Interior Symmetries and Multiple Eigenvalues for Homogeneous Networks

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Abstract

We analyze the impact of interior symmetries on the multiplicity of the eigenvalues of the Jacobian matrix at a fully synchronous equilibrium for the coupled cell systems associated to homogeneous networks. We consider also the special cases of regular and uniform networks. It follows from our results that the interior symmetries, as well as the *reverse interior symmetries* and *quotient interior symmetries*, of the network force the existence of eigenvalues with algebraic multiplicity greater than one. The proofs are based on the special form of the adjacency matrices of the networks induced by those interior symmetries.

Keywords: Coupled systems, interior symmetry, multiple eigenvalues.

1 Introduction

A coupled cell system is a set of individual dynamical systems, or *cells*, that are coupled together through interactions. Such systems can be represented by a directed graph, a *coupled cell network*, whose nodes correspond to cells and whose edges represent couplings. Two cells are called *identical*, if they have the same phase space and the same internal dynamics (cf. Golubitsky *et al.* [5]). Here we assume that the internal dynamics of a cell is modeled by a system of ordinary differential equations.

This paper is mainly concerned with *homogeneous networks*, which are coupled cell networks of identical cells that are identically coupled. A homogeneous network is called *regular*, if all the couplings (*arrows* or *edges*) are of the same type. The *valency* of a homogeneous network is the number of arrows that input to each cell.

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An important feature of homogeneous networks is that the diagonal subspace Δ , formed by setting all cell coordinates equal in the total phase space, is always flow-invariant by admissible coupled systems. Moreover, the restriction of these systems to Δ gives the set of all vector fields on Δ (cf. Theorem 5.2 of Golubitsky *et al.* [8]). Suppose that a homogeneous cell system admits a fully synchronous equilibrium in Δ . We say that the system undergoes a *local synchrony-breaking steady-state bifurcation*, if the synchronous equilibrium loses its stability and bifurcates to a state with less synchrony, as a bifurcation parameter crosses certain critical value.

It is known that a steady-state bifurcation occurs only if some real eigenvalue of the Jacobian at the equilibrium becomes zero, as the bifurcation parameter is varied. When a pair of complex conjugate eigenvalues of the Jacobian crosses the imaginary axis as the bifurcation parameter crosses the critical value, the system presents a *Hopf bifurcation*. In this case, under additional nondegeneracy conditions, the steady-state bifurcates to a periodic solution. In non-symmetric systems, the eigenvalues are generically simple, so the bifurcation analysis is straightforward. In the presence of symmetry, however, multiple eigenvalues appear even under generic assumptions, and in this case the bifurcation analysis problem turns to be highly degenerate (cf. Golubitsky *et al.* [6]).

It seems that for homogeneous cell systems, multiple eigenvalues occur more frequently than for general systems. More importantly, they are very often not forced by symmetry alone. Recall that a *symmetry* of a coupled cell network is a permutation on the set of cells and arrows such that it preserves the network structure. In [5], Golubitsky *et al.* introduced a less stringent form of symmetry for coupled cell networks – *interior symmetry*. Following Antoneli *et al.* [2], we say that a network G has an interior symmetry on a subset S of cells, if S together with all the arrows directed to it form a subnetwork that has a nontrivial symmetry. Notice that a symmetry of a network is a particular case of an interior symmetry, and by interior symmetry, we mean nontrivial interior symmetry.

The main goal of this paper is to address how interior symmetry may result in multiple critical eigenvalues at synchrony-breaking bifurcations. Without loss of generality, we assume the synchronous equilibrium is at the origin. As similar to the case of regular networks (cf. Leite *et al.* [10]), the Jacobian of a homogeneous coupled cell system at a fully-synchronized equilibrium at the origin is determined by the cell internal dynamics and the adjacency matrices of different types of arrows. More precisely, let G be an *n*-cell homogeneous network with *s* type of arrows, whose cell internal dynamics is *r*-dimensional. Thus, the total phase space is $(\mathbb{R}^r)^n$. Let A_l , l = 1, 2, ..., s, be the adjacency matrix of the *l*-th type of arrows in G, i.e. $A_l = [a_{ij}]_{1 \le i,j \le n}$ is a matrix whose entry a_{ij} is the number of the *l*-th type arrows connecting cell *j* to *i*. Let α be the linearized internal dynamics at the origin, β_l be the linearized internal coupling at the origin with the *l*-th type of input, for l = 1, 2, ..., s. Note that α and β_l are $r \times r$ matrices. Then, the Jacobian at the origin is of the form

$$J_{\boldsymbol{G}} = \alpha \otimes I_n + \beta_1 \otimes A_1 + \dots + \beta_m \otimes A_s.$$

In the case s = 1, G is a regular network, for which we write $\beta = \beta_1$ and $A_G = A_1$. Let μ_1, \ldots, μ_n be the eigenvalues of A_G . Then, as it is shown in Leite *et al.* [10] and Aguiar *et al.* [1], the eigenvalues of J_G are the union of the eigenvalues of the $r \times r$ matrices $\alpha + \mu_j \beta$, for $j = 1, \ldots, n$, including algebraic multiplicity. Thus, for regular networks, our problem reduces to understanding how interior symmetry may affect the multiplicity of the eigenvalues of A_G .

Examples show that interior symmetry often forces multiple eigenvalues, and frequently, with geometric multiplicity smaller than the algebraic multiplicity (cf. [1] and [10]).

In the case s > 1, there is generally no straightforward relation between the spectrum of $J_{\mathcal{G}}$ and that of the A_l 's (cf. Golubitsky *et al.* [4] where the product of two regular networks was considered). However, as we will see, using interior symmetry, it is possible to relate the multiplicities of the eigenvalues of $J_{\mathcal{G}}$ with those of the A_l 's.

The main reason why interior symmetry may result in multiple eigenvalues is that it imposes restrictions on the network structure and thus on the form of adjacency matrices of the network. For example, an interior symmetry (*ij*) on the set of cells $C = \{1, ..., n\}$ of a regular network G given by the permutation of cells *i* and *j* corresponds to the following constraints on the entries of A_G

$$a_{ii} = a_{ji}, \quad a_{ij} = a_{ji} \quad \text{and} \quad a_{ik} = a_{jk}, \text{ for all } k \in C \setminus \{i, j\}.$$

Interior symmetries are common for regular networks specially when they have at most one connection from one cell to another, as is the particular case of *uniform networks*. Following Stewart [13], we say that a network is *uniform*, if it has no multiple arrows or self-couplings. As we will see, interior symmetries force the existence of integer eigenvalues for the adjacency matrix of a regular network. It is known that for a regular network \mathcal{G} with valency v all the eigenvalues λ of $A_{\mathcal{G}}$ satisfy $||\lambda|| \leq v$. Thus, for example, if the network has valency v = 2, in the presence of interior symmetries, the eigenvalues -1, 0 and 1 will arise very often. Moreover, as we will see in Subsection 3.1.6, for regular uniform networks, interior symmetry forces the existence of eigenvalues for $A_{\mathcal{G}}$ in $\{-2, -1, 0, 1\}$.

We define other variations of interior symmetry in a network such as *reverse interior symmetry* and *quotient interior symmetry*, which may also result in multiple eigenvalues for the Jacobian at the origin of the corresponding coupled cell systems. A reverse interior symmetry is an interior symmetry of the *reverse network*, where the direction of arrows of G is reversed. A quotient interior symmetry is a short-hand notion of an interior symmetry of a *quotient network* of G, which is obtained by restricting G to a balanced equivalence relation on the cells. If a quotient network has a reverse interior symmetry, then we call this symmetry a *quotient reverse interior symmetry*.

The paper is organized as follows. In Section 2, we recall concepts from coupled cell networks, give preliminary definitions of various interior symmetries and collect a few results from linear algebra. In Subsection 3.1, we study how interior symmetry can give rise to multiple eigenvalues for the Jacobian at the origin of coupled cell systems associated to regular networks. We discuss several important groups of interior symmetries, such as the cyclic group \mathbb{Z}_k , the dihedral group \mathbb{D}_k , the alternating group \mathbb{A}_k and the symmetric group \mathbb{S}_k . In Subsection 3.2, we extend this discussion to homogeneous networks. We give some concluding remarks in Section 4. Throughout the paper, numerous examples will be used to illustrate our results.

2 Preliminaries

In this section, we summarize necessary concepts from coupled cell networks. We restrict our attention to homogeneous coupled cell networks since they are our main case of study. For more general definitions and results on coupled cell networks, we refer to Golubitsky *et al.* [7], Golubitsky *et al.* [8] and references therein.

Definition 2.1 A *coupled cell network* consists of a finite set $C = \{1, \dots, n\}$ of *nodes* or *cells* and a finite set $\mathcal{E} = \{(c, d) : c, d \in C\}$ of *edges* or *arrows* and two equivalence relations, \sim_C on cells in C and \sim_E on edges in \mathcal{E} , with the consistency condition: if $e_1 \sim_E e_2$, for $e_1 = (c_1, d_1) \in E$ and $e_2 = (c_2, d_2) \in E$, then $c_1 \sim_C c_2$ and $d_1 \sim_C d_2$. We write $\mathcal{G} = (C, \mathcal{E}, \sim_C, \sim_E)$.

For an edge $e = (c, d) \in \mathcal{E}$, *c* is called the *head cell* and *d* is called the *tail cell*; and *e* is called an *input edge* of *c*. The set of all input edges of *c* is called the *input set* of *c* and denoted by I(c). Two cells *c* and *d* in a network are said to be *input-equivalent*, if there is an edge-type preserving isomorphism $\beta : I(c) \rightarrow I(d)$ between their input sets. Note that the relation of input-equivalence refines the relation of cell-equivalence.

Definition 2.2 A *homogeneous* network is a coupled cell network with only one input-equivalence class. A *regular* network is a homogeneous network with only one edge-equivalence class. It follows that in a homogeneous network all cells are of identical type and receive the same number (per type) of input edges. This number, which is the cardinality of the input set, is called the *valency* of the network.

We follow the multiarrow formalism in Golubitsky *et al.* [8] and thus allow multiple arrows of the same type between two cells and self-coupling arrows. We call the networks without multiple arrows nor self-coupling arrows *uniform networks* (cf. Stewart [13]).

A coupled cell network can be represented graphically by a directed graph, where cells are placed at the nodes and arrows identify the connections. Alternatively, the architecture of a homogeneous network with *s* edge-equivalence classes can be given by *s* adjacency matrices A_1, A_2, \ldots, A_s . More precisely, the *l*-th adjacency matrix A_l of an *n*-cell homogeneous network G is an $n \times n$ matrix, whose (i, j)-entry equals to the number of the *l*-th type arrows directing from cell *j* to cell *i*.

Example 2.3 Consider the 5-cell homogeneous network G with 2 types of arrows and valency 4, which is shown in Figure 1. Let A_1 (resp. A_2) be the adjacency matrix of the arrows with

Figure 1: A homogeneous network \mathcal{G} with valency 4.

solid (resp. hollow) arrow head. Then,

$$A_{1} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}, \quad A_{2} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$
 (2.1)

Example 2.4 Consider the two subnetworks G_1, G_2 obtained from the network G in Example 2.3, by only keeping all arrows with solid (resp. hollow) arrow head. Then, G_1, G_2 are regular networks, as shown in Figure 2, with the adjacency matrix given by A_1, A_2 respectively. (cf. (2.1)).



Figure 2: Regular networks G_1, G_2 obtained from G in Figure 1.

Notice that G_1 is an example of a uniform network, while G_2 is not.

2.1 Symmetry and symmetric groups

We adapt and simplify the definition of a symmetry of a general coupled cell network in Antoneli *et al.* [3] to a symmetry of a homogeneous network.

Definition 2.5 Let $\mathcal{G} = (C, \mathcal{E}, \sim_C, \sim_E)$ be a homogeneous network. A *symmetry* of \mathcal{G} is a permutation σ on C such that there is a bijection between the edges ($\sigma(a), \sigma(b)$) and (a, b), which preserves the edge-equivalence relation \sim_E , for all $a, b \in C$.

Let G be an *n*-cell homogeneous network with *s* edge-equivalence classes, whose adjacency matrices are given by $A_1, A_2, ..., A_s$. Write $A_l = [a_{ij}^{(l)}]_{n \times n}$, for l = 1, 2, ..., s. Then, a permutation σ is a symmetry of G, if and only if

$$a_{ij}^{(l)} = a_{\sigma(i)\sigma(j)}^{(l)}, \quad \forall i, j = 1, 2, \dots, n, \ l = 1, 2, \dots, s.$$

It is clear that the set of all symmetries of an *n*-cell homogeneous network \mathcal{G} forms a group, which can be identified canonically with a subgroup of the *symmetric group* \mathbf{S}_n , that is defined as the group of all permutations of *n* symbols. Let $i_1, \ldots, i_k \in \mathbb{N}$ be distinct positive integers. We use the standard notation $(i_1 \ldots i_k)$ to denote a *k*-cycle in \mathbf{S}_n , which is a permutation σ defined by

$$\sigma: i_j \mapsto i_{j+1} \quad \text{for } j = 1, \dots k - 1,$$
$$i_k \mapsto i_1$$
$$l \mapsto l \quad \text{for } l \notin \{i_1, \dots, i_k\}.$$

A 2-cycle is called a *transposition*. Every permutation can be written as a product of simple transpositions. A permutation is called *even* (resp. *odd*), if it can be expressed as a product of an even (resp. odd) number of transpositions. The subset of S_n consisting of all even permutations is a subgroup called the *alternating group* A_n . A group generated by permutations $\sigma_1, \sigma_2, \ldots, \sigma_m$ will be denoted by $\langle \sigma_1, \sigma_2, \ldots, \sigma_m \rangle$.

Example 2.6 Consider the *k*-cycle $(1 \ 2 \dots k)$ in **S**_{*k*} and the cyclic group

$$\mathbb{Z}_k = \langle (1 \ 2 \ \dots \ k) \rangle$$

generated by the *k*-cycle. Let \mathcal{G} be a \mathbb{Z}_k -symmetric homogeneous network of *k* cells and A_1, A_2, \ldots, A_s be the adjacency matrices of \mathcal{G} . Then, every A_l is of the form

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1,k-1} & a_{1k} \\ a_{1k} & a_{11} & a_{12} & \dots & a_{1,k-1} \\ a_{1,k-1} & a_{1k} & a_{11} & \ddots & a_{1,k-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{12} & a_{13} & \dots & a_{1k} & a_{11} \end{bmatrix},$$
(2.2)

where every row vector is obtained by shifting the preceding row vector to the right by one element. $\hfill \Box$

A matrix of the form (2.2) is called a *circulant matrix*, which is often written as

$$circ(a_{11}, a_{12}, \ldots, a_{1k})$$

for a shorthand. Circulant matrices and their spectral information are needed for our later discussions. It is known that all circulant matrices of the form (2.2) share the same eigenvectors

$$v_j = (1, \omega_j, \omega_j^2, \dots, \omega_j^{k-1}), \text{ for } \omega_j = e^{\frac{2\pi i j}{k}}, j = 0, 1, \dots, k-1,$$
 (2.3)

which are eigenvectors of the following eigenvalues

$$\lambda_j = a_{11} + a_{12}\omega_j + a_{13}\omega_j^2 + \dots + a_{1k}\omega_j^{k-1}, \ j = 0, 1, \dots, k-1.$$
(2.4)

Another concept that we will need later is that of a *centrosymmetric matrix*, which is a matrix that is symmetric about its center. More formally,

Definition 2.7 A square matrix $A = [a_{ij}]_{n \times n}$ is called *centrosymmetric*, if the following relation is satisfied

$$a_{ij} = a_{(n+1-i)(n+1-j)}, \quad \forall i, j = 1, 2, \dots, n,$$

which is equivalent to the relation

A = JAJ,

where $J = [e_{ij}]_{n \times n}$ is the *exchange matrix*; that is, $e_{i,n+1-i} = 1$ and $e_{ij} = 0$ for all $j \neq n + 1 - i$, i = 1, 2, ..., n, i.e. it has 1 on the anti-diagonal and 0 elsewhere.

Example 2.8 Consider the network G in Example 2.3. The symmetry group of G is

$$\mathbb{Z}_2 = \langle (1\ 5)(2\ 4) \rangle.$$

The adjacency matrices A_l 's of any 5-cell homogeneous network having this symmetry are centrosymmetric matrices of the form

2.2 Interior symmetry

The concept of interior symmetry of a coupled cell network is a generalized notion of a symmetry of a coupled cell network. Roughly speaking, it is a permutation of the cells that preserves certain amount of input structure. The notion of interior symmetry was first introduced by Golubitsky *et al.* [5]. We adapt and simplify the definition in [5] to define an interior symmetry of a homogeneous network as follows.

Definition 2.9 Let $\mathcal{G} = (C, \mathcal{E}, \sim_C, \sim_E)$ be a homogeneous network. Let $\mathcal{S} \subseteq C$ be a subset. An *interior symmetry* of \mathcal{G} on \mathcal{S} is a permutation σ on C such that σ fixes every element in $C \setminus S$, and there is a bijection between edges ($\sigma(a), \sigma(b)$) and (a, b), which preserves edge-equivalence relation \sim_E , for $a \in S, b \in C$.

Note that in the case S = C, an interior symmetry on *C* is precisely a symmetry of *G*. In what follows, when referring to interior symmetry, we also include the case of symmetry.

Let \mathcal{G} be an *n*-cell homogeneous network with *s* edge-equivalence classes, whose adjacency matrices are given by A_1, A_2, \ldots, A_s . Write $A_l = [a_{ij}^{(l)}]_{n \times n}$, for $l = 1, \ldots, s$. Then, a permutation σ is an interior symmetry of \mathcal{G} on \mathcal{S} , if and only if

$$a_{ij}^{(l)} = a_{\sigma(i)\sigma(j)}^{(l)}, \quad \forall i \in \mathcal{S}, \ j \in C, \ l = 1, \dots, s.$$
 (2.5)

Following the formulation in Antoneli *et al.* [2], one can characterize the interior symmetry using symmetry of subnetworks. Given a network \mathcal{G} and a subset $\mathcal{S} \subseteq C$, define $\mathcal{G}_{\mathcal{S}} = (C, I(\mathcal{S}), \sim_C, \sim_E)$ to be the subnetwork of \mathcal{G} , whose set of cells is C (together with its cell-equivalence relation \sim_C) and whose set of arrows is the input set $I(\mathcal{S})$ of \mathcal{S} . The edge-equivalence relation on $I(\mathcal{S})$ is given by the restriction of the edge-equivalence \sim_E of \mathcal{E} to $I(\mathcal{S})$.

Proposition 2.10 (cf. [2]) Let G be a coupled cell network and $S \subseteq C$ be a subset of cells of the set of cells of G. Consider the network G_S as defined above. Then the group of interior symmetries of the network G on S can be canonically identified with the group of symmetries of the network G_S .

Example 2.11 Consider the homogeneous network G in Example 2.3. Let $S = \{2, 3, 4\}$. Then, the network G_S has an S_3 -symmetry, as shown in Figure 3. Thus, G has an interior symmetry S_3 on S.



Figure 3: An **S**₃-symmetric network $\mathcal{G}_{\mathcal{S}}$ for $\mathcal{S} = \{2, 3, 4\}$.

Indeed, adjacency matrices A_l 's of any 5-cell homogeneous networks with S_3 interior symmetry on $S = \{2, 3, 4\}$ are of form

a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	
a_{21}	<i>a</i> ₂₂	<i>a</i> ₂₃	<i>a</i> ₂₃	a_{25}	
a_{21}	<i>a</i> ₂₃	<i>a</i> ₂₂	<i>a</i> ₂₃	a_{25}	
a_{21}	<i>a</i> ₂₃	<i>a</i> ₂₃	<i>a</i> ₂₂	a_{25}	
a_{51}	a_{52}	<i>a</i> ₅₃	a_{54}	a ₅₅	

 \diamond

2.3 **Reverse interior symmetry**

We introduce a new concept of symmetry for coupled cell networks, the *reverse interior symmetry*. To do so, we need the notion of the *reverse network* \mathcal{G}^R of a coupled cell network \mathcal{G} , which is a network defined on the same set of cells, but with all the edges in the reversed direction.

Definition 2.12 Let $\mathcal{G} = (C, \mathcal{E}, \sim_C, \sim_E)$ be a coupled cell network. Define

$$\mathcal{E}^{\mathcal{R}} := \{ (d, c) : (c, d) \in \mathcal{E} \}.$$

and an equivalence relation \sim_{E^R} on \mathcal{E}^R by

$$(b,a) \sim_{E^R} (d,c) \Leftrightarrow (a,b) \sim_E (c,d).$$

The *reverse network* \mathcal{G}^R of \mathcal{G} is the network given by $\mathcal{G}^R = (\mathcal{C}, \mathcal{E}^R, \sim_C, \sim_{E^R})$.

Note that the adjacency matrices of \mathcal{G}^{R} are given by the transpose of the adjacency matrices of \mathcal{G} . Also, a reverse network of a homogeneous (resp. regular) network may not be homogeneous (resp. regular) again.

Definition 2.13 Let $\mathcal{G} = (C, \mathcal{E}, \sim_C, \sim_E)$ be a coupled cell network and \mathcal{G}^R be its reverse network. Let $\mathcal{S} \subseteq C$ be a subset. A permutation σ is called a *reverse interior symmetry* of \mathcal{G} on \mathcal{S} , if σ is an interior symmetry of \mathcal{G}^R on \mathcal{S} .

That is, the group of reverse interior symmetries of \mathcal{G} on \mathcal{S} can be canonically identified with the group of interior symmetries of \mathcal{G}^R on \mathcal{S} . Roughly speaking, a reverse interior symmetry is a permutation of the cells that preserves certain amount of output structure.

Let \mathcal{G} be a homogeneous network with *s* type of arrows whose adjacency matrices are A_1, A_2, \ldots, A_s . Then, a permutation σ is a reverse interior symmetry of \mathcal{G} on \mathcal{S} if and only if

$$a_{ij} = a_{\sigma(i)\sigma(j)}, \quad \forall i \in C \text{ and } \forall j \in S$$

for l = 1, ..., s.

Example 2.14 Consider the homogeneous network \mathcal{G} in Example 2.3. Then, the reverse network \mathcal{G}^R is as shown in Figure 4. It can be verified that \mathcal{G}^R has an interior symmetry (15) on $\mathcal{S} = \{1, 5\}$. Thus, (15) is a reverse interior symmetry of \mathcal{G} .

Note that a symmetry of a coupled cell network G is both an interior symmetry and a reverse interior symmetry of G, but the reverse may not be true.

Example 2.15 Consider the two networks in Figure 5, which are reverse to each other. Both networks have S_3 as an interior symmetry on $S = \{1, 2, 3\}$, thus S_3 is a reverse interior symmetry of both networks on S. However, neither network has an S_3 -symmetry.



Figure 4: The reverse network of the homogeneous network G in Figure 1.



Figure 5: Two networks that are reverse to each other.

2.4 Balanced equivalence relation

Given an equivalence relation \bowtie on the set of cells of a coupled cell network, we can color the nodes of the network in the following way: two cells *i*, *j* receive the same color precisely when they belong to the same \bowtie -equivalence class. The coloring is called *balanced*, or equivalently \bowtie is called a *balanced equivalence relation*, if any pair of cells with the same color have the same number and type of input arrows from cells of color *b*, for every *b*.

More formally,

Definition 2.16 (cf. [8]) Given a coupled cell network $\mathcal{G} = (C, \mathcal{E}, \sim_C, \sim_E)$, an equivalence relation \bowtie on the set *C* is called *balanced*, if for every $c, d \in C$ with $c \bowtie d$, there exists a bijection $\beta : I(c) \rightarrow I(d)$ between their input sets, which preserves the edge-equivalence relation \sim_E , and such that for all $i \in I(c)$, the tail cells of *i* and $\beta(i)$ are in the same \bowtie -class.

The next proposition states that every interior symmetry permutation determines a balanced equivalence relation.

Proposition 2.17 *Let* G *be an n-cell homogeneous network and* σ *be an interior symmetry of* G *on a subset* $S \subseteq C$. *If* \bowtie *is an equivalence relation on the cells* C *of* G *such that*

 $c \bowtie d \Leftrightarrow c, d$ belong to the same orbit under σ ,

then \bowtie *is balanced.*

Proof Let c, d be such that $c \bowtie d$. Then, $\sigma^m(c) = d$ for some $m \in \mathbb{N}$. Note that σ^m is an interior symmetry of \mathcal{G} on \mathcal{S} , for all $m \in \mathbb{N}$. Thus, by the definition of interior symmetry, there exists an edge-equivalence preserving bijection between the edges $(\sigma^m(c), \sigma^m(x))$ and (c, x), for every input arrows (c, x). Thus, there exists a bijection between the input sets of $d = \sigma^m(c)$ and c, which preserves the edge-equivalence relation. On the other hand, the tail cells x and $\sigma^m(x)$ are in the same orbit by σ , thus are in the same \bowtie -class. Therefore, \bowtie is a balanced equivalence relation.

Let $\Sigma_{\mathcal{S}}$ be the group of all interior symmetries of \mathcal{G} on a subset $\mathcal{S} \subseteq C$. Let $K \subseteq \Sigma_{\mathcal{S}}$ be a subgroup. By Proposition 2.17, every permutation in K determines a balanced equivalence relation on \mathcal{G} . In fact, the set of all these equivalence relations forms a sublattice of the total lattice of balanced equivalence relations on \mathcal{G} (cf. Stewart [12]). Moreover, the balanced equivalence relation \bowtie_K determined by the subgroup K is given by the join of all the equivalence relations determined by permutations in K and corresponds to the top element of this sublattice.

2.5 Quotient networks and quotient interior symmetry

Given a balanced equivalence relation \bowtie on a coupled cell network \mathcal{G} , a *quotient network* $\mathcal{G}_{\bowtie} = (C_{\bowtie'}, \mathcal{E}_{\bowtie'}, \sim_{C_{\bowtie'}}, \sim_{E_{\bowtie}})$ can be defined naturally as follows: the cells in C_{\bowtie} are the \bowtie -equivalence classes of the cells of \mathcal{G} and the edges in \mathcal{E}_{\bowtie} from quotient cell $[c]_{\bowtie}$ to quotient cell $[d]_{\bowtie}$, where $[c]_{\bowtie}$ denotes the \bowtie -equivalence class of c, are in correspondence with the edges (c', d') of \mathcal{G} , for all $c' \bowtie c$, $d' \bowtie d$. The cell-equivalence $\sim_{C_{\bowtie}}$ and edge-equivalence $\sim_{E_{\bowtie}}$ relations for \mathcal{G}_{\bowtie} are induced from those of \mathcal{G} . Since \bowtie is balanced, the quotient network \mathcal{G}_{\bowtie} is well-defined. See Golubitsky *et al.* [8].

Let \mathcal{G} be a homogeneous network of *n*-cells with *s* edge-equivalence classes whose adjacency matrices are A_1, A_2, \ldots, A_s . Let \bowtie be a balanced equivalence relation, which divides the cells of \mathcal{G} into *p* equivalence-classes. Then, \mathcal{G}_{\bowtie} is a homogeneous network of *p*-cells with *s* edgeequivalence classes. Denote the adjacency matrices of \mathcal{G}_{\bowtie} by $A_{1_{\bowtie}}, A_{2_{\bowtie}}, \ldots, A_{s_{\bowtie}}$. Let $A_{l_{\bowtie}} = [\bar{a}_{\alpha\beta}^{(l)}]_{p \times p}$. Then, for $\alpha = [i]_{\bowtie}, \beta = [j]_{\bowtie}$ in \mathcal{C}_{\bowtie} , we have (cf. Proposition 2.3, [1])

$$\bar{a}_{\alpha\beta}^{(l)} = \sum_{k \in [j]_{\bowtie}} a_{ik}^{(l)}.$$
(2.6)

Example 2.18 Let G be the homogeneous network in Example 2.3. As shown in Example 2.8 and Example 2.11, G has a symmetry $\mathbb{Z}_2 = \langle (15)(24) \rangle$ and an interior symmetry \mathbb{S}_3 on $S = \{2, 3, 4\}$. Consider $\bowtie_1 = \{\{1\}, \{2, 3, 4\}, \{5\}\}$ and $\bowtie_2 = \{\{1, 5\}, \{2, 4\}, \{3\}\}$. As seen in Subsection 2.4, both \bowtie_1, \bowtie_2 are balanced equivalence relations on G. Let G_1 (resp. G_2) be the quotient network induced by \bowtie_1 (resp. \bowtie_2). Then, the adjacency matrices of G_1 are

$$A_{1_{\bowtie_{1}}} = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}, \quad A_{2_{\bowtie_{1}}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

and the adjacency matrices of G_2 are

$$A_{1_{\bowtie_2}} = \left[\begin{array}{ccc} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{array} \right], \quad A_{2_{\bowtie_2}} = \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{array} \right]$$

The networks $\mathcal{G}_1, \mathcal{G}_2$ are shown in Figure 6.



Figure 6: Quotient networks for G in Figure 1 given by the **S**₃-interior symmetry (left) and $\mathbb{Z}_2 = \langle (15)(24) \rangle$ -symmetry (right).

Note that a quotient network of a uniform network is a regular network which may not be uniform in general.

One can also consider interior symmetry and reverse interior symmetry of quotient networks.

Definition 2.19 Let \mathcal{G} be a coupled cell network. We say that a permutation σ is a *quotient* (*interior*) symmetry of \mathcal{G} , if \mathcal{G} has a quotient network \mathcal{G}_{\bowtie_1} which has σ as an (interior) symmetry, for some balanced equivalence relation \bowtie_1 . Similarly, we say that a permutation γ is a *quotient reverse* (*interior*) symmetry of \mathcal{G} , if \mathcal{G} has a quotient network \mathcal{G}_{\bowtie_2} which has γ as a reverse (interior) symmetry, for some balanced equivalence relation \bowtie_2 .

Example 2.20 Based on Example 2.18, we conclude that the homogeneous network in Figure 1 has a quotient symmetry $\langle (15) \rangle$, since \mathcal{G}_1 is symmetric with respect to (15) in Figure 6 (left). \Box

In many cases, symmetric properties of the total network may be inherited by quotient networks. Yet, the following examples show that there may be no definite relation between the (interior) symmetry of the total network and the (interior) symmetry of its quotient networks.

Example 2.21 Consider the three-cell bidirectional ring pictured in Figure 7 which is S_3 -symmetric and whose quotient networks have no symmetry nor interior symmetry.



Figure 7: The three-cell bidirectional ring.

On the other hand, networks that quotient to (interior) symmetric networks tend to have (interior) symmetry. Consider the 5-cell networks given in Figures 9, 10 and 11. All of them have a quotient network which is isomorphic to the S_3 -symmetric network in Figure 7, for the balanced equivalence relation {{1}, {2,3}, {4,5}}. At the same time, they all have interior symmetries. More examples can be found in Aguiar *et al.* [1], where all the five cell regular networks admitting the three-cell bidirectional ring as a quotient network are listed.

However, there can exist networks without any symmetry nor interior symmetry which have an (interior) symmetric quotient network.

Example 2.22 Consider the 6-cell regular network in Figure 8. It can be directly verified that it



Figure 8: A regular network without symmetry nor interior symmetry with the S_3 -symmetric three-cell bidirectional ring as a quotient network.

has no nontrivial symmetry nor interior symmetry, but it quotients to the three-cell bidirectional ring, for the balanced equivalence relation $\{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$.

2.6 Direct sum decomposition of \mathbb{R}^n

Let \mathcal{G} be an *n*-cell homogeneous network with adjacency matrices A_1, A_2, \ldots, A_s and \bowtie be a balanced equivalence relation on \mathcal{G} . As seen in the previous subsection, there is an associated quotient network \mathcal{G}_{\bowtie} , whose adjacency matrices are given by $A_{1_{\bowtie}}, A_{2_{\bowtie}}, \ldots, A_{s_{\bowtie}}$ (cf. (2.6)). Based on results on regular networks (cf. Section 4 of Golubitsky *et al.* [5]), one can show that \bowtie induces a direct sum decomposition of \mathbb{R}^n such that every A_l has a form of block matrix containing $A_{l_{\bowtie}}$, for $l = 1, 2, \ldots, s$ (cf. Theorem 2.9 in Aguiar *et al.* [1] for regular networks).

More precisely, given a balanced equivalence relation >>, define

$$\Delta_{\bowtie}(\mathbb{R}^n) = \{ x \in \mathbb{R}^n : x_c = x_d \text{ if } c \bowtie d, \forall c, d \in C \},\$$

which is a linear subspace of \mathbb{R}^n . Then, $\Delta_{\bowtie}(\mathbb{R}^n)$ is A_l -invariant, for every l = 1, 2, ..., s, since \bowtie is balanced (cf. Theorem 4.3 in Golubitsky *et al.* [8]). Let $I_1, ..., I_p$ be the \bowtie -equivalence classes of order greater than one and $I = \bigcup_{l=1}^p I_l$. Define

$$W = \{x \in \mathbb{R}^n : x_j = 0 \forall j \in C \setminus I \text{ and } \sum_{i \in I_l} x_i = 0 \text{ for } 1 \le l \le p\}$$
(2.7)

$$U = \Delta_{\bowtie}(\mathbb{R}^n). \tag{2.8}$$

Note that if \bowtie is defined by an interior symmetry σ (cf. Subsection 2.4), then both W and U are σ -invariant subspaces. Since $W \cap U = \{0\}$, we can decompose \mathbb{R}^n as a direct sum

$$\mathbb{R}^n = W \oplus U. \tag{2.9}$$

Then, with respect to a basis adapted to (2.9), every adjacency matrix A_l has a block form

$$A_l = \left[\begin{array}{cc} A & 0 \\ C & A_{l_{\mathsf{bs}}} \end{array} \right],$$

where $A_{l_{\bowtie}}$ is the *l*-th matrix of the quotient network \mathcal{G}_{\bowtie} associated to the balanced equivalence relation \bowtie .

3 Interior symmetries and multiple eigenvalues

In this section, we analyze how the interior symmetry of a homogeneous network may affect the multiplicity of eigenvalues of the Jacobian at a fully-synchronized equilibrium of the associated coupled cell system. We start with the case for regular networks in Subsection 3.1, and then generalize the results to homogeneous networks in Subsection 3.2.

Beyond the notion of interior symmetry introduced by Golubitsky *et al.* [5], we defined in Section 2 two further concepts of interior symmetry: the *reverse interior symmetry*, which is the interior symmetry of the reverse network and the *quotient interior symmetry*, which is the symmetry of a quotient network.

The results presented in the following two subsections are stated for interior symmetry, but they can be easily extended for reverse interior symmetry and quotient interior symmetry. This follows from the fact that all the arguments we will use are based on the special form of the adjacency matrices of the networks, which is forced by interior symmetry. Since analogous form of adjacency matrices can be also induced by reverse interior symmetry and quotient interior symmetry, the results also apply to networks with reverse interior symmetry and quotient interior symmetry. More technically, note that each adjacency matrix A_l , for $l = 1, \ldots, s$, of a homogeneous network \mathcal{G} corresponds to the transpose of the adjacency matrix A_1^R of the reverse network \mathcal{G}^{R} . Thus, the eigenvalues of A_{l} coincide with those of A_{l}^{R} . Consequently, multiple eigenvalues of A_l may appear not only due to the interior symmetry of \mathcal{G}_l , but also due to its reverse interior symmetry. As seen in Subsection 2.6, for each quotient network \mathcal{G}_{\bowtie} there is a special basis such that each adjacency matrix A_l , for $l = 1, \ldots, s$, of \mathcal{G} has a block lower-triangular form with the adjacency matrix $A_{l_{id}}$ of the quotient network at one of the diagonal blocks. Thus, the eigenvalues of the adjacency matrix $A_{l_{\omega}}$ of a quotient network \mathcal{G}_{ω} are also eigenvalues of A_l . Therefore, multiple eigenvalues of A_l may appear not only due to the interior symmetry of *G*, but also due to its quotient interior symmetry.

In summary, from the results presented in the following two subsections, it follows that the interior symmetries, reverse interior symmetries and quotient interior symmetries of regular and homogeneous networks favor multiple eigenvalues of the Jacobian matrix at a fullysynchronized equilibrium for the associated coupled cell systems.

3.1 Regular networks

Let \mathcal{G} be an *n*-cell regular network with *r*-dimensional cell internal dynamics. Let μ_1, \ldots, μ_n be the eigenvalues of the adjacency matrix $A_{\mathcal{G}}$ of \mathcal{G} . As it is shown in Leite *et al.* [10] and Aguiar *et al.* [1], the eigenvalues of the Jacobian $J_{\mathcal{G}}$ of the associated coupled systems at a fully-synchronized equilibrium are the union of the eigenvalues of the *r* × *r* matrices

$$\alpha + \mu_j \beta$$
, for $j = 1, \dots, n$

including algebraic multiplicity.

Remark 3.1 It follows that if A_G has one eigenvalue with multiplicity m_a , then J_G has r eigenvalues with multiplicity at least m_a (note that it can also happen that some of the r eigenvalues are equal).

As mentioned before, interior symmetry imposes restrictions on the network structure and thus on the entries of the adjacency matrix. By Remark 3.1, to analyze the effect of interior

symmetries on the multiplicity of the eigenvalues of the Jacobian, it is sufficient to concentrate on the influence of interior symmetries on the multiplicity of the eigenvalues of A_G .

As we will see, interior symmetries force the existence of integer eigenvalues for the adjacency matrix $A_{\mathcal{G}}$ of a regular network \mathcal{G} . Moreover, all the eigenvalues λ of $A_{\mathcal{G}}$ satisfy $\|\lambda\| \leq v$, where v is the valency of \mathcal{G} . Thus, for regular networks with valency 2, the eigenvalues -1, 0and 1 will arise very often, in the presence of interior symmetry.

3.1.1 Product interior symmetry

We show that the case of product interior symmetries can be inferred from their component symmetries.

Let G be an *n*-cell regular network having interior symmetry groups Σ_{S_j} , for j = 1, ..., r, on disjoint subsets S_j of cells of G. We say that G has a *product interior symmetry*

$$\Sigma_{\mathcal{S}} = \Sigma_{\mathcal{S}_1} \times \ldots \times \Sigma_{\mathcal{S}_{r'}}$$

where $S = \bigcup_{j=1}^{r} S_j$. Let \bowtie_j be the balanced equivalence relation induced by Σ_{S_j} , for j = 1, ..., r. Then, the balanced equivalence relation \bowtie induced by Σ_S is given by

$$c \bowtie d \Leftrightarrow c \bowtie_j d$$
 for some j . (3.10)

Set $U = \Delta_{\bowtie}(\mathbb{R}^n)$. Let $I_1^j, I_2^j, \dots, I_{p_j}^j$ be the \bowtie_j -equivalence classes of order greater than one and $I^j = \bigcup_{l=1}^{p_j} I_l^j$. Define

$$W_j = \{x \in \mathbb{R}^n : x_i = 0 \ \forall i \in C \setminus I^j \text{ and } \sum_{i \in I_l^j} x_i = 0 \text{ for } 1 \le l \le p_j\}, \quad j = 1, 2..., r.$$

Let $I = \bigcup_{j=1}^{r} I^{j}$. Note that dim $W_{j} = |S_{j}| - p_{j}$, dim $U = |C \setminus I| + \sum_{j=1}^{r} p_{j}$ and $W_{i} \cap W_{j} = \{0\}$, $U \cap W_{j} = \{0\}$, for $i \neq j, j = 1, 2, ..., r$. Thus, we have

$$\mathbb{R}^n = W_1 \oplus \ldots \oplus W_r \oplus U. \tag{3.11}$$

Theorem 3.2 Let \mathcal{G} be an n-cell regular network having a product interior symmetry $\Sigma_{\mathcal{S}} = \Sigma_{\mathcal{S}_1} \times \ldots \times \Sigma_{\mathcal{S}_r}$ on disjoint subsets \mathcal{S}_j of cells of \mathcal{G} . Then, with respect to the decomposition (3.11), the adjacency matrix $A_{\mathcal{G}}$ of \mathcal{G} takes the form

$\begin{bmatrix} A^1 \end{bmatrix}$	0	•••	0	0	1
0	A^2	•••	0	0	
:	÷	·	÷	÷	,
0	0	• • •	A^r	0	
B_1	B_2	•••	B_r	A_{\bowtie}	

where A^j is a matrix of order $(|S_j| - p_j) \times (|S_j| - p_j)$ for j = 1, ..., r, and A_{\bowtie} is the adjacency matrix of the quotient network associated with \bowtie (cf. 3.10)).

Proof Let *W* be the linear subspace induced by \bowtie (cf. (2.7)). Note that $W = W_1 \oplus ... \oplus W_r$. Then, as discussed in Section 2.6, with respect to the decomposition

$$\mathbb{R}^n = W \oplus U,$$

 A_{G} takes the form

$$\left[\begin{array}{cc} A & 0 \\ C & A_{\bowtie} \end{array}\right].$$

It remains to show that *A* is a block matrix of diagonal form, with respect to the dimensions of the W_j 's, j = 1, ..., r. Observe that to show that the entries of the *j*-th column of *A* are all zeros except those on the diagonal block, it is enough to show that $(W_j \oplus U)$ is $A_{\mathcal{G}}$ -invariant. Let $x \in W_j$ for a $j \in \{1, 2, ..., r\}$ and $y = A_{\mathcal{G}}x$. We need to show that $y \in W_j \oplus U$, i.e.

$$y_i = y_l, \quad \forall i \bowtie_k l, \ \forall k \neq j.$$

Since $x \in W_j$, the *i*-th component x_i of x is zero except when $i \in S_j$. Thus, the value of y_i (resp. y_l) depends only on the (i, m)-th (resp. (l, m)-th) entries of A_G , where $m \in S_j$. When $i \bowtie_k l$ and $k \neq j$, we have $i, l \notin S_j$. Thus, the (i, m)-th entry of A_G is equal to the (l, m)-th entry of A_G , for all $m \in S_j$. It follows that $y_i = y_l$, for all $i \bowtie_k l, k \neq j$.

Therefore, we have $A_{\mathcal{G}}W_j \subseteq W_j \oplus U$. Combined with the fact $A_{\mathcal{G}}U \subseteq U$, we conclude that $(W_j \oplus U)$ is $A_{\mathcal{G}}$ -invariant, for j = 1, 2, ..., r.

Corollary 3.3 Under the assumptions of Theorem 3.2, we have that the set of eigenvalues of the adjacency matrix $A_{\mathcal{G}}$ of \mathcal{G} is given by the disjoint union of the set of eigenvalues of A^{j} and the set of eigenvalues of A_{\bowtie} , for j = 1, 2, ..., r.

Taking into account Theorem 3.2 and Corollary 3.3, in what follows, we shall concentrate on interior symmetry groups that cannot be written as a product of subgroups. We will certainly not consider here all subgroups of S_n with this property, as the number of subgroups increases exponentially with n (cf. Holt [9] for an enumeration of subgroups and conjugacy classes of the subgroups of S_n , for $n \le 18$).

In this paper, we will be primarily interested in the following subgroups of S_n :

- (i) the symmetric groups $\mathbf{S}_k = \langle (i_1 \dots i_k), (i_1 i_2) \rangle$, with $2 \le k \le n$;
- (ii) the alternating groups \mathbf{A}_k , with $2 \le k \le n$;
- (iii) the dihedral groups $\mathbf{D}_k = \langle (i_1 \dots i_k), (i_2 i_k)(i_3 i_{k-1}) \dots (i_j i_{k+2-j}) \rangle$, with $2 \le k \le n$;
- (iv) the cyclic groups $\mathbb{Z}_k = \langle (i_1 \dots i_k) \rangle$, with $2 \le k \le n$.

Note that $\mathbf{S}_2 \simeq \mathbf{D}_2 \simeq \mathbb{Z}_2$ and $\mathbf{S}_3 \simeq \mathbf{D}_3$.

3.1.2 S_k- and A_k-Interior symmetry

We show that

Theorem 3.4 Let \mathcal{G} be an n-cell regular network having an interior symmetry group \mathbf{S}_k or \mathbf{A}_k on a subset $\mathcal{S} \subseteq C$ of k cells of \mathcal{G} , for $2 \leq k \leq n$. Let i and j be any two different cells in \mathcal{S} . Then, the adjacency matrix $A_{\mathcal{G}} = [a_{\alpha\beta}]_{1 \leq \alpha, \beta \leq n}$ of \mathcal{G} has the eigenvalue $a_{ii} - a_{ij}$ with algebraic multiplicity at least k - 1. As a result, the Jacobian $J_{\mathcal{G}}$ has r eigenvalues with algebraic multiplicity at least k - 1.

Proof Without loss of generality, we can assume $S = \{1, ..., k\}$. First notice that for any $i, j, l, m \in S$, the product $(i \ j)(l \ m)$ of two transpositions is an element in $\mathbf{A}_k \subset \mathbf{S}_k$. Since \mathcal{G} has an interior symmetry \mathbf{S}_k (resp. \mathbf{A}_k), the entries of $A_{\mathcal{G}}$ satisfy (cf. (2.5))

$$\begin{array}{ll} a_{ii} = a_{jj}, & \forall i, j \in \mathcal{S} \\ a_{il} = a_{jm}, & \forall i, j, l, m \in \mathcal{S} \text{ with } i \neq l \text{ and } j \neq m \\ a_{il} = a_{jl}, & \forall i, j, \in \mathcal{S} \text{ and } \forall l \in \mathcal{C} \setminus \mathcal{S}. \end{array}$$

Consider the balanced equivalence relation \bowtie induced by \mathbf{S}_k (resp. \mathbf{A}_k)

$$= \{\{1, 2, \dots, k\}, \{k+1\}, \dots, \{n\}\}.$$

Let *W*, *U* be given by (2.7)–(2.8). Then, with respect to (2.9), the adjacency matrix A_G takes the form

$$\left[\begin{array}{cc} A & 0 \\ C & A_{\bowtie} \end{array}\right]'$$

where *A* is a scalar matrix of order (k-1) with the element $(a_{11} - a_{12})$ on the diagonal. Thus, the adjacency matrix $A_{\mathcal{G}}$ has the eigenvalue $(a_{11} - a_{12})$ with algebraic multiplicity at least (k-1). It follows from Remark 3.1 that the Jacobian $J_{\mathcal{G}}$ has *r* eigenvalues with algebraic multiplicity at least k-1.

Example 3.5 Let \mathcal{G} be a 5-cell regular network that quotients to the three-cell bidirectional ring \mathcal{R} (cf. Figure 7). Examples of \mathcal{G} are networks given in Figures 9, 10 and 11. By Theorem 3.4, the adjacency matrix of \mathcal{R} has -1 as an eigenvalue with algebraic multiplicity 2, as a result of the S_3 (interior) symmetry of \mathcal{R} . Thus, due to the S_3 quotient interior symmetry of \mathcal{G} , the adjacency matrix of \mathcal{G} has -1 as an eigenvalue with algebraic multiplicity at least 2.

3.1.3 D_{*k*}**-Interior symmetry**

We prove the following

Theorem 3.6 Let \mathcal{G} be an n-cell regular network having an interior symmetry group \mathbf{D}_k for some $k \in \{3, ..., n\}$. Set

$$m = \begin{cases} (k-1)/2, & \text{if } k \text{ is odd} \\ k/2, & \text{if } k \text{ is even.} \end{cases}$$

Then, the adjacency matrix $A_{\mathcal{G}} = [a_{ij}]_{1 \le i,j \le n}$ of \mathcal{G} has m eigenvalues with algebraic multiplicity at least 2, if k is odd; $A_{\mathcal{G}}$ has (m-1) eigenvalues with algebraic multiplicity at least 2, if k is even. As a result, if k is odd (resp. even), then the Jacobian $J_{\mathcal{G}}$ has mr (resp. (m-1)r) eigenvalues with algebraic multiplicity at least 2.

The following lemma will be needed for the proof of Theorem 3.6.

Lemma 3.7 Let $m \in \mathbb{N}$. Consider the following two matrices of order $m \times m$

$$B_{1} = \begin{bmatrix} a_{11} - a_{13} & a_{12} - a_{14} & a_{13} - a_{15} & \cdots & a_{1,m-1} - a_{1,m+1} & a_{1m} - a_{1,m+1} \\ a_{12} - a_{14} & a_{11} - a_{15} & a_{12} - a_{16} & \cdots & a_{1,m-2} - a_{1,m+1} & a_{1,m-1} - a_{1m} \\ a_{13} - a_{15} & a_{12} - a_{16} & a_{11} - a_{17} & \cdots & a_{1,m-3} - a_{1m} & a_{1,m-2} - a_{1,m-1} \\ & & & & & & & & \\ a_{1,m-1} - a_{1,m+1} & a_{1,m-2} - a_{1,m+1} & a_{1,m-3} - a_{1m} & \cdots & a_{11} - a_{14} & a_{12} - a_{13} \\ a_{1m} - a_{1,m+1} & a_{1,m-1} - a_{1m} & a_{1,m-2} - a_{1,m-1} & \cdots & a_{12} - a_{13} & a_{11} - a_{12} \end{bmatrix}$$

$$(3.12)$$

$$B_{2} = \begin{bmatrix} a_{11} + a_{13} & a_{12} + a_{14} & a_{13} + a_{15} & \cdots & a_{1,m-1} + a_{1,m+1} & a_{1m} + a_{1,m+1} \\ a_{12} + a_{14} & a_{11} + a_{15} & a_{12} + a_{16} & \cdots & a_{1,m-2} + a_{1,m+1} & a_{1,m-1} + a_{1m} \\ a_{13} + a_{15} & a_{12} + a_{16} & a_{11} + a_{17} & \cdots & a_{1,m-3} + a_{1m} & a_{1,m-2} + a_{1,m-1} \\ & & & & & & & & & & & \\ a_{1,m-1} + a_{1,m+1} & a_{1,m-2} + a_{1,m+1} & a_{1,m-3} + a_{1m} & \cdots & a_{11} + a_{14} & a_{12} + a_{13} \\ a_{1m} + a_{1,m+1} & a_{1,m-1} + a_{1m} & a_{1,m-2} + a_{1,m-1} & \cdots & a_{12} + a_{13} & a_{11} + a_{12} \end{bmatrix} \\ + \begin{bmatrix} -2a_{12} & -2a_{12} & \cdots & -2a_{12} & -2a_{12} \\ -2a_{13} & -2a_{13} & \cdots & -2a_{13} & -2a_{13} \\ -2a_{14} & -2a_{14} & \cdots & -2a_{14} & -2a_{14} \\ & & & & \\ -2a_{1m} & -2a_{1m} & \cdots & -2a_{1m} & -2a_{1m} \\ -2a_{1,m+1} & -2a_{1,m+1} & \cdots & -2a_{1,m+1} \end{bmatrix} .$$

$$(3.13)$$

Then, B_1 *and* B_2 *are similar.*

Proof Notice that any matrix $M = (x_{ij})_{m \times m}$ is similar to the matrix $(x_{m-i+1,m-j+1})$ by exchanging rows R_i with R_{m-i+1} and exchanging columns C_i with C_{m-i+1} for $1 \le i \le m$. We will denote by \tilde{B}_1 the matrix obtained in this way from B_1 .

For r = 1, 2, ..., m - 1, denote by O_r the row operation

$$R_r \quad \rightsquigarrow \quad R_r + R_{r+1} + \cdots + R_m,$$

where the *r*-th row is replaced by the sum of the *j*-th row for $r \le j \le m$. It suffices to show that

$$O_{m-1}O_{m-2}\cdots O_2O_1B_2O_1^{-1}O_2^{-1}\cdots O_{m-2}^{-1}O_{m-1}^{-1} = \tilde{B}_1.$$
(3.14)

Write $B_2 = (b_{ij})_{m \times m}$ and denote by $C = (c_{ij})_{m \times m}$ the left hand side of (3.14). We first show that

$$c_{ij} = \begin{cases} \sum_{p=i}^{m} b_{p1}, & \text{if } j = 1, \\ \sum_{p=i}^{m} (b_{pj} - b_{p,j-1}), & \text{if } 1 < j \le m. \end{cases}$$
(3.15)

Notice that O_r^{-1} represents the column operations

$$C_{r+1} \rightsquigarrow C_{r+1} - C_r, \quad C_{r+2} \rightsquigarrow C_{r+2} - C_r, \cdots, \quad C_m \rightsquigarrow C_m - C_r.$$

Thus, it is clear that column operations and *r*-th row operations O_r for $r \neq i$, do not change the value of (i, 1)-th element. Thus, c_{i1} is equal to the (i, 1)-th element of O_iB_2 , i.e.

$$c_{i1} = \sum_{p=i}^m b_{p1}.$$

Assume j > 1. Then, column operations for $i \ge j$ and row operations for $i \ne j$ do not change the value of (i, j)-th element. Thus, c_{ij} is equal to the (i, j)-th element of $O_i B_2 O_1^{-1} O_2^{-1} \cdots O_{j-1}^{-1}$. We need to differentiate the cases i < j and $i \ge j$, since it determines the order of the operations. *Case I.* $i \ge j$. Let $c_{ij}^{(l)}$ denote the (i, j)-th element of $B_2 O_1^{-1} O_2^{-1} \cdots O_{j-l}^{-1}$ for $1 \le l \le j - 1$. Then,

$$c_{ij} = \sum_{p=i}^{m} c_{pj}^{(1)} = \sum_{p=i}^{m} (c_{pj}^{(2)} - c_{p,j-1}^{(2)})$$

= $\sum_{p=i}^{m} ((c_{pj}^{(3)} - c_{p,j-2}^{(3)}) - (c_{p,j-1}^{(3)} - c_{p,j-2}^{(3)})) = \sum_{p=i}^{m} (c_{pj}^{(3)} - c_{p,j-1}^{(3)}) = \cdots$
= $\sum_{p=i}^{m} (c_{pj}^{(j-1)} - c_{p,j-1}^{(j-1)}) = \sum_{p=i}^{m} (b_{pj} - b_{p,j-1})$

Case II. i < j.

Let $c_{ij}^{(l)}$ denote the (i, j)-th element of $O_i B_2 O_1^{-1} O_2^{-1} \cdots O_{j-l-1}^{-1}$ for $1 \le l \le j - i$, and $c_{ij}^{(l)}$ denote the (i, j)-th element of $B_2 O_1^{-1} O_2^{-1} \cdots O_{j-l}^{-1}$ for $j - i + 1 \le l \le j - 1$. Then,

$$\begin{aligned} c_{ij} &= c_{ij}^{(1)} - c_{i,j-1}^{(1)} \\ &= (c_{ij}^{(2)} - c_{i,j-2}^{(2)}) - (c_{i,j-1}^{(2)} - c_{i,j-2}^{(2)}) = c_{ij}^{(2)} - c_{i,j-1}^{(2)} = \cdots \\ &= c_{ij}^{(j-i)} - c_{i,j-1}^{(j-i)} \\ &= \sum_{p=i}^{m} (c_{pj}^{(j-i+1)} - c_{p,j-1}^{(j-i+1)}) = \sum_{p=i}^{m} ((c_{pj}^{(j-i+2)} - c_{p,j-2}^{(j-i+2)}) - (c_{p,j-1}^{(j-i+2)} - c_{p,j-2}^{(j-i+2)})) \\ &= \sum_{p=i}^{m} (c_{pj}^{(j-i+2)} - c_{p,j-1}^{(j-i+2)}) = \cdots \\ &= \sum_{p=i}^{m} (c_{pj}^{(j-1)} - c_{p,j-1}^{(j-1)}) = \sum_{p=i}^{m} (b_{pj} - b_{p,j-1}) \end{aligned}$$

Therefore, (3.15) is proved. It remains to show $C = \tilde{B}_1$. Recall that a_{ij} denotes the (i, j)-th element of the adjacency matrix A_G . Consider the vector

$$v = (a_{11}, a_{12}, a_{13}, \dots, a_{1m}, a_{1,m+1}, a_{1,m+1}, a_{1m}, \dots, a_{13}, a_{12})^{T}$$

and the shifting operator ρ

$$\rho v = (a_{12}, a_{11}, a_{12}, a_{13}, \dots, a_{1m}, a_{1,m+1}, a_{1,m+1}, a_{1m}, \dots, a_{13})^T.$$

Notation: In the rest of the proof of Lemma 3.7, we use v_p to denote the p'-th element of v, with $p' = p \pmod{k}$, for $p \in \mathbb{Z}$ and $v \in \mathbb{R}^k$. Also, we use a short-hand notation of $(\rho v)_p$ by ρv_p .

Due to the symmetric form of *v*, we have

$$v_{m+q} = v_{m-q+3}, \quad q \in \mathbb{Z},$$
 (3.16)

and

$$v_p = \rho v_{p+1}, \quad v_p = \rho^{-1} v_{p-1}, \quad p \in \mathbb{Z}.$$
 (3.17)

In terms of *v* and ρ , the matrix B_1 consists of the first *m* rows of the matrix

$$(v - \rho^{-2}v, \rho v - \rho^{-3}v, \rho^{2}v - \rho^{-4}v, \dots, \rho^{m-2}v - \rho^{-m}v, \rho^{m-1}v - \rho^{-(m+1)}v)$$

and B_2 consists of the first *m* rows of the matrix

$$(v + \rho^{-2}v, \rho v + \rho^{-3}v, \rho^{2}v + \rho^{-4}v, \dots, \rho^{m-2}v + \rho^{-m}v, \rho^{m-1}v + \rho^{-(m+1)}v) - 2(\rho^{-1}v, \dots, \rho^{-1}v).$$

Assume that $1 \le i \le m$, $1 < j \le m$. By (3.15), we have

$$c_{ij} = \sum_{p=i}^{m} (b_{pj} - b_{p,j-1}) = \sum_{p=i}^{m} ((\rho^{j-1}v_p + \rho^{-j-1}v_p) - (\rho^{j-2}v_p + \rho^{-j}v_p))$$

$$\stackrel{(3.17)}{=} \rho^{j-1}v_i - \rho^{-j}v_i + \rho^{-j-1}v_m - \rho^{j-2}v_m$$

$$\stackrel{(3.17)}{=} v_{i-j+1} - v_{i+j} + v_{m+j+1} - v_{m-j+2}$$

$$\stackrel{(3.16)}{=} v_{i-j+1} - v_{i+j}.$$
(3.18)

On the other hand, the (i, j)-th element of \tilde{B}_1 is equal to the (m - i + 1, m - j + 1)-th element of B_1 , which equals to

$$\rho^{m-j}v_{m-i+1} - \rho^{-m+j-2}v_{m-i+1} \stackrel{(3.17)}{=} v_{j-i+1} - v_{2m-i-j+3}.$$
(3.19)

By (3.16), we also have

$$v_{i-j+1} = v_{k+i-j+1} = v_{2m+1+i-j+1} = v_{m+(m+i-j+2)} \stackrel{(3.16)}{=} v_{j-i+1}, \tag{3.20}$$

and

$$v_{2m-i-j+3} = v_{m+(m-i-j+3)} \stackrel{(3.16)}{=} v_{i+j}.$$
(3.21)

It follows from (3.18)–(3.21) that the (i, j)-th element of *C* coincides with the (i, j)-th element of \tilde{B}_1 , for $1 \le i \le m, 1 < j \le m$.

The case of j = 1 can be similarly proved. By (3.15), we have

$$c_{i1} = \sum_{p=i}^{m} b_{p1} = \sum_{p=i}^{m} (v_p + \rho^{-2} v_p - 2\rho^{-1} v_p)$$

$$\stackrel{(3.17)}{=} v_i - \rho^{-1} v_i + \rho^{-2} v_m - \rho^{-1} v_m \stackrel{(3.17)}{=} v_i - v_{i+1} + v_{m+2} - v_{m+1}$$

$$\stackrel{(3.16)}{=} v_i - v_{i+1} \stackrel{(3.20)-(3.21)}{=} v_{2-i} - v_{2m-i+2} \stackrel{(3.17)}{=} \rho^{m-1} v_{m-i+1} - \rho^{-m-1} v_{m-i+1},$$

which is the (i, 1)-th element of \tilde{B}_1 .

Consequently, we showed that $C = \tilde{B}_1$ and thus (3.14) holds.

Lemma 3.7.

Proof of Theorem 3.6 Without loss of generality, assume G has an interior symmetry D_k on the cells $\{1, \ldots, k\}$. Due to this interior symmetry, the entries of A_G satisfy

$a_{ij} = a_{l(j+l-i) \pmod{k}}$	for $i, j, l \in \{1,, k\}$,
$a_{ij} = a_{lj},$	for $i, l \in \{1,, k\}$ and $j \in \{k + 1,, n\}$,
$a_{1j} = a_{1(k-j+2)},$	for $j \in \{2,, m, m + 1\}$.

Thus, A_G has the form

$$A_{\mathcal{G}} = \begin{bmatrix} A & D \\ E & F \end{bmatrix}, \tag{3.22}$$

where *D* is a $k \times (n - k)$ matrix with all rows equal and *A* is a (symmetric) circulant matrix

$$A = \begin{cases} circ(a_{11}a_{12}a_{13}\dots a_{1m+1}a_{1m+1}\dots a_{13}a_{12}), & \text{if } k \text{ is odd,} \\ circ(a_{11}a_{12}a_{13}\dots a_{1m}a_{1m+1}a_{1m}\dots a_{13}a_{12}), & \text{if } k \text{ is even.} \end{cases}$$

It follows from (2.4) that the eigenvalues λ_j , j = 0, ..., k - 1 of A are real and satisfy $\lambda_j = \lambda_{k-j}$, for j = 1, ..., m. That is, A has m eigenvalues with algebraic multiplicity at least 2, if k is odd; A has (m - 1) eigenvalues with algebraic multiplicity at least 2, if k is even. Our goal is to prove the same property for A_G .

Case I: Assume that *k* is odd.

Consider the balanced equivalence relation $\bowtie = \{\{1, 2, ..., k\}, \{k + 1\}, ..., \{n\}\}$ induced by **D**_k. Motivated by the direct sum decomposition (2.9), we define a basis $\mathcal{B} = \{b_1, b_2, ..., b_n\}$ in \mathbb{R}^n by

$$b_{i} = \begin{cases} e_{i+1} - e_{k-i+1}, & \text{for } 1 \leq i \leq m \\ -2e_{1} + e_{i-m+1} + e_{k-i+m+1}, & \text{for } m+1 \leq i \leq 2m \\ e_{1} + e_{2} + \dots + e_{k}, & \text{for } i = k \\ e_{i}, & \text{for } k+1 \leq i \leq n, \end{cases}$$
(3.23)

where $\{e_1, e_2, \ldots, e_n\}$ denote the standard basis in \mathbb{R}^n (cf. Example 3.8 for k = 7). Then, the adjacency matrix A_G in the basis \mathcal{B} has the form

$$\mathcal{B}^{-1}A_{\mathcal{G}}\mathcal{B} = \begin{bmatrix} B_1 & 0 & \\ 0 & B_2 & 0 \\ \hline C & A_{\bowtie} \end{bmatrix},$$

where B_1 , B_2 are matrices of order $m \times m$ given by (3.12)–(3.13) and A_{\bowtie} is the adjacency matrix of the quotient network induced by \bowtie . By Lemma 3.7, B_1 and B_2 are similar matrices, thus have the same eigenvalues. Consequently, $A_{\mathcal{G}}$ has m eigenvalues of multiplicity at least 2. It follows from Remark 3.1 that the Jacobian $J_{\mathcal{G}}$ has mr eigenvalues with algebraic multiplicity at least 2.

Notice that we can obtain an "optimal" basis $\tilde{\mathcal{B}}$ by applying the operations specified in the proof of Lemma 3.7 to \mathcal{B} , so that $A_{\mathcal{G}}$ has two copies of B_1 lying on the diagonal. More precisely, let $R = O_{m-1}O_{m-2}\cdots O_2O_1$ be the total row operation on B_2 and S the total row switching operation such that $SB_1S^{-1} = \tilde{B}_1$. Then, we have

$$SRB_2R^{-1}S^{-1} = B_1$$

Set

$$O = \begin{bmatrix} I_m & 0 & 0 \\ 0 & SR & 0 \\ 0 & 0 & I_{n-2m} \end{bmatrix},$$

where I_i stands for the identity matrix of order $i \times i$. Define a new basis by

$$\tilde{\mathcal{B}} = \mathcal{B}O^{-1}$$

Then, the adjacency matrix $A_{\mathcal{G}}$ has the form

$$\tilde{\mathcal{B}}^{-1}A_{\mathcal{G}}\tilde{\mathcal{B}} = \begin{bmatrix} B_1 & 0 & \\ 0 & B_1 & 0 \\ \hline C' & A_{\bowtie} \end{bmatrix}$$

A precise formula of $\tilde{\mathcal{B}} = \{\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n\}$ is given by

$$\tilde{b}_{i} = \begin{cases} e_{i+1} - e_{k-i+1}, & \text{for } 1 \leq i \leq m \\ -e_{k-i} + e_{k-i+1} + e_{i+1} - e_{i+2}, & \text{for } m+1 \leq i \leq 2m-1 \\ -2e_{1} + e_{2} + e_{k}, & \text{for } i = 2m \\ e_{1} + e_{2} + \dots + e_{k}, & \text{for } i = 2m+1 = k \\ e_{i}, & \text{for } k+1 \leq i \leq n, \end{cases}$$
(3.24)

(cf. Example 3.8 for k = 7).

Case II: Assume that *k* is even.

Similar to the case of odd k, we try to find an optimal basis for the diagonal form of $A_{\mathcal{G}}$. Motivated by the direct sum decomposition (2.9), define the following basis \mathcal{B}

$$b_{i} = \begin{cases} e_{i+1} - e_{k-i+1}, & \text{for } 1 \leq i \leq m-1 \\ e_{1} - e_{2} + e_{3} - e_{4} \cdots + e_{k-1} - e_{k}, & \text{for } i = m \\ -2e_{1} + e_{i-m+1} + e_{k-i+m+1}, & \text{for } m+1 \leq i \leq 2m-1 \\ e_{1} + e_{2} + \cdots + e_{k}, & \text{for } i = 2m = k \\ e_{i}, & \text{for } k \leq i \leq n, \end{cases}$$
(3.25)

(cf. Example 3.9 for k = 8). Then, the adjacency matrix $A_{\mathcal{G}}$ in the basis \mathcal{B} has the form

$$\mathcal{B}^{-1}A_{\mathcal{G}}\mathcal{B} = \begin{bmatrix} \begin{array}{c|c} B_1 & 0 \\ 0 & a & * \\ 0 & 0 & B_2 \\ \hline C & A_{\bowtie} \end{bmatrix},$$

where B_1, B_2 are matrices of order $(m - 1) \times (m - 1)$,

$$a = a_{11} - 2a_{12} + 2a_{13} - 2a_{1,4} + \dots + (-1)^{m-1} 2a_{1,m} + (-1)^m a_{1,m+1}$$
(3.26)

and A_{\bowtie} is the adjacency matrix of the quotient network. More precisely,

$$B_{1} = \begin{bmatrix} a_{11} - a_{13} & a_{12} - a_{14} & a_{13} - a_{15} & \cdots & a_{1,m-1} - a_{1,m+1} \\ a_{12} - a_{14} & a_{11} - a_{15} & a_{12} - a_{16} & \cdots & a_{1,m-2} - a_{1,m} \\ a_{13} - a_{15} & a_{12} - a_{16} & a_{11} - a_{17} & \cdots & a_{1,m-3} - a_{1,m-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1,m-1} - a_{1,m+1} & a_{1,m-2} - a_{1,m} & a_{1,m-3} - a_{1,m-1} & \cdots & a_{11} - a_{13} \end{bmatrix}$$

$$B_{2} = \begin{bmatrix} a_{11} + a_{13} & a_{12} + a_{14} & a_{13} + a_{15} & \cdots & a_{1,m-1} + a_{1,m+1} \\ a_{12} + a_{14} & a_{11} + a_{15} & a_{12} + a_{16} & \cdots & a_{1,m-2} + a_{1,m} \\ a_{13} + a_{15} & a_{12} + a_{16} & a_{11} + a_{17} & \cdots & a_{1,m-3} + a_{1,m-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1,m-1} + a_{1,m+1} & a_{1,m-2} + a_{1,m} & a_{1,m-3} + a_{1,m-1} & \cdots & a_{11} + a_{13} \end{bmatrix} \\ + \begin{bmatrix} 2a_{1m} - 2a_{1,m+1} & 2a_{1,m-1} - 2a_{1,m+1} & \cdots & 2a_{12} - 2a_{1,m+1} \\ -2a_{1m} + 2a_{1,m+1} & -2a_{1,m-1} + 2a_{1,m+1} & \cdots & -2a_{12} + 2a_{1,m+1} \\ 2a_{1m} - 2a_{1,m+1} & 2a_{1,m-1} - 2a_{1,m+1} & \cdots & 2a_{12} - 2a_{1,m+1} \\ \cdots & \cdots & \cdots \\ (-1)^{m}(2a_{1m} - 2a_{1,m+1}) & (-1)^{m}(2a_{1,m-1} - 2a_{1,m+1}) & \cdots & (-1)^{m}(2a_{12} - 2a_{1,m+1}) \end{bmatrix} \\ + \begin{bmatrix} -2a_{12} & -2a_{12} & \cdots & -2a_{12} \\ -2a_{13} & -2a_{13} & \cdots & -2a_{13} \\ -2a_{14} & -2a_{14} & \cdots & -2a_{14} \\ \cdots & \cdots & \cdots \\ -2a_{1m} & -2a_{1m} & \cdots & -2a_{1m} \end{bmatrix}$$

Analog to Lemma 3.7, one can show that B_1 and B_2 are similar. Indeed, denote by O_r the row operation

$$R_r \quad \rightsquigarrow \quad R_r + 2R_{r+1} - 2R_{r+2} + \dots + (-1)^{m-r} 2R_{m-1}, \quad r = 1, 2, \dots, m-2.$$

Then,

$$O_1 O_2 \cdots O_{m-3} O_{m-2} B_2 O_{m-2}^{-1} O_{m-3}^{-1} \cdots O_2^{-1} O_1^{-1} = B_1.$$
(3.27)

By applying the operations specified in (3.27) to \mathcal{B} , we can obtain a new basis $\tilde{\mathcal{B}}$. Let $R = O_1 O_2 \cdots O_{m-3} O_{m-2}$. Define

$$O = \begin{bmatrix} I_m & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & I_{n-2m+1} \end{bmatrix}, \qquad \tilde{\mathcal{B}} = \mathcal{B}O^{-1}.$$

Then, the adjacency matrix $A_{\mathcal{G}}$ in the basis $\tilde{\mathcal{B}}$ has the form

$$\tilde{\mathcal{B}}^{-1}A_{\mathcal{G}}\tilde{\mathcal{B}} = \begin{bmatrix} \begin{array}{c|c} B_1 & 0 \\ 0 & a & *' \\ 0 & 0 & B_1 \\ \hline C'' & A_{\bowtie} \end{bmatrix}.$$

It follows that $A_{\mathcal{G}}$ has (m-1) eigenvalues of multiplicity at least 2 and thus, by Remark 3.1, the Jacobian $J_{\mathcal{G}}$ has (m-1)r eigenvalues with algebraic multiplicity at least 2.

A precise formula of $\tilde{\mathcal{B}} = \{\tilde{b}_1, \dots, \tilde{b}_n\}$ is given by

$$\tilde{b}_{i} = \begin{cases} e_{i+1} - e_{k-i+1}, & \text{for } 1 \leq i \leq m-1 \\ e_{1} - e_{2} + e_{3} - e_{4} \cdots + e_{k-1} - e_{k}, & \text{for } i = m \\ (-1)^{i-m} 2(e_{1} - e_{2} + \cdots + (-1)^{i-m-1} e_{i-m}) + e_{i-m+1} + e_{k-i+m+1}, & \text{for } m+1 \leq i \leq 2m-1 \\ e_{1} + e_{2} + \cdots + e_{k}, & \text{for } i = 2m = k \\ e_{i}, & \text{for } k+1 \leq i \leq n, \end{cases}$$

$$(3.28)$$

(cf. Example 3.9 for k = 8).

Theorem 3.6.

Example 3.8 Let G be a 9-cell regular network with an interior symmetry D_7 on the cells $\{1, 2, 3, 4, 5, 6, 7\}$. Then, with respect to the basis (cf. (3.23))

$$\mathcal{B} = \begin{bmatrix} 0 & 0 & 0 & -2 & -2 & -2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

the adjacency matrix $A_{\ensuremath{\mathcal{G}}}$ has the form

$$\mathcal{B}^{-1}A_{\mathcal{G}}\mathcal{B} = \begin{bmatrix} B_1 & 0 & \\ 0 & B_2 & 0 \\ \hline C_1 & C_2 & A_{\bowtie} \end{bmatrix},$$

where

$$B_{1} = \begin{bmatrix} a_{11} - a_{13} & a_{12} - a_{14} & a_{13} - a_{14} \\ a_{12} - a_{14} & a_{11} - a_{14} & a_{12} - a_{13} \\ a_{13} - a_{14} & a_{12} - a_{13} & a_{11} - a_{12} \end{bmatrix},$$

$$B_{2} = \begin{bmatrix} a_{11} - 2a_{12} + a_{13} & -a_{12} + a_{14} & a_{13} - 2a_{12} + a_{14} \\ a_{12} - 2a_{13} + a_{14} & a_{11} - 2a_{13} + a_{14} & a_{12} - a_{13} \\ a_{13} - a_{14} & a_{12} - 2a_{14} + a_{13} & a_{11} - 2a_{14} + a_{12} \end{bmatrix},$$

$$A_{\bowtie} = \begin{bmatrix} a_{11} + 2a_{12} + 2a_{13} + 2a_{14} & a_{18} & a_{19} \\ a_{81} + a_{82} + a_{83} + a_{84} + a_{85} + a_{86} + a_{87} & a_{88} & a_{89} \\ a_{91} + a_{92} + a_{93} + a_{94} + a_{95} + a_{96} + a_{97} & a_{98} & a_{99} \end{bmatrix},$$

and

$$C_{1} = \begin{bmatrix} 0 & 0 & 0 \\ a_{82} - a_{87} & a_{83} - a_{86} & a_{84} - a_{85} \\ a_{92} - a_{97} & a_{93} - a_{96} & a_{94} - a_{95} \end{bmatrix},$$

$$C_{2} = \begin{bmatrix} 0 & 0 & 0 \\ -2a_{81} + a_{82} + a_{87} & -2a_{81} + a_{83} + a_{86} & -2a_{81} + a_{84} + a_{85} \\ -2a_{91} + a_{92} + a_{97} & -2a_{91} + a_{93} + a_{96} & -2a_{91} + a_{94} + a_{95} \end{bmatrix}.$$

Consider a new basis (cf. (3.24))

$$\tilde{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then, the adjacency matrix $A_{\mathcal{G}}$ is of form

$$\tilde{\mathcal{B}}^{-1}A_{\mathcal{G}}\tilde{\mathcal{B}} = \begin{bmatrix} B_1 & 0 & \\ 0 & B_1 & 0 \\ \hline C'_1 & C'_2 & A_{\bowtie} \end{bmatrix},$$

where

$$C_{1}' = \begin{bmatrix} 0 & 0 & 0 \\ a_{82} - a_{87} & a_{83} - a_{86} & a_{84} - a_{85} \\ a_{92} - a_{97} & a_{93} - a_{96} & a_{94} - a_{95} \end{bmatrix},$$

$$C_{2}' = \begin{bmatrix} 0 & 0 & 0 \\ -a_{83} + a_{84} + a_{85} - a_{86} & -a_{82} + a_{83} + a_{86} - a_{87} & -2a_{81} + a_{82} + a_{87} \\ -a_{93} + a_{94} + a_{95} - a_{96} & -a_{92} + a_{93} + a_{96} - a_{97} & -2a_{91} + a_{92} + a_{97} \end{bmatrix}.$$

Example 3.9 Let \mathcal{G} be a 10-cell regular network with an interior symmetry \mathbf{D}_8 on the cells {1, 2, 3, 4, 5, 6, 7, 8}. Then, with respect to the basis (cf. (3.25))

the adjacency matrix $A_{\mathcal{G}}$ has the form

$$\mathcal{B}^{-1}A_{\mathcal{G}}\mathcal{B} = \begin{bmatrix} \frac{B_1 & 0}{0 & a & a_1 & 0} \\ 0 & 0 & B_2 & \\ \hline C_1 & C_2 & C_3 & A_{\bowtie} \end{bmatrix},$$

where $a = a_{11} - 2a_{12} + 2a_{13} - 2a_{1,4} + a_{15}$,

$$a_{1} = [2a_{14} - 2a_{15}, 2a_{13} - 2a_{15}, 2a_{12} - 2a_{15}],$$
$$B_{1} = \begin{bmatrix} a_{11} - a_{13} & a_{12} - a_{14} & a_{13} - a_{15} \\ a_{12} - a_{14} & a_{11} - a_{15} & a_{12} - a_{14} \\ a_{13} - a_{15} & a_{12} - a_{14} & a_{11} - a_{13} \end{bmatrix},$$

$$B_{2} = \begin{bmatrix} a_{11} - 2a_{12} + a_{13} + 2a_{14} - 2a_{15} & -a_{12} + 2a_{13} + a_{14} - 2a_{15} & a_{13} - a_{15} \\ a_{12} - 2a_{13} - a_{14} + 2a_{15} & a_{11} - 4a_{13} + 3a_{15} & -a_{12} - 2a_{13} + a_{14} + 2a_{15} \\ a_{13} - a_{15} & a_{12} + 2a_{13} - a_{14} - 2a_{15} & a_{11} + 2a_{12} + a_{13} - 2a_{14} - 2a_{15} \end{bmatrix},$$

$$A_{\bowtie} = \begin{bmatrix} a_{11} + 2a_{12} + 2a_{13} + 2a_{14} + a_{15} & a_{19} & a_{1,10} \\ a_{91} + a_{92} + a_{93} + a_{94} + a_{95} + a_{96} + a_{97} + a_{98} & a_{99} & a_{9,10} \\ a_{10,1} + a_{10,2} + a_{10,3} + a_{10,4} + a_{10,5} + a_{10,6} + a_{10,7} + a_{10,8} & a_{10,9} & a_{10,10} \end{bmatrix},$$

and

$$C_{1} = \begin{bmatrix} 0 & 0 & 0 \\ a_{92} - a_{98} & a_{93} - a_{97} & a_{94} - a_{96} \\ a_{10,2} - a_{10,8} & a_{10,3} - a_{10,7} & a_{10,4} - a_{10,6} \end{bmatrix},$$

$$C_{2} = \begin{bmatrix} 0 \\ a_{91} - a_{92} + a_{93} - a_{94} + a_{95} - a_{96} + a_{97} - a_{98} \\ a_{10,1} - a_{10,2} + a_{10,3} - a_{10,4} + a_{10,5} - a_{10,6} + a_{10,7} - a_{10,8} \end{bmatrix},$$

$$C_{3} = \begin{bmatrix} 0 & 0 \\ -2a_{91} + a_{92} + a_{98} & -2a_{91} + a_{93} + a_{97} & -2a_{91} + a_{94} + a_{96} \\ -2a_{10,1} + a_{10,2} + a_{10,8} & -2a_{10,1} + a_{10,3} + a_{10,7} & -2a_{10,1} + a_{10,4} + a_{10,6} \end{bmatrix}.$$

Consider a new basis (cf. (3.28))

Then, the adjacency matrix $A_{\mathcal{G}}$ is of form

$$\tilde{\mathcal{B}}^{-1}A_{\mathcal{G}}\tilde{\mathcal{B}} = \begin{bmatrix} \begin{array}{c|c} B_{1} & 0 \\ 0 & a & a_{1}' \\ 0 & 0 & B_{1} \\ \hline C_{1}' & C_{2}' & C_{3}' \\ \end{array} \end{bmatrix}.$$

where $a'_1 = [2a_{14} - 2a_{15}, 2a_{13} - 4a_{14} + 2a_{15}, 2a_{12} - 4a_{13} + 4a_{14} - 2a_{15}]$,

$$C_{1}' = \begin{bmatrix} 0 & 0 & 0 \\ a_{92} - a_{98} & a_{93} - a_{97} & a_{94} - a_{96} \\ a_{10,2} - a_{10,8} & a_{10,3} - a_{10,7} & a_{10,4} - a_{10,6} \end{bmatrix},$$

$$C_{2}' = \begin{bmatrix} 0 & 0 \\ a_{91} - a_{92} + a_{93} - a_{94} + a_{95} - a_{96} + a_{97} - a_{98} \\ a_{10,1} - a_{10,2} + a_{10,3} - a_{10,4} + a_{10,5} - a_{10,6} + a_{10,7} - a_{10,8} \end{bmatrix},$$

$$C_{3}' = \begin{bmatrix} 0 & 0 & 0 \\ -2a_{91} + a_{92} + a_{98} & 2a_{91} - 2a_{92} + a_{93} + a_{97} - 2a_{98} & -2a_{91} + 2a_{92} - 2a_{93} + a_{94} + a_{96} - 2a_{97} + 2a_{98} \\ -2a_{10,1} + a_{10,2} + a_{10,8} & 2a_{10,1} - 2a_{10,2} + a_{10,3} + a_{10,7} - 2a_{10,8} & -2a_{10,1} + 2a_{10,2} - 2a_{10,3} + a_{10,4} + a_{10,6} - 2a_{10,7} + 2a_{10,8} \end{bmatrix}. \Box$$

3.1.4 $\Sigma_{\mathcal{S}}$ -Interior symmetry with $\mathbf{D}_k \subseteq \Sigma_{\mathcal{S}} \subseteq \mathbf{S}_k$

In this subsection, we consider regular networks \mathcal{G} with an interior symmetry group $\Sigma_{\mathcal{S}}$ with $\mathbf{D}_k \subseteq \Sigma_{\mathcal{S}} \subseteq \mathbf{S}_k$. Besides the result of Theorem 3.6 that applies to \mathcal{G} , we show that the multiplicity

of the eigenvalues of the adjacency matrix $A_{\mathcal{G}}$ can be directly analyzed using the eigenvalues of the circulant part A of $A_{\mathcal{G}}$.

As shown in Subsection 3.1.3, the adjacency matrix of a regular network having an interior symmetry at least D_k is of the form

$$A_{\mathcal{G}} = \begin{bmatrix} A & D \\ E & F \end{bmatrix}, \tag{3.29}$$

where *D* is a $k \times (n - k)$ matrix with all rows equal and *A* is a circulant matrix of order $k \times k$ being of the form

$$A = \begin{cases} circ(a_{11}a_{12}a_{13}\dots a_{1m+1}a_{1m+1}\dots a_{13}a_{12}), & \text{if } k \text{ is odd,} \\ circ(a_{11}a_{12}a_{13}\dots a_{1m}a_{1m+1}a_{1m}\dots a_{13}a_{12}), & \text{if } k \text{ is even.} \end{cases}$$
(3.30)

It follows from (2.4) that the eigenvalues λ_j , j = 0, ..., k - 1 of A are real and satisfy $\lambda_j = \lambda_{k-j}$, for j = 1, ..., m. That is, A has m eigenvalues with algebraic multiplicity at least 2, if k is odd; A has (m - 1) eigenvalues with algebraic multiplicity at least 2, if k is even. In Theorem 3.6 we proved the same property for A_G .

Now, using the proof of Theorem 3.6, we show that

Theorem 3.10 Let \mathcal{G} be an *n*-cell regular network with an interior symmetry $\Sigma_{\mathcal{S}}$ such that $\mathbf{D}_k \subseteq \Sigma_{\mathcal{S}} \subseteq \mathbf{S}_k$. Let $A_{\mathcal{G}}$ be the adjacency matrix of \mathcal{G} , A be given by (3.30) and λ_j be eigenvalues of A, for $j = 0, \ldots, k-1$ given by (2.4). Then, there exists a basis \mathcal{B} of \mathbb{R}^n , which is independent of entries of $A_{\mathcal{G}}$, such that

$$\mathcal{B}^{-1}A_{\mathcal{G}}\mathcal{B} = \begin{bmatrix} \Lambda & 0\\ C & A_{\bowtie} \end{bmatrix}, \quad for \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_{k-1} \end{bmatrix}, \quad (3.31)$$

where A_{\bowtie} is the adjacency matrix of the quotient network induced by $\Sigma_{\mathcal{S}}$.

Proof Consider the *k*-cell regular network \mathcal{G}_o whose adjacency matrix is given by *A* in (3.29). Since \mathcal{G} is $\Sigma_{\mathcal{S}}$ -interior symmetric, \mathcal{G}_o is $\Sigma_{\mathcal{S}}$ -symmetric. Let $\tilde{\mathcal{B}}_o$ be a basis in \mathbb{R}^k given by (3.24) for odd *k* and (3.28) for even *k*. As shown in the proof of Theorem 3.6, we have

$$\tilde{\mathcal{B}}_{o}^{-1}A\tilde{\mathcal{B}}_{o} = \left[\begin{array}{cc} M & 0 \\ 0 & \lambda_{0} \end{array} \right],$$

where *M* is a matrix of order $(k - 1) \times (k - 1)$ of form

$$M = \begin{bmatrix} B_1 & 0 \\ 0 & B_1 \end{bmatrix} \text{ or } \begin{bmatrix} B_2 & 0 \\ 0 & a & *' \\ 0 & 0 & B_2 \end{bmatrix},$$

for odd *k* or even *k*, respectively. Let v_j be the eigenvector of λ_j given by (2.3), for j = 0, 1, ..., k-1. Set $V = \{v_1, ..., v_{k-1}, v_0\}$. Then,

$$V^{-1}AV = \left[\begin{array}{cc} \Lambda & 0\\ 0 & \lambda_0 \end{array} \right].$$

Thus, we have

$$\left(\tilde{\mathcal{B}}_{o}^{-1}V\right)^{-1}\left[\begin{array}{cc}M&0\\0&\lambda_{0}\end{array}\right]\left(\tilde{\mathcal{B}}_{o}^{-1}V\right)=\left[\begin{array}{cc}\Lambda&0\\0&\lambda_{0}\end{array}\right].$$

Moreover, it can be verified that

$$\tilde{\mathcal{B}}_o^{-1}V = \left[\begin{array}{cc} X & 0 \\ 0 & 1 \end{array} \right],$$

for a matrix X of order $(k - 1) \times (k - 1)$. Consequently, we have

$$X^{-1}MX = \Lambda.$$

On the other hand, let $\tilde{\mathcal{B}}$ be a basis in \mathbb{R}^n given by (3.24) for odd *k* and (3.28) for even *k*. Then,

$$\tilde{\mathcal{B}}^{-1}A_{\mathcal{G}}\tilde{\mathcal{B}} = \begin{bmatrix} M & 0\\ C' & A_{\bowtie} \end{bmatrix}.$$

Now set

$$\tilde{X} = \begin{bmatrix} X & 0 \\ 0 & I_{n-k+1} \end{bmatrix}, \quad \mathcal{B} := \tilde{\mathcal{B}}\tilde{X}.$$

Then, \mathcal{B} is a basis such that (3.31) holds, for C = C'X. Moreover, \mathcal{B} is also independent of the entries of A_G , since both $\tilde{\mathcal{B}}_o$ and V are independent of the entries of A_G .

Consequently, the influence of Σ_S on the eigenvalues of A_G , and thus of J_G can be directly examined by looking at eigenvalues of A.

Example 3.11 Let k = 12, m = 6 and n > 12. Consider an *n*-cell regular network \mathcal{G} with an interior symmetry at least \mathbf{D}_{12} on the set of cells $\{1, 2, ..., 12\}$. Let $A_{\mathcal{G}}$ be the adjacency matrix, A be the circulant part of $A_{\mathcal{G}}$ (cf. (3.29)) and λ_j be the eigenvalues of A, for j = 0, 1, ..., 11. By (2.4), $\lambda_j = \lambda_{12-j}$, for j = 1, 2, ..., 5 and, denoting by $\lambda_{i,j}$ both the eigenvalues λ_i and λ_j , we have

$$\lambda_{0} = a_{11} + 2a_{12} + 2a_{13} + 2a_{14} + 2a_{15} + 2a_{16} + a_{17}(=a_{\bowtie})$$

$$\lambda_{1,11} = a_{11} + r_{1}a_{12} + r_{2}a_{13} - r_{2}a_{15} - r_{1}a_{16} - a_{17}$$

$$\lambda_{2,10} = a_{11} + r_{2}a_{12} - r_{2}a_{13} - 2a_{14} - r_{2}a_{15} + r_{2}a_{16} + a_{17}$$

$$\lambda_{3,9} = a_{11} - 2a_{13} + 2a_{15} - a_{17}$$

$$\lambda_{4,8} = a_{11} - r_{2}a_{12} - r_{2}a_{13} + 2a_{14} - r_{2}a_{15} - r_{2}a_{16} + a_{17}$$

$$\lambda_{5,7} = a_{11} - r_{1}a_{12} + r_{2}a_{13} - r_{2}a_{15} + r_{1}a_{16} - a_{17}$$

$$\lambda_{6} = a_{11} - 2a_{12} + 2a_{13} - 2a_{14} + 2a_{15} - 2a_{16} + a_{17}(=a)$$

where $r_1 = 2\text{Re}\omega_1 = \sqrt{3}$, $r_2 = 2\text{Re}\omega_1^2 = 1$. Note that

- (i) if $a_{12} = a_{16}$, then $\lambda_{1,11} = \lambda_{5,7}$;
- (ii) if $a_{12} = a_{16}$ and $a_{13} = a_{15}$, then $\lambda_{1,11} = \lambda_{3,9} = \lambda_{5,7}$;
- (iii) if $a_{12} = a_{13} = a_{14} = a_{16} = a_{17}$ and $a_{11} = a_{15}$, then $\lambda_{1,11} = \lambda_{2,10} = \lambda_{5,7}$ and $\lambda_{3,9} = \lambda_6$;
- (iv) if $a_{11} = a_{12} = a_{13} = a_{15} = a_{16} = a_{17}$, then $\lambda_{1,11} = \lambda_{3,9} = \lambda_{5,7}$ and $\lambda_{2,10} = \lambda_6$;
- (v) if $a_{12} = a_{13} = a_{15} = a_{16}$ and $a_{14} = a_{17}$, then $\lambda_{1,11} = \lambda_{2,10} = \lambda_{3,9} = \lambda_{5,7}$;

(vi) if $a_{12} = a_{13} = a_{14} = a_{15} = a_{16} = a_{17}$, then $\lambda_{1,11} = \lambda_{2,10} = \lambda_{3,9} = \lambda_{4,8} = \lambda_{5,7} = \lambda_6$.

Thus, by Theorem 3.10, the following holds for any *n*-cell regular network \mathcal{G} with $n \ge 12$ having a $\Sigma_{\mathcal{S}}$ -interior symmetry:

- (i) if $\Sigma_{S} = \langle \mathbf{D}_{12}, (2\ 6\ 8\ 12)(3\ 11)(4\ 10)(5\ 9) \rangle$, then $A_{\mathcal{G}}$ has 3 eigenvalues of multiplicity at least 2 and 1 eigenvalue of multiplicity at least 4;
- (ii) if $\Sigma_{\mathcal{S}} = \langle \mathbf{D}_{12}, (2\ 6\ 8\ 12)(3\ 5\ 9\ 11)(4\ 10) \rangle$, then $A_{\mathcal{G}}$ has 2 eigenvalues of multiplicity at least 2 and 1 eigenvalue of multiplicity at least 6;
- (iii) if $\Sigma_{S} = \langle \mathbf{D}_{12}, (159)(234678101112) \rangle$, then $A_{\mathcal{G}}$ has 1 eigenvalue of multiplicity at least 2, 1 eigenvalue of multiplicity at least 3 and 1 eigenvalue of multiplicity at least 6;
- (iv) if $\Sigma_{S} = \langle \mathbf{D}_{12}, (4\ 10)(1\ 2\ 3\ 5\ 6\ 7\ 8\ 9\ 11\ 12) \rangle$, then $A_{\mathcal{G}}$ has 1 eigenvalue of multiplicity at least 2, 1 eigenvalue of multiplicity at least 3 and 1 eigenvalue of multiplicity at least 6;
- (v) if $\Sigma_{S} = \langle \mathbf{D}_{12}, (2\,3\,5\,6\,8\,9\,11\,12)(4\,7\,10) \rangle$, then $A_{\mathcal{G}}$ has 1 eigenvalue of multiplicity at least 2 and 1 eigenvalue of multiplicity at least 8;
- (vi) if $\Sigma_{\mathcal{S}} = \langle \mathbf{D}_{12}, (2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12) \rangle = \mathbf{S}_{12}$, then $A_{\mathcal{G}}$ has 1 eigenvalue of multiplicity at least 11.

3.1.5 \mathbb{Z}_k -Interior symmetry

Consider an *n*-cell regular network \mathcal{G} with adjacency matrix $A_{\mathcal{G}} = [a_{ij}]_{1 \le i,j \le n}$, which has an interior symmetry \mathbb{Z}_k for some $3 \le k < n$ on some subset of *k* cells which, up to a reordering of the cells, we can assume to be the first *k* cells. Then,

$$A_{\mathcal{G}} = \left[\begin{array}{cc} A & D \\ E & F \end{array} \right],$$

where *D* is a $k \times (n - k)$ matrix with all rows equal and *A* is a circulant matrix

$$A = circ(a_{11}, a_{12}, a_{13}, \ldots, a_{1k}).$$

Examples show that in general, A_G does not have multiple eigenvalues due to \mathbb{Z}_k -interior symmetry. In fact, even with additional equalities on $\{a_{12}, a_{13}, \ldots a_{1k}\}$, as long as the resulting symmetry is less than \mathbf{D}_k , A_G seems to be free of multiple eigenvalues in general.

3.1.6 Cyclic interior symmetry of regular uniform networks

Despite of the fact that cyclic interior symmetries are not sufficient for the adjacency matrix of regular networks to have multiple eigenvalues, this may become different if they are uniform networks.

Recall that *uniform regular networks* are regular networks without multiple arrows nor selfcoupling arrows (cf. Stewart [13]). In the next two subsections, we analyze two particular types of cyclic interior symmetry groups and show their influence on the multiplicity of eigenvalues of adjacency matrices of uniform networks. As we will see, for regular uniform networks, interior symmetry forces the existence of eigenvalues in $\{-2, -1, 0, 1\}$.

$\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ -Interior symmetry

We show that

Theorem 3.12 Let \mathcal{G} be an n-cell regular network with a product interior symmetry $\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ on r disjoint subsets $\mathcal{S}_k = \{i_k, j_k\}$ of cells of \mathcal{G} , for $k = 1, 2, \ldots, r$. Then, the adjacency matrix $A_{\mathcal{G}} = [a_{ij}]_{1 \le i, j \le n}$ of \mathcal{G} has r eigenvalues $(a_{i_k i_k} - a_{i_k j_k})$, for $k = 1, 2, \ldots, r$. Moreover, if \mathcal{G} is a uniform network, then $(a_{i_k i_k} - a_{i_k j_k}) \in \{-1, 0, 1\}$, for $k = 1, 2, \ldots, r$.

Proof Without loss of generality, we assume $S_k = \{2k - 1, 2k\}$ for k = 1, 2, ..., r. Then,

$$\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2 = \langle (1 \ 2), \ldots, (2r - 1 \ 2r) \rangle.$$

Due to this interior symmetry, the entries of A_G satisfy

 $a_{ii} = a_{i+1,i+1}, \quad a_{i,i+1} = a_{i+1,i} \quad \text{and} \quad a_{il} = a_{i+1,l},$

for all i = 1, 3, ..., 2r - 1 and for all $l \neq i, i + 1$.

Consider the balanced equivalence relation

$$\bowtie = \{\{1, 2\}, \{3, 4\}, \dots, \{2r - 1, 2r\}, \{2r + 1\}, \dots, \{n\}\}\$$

induced by $\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ and the basis $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$ given by

$$b_{k} = \begin{cases} e_{2k-1} - e_{2k}, & \text{if } 1 \le k \le r \\ e_{2(k-r)-1} + e_{2(k-r)}, & \text{if } r+1 \le k \le 2r \\ e_{k}, & \text{if } 2r+1 \le k \le n, \end{cases}$$

adapted to the decomposition in (3.11). It follows from Theorem 3.2 that

$$\mathcal{B}^{-1}A_{\mathcal{G}}\mathcal{B} = \left[\begin{array}{cc} A & 0\\ B & A_{\bowtie} \end{array}\right],$$

where

	$a_{11} - a_{12}$	0	0	•••	0]
	0	$a_{33} - a_{34}$	0		0
<i>A</i> =	0	0	$a_{55} - a_{56}$	•••	0
		•••	•••	•••	
	0	0	0	•••	$a_{2r-1,2r-1} - a_{2r-1,2r}$

Thus, $(a_{2k-1,2k-1} - a_{2k-1,2k})$ are eigenvalues of A_G , for k = 1, 2, ..., r.

If G is a uniform network, then $a_{ij} \in \{0, 1\}$ and consequently, $(a_{2k-1,2k-1} - a_{2k-1,2k}) \in \{-1, 0, 1\}$, for k = 1, 2, ..., r.

Corollary 3.13 Let G be an n-cell uniform network with a product interior symmetry $\mathbb{Z}_2 \times ... \times \mathbb{Z}_2$ on r disjoint subsets S_k of cells of G, for k = 1, 2, ..., r. Assume that $r \ge 4$. Then, the adjacency matrix A_G of G has at least one multiple eigenvalue.

Proof By Theorem 3.12, $A_{\mathcal{G}} = [a_{ij}]_{1 \le i,j \le n}$ has *r* eigenvalues $\lambda_k := a_{i_k i_k} - a_{i_k j_k} \in \{-1, 0, 1\}$, for k = 1, 2, ..., r. Thus, if $r \ge 4$, values of λ_k 's must be duplicated for some *k*.



Figure 9: The 5-cell uniform network G in Example 3.14.

Example 3.14 Let \mathcal{G} be the 5-cell uniform network given in Figure 9 and $A_{\mathcal{G}} = [a_{ij}]_{5\times 5}$ be the adjacency matrix.

The network \mathcal{G} has an interior symmetry group $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle (2 \ 3), (4 \ 5) \rangle$. It follows from Theorem 3.12 that $a_{22} - a_{23} = 0$ and $a_{44} - a_{45} = 0$ are eigenvalues of $A_{\mathcal{G}}$. Thus, 0 is an eigenvalue of algebraic multiplicity at least 2 for $A_{\mathcal{G}}$.

Moreover, consider the balanced equivalence relation $\bowtie = \{\{1\}, \{2, 3\}, \{4, 5\}\}$ induced by $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle (2 \ 3), (4 \ 5) \rangle$. Then, the quotient network \mathcal{G}_{\bowtie} has an interior symmetry \mathbf{S}_3 on the set $C_{\bowtie} = \{[1]_{\bowtie}, [2]_{\bowtie}, [3]_{\bowtie}\}$, for $[1]_{\bowtie} = \{1\}, [2]_{\bowtie} = \{2, 3\}$ and $[3]_{\bowtie} = \{4, 5\}$. Let $A_{\bowtie} = [\bar{a}_{ij}]_{3\times 3}$ be the adjacency matrix of \mathcal{G}_{\bowtie} . By Theorem 3.4, $\bar{a}_{11} - \bar{a}_{12} = -1$ is an eigenvalue of algebraic multiplicity at least 2 for the adjacency matrix A_{\bowtie} . Thus, by Theorem 3.2 and Corollary 3.3, -1 is an eigenvalue of algebraic multiplicity 2 for $A_{\mathcal{G}}$.

Lastly, the remaining eigenvalue of A_G is given by the valency 2 of the network.

Example 3.15 Let \mathcal{G} be the 5-cell uniform network given in Figure 10 and $A_{\mathcal{G}} = [a_{ij}]_{5\times 5}$ be the adjacency matrix. Using Theorem 3.2, Corollary 3.3 and Theorem 3.12, we show that besides



Figure 10: The 5-cell uniform network G in Example 3.15.

the valency 2 of the network, $A_{\mathcal{G}}$ has 0 and -1 as eigenvalues, both with algebraic multiplicity 2.

We first consider the interior symmetry group $\mathbb{Z}_2 = \langle (2 \ 3) \rangle$ of \mathcal{G} . Then, the eigenvalues of $A_{\mathcal{G}}$ are $a_{22} - a_{23} = 0$ with algebraic multiplicity at least 1 and those of the quotient network \mathcal{G}_{\bowtie_1} induced by the balanced equivalence relation $\bowtie_1 = \{[1]_{\bowtie_1}, [2]_{\bowtie_1}, [3]_{\bowtie_1}, [4]_{\bowtie_1}\}$, for $[1]_{\bowtie_1} = \{1\}, [2]_{\bowtie_1} = \{2, 3\}, [3]_{\bowtie_1} = \{4\}$ and $[4]_{\bowtie_1} = \{5\}$.

The quotient network \mathcal{G}_{\bowtie_1} in turn, has an interior symmetry $\mathbb{Z}_2 = \langle ([2]_{\bowtie_1} \ [4]_{\bowtie_1}) \rangle$. Let $A_{\bowtie_1} = (\bar{a}_{ij}^1)_{4\times 4}$ be the adjacency matrix of \mathcal{G}_{\bowtie_1} . Then, the eigenvalues of A_{\bowtie_1} are $\bar{a}_{22}^1 - \bar{a}_{24}^1 = (a_{22} + a_{23}) - a_{25} = -1$ with algebraic multiplicity at least 1 and those of the quotient network \mathcal{G}_{\bowtie_2}

induced by the balanced equivalence relation $\bowtie_2 = \{[1]_{\bowtie_2}, [2]_{\bowtie_2}, [3]_{\bowtie_2}\}, \text{ for } [1]_{\bowtie_2} = \{[1]_{\bowtie_1}\}, [2]_{\bowtie_2} = \{[2]_{\bowtie_1}, [4]_{\bowtie_1}\} \text{ and } [3]_{\bowtie_2} = \{[3]_{\bowtie_1}\}.$

Further, the quotient network \mathcal{G}_{\bowtie_2} has an interior symmetry $\mathbb{Z}_2 = \langle ([1]_{\bowtie_2} [3]_{\bowtie_2}) \rangle$. Let $A_{\bowtie_2} = (\bar{a}_{ij}^2)_{3\times 3}$ be the adjacency matrix of \mathcal{G}_{\bowtie_2} . Then, the eigenvalues of A_{\bowtie_2} are $\bar{a}_{11}^2 - \bar{a}_{13}^2 = a_{11} - a_{14} = -1$ with algebraic multiplicity at least 1 and those of the quotient network \mathcal{G}_{\bowtie_3} for the balanced equivalence relation $\bowtie_3 = \{[1]_{\bowtie_3}, [2]_{\bowtie_3}\}$, with $[1]_{\bowtie_3} = \{[1]_{\bowtie_2}, [3]_{\bowtie_2}\}$ and $[2]_{\bowtie_3} = \{[2]_{\bowtie_2}\}$.

The quotient network \mathcal{G}_{\bowtie_3} also has an interior symmetry $\mathbb{Z}_2 = \langle ([1]_{\bowtie_3} [2]_{\bowtie_3}) \rangle$. Let $A_{\bowtie_3} = (\bar{a}_{1j}^3)_{2\times 2}$ be the adjacency matrix of \mathcal{G}_{\bowtie_3} . By Theorem 3.12, $\bar{a}_{11}^3 - \bar{a}_{12}^3 = (a_{11} + a_{14}) - (a_{12} + a_{13} + a_{15}) = 0$ is an eigenvalue of algebraic multiplicity 1 for A_{\bowtie_3} .

*V*₄-Interior symmetry

We discuss in this subsection *n*-cell uniform networks with an interior symmetry group

$$V_4 := \langle (i j)(k l), (i k)(j l) \rangle \subset \mathbf{S}_n,$$

where i, j, k, l are distinct cells of G.

Theorem 3.16 Let \mathcal{G} be an n-cell uniform network having an interior symmetry group $V_4 = \langle (i j)(k l), (i k)(j l) \rangle \subset$ \mathbf{S}_n on a subset $\{i, j, k, l\}$ of cells of \mathcal{G} . Then, the adjacency matrix $A_{\mathcal{G}} = [a_{\alpha\beta}]_{1 \leq \alpha, \beta \leq n}$ has the 3 eigenvalues

$$-a_{ij} + a_{ik} - a_{il},$$

 $-a_{ij} - a_{ik} + a_{il},$
 $a_{ij} - a_{ik} - a_{il},$

which take value in $\{-2, -1, 0, 1\}$ *.*

Proof Due to the interior symmetry $\mathbb{Z}_2 = \langle (i \ j)(k \ l) \rangle$, the entries of A_G satisfy

$$a_{ii} = a_{jj}, \quad a_{ij} = a_{ji}, \quad a_{ik} = a_{jl}, \quad a_{il} = a_{jk} \quad \text{and} \quad a_{im} = a_{jm}, \quad \forall m \neq i, j, k, l$$

 $a_{kk} = a_{ll}, \quad a_{kl} = a_{lk}, \quad a_{ki} = a_{lj}, \quad a_{kj} = a_{li} \quad \text{and} \quad a_{km} = a_{lm}, \quad \forall m \neq i, j, k, l$

Due to the interior symmetry $\mathbb{Z}_2 = \langle (i \ k)(j \ l) \rangle$, they satisfy

$$a_{ii} = a_{kk}$$
, $a_{ik} = a_{ki}$, $a_{ij} = a_{kl}$, $a_{il} = a_{kj}$ and $a_{im} = a_{km}$, $\forall m \neq i, j, k, l$
 $a_{jj} = a_{ll}$, $a_{jl} = a_{lj}$, $a_{ji} = a_{lk}$, $a_{jk} = a_{li}$ and $a_{jm} = a_{lm}$, $\forall m \neq i, j, k, l$

Thus, due to the interior symmetry V_4 , the entries of A_G satisfy

$$a_{ii} = a_{jj} = a_{kk} = a_{ll}, \quad a_{ij} = a_{ji} = a_{kl} = a_{lk}, \quad a_{ik} = a_{jl} = a_{ki} = a_{lj}, a_{il} = a_{jk} = a_{kj} = a_{li}, \quad a_{im} = a_{jm} = a_{km} = a_{lm}, \quad \forall m \neq i, j, k, l.$$
(3.32)

Without loss of generality, we assume i = 1, j = 2, k = 3 and l = 4. Let \bowtie be the balanced equivalence relation induced by $\mathbb{Z}_2 = \langle (1, 2)(3, 4) \rangle$, i.e.

$$\bowtie = \{\{1, 2\}, \{3, 4\}, \{5\}, \dots, \{n\}\}.$$

Let W, U be given by (2.7)–(2.8). Then, we have

$$\mathbb{R}^n = W \oplus U.$$

A basis $\mathcal{B} = \{b_1, b_2, \dots, b_n\}$ adapted to this decomposition is given by

$$b_{k} = \begin{cases} e_{2k-1} - e_{2k}, & \text{if } 1 \le k \le 2\\ e_{2(k-2)-1} + e_{2(k-2)}, & \text{if } 3 \le k \le 4\\ e_{k}, & \text{if } 5 \le k \le n. \end{cases}$$

Then, the adjacency matrix A_G in the basis $\mathcal B$ has the form

$$\mathcal{B}^{-1}A_{\mathcal{G}}\mathcal{B} = \left[\begin{array}{cc} A & 0 \\ C & A_{\bowtie} \end{array} \right],$$

where

$$A = \left[\begin{array}{cc} a_{11} - a_{12} & a_{13} - a_{14} \\ a_{13} - a_{14} & a_{11} - a_{12} \end{array} \right].$$

Since *G* is uniform, we have $a_{11} = 0$. Thus, *A* has eigenvalues $(-a_{12} \pm (a_{13} - a_{14}))$, which are also eigenvalues of A_G . Similarly, using symmetry $\mathbb{Z}_2 = \langle (13)(24) \rangle$, one can show that A_G has eigenvalues $(-a_{13} \pm (a_{12} - a_{14}))$. Thus, altogether A_G has the following 3 eigenvalues

$$-a_{12} + a_{13} - a_{14}$$
, $-a_{12} - a_{13} + a_{14}$, $a_{12} - a_{13} - a_{14}$,

which take value in $\{-2, -1, 0, 1\}$, since $a_{\alpha\beta} \in \{0, 1\}$.

Corollary 3.17 Let \mathcal{G} be an n-cell uniform network with adjacency matrix $A_{\mathcal{G}} = [a_{\alpha\beta}]_{1 \le \alpha, \beta \le n}$ having an interior symmetry group $V_4 \rtimes \mathbb{Z}_2 = \langle (i \ j)(k \ l), (i \ k)(j \ l), (a \ b) \rangle \subset \mathbf{S}_n$ on a subset $\mathcal{S} = \{i, j, k, l\}$ of cells of \mathcal{G} , with $a \ne b$ in \mathcal{S} . If $(a \ b) = (i \ j)$ or $(a \ b) = (k \ l)$ then $-a_{ij} \in \{-2, -1, 0, 1\}$ is an eigenvalue of $A_{\mathcal{G}}$ with algebraic multiplicity at least 2. Analogously, if $(a \ b) = (i \ k)$ or $(a \ b) = (j \ l)$, then $-a_{ik} \in \{-2, -1, 0, 1\}$ is an eigenvalue of $A_{\mathcal{G}}$ with algebraic multiplicity at least 2; if $(a \ b) = (i \ l)$ or $(a \ b) = (j \ k)$, then $-a_{il} \in \{-2, -1, 0, 1\}$ is an eigenvalue of $A_{\mathcal{G}}$ with algebraic multiplicity at least 2.

Proof Consider $A_{\mathcal{G}}$ as a network having V_4 as an interior symmetry group. Then, by Theorem 3.16, $A_{\mathcal{G}}$ has the following 3 eigenvalues

$$-a_{ij} + a_{ik} - a_{il}, \quad -a_{ij} - a_{ik} + a_{il}, \quad a_{ij} - a_{ik} - a_{il}.$$

We only give the proof for the case of $(a \ b) = (i \ j)$ or $(a \ b) = (k \ l)$. The other two cases can be proved in a similar way. Due to the interior symmetry $(i \ j)$ or $(k \ l)$, we have $a_{ik} = a_{il}$. Thus, $-a_{ij}$ is an eigenvalue of algebraic multiplicity 2 in $\{-2, -1, 0, 1\}$.

Example 3.18 Let \mathcal{G} be the 5-cell uniform network given in Figure 11 and $A_{\mathcal{G}} = [a_{ij}]_{1 \le i,j \le 5}$ be the adjacency matrix.

The network *G* has an interior symmetry group $V_4 \rtimes \mathbb{Z}_2 = \langle (2 \ 3)(4 \ 5), (2 \ 4)(3 \ 5), (a \ b) \rangle$, on the set of cells $S = \{2, 3, 4, 5\}$, for $(a \ b) = (2 \ 5)$, as well as for $(a \ b) = (3 \ 4)$. By Corollary 3.17, the adjacency matrix A_G has the eigenvalue $-a_{25} = -1$ with algebraic multiplicity at least 2.

In fact, using Theorem 3.2, Corollary 3.3 and Theorem 3.4, we can show that the algebraic multiplicity of the eigenvalue -1 is at least 3. Note that \mathcal{G} has an interior symmetry group $\mathbb{Z}_2 = \langle (2 \ 3)(4 \ 5) \rangle$. Let $\bowtie = \{\{1\}, \{2, 3\}, \{4, 5\}\}$. Then, with respect to the basis

$$b = ((0, 1, -1, 0, 0), (0, 0, 0, 1, -1), (1, 0, 0, 0, 0), (0, 1, 1, 0, 0), (0, 0, 0, 1, 1)),$$



Figure 11: The 5-cell uniform network G in Example 3.18.

 A_G is given by

$$\left[\begin{array}{cc} A & 0 \\ C & A_{\bowtie} \end{array}\right]$$

where A_{\bowtie} is the adjacency matrix of the quotient network associated with \bowtie and

$$A = \left[\begin{array}{ccc} a_{22} - a_{23} & a_{24} - a_{25} \\ a_{42} - a_{43} & a_{44} - a_{45} \end{array} \right].$$

Since the quotient network is isomorphic to the (**S**₃-symmetric) three-cell bidirectional ring in Figure 7, by Theorem 3.4, A_{\bowtie} has $a_{11} - a_{12} = -1$ as an eigenvalue with algebraic multiplicity at least 2. On the other hand, due to the fact that \mathcal{G} also has an interior symmetry (2 4)(3 5) and (3 4), we have $a_{23} = a_{24} = a_{42} = a_{45}$ and $a_{25} = a_{43}$. Also, since \mathcal{G} is uniform, $a_{22} = a_{44} = 0$. Thus, the eigenvalues of A are $-a_{23} \pm (a_{23} - a_{25})$, one of which is equal to $-a_{25} = -1$. Therefore, -1 is an eigenvalue of $A_{\mathcal{G}}$ with algebraic multiplicity at least 3.

3.2 Homogeneous networks

We generalize our results on regular networks to homogeneous networks. Recall that a *homogeneous network* is a coupled cell network in which all cells are identical but which may have multiple type of arrows. Let G be an *n*-cell homogeneous network with *s* types of arrows, whose adjacency matrices are A_1, \ldots, A_s . Let *r* be the dimension of the cell internal dynamics. Then, the Jacobian at a fully-synchronized equilibrium has the form

$$J_{\mathcal{G}} = \alpha \otimes I_n + \sum_{l=1}^{s} \beta_l \otimes A_l, \qquad (3.33)$$

where α is the linearized internal dynamics and β_l is the linearized internal coupling for the *l*-th type arrow, for l = 1, ..., s.

3.2.1 S_k- and A_k-Interior symmetry

We show that

Theorem 3.19 Let \mathcal{G} be an n-cell homogeneous network with s types of arrows with adjacency matrices A_1, \ldots, A_s . Let $J_{\mathcal{G}}$ be given by (3.33). Assume that all matrices A_l , $l = 1, \ldots, s$ have an interior symmetry \mathbf{S}_k or \mathbf{A}_k , on the same subset $S \subseteq C$ of k cells of \mathcal{G} , for some $k \in \{3, \ldots, n\}$. Then, the Jacobian $J_{\mathcal{G}}$ has r eigenvalues of algebraic multiplicity at least k - 1.

Proof Without loss of generality, we assume $S = \{1, ..., k\}$. Write $A_l = [a_{ij}^{(l)}]_{1 \le i,j \le n}$, for l = 1, ..., s. It follows from the proof of Theorem 3.4 that there is a basis \mathcal{B} such that

$$\mathcal{B}^{-1}A_l\mathcal{B} = \begin{bmatrix} \bar{A}_l & 0\\ C_l & A_{\bowtie_l} \end{bmatrix}, \qquad \forall l = 1, \dots, s_l$$

where \bar{A}_l is a scalar matrix of order k - 1, being equal to $(a_{11}^{(l)} - a_{12}^{(l)})I_{k-1}$. For convenience, we denote $a_l := a_{11}^{(l)} - a_{12}^{(l)}$.

Let $\hat{\mathcal{B}} = I_r \otimes \mathcal{B}$. Then, we have

$$\hat{\mathcal{B}}^{-1} J_{\mathcal{G}} \hat{\mathcal{B}} = \begin{bmatrix} \alpha \otimes I_{k-1} + \sum_{l=1}^{s} \beta_{l} \otimes \bar{A}_{l} & 0 \\ \sum_{l=1}^{s} \beta_{l} \otimes C_{l} & \alpha \otimes I_{n-k+1} + \sum_{l=1}^{s} \beta_{l} \otimes A_{\bowtie_{l}} \end{bmatrix}$$
$$= \begin{bmatrix} \left(\alpha + \sum_{l=1}^{s} a_{l} \beta_{l}\right) \otimes I_{k-1} & 0 \\ \sum_{l=1}^{s} \beta_{l} \otimes C_{l} & \alpha \otimes I_{n-k+1} + \sum_{l=1}^{s} \beta_{l} \otimes A_{\bowtie_{l}} \end{bmatrix}.$$

Thus, every eigenvalue of $\alpha + \sum_{l=1}^{s} a_l \beta_l$ is an eigenvalue of J_G with algebraic multiplicity at least k - 1. Therefore, J_G has r eigenvalues of multiplicity at least k - 1.

3.2.2 *D_k*-Interior Symmetry

We show that

Theorem 3.20 Let G be an n-cell homogeneous network with s types of arrows with adjacency matrices A_1, \ldots, A_s . Let J_G be given by (3.33). Assume that all matrices A_l , $l = 1, \ldots, s$ have an interior symmetry \mathbf{D}_k , on the same subset $S \subseteq C$ of k cells of G, for some $k \in \{3, \ldots, n\}$. Set

$$m = \begin{cases} (k-1)/2, & \text{if } k \text{ is odd,} \\ k/2, & \text{if } k \text{ is even.} \end{cases}$$

Then, J_G has mr eigenvalues with algebraic multiplicity at least 2, if k is odd; J_G has (m-1)r eigenvalues with algebraic multiplicity at least 2, if k is even.

Proof For simplicity, we present the proof for s = 2. The general case can be proved analogously.

Without loss of generality, we assume that the D_k -interior symmetry is on the cells $S = \{1, ..., k\}$. It follows from the proof of Theorem 3.6 that the adjacency matrices A_1 and A_2 can be diagonalized to a "double-block" form using the same basis $\tilde{\mathcal{B}}$ given by (3.24) for odd k and (3.28) for even k.

By applying this basis to A_l , l = 1, 2 in case of odd k, we have

$$\tilde{\mathcal{B}}^{-1}A_l\tilde{\mathcal{B}} = \begin{bmatrix} B_l & 0 & \\ 0 & B_l & \\ \hline C'_l & A_{\bowtie_l} \end{bmatrix}, \quad l = 1, 2$$

where B_l is a matrix of order $m \times m$. Consider the following basis for J_G

$$\hat{\mathcal{B}} = I_r \otimes \tilde{\mathcal{B}}.$$

Then, we have

$$\hat{\mathcal{B}}^{-1} J_{\mathcal{G}} \hat{\mathcal{B}} = \alpha \otimes I_n + \beta_1 \otimes \begin{bmatrix} B_1 & 0 & \\ 0 & B_1 & 0 \\ \hline C'_1 & A_{\bowtie_1} \end{bmatrix} + \beta_2 \otimes \begin{bmatrix} B_2 & 0 & \\ 0 & B_2 & 0 \\ \hline C'_2 & A_{\bowtie_2} \end{bmatrix}$$
$$= \begin{bmatrix} \alpha \otimes I_m + \beta_1 \otimes B_1 + \beta_2 \otimes B_2 & 0 & \\ \hline 0 & \alpha \otimes I_m + \beta_1 \otimes B_1 + \beta_2 \otimes B_2 & 0 & \\ \hline \beta_1 \otimes C'_1 + \beta_2 \otimes C'_2 & \alpha \otimes I_{n-2m} + \beta_1 \otimes A_{\bowtie_1} + \beta_2 \otimes A_{\bowtie_2} \end{bmatrix}.$$

Thus, every eigenvalue of $(\alpha \otimes I_m + \beta_1 \otimes B_1 + \beta_2 \otimes B_2)$ is also an eigenvalue of J_G . Therefore, J_G has *mr* eigenvalues with algebraic multiplicity at least 2.

In the case *k* is even, we have

$$\tilde{\mathcal{B}}^{-1}A_i\tilde{\mathcal{B}} = \begin{bmatrix} \begin{array}{c|c} B_i & 0 \\ \hline 0 & a_i & *_i \\ \hline 0 & 0 & B_i \\ \hline C_i^{\prime\prime} & A_{\bowtie_i} \end{bmatrix}, \quad i = 1, 2$$

where B_i is a matrix of order $(m-1) \times (m-1)$. Consider again the basis $\hat{\mathcal{B}} = I_r \otimes \tilde{\mathcal{B}}$. Then,

$$\hat{\mathcal{B}}^{-1} J_{\mathcal{G}} \hat{\mathcal{B}} = \alpha \otimes I_n + \beta_1 \otimes \begin{bmatrix} \frac{B_1}{0} & 0 \\ 0 & B_1 & 0 \\ \hline C_1'' & A_{\bowtie_1} \end{bmatrix} + \beta_2 \otimes \begin{bmatrix} \frac{B_2}{0} & 0 \\ 0 & B_2 & 0 \\ \hline C_2'' & A_{\bowtie_2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\alpha \otimes I_{m-1} + \beta_1 \otimes B_1 + \beta_2 \otimes B_2}{0} & 0 \\ \hline 0 & \alpha \otimes I_{m-1} + \beta_1 \otimes B_1 + \beta_2 \otimes B_2 \end{bmatrix} \begin{bmatrix} \alpha + \beta_1 \otimes a_1 + \beta_2 \otimes a_2 & \beta_1 \otimes *_1 + \beta_2 \otimes *_2 \\ \hline 0 & \alpha \otimes I_{m-1} + \beta_1 \otimes B_1 + \beta_2 \otimes B_2 \end{bmatrix} \begin{bmatrix} \alpha \otimes I_{m-2m+1} + \beta_1 \otimes A_{\bowtie_1} + \beta_2 \otimes A_{\bowtie_2} \end{bmatrix}$$

Thus, every eigenvalue of $(\alpha \otimes I_{m-1} + \beta_1 \otimes B_1 + \beta_2 \otimes B_2)$ is also an eigenvalue of J_G . Therefore, J_G has (m-1)r eigenvalues with algebraic multiplicity at least 2.

3.2.3 $\Sigma_{\mathcal{S}}$ -Interior symmetry with $\mathbf{D}_k \subseteq \Sigma_{\mathcal{S}} \subseteq \mathbf{S}_k$

Let \mathcal{G} be an *n*-cell homogeneous network with *s* types of arrows with adjacency matrices A_1, \ldots, A_s . Assume that every A_l , for $l = 1, \ldots, s$, has an interior symmetry $\Sigma_{\mathcal{S}}^l$ on the same subset $\mathcal{S} \in \mathcal{C}$ such that $\mathbf{D}_k \subseteq \Sigma_{\mathcal{S}}^l \subseteq \mathbf{S}_k$. Let \bar{A}_l denote the upper left $k \times k$ -submatrix of A_l , $l = 1, \ldots, s$ (cf. (3.29)). Then, \bar{A}_l is a circulant matrix of the form (3.30). We show that the multiplicity of the eigenvalues of $J_{\mathcal{G}}$ can be directly analyzed by the eigenvalues of $\bar{A}_1, \ldots, \bar{A}_s$.

Theorem 3.21 Let G be an n-cell homogeneous network with s types of arrows, where every adjacency matrix A_l has an interior symmetry Σ_{S}^{l} on the same subset $S \subseteq C$ such that $\mathbf{D}_k \subseteq \Sigma_{S}^{l} \subseteq \mathbf{S}_k$, for l = 1, 2, ..., s. Let \overline{A}_l be the upper left $k \times k$ -submatrix of A_l , for l = 1, 2, ..., s. Let $\lambda_j^{(l)}$ be the j-th eigenvalue of \overline{A}_l , for j = 0, 1, ..., k-1, l = 1, 2, ..., s (cf. (2.4)). Then, every eigenvalue of $(\alpha + \sum_{l=1}^{s} \lambda_j^{(l)} \beta_l)$ is an eigenvalue of J_G , for j = 1, ..., k-1.

Proof We only present the proof for the case s = 2. The general case can be proved analogously. Without loss of generality we assume $S = \{1, ..., k\}$.

Let \mathcal{B} be the basis given by Theorem 3.10. Then, we have

$$\mathcal{B}^{-1}A_{l}\mathcal{B} = \begin{bmatrix} \Lambda_{l} & 0\\ C_{l} & Q_{l} \end{bmatrix}, \text{ for } \Lambda_{l} = \begin{bmatrix} \lambda_{1}^{(l)} & 0 & \cdots & 0\\ 0 & \lambda_{2}^{(l)} & \cdots & 0\\ \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_{k-1}^{(l)} \end{bmatrix}, l = 1, 2.$$

Consider the basis $\hat{\mathcal{B}} = I_r \otimes \mathcal{B}$. Then,

$$\hat{\mathcal{B}}^{-1} J_{\mathcal{G}} \hat{\mathcal{B}} = \begin{bmatrix} \alpha \otimes I_{k-1} + \beta_1 \otimes \Lambda_1 + \beta_2 \otimes \Lambda_2 & 0\\ \beta_1 \otimes C_1 + \beta_2 \otimes C_2 & \alpha \otimes I_{n-k+1} + \beta_1 \otimes Q_1 + \beta_2 \otimes Q_2 \end{bmatrix}$$

Let $u \in \mathbb{R}^r$ and v_j be the eigenvector of $\lambda_j^{(1)}$ and $\lambda_j^{(2)}$ given by (2.3), for some $j \in \{1, 2, ..., k-1\}$. Then,

$$(\alpha \otimes I_{k-1} + \beta_1 \otimes \Lambda_1 + \beta_2 \otimes \Lambda_2)(u \otimes v_j) = \alpha u \otimes v_j + \beta_1 u \otimes \Lambda_1 v_j + \beta_2 u \otimes \Lambda_2 v_j$$
$$= \alpha u \otimes v_j + \beta_1 u \otimes \lambda_j^{(1)} v_j + \beta_2 u \otimes \lambda_j^{(2)} v_j$$
$$= (\alpha + \lambda_i^{(1)} \beta_1 + \lambda_i^{(2)} \beta_2)u \otimes v_j.$$

Thus, every eigenvalue of $(\alpha + \lambda_j^{(1)}\beta_1 + \lambda_j^{(2)}\beta_2)$ is an eigenvalue of $(\alpha \otimes I_{k-1} + \beta_1 \otimes \Lambda_1 + \beta_2 \otimes \Lambda_2)$, which is also an eigenvalue of $J_{\mathcal{G}}$.

Example 3.22 Let n > 12. Let G be an *n*-cell homogeneous network with 2 types of arrows whose adjacency matrices A_1, A_2 have an interior symmetry (respectively)

$$\Sigma_{\mathcal{S}}^{1} = \langle \mathbf{D}_{12}, (1\,5\,9)(2\,3\,4\,6\,7\,8\,10\,11\,12) \rangle,$$

$$\Sigma_{\mathcal{S}}^{2} = \langle \mathbf{D}_{12}, (4\,10)(1\,2\,3\,5\,6\,7\,8\,9\,11\,12) \rangle,$$

on cells {1, 2, ..., 12}. Note that $\mathbf{D}_{12} \subset \Sigma_{\mathbf{S}}^{i} \subset \mathbf{S}_{12}$, for i = 1, 2. An example of A_1, A_2 is

	(1	0	0	0	1	0	0	0	1	0	Ũ	0	0) ((0	0	0	1	0	0	0	0	0	1	0	0	0)
	0	1	0	0	0	1	0	0	0	1	0	0	0		0	0	0	0	1	0	0	0	0	0	1	0	0
	0	0	1	0	0	0	1	0	0	0	1	0	0		0	0	0	0	0	1	0	0	0	0	0	1	0
	0	0	0	1	0	0	0	1	0	0	0	1	0		1	0	0	0	0	0	1	0	0	0	0	0	0
	1	0	0	0	1	0	0	0	1	0	0	0	0		0	1	0	0	0	0	0	1	0	0	0	0	0
	0	1	0	0	0	1	0	0	0	1	0	0	0		0	0	1	0	0	0	0	0	1	0	0	0	0
$A_1 =$	0	0	1	0	0	0	1	0	0	0	1	0	0	$,A_{2} =$	0	0	0	1	0	0	0	0	0	1	0	0	0
	0	0	0	1	0	0	0	1	0	0	0	1	0		0	0	0	0	1	0	0	0	0	0	1	0	0
	1	0	0	0	1	0	0	0	1	0	0	0	0		0	0	0	0	0	1	0	0	0	0	0	1	0
	0	1	0	0	0	1	0	0	0	1	0	0	0		1	0	0	0	0	0	1	0	0	0	0	0	0
	0	0	1	0	0	0	1	0	0	0	1	0	0		0	1	0	0	0	0	0	1	0	0	0	0	0
	0	0	0	1	0	0	0	1	0	0	0	1	0		0	0	1	0	0	0	0	0	1	0	0	0	0
	0	0	1	0	0	0	0	0	0	2	0	0	0		0	0	0	0	1	1	0	0	0	0	0	0	0)

The corresponding network is then as shown in Figure 12, where the arrows with solid (resp. hollow) head depict connections given by A_1 (resp. A_2).



Figure 12: The 12-cell homogeneous network G in Example 3.22 with interior symmetries $\Sigma_{S}^{1} = \langle \mathbf{D}_{12}, (1 5 9)(2 3 4 6 7 8 10 11 12) \rangle$ and $\Sigma_{S}^{2} = \langle \mathbf{D}_{12}, (4 10)(1 2 3 5 6 7 8 9 11 12) \rangle$, for $S = \{1, ..., 12\}$.

Let \bar{A}_l be the upper left 12 × 12-submatrix of A_l and $\lambda_j^{(l)}$ be the *j*-th eigenvalue of \bar{A}_l (cf.(2.4)), for j = 0, 1, ..., 11, l = 1, 2. Then, we have (cf. Example 3.11 (iii)-(iv))

$$\lambda_{1,11}^{(1)} = \lambda_{2,10}^{(1)} = \lambda_{5,7}^{(1)}, \quad \lambda_{3,9}^{(1)} = \lambda_6^{(1)}$$

and

$$\lambda_{1,11}^{(2)} = \lambda_{3,9}^{(2)} = \lambda_{5,7}^{(2)}, \quad \lambda_{2,10}^{(2)} = \lambda_6^{(2)}$$

Thus, by Theorem 3.21, for every homogeneous network \mathcal{G} with interior symmetries $\Sigma_{\mathcal{S}}^1$ and $\Sigma_{\mathcal{S}'}^2$ every eigenvalue of $(\alpha + \lambda_{1,5,7,11}^{(1)}\beta_1 + \lambda_{1,5,7,11}^{(2)}\beta_2)$ is an eigenvalue of $J_{\mathcal{G}}$ of multiplicity at least 4; every eigenvalue of $(\alpha + \lambda_{2,10}^{(1)}\beta_1 + \lambda_{2,10}^{(2)}\beta_2)$ is an eigenvalue of $J_{\mathcal{G}}$ of multiplicity at least 2; every eigenvalue of $(\alpha + \lambda_{3,9}^{(1)}\beta_1 + \lambda_{3,9}^{(2)}\beta_2)$ is an eigenvalue of $J_{\mathcal{G}}$ of multiplicity at least 2; every eigenvalue of $(\alpha + \lambda_{4,8}^{(1)}\beta_1 + \lambda_{3,9}^{(2)}\beta_2)$ is an eigenvalue of $J_{\mathcal{G}}$ of multiplicity at least 2; every eigenvalue of $(\alpha + \lambda_{4,8}^{(1)}\beta_1 + \lambda_{4,8}^{(2)}\beta_2)$ is an eigenvalue of $J_{\mathcal{G}}$ of multiplicity at least 2; every eigenvalue of $(\alpha + \lambda_{4,8}^{(1)}\beta_1 + \lambda_{4,8}^{(2)}\beta_2)$ is an eigenvalue of $J_{\mathcal{G}}$ of multiplicity at least 2.

4 Conclusions

Interior symmetry may be viewed as an appropriate generalization of symmetry for coupled cell networks. Besides the original concept of interior symmetry, we introduced further notions including quotient interior symmetry, reverse interior symmetry and quotient reverse interior symmetry.

For homogeneous coupled cell systems, we analyzed how multiple eigenvalues of the Jacobian at fully-synchronized equilibria may occur due to these different types of interior symmetry. The groups of interior symmetry that we focused on are symmetric groups, alternating groups, dihedral groups, cyclic groups and their products.

Based on our analysis, we concluded that the eigenvalue multiplicity of the Jacobian is sensitively dependent on the interior symmetric properties of the underlying network structure, and that symmetry alone is not sufficient to explain.

Indeed, in the examples we present throughout the paper, all the multiple eigenvalues are a consequence of an interior symmetry, in one form or another. In the case of uniform networks, already a relative weak interior symmetry may be sufficient to give rise to multiple eigenvalues.

Since, very easily, a homogeneous network has some type of interior symmetry we can say that multiple eigenvalues of the Jacobian at a fully synchronous equilibrium are frequent for homogeneous coupled cell systems.

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