi](x)|^2 x^{2\gamma-1}dx$$

$$\leq \text{const.}\int_0^\infty \frac{\tau}{\sinh 2\pi\tau}|\psi(\tau)|^2 d\tau < \infty.$$

Thus the norm (1.11) of the Lebedev operator (4.1) is finite for any $\gamma\in[0,1/2]$ and in particular, for $\gamma=0$. If we let $\psi=\theta_\tau$ (see (3.29)), which evidently satisfies the above conditions, then we consider the integral

$$\int_0^\infty g(x)\overline{[KI\theta_\tau](x)}\frac{dx}{x} = 2\sqrt{\pi}\int_0^\infty g(x)\int_0^\tau K_{i\xi}(x)\left[I_{i\xi}(x)+I_{-i\xi}(x)\right]\frac{\xi d\xi}{\cosh\xi}\frac{dx}{x}. \tag{4.18}$$

The left-hand side of (4.17) converges absolutely due to Schwarz's inequality if we show that $g(x)\in L_2(\mathbf{R}_+;x^{-1}dx)$. Furthermore, by virtue of (1.13), (3.4), (4.15), (4.16) and the parallelogram identity we will prove the equality

$$\int_0^\infty g(x)\int_0^\tau K_{i\xi}(x)\left[I_{i\xi}(x)+I_{-i\xi}(x)\right]\frac{\xi d\xi}{\cosh\xi}\frac{dx}{x} = -\pi\sqrt{\pi}\int_0^\tau \frac{\xi}{\sinh 2\pi\xi}f(\xi)d\xi. \tag{4.19}$$

After differentiation with respect to τ in the latter equality we will arrive at the desired inversion formula of the Lebedev transformation (4.14)

$$f(\tau) = -\frac{\sinh 2\pi\tau}{\pi\sqrt{\pi}\tau}\frac{d}{d\tau}\int_0^\infty g(x)\int_0^\tau K_{i\xi}(x)\left[I_{i\xi}(x)+I_{-i\xi}(x)\right]\frac{\xi d\xi}{\cosh\xi}\frac{dx}{x} \tag{4.20}$$

and establish a solution of the Neumann problem (4.10). If also $g(x)\in L_1((0,1);x^{-3/2}dx)$ then via the uniform estimate (see [2])

$$\left|K_{i\xi}(x)\left[I_{i\xi}(x)+I_{-i\xi}(x)\right]\right| = O(x^{-1/2}), x>0,$$

it allows to differentiate with respect to τ through the sign of the integral in (4.20) to get after simple manipulations that

$$f(\tau) = -\frac{2\sinh\pi\tau}{\pi\sqrt{\pi}} \int_0^\infty K_{i\tau}(x)\left[I_{i\tau}(x) + I_{-i\tau}(x)\right] g(x)\frac{dx}{x}. \tag{4.21}$$

So it remains to show that under the condition $\tau f(\tau) \in L^*(\mathbf{R}_+)$ the Lebedev transform (4.14) is of the space $L_2(\mathbf{R}_+; x^{-1}dx)$. In fact, employing (4.15), (1.13) and the above estimate of the ratio of gamma-functions we have for each $A > 0$

$$\int_0^\infty |g(x)|^2 x^{2\gamma-1}dx = \frac{1}{2\pi}\int_{-\infty}^\infty \left|\frac{\Gamma(1+(\gamma+it)/2)}{\Gamma((1+\gamma+it)/2)}[KLf]^{\mathcal{M}}(\gamma+it)\right|^2 dt$$

$$= \frac{1}{2\pi}\left(\int_{|t|\le A} + \int_{|t|>A}\right)\left|\frac{\Gamma(1+(\gamma+it)/2)}{\Gamma((1+\gamma+it)/2)}[KLf]^{\mathcal{M}}(\gamma+it)\right|^2 dt, \gamma \in \left[0, \frac{1}{2}\right]. \tag{4.22}$$

Further, since the gamma-ratio is continuous with respect to t we find with (3.4) and (3.20) that

$$\int_{|t|\le A}\left|\frac{\Gamma(1+(\gamma+it)/2)}{\Gamma((1+\gamma+it)/2)}[KLf]^{\mathcal{M}}(\gamma+it)\right|^2 dt \le \text{const.}\int_{-\infty}^\infty \left|[KLf]^{\mathcal{M}}(\gamma+it)\right|^2 dt$$

$$= \text{const.}\int_0^\infty |[KLf](x)|^2 x^{2\gamma-1}dx < \infty, \gamma \in \left[0, \frac{1}{2}\right].$$

On the other hand, by using (2.5) and (3.15) it is not difficult to arrive at the estimate

$$\left|[KLf]^{\mathcal{M}}(\gamma+it)\right| \le \text{const.}|\Gamma(\gamma+it)| \int_0^\infty \frac{|(F_c;\tau f(\tau))(2y)|}{\cosh^\gamma y}dy$$

$$\le \text{const.}|\Gamma(\gamma+it)|\ ||\tau f(\tau)||_{L^*}.$$

Consequently, the second integral on the right-hand side of (4.22) can be treated as follows

$$\int_{|t|>A}\left|\frac{\Gamma(1+(\gamma+it)/2)}{\Gamma((1+\gamma+it)/2)}[KLf]^{\mathcal{M}}(\gamma+it)\right|^2 dt \le \text{const.}\int_{|t|>A}|t|\left|[KLf]^{\mathcal{M}}(\gamma+it)\right|^2 dt$$

$$\le \text{const.}\int_{|t|>A}|t|\,|\Gamma(\gamma+it)|^2\, dt < \infty, \gamma \in \left[0, \frac{1}{2}\right].$$

Combining these estimates we immediately obtain that the left-hand side of (4.22) is finite for all $\gamma \in [0, 1/2]$ and in particular for $\gamma = 0$. Thus $g(x) \in L_2(\mathbf{R}_+; x^{-1}dx)$.

Hence, returning to the integral on the left-hand side of (4.18) and invoking (1.13), (4.15), (4.17), (3.4) we deduce by straightforward calculations

$$\int_0^\infty g(x)\overline{[KI\theta_\tau](x)}\frac{dx}{x} = -\frac{1}{2\pi}\int_{-\infty}^\infty \frac{\Gamma(1+it/2)}{\Gamma((1+it)/2)}[KLf]^{\mathcal{M}}(it)\frac{\Gamma((1+it)/2)}{\Gamma(1+it/2)}\overline{[KL\theta_\tau]^{\mathcal{M}}(it)}dt$$

$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty}[KLf]^{\mathcal{M}}(it)\overline{[KL\theta_\tau]^{\mathcal{M}}(it)}dt = -\int_0^{\infty}[KLf](x)\overline{[KL\theta_\tau](x)}\frac{dx}{x}$$

$$= -\pi\sqrt{\pi}\int_0^{\tau}\frac{\xi}{\sinh 2\pi\xi}f(\xi)d\xi.$$

Combining with (4.18) we prove equality (4.19). Hence we get representations (4.20) and (4.21), respetively. We finally summarize the results of this section by the following

Theorem 4. *Integral* (4.5) *is a solution of the weighted Neumann problem* (4.10) *with a bounded continuous function* $g(x), x > 0$*, which is represented by the Lebedev integral* (4.14) *of an arbitrary function* $f(\tau)$*, such that* $\tau f(\tau) \in L^*(\mathbf{R}_+)$*. A function* f *can be uniquely determined in terms of the given function* g *by integrals* (4.20) *or* (4.21) *under additional condition* $g \in L_1((0,1); x^{-3/2}dx)$.

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