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ON MODULI SPACES OF HITCHIN PAIRS

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Abstract. Let X be a compact Riemann surface X of genus at–least two. Fix a holomorphic line bundle L over X. Let $\mathcal{M}$ be the moduli space of Hitchin pairs $(E\,,\phi\in H^0(End_0(E)\otimes L))$ over X of rank r and fixed determinant of degree d. The following conditions are imposed:

- $\deg(L)\geq 2g-2$, $r\geq 2$, and $L^{\otimes r}\neq K_X^{\otimes r}$,
- $(r,d)=1$, and
- if $g=2$ then $r\geq 6$, and if $g=3$ then $r\geq 4$.

We prove that that the isomorphism class of the variety $\mathcal{M}$ uniquely determines the isomorphism class of the Riemann surface X. Moreover, our analysis shows that $\mathcal{M}$ is irreducible (this result holds without the additional hypothesis on the rank for low genus).

1. Introduction

The classical Torelli theorem says that the isomorphism class of a smooth complex projective curve is uniquely determined by the isomorphism class of its polarized Jacobian (the polarization is given by a theta divisor). This means that if $(\mathrm{Jac}(X)\,,\theta)\cong(\mathrm{Jac}(X')\,,\theta')$, then $X\cong X'$. Given any moduli space associated to a smooth projective curve, the corresponding Torelli question asks whether the isomorphism class of the moduli space uniquely determines the isomorphism class of the curve. The answer is affirmative in many situations. For instance, any moduli space of vector bundles with fixed determinant with degree and rank coprime; this was proved by Mumford and Newstead [MN] for rank two, and extended to any rank by Narasimhan and Ramanan [NR]. They show that the second intermediate Jacobian of the moduli space is isomorphic to the Jacobian of the curve. Since the Picard group of the moduli space is $\mathbb{Z}$, the second intermediate Jacobian has a canonical polarization. This reduces the question to the original Torelli theorem. This result for vector bundles has been crucial for proving Torelli theorems for moduli spaces of vector bundles with additional structures, for example, a Higgs field or a section (see [BG], [Mu]).

Let X be a compact connected Riemann surface of genus g, with $g\geq 2$. Fix a holomorphic line bundle L over X. An *L–twisted Higgs bundle* or a *Hitchin pair* consists of a holomorphic vector bundle $E\longrightarrow X$ and a section $\phi\in H^0(X,End(E)\otimes L)$. There is an appropriate notion of (semi)stability of these objects. We recall that

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if $L = K_X$, a Hitchin pair is a Higgs bundle. Higgs bundles were introduced by Hitchin in [Hi], where the moduli spaces of Higgs bundles were constructed using gauge theoretic methods. Nitsure in [Ni] gave a GIT construction of a coarse moduli scheme $\mathcal{M}(r, d, L)$ of S–equivalence classes of semistable Hitchin pairs of rank r and degree d. This moduli space is a normal quasi–projective variety.

The determinant map on $\mathcal{M}(r, d, L)$ is defined as follows:

$$\begin{aligned} \det : \mathcal{M}(r, d, L) &\longrightarrow \mathcal{M}(1, d, L) \cong \mathrm{Jac}^d(X) \times H^0(X, L) \\ (E, \phi) &\longmapsto (\Lambda^r E, \mathrm{Tr}\,(\phi))\,. \end{aligned} \tag{1}$$

Fix a line bundle ξ over X of degree d. The moduli space of Hitchin pairs with fixed determinant ξ is then the preimage $\mathcal{M}_\xi(r, d, L) := \det^{-1}(\xi, 0)$.

Our goal will be to prove the following.

Theorem 1.1. *Let $L \longrightarrow X$ be a fixed line bundle and let $\mathcal{M}_{\xi,X}(r, d, L)$ be the moduli space of Hitchin pairs of fixed determinant $\xi \longrightarrow X$ of degree d. Assume that*

- *$\deg(L) \geq 2g - 2$, $r \geq 2$, and $L^{\otimes r} \neq K_X^{\otimes r}$,*
- *$(r, d) = 1$, and*
- *if $g = 2$ then $r \geq 6$, and if $g = 3$ then $r \geq 4$.*

If $\mathcal{M}_{\xi,X}(r, d, L) \cong \mathcal{M}_{\xi',X'}(r, d, L')$, where the Riemann surface X' and the line bundles L' and ξ' are also of the above type, then X is isomorphic to X'.

Let $\mathcal{N}_\xi(r, d)$ (respectively, $\mathcal{N}_{\xi'}(r, d)$) be the moduli space of stable vector bundles E over X (respectively, X') of rank r and $\Lambda^r E = \xi$ (respectively, $\Lambda^r E = \xi'$). Theorem 1.1 is proved by showing that if $\mathcal{M}_{\xi,X}(r, d, L) \cong \mathcal{M}_{\xi',X'}(r, d, L)$, then $\mathcal{N}_\xi(r, d) \cong \mathcal{N}_{\xi'}(r, d)$. In fact, the strategy will be to show that the open subset $\mathcal{U} \subset \mathcal{M}_\xi(r, d, L)$ consisting of pairs $(E\,, \phi)$ such that E is stable is actually a vector bundle over the moduli space $\mathcal{N}_\xi(r, d)$. This open subset is proven to be of codimension large enough to induce an isomorphism of the second intermediate Jacobians $\mathrm{Jac}^2(\mathcal{M}_\xi(r, d, L)) \xrightarrow{\sim} \mathrm{Jac}^2(\mathcal{U})$. On the other hand, $\mathrm{Jac}^2(\mathcal{U}) \cong \mathrm{Jac}^2(\mathcal{N}_\xi(r, d))$. Then using the fact that $\mathrm{Pic}(\mathcal{U}) = \mathbb{Z}$ we construct a natural polarization on $\mathrm{Jac}^2(\mathcal{U})$; this is done following the method in [Mu, Section 6]. This polarization is taken to the natural polarization on $\mathrm{Jac}^2(\mathcal{N}_\xi(r, d))$. This proves Theorem 1.1 using the earlier mentioned result of [MN], [NR]. The details are given in Section 6 below.

As a byproduct of our computations we also obtain the following theorem (proved in Section 5 below):

Theorem 1.2. *Let $L \longrightarrow X$ be a fixed line bundle and let $\mathcal{M}_{\xi,X}(r, d, L)$ be the moduli space of Hitchin pairs of fixed determinant $\xi \longrightarrow X$ of degree d. Assume that*

- *$\deg(L) \geq 2g - 2$, $r \geq 2$, and $L^{\otimes r} \neq K_X^{\otimes r}$, and*
- *$(r, d) = 1$.*

Then the moduli space $\mathcal{M}_\xi(r, d, L)$ is irreducible.

Theorem 1.2 was proved in [Ni] under the assumption that $r = 2$.

2. Hitchin pairs

As before, X is a compact connected Riemann surface of genus g, with $g \geq 2$, and $L \longrightarrow X$ is a holomorphic line bundle. Fix integers $r \geq 2$ and d. We consider Hitchin pairs $(E, \phi \in H^0(End(E) \otimes L))$ with $\mathrm{rk}\,(E) = r$ and $\deg(E) = d$, as described in the Introduction. Recall that the slope of E is $\mu(E) = \deg(E)/\,\mathrm{rk}\,(E)$.

The following result is stated without proof in Remark 1.2.2 of [Bo].

Proposition 2.1. *There is a universal vector bundle $\mathbb{E}$ on $\mathcal{M}(r, d, L) \times X$ (respectively, $\mathcal{M}_\xi(r, d, L) \times X$) whenever $(r, d) = 1$.*

Proof. By [Ni], the moduli space $\mathcal{M}(r, d, L)$ is the GIT quotient of an appropriate Quot–scheme Q by $\mathrm{GL}_N(\mathbb{C})$ (for suitable N), and there is a universal vector bundle $\mathbb{E}$ over $X \times Q$ ([Ni, Proposition 3.6]). Moreover, the isotropy for the action of $\mathrm{GL}_N(\mathbb{C})$ on a stable point of Q is $\mathbb{C}^*$ (the center of $\mathrm{GL}_N(\mathbb{C})$), and $\mathbb{C}^*$ is contained in the isotropy subgroup of each point of Q. Also, the universal vector bundle $\mathbb{E}$ on $X \times Q$ is equipped with a lift of the action of $\mathrm{GL}_N(\mathbb{C})$.

There is a fixed integer δ such that for any c in the center of $\mathrm{GL}_N(\mathbb{C})$, the action of c on a fiber of $\mathbb{E}$ is multiplication by c^δ. (As pointed out in Remark 1.2.3 of [Bo], the fact that δ may be non-zero is the reason why $\mathbb{E}$ does not, in general, descend to the quotient.)

Fix a point x_0 of X. We have two line bundles on Q. The first line bundle L_1 is the top exterior power of the restriction of $\mathbb{E}$ to $x_0 \times Q$. The second line bundle L_2 is the Quillen determinant line bundle for the family, i.e.,

$$L_2 = \mathrm{Det}(\mathbb{E}) = \det(R^0 f_* \mathbb{E}) \otimes \det(R^1 f_* \mathbb{E})^*$$

where f is the projection of $X \times Q$ to Q.

Both these line bundles are equipped with a lift of the action of $\mathrm{GL}_N(\mathbb{C})$: for L_1, any $c \in \mathbb{C}^*$ acts as multiplication by c^r, and for L_2 any $c \in \mathbb{C}^*$ acts as multiplication by c^e, where

$$e = d + r(1 - g)$$

is the Euler characteristic of $\mathbb{E}$ restricted to a fiber of f. Since r and e are coprime, we can express $-\delta$ (defined above) as

$$-\delta = ar + be\,,$$

where a and b are integers.

Now replace the universal bundle $\mathbb{E}$ by

$$\mathbb{E}' := \mathbb{E} \otimes (L_1)^{\otimes a} \otimes (L_2)^{\otimes b}\,.$$

It follows from our construction that $\mathbb{C}^*$ acts trivially on the fibers of $\mathbb{E}'$. Hence $\mathbb{E}'$ descends to $X \times \mathcal{M}(r, d, L)$. $\square$

3. Deformation theory for Hitchin pairs

The infinitesimal deformations of a Hitchin pair (E,ϕ) are given by the first hypercohomology of the complex

$$C^\bullet = C^\bullet(E,\phi)\colon End(E) \xrightarrow{\mathrm{ad}\,(\phi)} End(E)\otimes L \tag{2}$$

where $\mathrm{ad}\,(\phi)(s) = \phi\circ s - (s\otimes 1_L)\circ\phi$ (see, for example, [BR], [Bo]). Therefore, Proposition 2.1 implies the following result:

Proposition 3.1. *For any stable Hitchin pair $(E,\phi)\in\mathcal{M}(r,d,L)$, the Zariski tangent space $T_{(E,\phi)}\mathcal{M}(r,d,L)$ to $\mathcal{M}(r,d,L)$ at the point (E,ϕ) is canonically isomorphic to $\mathbb{H}^1(C^\bullet)$.*

We have a long exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathbb{H}^0(C^\bullet)\longrightarrow H^0(X,End(E))\xrightarrow{\mathrm{ad}\,(\phi)} H^0(X,End(E)\otimes L)\\ &\longrightarrow \mathbb{H}^1(C^\bullet)\longrightarrow H^1(X,End(E))\xrightarrow{\mathrm{ad}\,(\phi)} H^1(X,End(E)\otimes L)\\ &\longrightarrow \mathbb{H}^2(C^\bullet)\longrightarrow 0 \end{aligned}\tag{3}$$

[BR, Remark 2.7]. In particular, $\mathbb{H}^0(C^\bullet)$ can be naturally identified with the space of global endomorphisms $\mathrm{End}\,(E,\phi)$.

Similarly, in the case of a Hitchin pair (E,ϕ) with fixed determinant $\Lambda^r E=\xi$, the deformation complex is

$$C_0^\bullet \,:\, End_0(E)\xrightarrow{\mathrm{ad}\,(\phi)} End_0(E)\otimes L\,, \tag{4}$$

where $End_0(E)\subset End(E)$ is the subbundle of rank r^2-1 given by the sheaf of trace–free endomorphisms.

We shall need the following standard lemma.

Lemma 3.2. *Let (E,ϕ) and (E',ϕ') be semistable Hitchin pairs. If $\mathrm{Hom}\,((E,\phi),(E',\phi'))\neq 0$, then $\mu(E)\le\mu(E')$.*

If, moreover, (E,ϕ) and (E',ϕ') are stable, then any non-zero $\psi\in\mathrm{Hom}\,((E,\phi),(E',\phi'))$ is an isomorphism.

Proof. Take any non-zero $\psi\in\mathrm{Hom}\,((E,\phi),(E',\phi'))$, so $\psi\colon E\to E'$ is a homomorphism such that

$$(\psi\otimes\mathrm{Id}_L)\circ\phi=\phi'\circ\psi\;\in\;H^0(X,\,Hom(E\,,E'\otimes L))\,.$$

Then the subsheaf $\ker(\psi)\subset E$ is ϕ–invariant and the subsheaf $\mathrm{im}(\psi)\subset E'$ is ϕ'–invariant. Now the lemma follows form the conditions of semistability and stability. □

Proposition 3.3. *Let (E,ϕ) be a stable Hitchin pair. Then*

$$\mathbb{H}^0(C^\bullet(E,\phi))\cong\mathbb{C}\,.$$

If, moreover, we assume that $L = M \otimes K$ for a line bundle M satisfying $\deg(M) \geq 0$, *then*

$$\mathbb{H}^2(C^\bullet(E,\phi)) \cong \begin{cases} 0 & \textit{if } E \ncong E \otimes M\,, \\ \mathbb{C} & \textit{if } E \cong E \otimes M\,. \end{cases}$$

Proof. Since $\mathbb{H}^0(C^\bullet(E,\phi)) \cong \mathrm{End}\,(E,\phi)$, the first statement is immediate from Lemma 3.2.

To prove the second statement, consider the by Serre duality $\mathbb{H}^2(C^\bullet(E,\phi)) = \mathbb{H}^0((C^\bullet)^\vee \otimes K_X)^*$, where

$$(C^\bullet)^\vee \otimes K_X \colon \mathit{End}(E) \otimes M^{-1} \xrightarrow{-\mathrm{ad}\,(\phi)} \mathit{End}(E) \otimes M^{-1}L\,.$$

But $\mathbb{H}^0((C^\bullet)^\vee \otimes K_X)$ is isomorphic to the space of global homomorphisms of Hitchin pairs $\mathrm{Hom}\,((E,\phi),(E \otimes M^{-1}, \phi \otimes 1_{M^{-1}}))$ (cf. [GK]). Hence the second statement also follows from Lemma 3.2. □

For the remainder of the paper we assume that

- $\deg(L) \geq 2g-2$,
- $L^{\otimes r} \neq K_X^{\otimes r}$, and
- r is coprime to d.

Proposition 3.4. *All the irreducible components of the moduli spaces $\mathcal{M}(r,d,L)$ and $\mathcal{M}_\xi(r,d,L)$ are smooth.*

Proof. By Lemma 3.2, the automorphism group of a stable Hitchin pair coincides with the center $\mathbb{C}^*$ of $\mathrm{GL}_r(\mathbb{C})$. Moreover, under our assumptions, Proposition 3.3 gives the vanishing $\mathbb{H}^2(C^\bullet) = 0$. Hence the result follows from the existence of a universal family (Proposition 2.1) and Theorem 3.1 of [BR]. □

Proposition 3.5. *For any stable Hitchin pair (E,ϕ),*

$$\dim T_{(E,\phi)}\mathcal{M}(r,d,L) = r^2(\deg L) + 1\,,$$

and every irreducible component of $\mathcal{M}(r,d,L)$ is smooth of this dimension.

Proof. The Euler characteristic of the complex $C^\bullet$ is the following

$$\chi(C^\bullet) = \dim \mathbb{H}^0(C^\bullet) - \dim \mathbb{H}^1(C^\bullet) + \dim \mathbb{H}^2(C^\bullet) = \chi(\mathit{End}(E)) - \chi(\mathit{End}(E) \otimes L)\,.$$

Hence,

$$\dim \mathbb{H}^1(C^\bullet) = \dim \mathbb{H}^0(C^\bullet) + \dim \mathbb{H}^2(C^\bullet) + r^2 \deg(L)\,. \tag{5}$$

Thus, by Proposition 3.3 we have $\dim \mathbb{H}^1(C^\bullet) = r^2(\deg L) + 1$. The rest follows from Propositions 3.1 and 3.4. □

Proposition 3.6. *For any stable Hitchin pair (E,ϕ) of fixed determinant $\Lambda^r E = \xi$,*

$$\dim T_{(E,\phi)}\mathcal{M}_\xi(r,d,L) = (r^2-1)\deg L\,,$$

and every irreducible component of $\mathcal{M}_\xi(r,d,L)$ is smooth of this dimension.

Proof. The proof is same as that of Proposition 3.5 after considering the fixed determinant deformation complex (4). □

Let

$$\mathcal{U} \subset \mathcal{M}_\xi(r,d,L) \tag{6}$$

be the Zariski open subset parametrizing all pairs $(E\,,\phi)$ such that the underlying vector bundle E is stable. Let $\mathcal{N}_\xi(r,d)$ be the moduli space of stable vector bundles E of rank r with $\Lambda^r E = \xi$. We have a forgetful map $f : \mathcal{U} \longrightarrow \mathcal{N}_\xi(r,d)$ defined by $(E\,,\phi) \longmapsto E$.

Lemma 3.7. *The above forgetful map*

$$f : \mathcal{U} \longrightarrow \mathcal{N}_\xi(r,d)$$

makes $\mathcal{U}$ an algebraic vector bundle over $\mathcal{N}_\xi(r,d)$.

Proof. Take any $(E\,,\phi) \in \mathcal{U}$. We have a map

$$H^0(X, End_0(E)\otimes L) \longrightarrow f^{-1}(E)$$

defined by $\psi \longmapsto (E\,,\psi)$, where f is the map in the statement of the lemma.

Since E is stable, $H^0(X, \mathrm{End}(E)) = \mathbb{C}\cdot \mathrm{Id}_E$. Hence the above map identifies the fiber $f^{-1}(E)$ with $H^0(X, End_0(E)\otimes L)$.

The given conditions that E is stable, $\deg(L) \geq 2g-2$, and $L^{\otimes r} \neq K_X^{\otimes r}$, imply that

$$\dim H^0(End_0(E)\otimes L) = (r^2-1)\cdot(\deg(L)+1-g)\,,$$

in particular, this dimension is independent of E. Therefore, f makes $\mathcal{U}$ into an algebraic vector bundle over $\mathcal{N}_\xi(r,d)$. □

4. Bialynicki–Birula and Bott–Morse stratifications

It is an important feature of the moduli space of Hitchin pairs that it admits an action of $\mathbb{C}^*$:

$$\begin{aligned} \mathbb{C}^* \times \mathcal{M}(r,d,L) &\longrightarrow \mathcal{M}(r,d,L) \\ (t,(E,\phi)) &\longmapsto (E,t\phi). \end{aligned}$$

This action gives rise to a stratification of the moduli space, which can be interpreted from a Morse theoretic point of view. Next we recall how this comes about. We start by recalling another interpretation of the moduli space from a gauge theoretic point of view.

The notion of stability for a twisted Higgs bundle $(E\,,\phi)$ is related to the existence of a special Hermitian metric on E. To explain this, fix a Kähler form on X. Let Λ be the contraction of differential forms on X with the Kähler form. Fix a Hermitian structure on L such that the curvature of the Chern connection (the unique connection compatible with both the holomorphic and Hermitian structures) is a constant scalar multiple of the Kähler form. Using this Hermitian structure on L, the dual line bundle $L^\vee$ is identified with the C^∞ line bundle $\overline{L}$. Let $E \longrightarrow X$ be a holomorphic

Hermitian vector bundle and ϕ a holomorphic section of $\operatorname{End}(E) \otimes L$. Using the identification of $\overline{L}$ with $L^\vee$, the adjoint ϕ^* is a section of $\operatorname{End}(E) \otimes L^\vee$.

A semistable L–twisted Higgs bundle is called *polystable* if it is a direct sum of stable L–twisted Higgs bundles, all of the same slope. The following Theorem is due to Li [Li]; it also follows from the general results of [BGM].

Theorem 4.1. *Let (E, ϕ) be a L–twisted Higgs bundle. The existence of a Hermitian metric h on E satisfying*

$$\Lambda F_h + [\phi \,, \phi^*] \;=\; \lambda \,\mathrm{Id} \;, \tag{7}$$

for some $\lambda \in \mathbb{R}$, is equivalent to the polystability of (E, ϕ).

Here F_h is the curvature of the Chern connection on E. The constant λ is determined by the slope of E. The smooth section $[\phi \,, \phi^*]$ of $\operatorname{End}(E)$ is the contraction of the section $\phi\phi^* - \phi^*\phi$ of $\operatorname{End}(E) \otimes L \otimes L^\vee$.

Fix a C^∞ vector bundle $E \longrightarrow X$ of degree d, and fix a Hermitian structure h on E. Let $\mathcal{A}$ is the space of all unitary connections on (E, h). The equation in (7) corresponds to the moment map for the action of the unitary group on the product Kähler manifold $\mathcal{A} \times End(E)$ (whose Kähler metric is induced by the Hermitian metrics on E and L). The moduli space of stable L–twisted Higgs bundles is then obtained as the Kähler quotient, and hence the moduli space inherits a Kähler structure.

The restriction of the $\mathbb{C}^*$-action to $S^1 \subset \mathbb{C}^*$ preserves the induced Kähler form on $\mathcal{M}(r, d, L)$. Thus we have a Hamiltonian action of the circle and the associated moment map is

$$\begin{aligned} \mu : \mathcal{M}(r, d, L) &\longrightarrow \mathbb{R} \\ (E, \phi) &\longmapsto \frac{1}{2} ||\phi||^2 \,. \end{aligned}$$

It has a finite number of critical submanifolds, and L-twisted Higgs bundles of the form $(V, 0)$ are the absolute minima.

Let F be the fixed point set for the $\mathbb{C}^*$–action on $\mathcal{M}(r, d, L)$. This fixed point set is a disjoint union of connected components which we denote by F_λ, so $F = \bigcup_\lambda F_\lambda$. For any component F_λ, define

$$U^+_\lambda = \{p \in M;\, \lim_{t \to 0} tp \in F_\lambda\}$$

and

$$U^-_\lambda = \{p \in M;\, \lim_{t \to \infty} tp \in F_\lambda\}.$$

The sets U^+_λ are strata for $\mathcal{M}(r, d, L)$; the resulting stratification is called the *Bialynicki–Birula stratification.*

Considering the moment map μ as a Morse function, we obtain another stratification. To a critical submanifold F_λ we can assign an unstable manifold, also called *upwards Morse flow,*

$$\widetilde{U}^+_\lambda = \{x \in \mathcal{M}_\xi(r, d, L);\, \lim_{t \to -\infty} \psi_t(x) \longmapsto F_\lambda\} \,,$$

where ψ_t is the gradient flow for μ. Similarly, we have the stable manifold

$$\widetilde{U}_\lambda^- = \{x \in \mathcal{M}_\xi(r,d,L);\, \lim_{t\to\infty} \psi_t(x) \longmapsto F_\lambda\}$$

which is known as the *downwards Morse flow*. Now, $\{\widetilde{U}_\lambda^+\}_\lambda$ give a stratification, which is called the *Morse stratification*.

In [Ki, Theorem 6.16], Kirwan proves that the stratifications $\widetilde{U}^+$ and U^+ coincide, and similarly $\widetilde{U}^- = U^-$ (cf. Hausel [Ha] for the first application of this result in the context of moduli of Higgs bundles).

Let

$$h \,:\, \mathcal{M}_\xi(r,d,L) \longrightarrow \bigoplus_{i=2}^{r} H^0(X,\, L^{\otimes i}) \tag{8}$$

be the *Hitchin map* defined by $(E\,,\phi) \longmapsto \sum_{i=2}^r \mathrm{Tr}\,(\Lambda^i\phi)$. The inverse image $\mathcal{H}^{-1}(0) \subset \mathcal{M}_\xi(r,d,L)$ is called the *nilpotent cone*.

The following result was observed by Hausel [Ha] in the context of Higgs bundles. It generalizes to L-twisted Higgs bundles with essentially the same proof.

Proposition 4.2. *The nilpotent cone coincides with the downwards Morse flow.*

Proof. First note that by [Ki, Theorem 6.16], the downwards Morse flow coincides with

$$\bigcup_\lambda \{p \in \mathcal{M}_\xi(r,d,L);\, \lim_{t\to\infty} tp \in F_\lambda\}\,,$$

where F_λ are the set of components of the fixed point set of the $\mathbb{C}^*$–action on our moduli space $\mathcal{M}_\xi(r,d,L)$. For the Hitchin map h in (8),

$$h(\lim_{\lambda\to\infty} \lambda p) \;=\; \lim_{\lambda\to\infty} (\lambda h(p))$$

this implies that $h(p) = 0$ for any point p in the downwards Morse flow. Hence, the downwards Morse flow is contained in the nilpotent cone.

To prove the converse, recall that the Hitchin map h is proper [Hi], [Ni]. Hence for any point p of the nilpotent cone, the $\mathbb{C}^*$–orbit of p is compact. So the limit $\lim_{\lambda\to\infty} \lambda p$ exists, hence p is in F_λ. □

5. A codimension computation

Let

$$\mathcal{S} = \{(E,\phi);\, E \;\text{is not stable}\} \,\subset\, \mathcal{M}_\xi(r,d,L)$$

be the subscheme of the moduli space $\mathcal{M}_\xi(r,d,L)$ that parametrizes all Hitchin pairs $(E\,,\phi)$ such that the underlying vector bundle E is not semistable (since r is coprime to d, the vector bundle E is semistable if and only if it is stable). Also, consider the following set

$$\mathcal{S}' = \{(E,\phi);\, \lim_{t\to 0}(E,t\phi) \notin \mathcal{N}_\xi(r,d)\}\,,$$

where $\mathcal{N}_\xi(r,d)$ is the moduli space of stable vector bundles $E \longrightarrow X$ with rank r and $\Lambda^r E \cong \xi$.

Proposition 5.1. *The equality $\mathcal{S}' = \mathcal{S}$ holds.*

Proof. Take any Hitchin pair $(E\,,\phi)$ such that the underlying vector bundle E is stable. Then

$$\lim_{t\to 0}(E, t\phi) = (E\,,0) \in \mathcal{N}_\xi(r,d)\,.$$

Hence $\mathcal{S}' \subset \mathcal{S}$.

To prove the converse, take any $(E\,,\phi) \in S$. Since E is not semistable, it has a unique Harder–Narasimhan filtration

$$E = E_m \supset E_{r-1} \supset \cdots E_1 \supset E_0 = 0\,.$$

We recall that E_i/E_{i-1} is semistable for all $i \in [1\,,m$, and furthermore, $\mu(E_i/E_{i-1}) > \mu(E_{i+1}/E_i)$. Following Atiyah–Bott, [AB], set $D_i = E_i/E_{i-1}$, $n_i = \mathrm{rk}\,(D_i)$, $k_i = \deg(D_i)$ and $\mu_i = k_i/n_i$. Note that $\sum_{i=1}^m n_i = r$, and $\sum_{i=1}^m k_i = d$. Consider the non-increasing sequence $(\mu_1, \ldots, \mu_r)$, where each μ_i is repeated n_i times. The vector $(\mu_1, \ldots, \mu_r)$ is called the *type* of E.

We will analyze the limit of $(E, t\phi)$ as $t \to 0$. For that we will recall from [AB], [Sh] a partial ordering of types. For any type $\underline{\delta} := (\delta_1, \ldots, \delta_r)$, consider the polygon in $\mathbb{R}^2$ traced by the points $(\sum_{i=1}^b \delta_i\,, b)$, $1 \leq b \leq r$, and the point $(0\,,0)$. For another type $\underline{\delta}'$, we say that $\underline{\delta} \geq \underline{\delta}'$ if the polygon for $\underline{\delta}$ is above the polygon for $\underline{\delta}'$. This ordering is complete when $r = 2$.

In a family of vector bundles, the type increases under specialization; see [Sh, p. 18, Theorem 3] for the precise statement. Consider the map

$$\tau\,:\,\mathbb{C}^* \longrightarrow \mathcal{M}_\xi(r,d,L)$$

that sends any t to the point representing the Hitchin pair $(E, t\phi)$. The Hitchin map h in (8) is proper, [Hi], [Ni], and $\lim_{t\to 0} h((E, t\phi)) = 0$. Hence the above map τ extends to a map

$$\widehat{\tau}\,:\,\mathbb{C} \longrightarrow \mathcal{M}_\xi(r,d,L)\,.$$

Let $\mathbb{E}$ be a universal vector bundle over $\mathcal{M}_\xi(r,d,L)\times X$ (see Proposition 2.1). Consider the family of vector bundles $(\widehat{\tau}\times \mathrm{Id}_X)^*\mathbb{E}$ parametrized by $\mathbb{C}$. For any $t \in \mathbb{C}^*$, the vector bundle $((\widehat{\tau}\times \mathrm{Id}_X)^*\mathbb{E})|_{\{t\}\times X} = E$ is not semistable. Hence from [Sh, p. 18, Theorem 3] it follows that $((\widehat{\tau}\times \mathrm{Id}_X)^*\mathbb{E})|_{\{0\}\times X}$ is not semistable. In other words, $(E,\phi) \in \mathcal{S}'$. This completes the proof. $\square$

Now we are in a position to estimate the codimension of $\mathcal{S}$. We use the notation of Section 4. Let $N := h^{-1}(0)$ be the nilpotent cone. From Proposition 4.2 we know that $N = \bigcup_\lambda F_\lambda$.

Now the complex dimension of the upwards Morse flow is:

$$\dim(U_\lambda^+) = \dim(T\mathcal{M}_\xi|_{F_\lambda})_{>0} + \dim(F_\lambda)$$

because

$$\dim(U_\lambda^+) + \dim(T\mathcal{M}_\xi|_{F_\lambda})_{<0} = \dim \mathcal{M}_\xi\,.$$

Since $\mathcal{S}' = \mathcal{S}$, from Bott–Morse theory we know that $\mathcal{S}' = \bigcup_{\lambda\neq 0} U_\lambda^+$, so $\operatorname{codim}\mathcal{S} = \min_{\lambda\neq 0}\operatorname{codim} U_\lambda^+$. Finally

$$\operatorname{codim}(U_\lambda^+) = \dim(T\mathcal{M}_\xi|_{F_\lambda})_{<0}$$

which is half the Morse index[1], at F_λ, for the perfect Morse function μ.

The critical points of the Bott–Morse function μ are exactly the fixed points of the $\mathbb{C}^*$–action on $\mathcal{M}_\xi(r,d,L)$. If (E,ϕ) corresponds to a fixed point, then it is a *Hodge bundle*, i.e., it is of the form

$$E = E_0 \oplus E_1 \oplus \cdots \oplus E_m, \tag{9}$$

with $\phi(E_i) \subset E_{i+1}\otimes L$ (see [Hi], [Si2]). Consequently, the deformation complex in (2) decomposes as

$$C^\bullet(E,\phi) = \bigoplus_k C_k^\bullet(E,\phi)\,,$$

where for each k,

$$C_k^\bullet(E,\phi) : \bigoplus_{j-i=k}\operatorname{Hom}(E_i,E_j) \xrightarrow{\operatorname{ad}(\phi)} \bigoplus_{j-i=k+1}\operatorname{Hom}(E_i,E_j)\otimes L\,.$$

Therefore, the tangent space to $\mathcal{M}_\xi(r,d,L)$ at the point $(E\,,\phi)$ has a decomposition

$$\mathbb{H}^1(C^\bullet(E,\phi)) = \bigoplus_k \mathbb{H}^1(C_k^\bullet(E,\phi))$$

(see Proposition 3.1), and half the Morse index at (E,ϕ) is

$$\sum_{k>0}\dim\mathbb{H}^1(C_k^\bullet(E,\phi)) \;=\; \sum_{k>0} -\chi(C_k^\bullet(E,\phi)) \tag{10}$$

(note that the above equality follows from Proposition 3.3).

We will estimate $\chi(C_k^\bullet(E,\phi))$ first, using a similar argument to the one given in [BGG, Lemma 3.11]. Define

$$C_k := \bigoplus_{j-i=k}\operatorname{Hom}(E_i,E_j)$$

and $\Phi_k := \operatorname{ad}(\phi)|_{C_k}$. Then we have the homomorphism

$$\Phi_k \,:\, C_k \longrightarrow C_{k+1}\otimes L\,.$$

Proposition 5.2. *Let (E,ϕ) be a stable Hitchin pair which corresponds to a fixed point of the $\mathbb{C}^*$–action. Then*

$$\chi(C_k^\bullet(E,\phi)) \le (1-g)(\operatorname{rk}(C_k) - \operatorname{rk}(C_{k+1})) - \deg(L)(\operatorname{rk}(C_{k+1}) - \operatorname{rk}(\Phi_k))\,.$$

Proof. Note that

$$\chi(C_k^\bullet(E,\phi)) = \deg(C_k) - \deg(C_{k+1}) - \operatorname{rk}(C_{k+1})\deg(L) + (\operatorname{rk}(C_k) - \operatorname{rk}(C_{k+1}))(1-g)\,, \tag{11}$$

[1]The Morse index is the real dimension which is twice the complex dimension.

so we will first bound $\deg(C_k) - \deg(C_{k+1})$. For that consider the short exact sequences

$$0 \longrightarrow \ker(\Phi_k) \longrightarrow C_k \longrightarrow \mathrm{Im}\,(\Phi_k) \longrightarrow 0 \tag{12}$$

$$0 \longrightarrow \mathrm{Im}\,(\Phi_k) \longrightarrow C_{k+1} \otimes L \longrightarrow \mathrm{coker}\,(\Phi_k) \longrightarrow 0\,. \tag{13}$$

From these,

$$\deg(C_k) - \deg(C_{k+1}) \;=\; \deg(\ker(\Phi_k)) + \deg(L)\,\mathrm{rk}\,(C_{k+1}) - \deg(\mathrm{coker}\,(\Phi_k))\,. \tag{14}$$

Clearly $\ker(\Phi_k) \subset End_0(E)$. In Lemma 5.3 it was proved that if the pair (E, ϕ) is stable, then $(End_0(E), \Phi_k)$ is a semistable pair; so we obtain

$$\deg(\ker(\Phi_k)) \leq 0\,. \tag{15}$$

In view of (11), (14) and (15), to prove the proposition it suffices to show that

$$-\deg(\mathrm{coker}\,(\Phi_k)) \;\geq\; -\deg(L)(\mathrm{rk}\,(C_{k+1}) - \mathrm{rk}\,(\Phi_k))\,. \tag{16}$$

Consider the dual homomorphism $\Phi_k^t \,:\, C_{k+1}^* \otimes L^{-1} \;\longrightarrow\; C_k^*$ of Φ_k. Let

$$C_{k+1} \otimes L \;\longrightarrow\; \ker(\Phi_k^t)^* \tag{17}$$

be the dual of the inclusion map

$$\ker(\Phi_k^t) \;\hookrightarrow\; C_{k+1}^* \otimes L^{-1}\,.$$

Note that the homomorphism in (17) vanishes identically on the image $\mathrm{Im}(\Phi_k)$. So we get a homomorphism

$$f \,:\, \mathrm{coker}\,(\Phi_k) \;\longrightarrow\; \ker(\Phi_k^t)^*$$

which is evidently surjective. Note that $\ker(\Phi_k^t)$ is a subbundle of $C_{k+1}^* \otimes L^{-1}$. We have a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Im}\,(\Phi_k) \arrow[r] \arrow[dr] & C_{k+1} \otimes L \arrow[r] \arrow[d] & \mathrm{coker}\,(\Phi_k) \arrow[r] \arrow[dl, "f"] & 0 \\
 & & (\ker(\Phi_k^t))^* & &
\end{tikzcd}$$

and a short exact sequence

$$0 \longrightarrow \ker(f) \longrightarrow \mathrm{coker}\,(\Phi_k) \longrightarrow (\ker(\Phi_k^t))^* \longrightarrow 0\,. \tag{18}$$

The kernel of f is a torsion subsheaf of $\mathrm{coker}\,(\Phi_k)$ (note that $\mathrm{coker}\,(\Phi_k)$ need not be locally free), hence from (18) we conclude that

$$\deg(\mathrm{coker}\,(\Phi_k)) \geq \deg(\ker(\Phi_k^t)^*)\,. \tag{19}$$

As $\ker(\Phi_k^t)$ is a subbundle of $C_{k+1}^* \otimes L^{-1}$, from (19),

$$-\deg(\mathrm{coker}\,(\Phi_k)) \leq \deg(\ker(\Phi_k^t))\,. \tag{20}$$

We have an isomorphism $C_k^* \otimes L^{-1} \cong C_{-k} \otimes L^{-1}$ and hence a commutative diagram

$$\begin{array}{ccc} C_{k+1}^* \otimes L^{-1} & \xrightarrow{\Phi_k^t} & C_k^* \\ \cong\Big\downarrow & & \Big\downarrow\cong \\ C_{-k-1} \otimes L^{-1} & \xrightarrow{-\Phi_{-k-1}\otimes 1_{L^{-1}}} & C_{-k} \end{array}$$

so $\ker(\Phi_k^t) = \ker(\Phi_{-k-1}) \otimes L^{-1}$. Hence

$$\deg(\ker(\Phi_k^t)) \;=\; \deg(\ker(\Phi_{-k-1})) - \deg(L)\,\mathrm{rk}\,(\ker(\Phi_{-k-1}))\,,$$

and then

$$\deg(\ker(\Phi_k^t) \;\leq\; -\deg(L)\,\mathrm{rk}\,(\ker(\Phi_{-k-1}))\,. \tag{21}$$

Notice that $\mathrm{rk}\,(\Phi_{-k-1}) = \mathrm{rk}\,(\Phi_k^t) = \mathrm{rk}\,(\Phi_k)$ and $\mathrm{rk}\,(C_{k+1}) = \mathrm{rk}\,(C_{-k-1}^*) = \mathrm{rk}\,(C_{-k-1})$. Then

$$\mathrm{rk}\,(\ker(\Phi_{-k-1})) = \mathrm{rk}\,(C_{k+1}) - \mathrm{rk}\,(\Phi_k)\,. \tag{22}$$

Note that (20), (21) and (22) together imply (16). This completes the proof of the proposition. □

The correspondence in Theorem 4.1 gives a differential geometric proof of the following lemma which was used above in the proof of Proposition 5.2.

Lemma 5.3. *Let (E, ϕ) be a stable pair, then the pair $(End(E), \mathrm{ad}\,(\phi))$ is semistable.*

Proof. An irreducible solution of the equations in (7) provides us with a semistable Hitchin pair. Now it is easy to see that the tensor product of two solutions is again a solution to the equations, and the same is with the dual. So if h is a Hermitian structure on E that satisfies the equation in Theorem 4.1, then $(End(E) = E^* \otimes E, \mathrm{ad}\,(\phi))$ with the induced metric also satisfies the equation in Theorem 4.1. □

Proposition 5.4. *The codimension of the subset $\mathcal{S}$ of $\mathcal{M}(r, d, L)$ consisting of pairs for which the underlying bundle is not stable, is greater than or equal to $(g-1)(r-1)$. Hence, it is greater than or equal to 4 whenever one of the following occur*

- *$g = 2$ and $r \geq 6$,*
- *$g = 3$ and $r \geq 4$,*
- *$g \geq 4$ and $r \geq 2$.*

Proof. From Proposition 5.2 we need to compute the following sum

$$\mathrm{codim}\,(\mathcal{S}) \;=\; \sum_{k>0} -\chi(C_k^\bullet)$$

$$\geq \sum_{k>0} \deg(L)(\mathrm{rk}\,(C_{k+1}) - \mathrm{rk}\,(\Phi_k)) + (g-1)(\mathrm{rk}\,(C_k) - \mathrm{rk}\,(C_{k+1}))$$

$$= \sum_{k>0} (2g-2+l)(\mathrm{rk}\,(C_{k+1}) - \mathrm{rk}\,(\Phi_k)) + (g-1)(\mathrm{rk}\,(C_k) - \mathrm{rk}\,(C_{k+1}))$$

with $l \geq 0$ since we are assuming that $\deg(L) \geq K$. Hence,

$$= \sum_{k>0}(g-1)(\mathrm{rk}\,(C_{k+1}) + \mathrm{rk}\,(C_k) - 2\,\mathrm{rk}\,(\Phi_k)) + l(\mathrm{rk}\,(C_{k+1}) - \mathrm{rk}\,(\Phi_k))$$
$$\geq (g-1)\,\mathrm{rk}\,(C_1)$$

(we use that $\mathrm{rk}\,(C_{k+1}) - \mathrm{rk}\,(\Phi_k) \geq 0$). We now just need to estimate $\mathrm{rk}\,(C_1)$, so let r_i be the rank of E_i in the Hodge decomposition. Since $k = 1$, we need to estimate $r_1 r_2 + r_2 r_3 + \cdots + r_{m-1} r_m$, where m was the top index in (9), but it is clearly bigger than or equal to $r - 1$. □

Proof of Theorem 1.2. The subset $\mathcal{U}$ of L-twisted Higgs bundles for which the underlying bundle is stable is the complement of $\mathcal{S}$. It follows from Lemma 3.7 that $\mathcal{U}$ retracts onto $\mathcal{N}_\xi(r, d)$, which is well known to be connected. Hence Proposition 5.4 and Proposition 3.5 together imply that the moduli space $\mathcal{M}_\xi(r, d, L)$ is irreducible. □

6. Torelli theorem

We recall the notion of s–th intermediate Jacobian which will be central in our study. Let M be a complex projective manifold. Let

$$H^n(M, \mathbb{C}) \;=\; \bigoplus_{p+q=n} H^{p,q}(M)$$

be the Hodge decomposition. Set $n = 2s - 1$, and define

$$V_s \;:=\; H^{s-1,s}(M) \oplus \cdots \oplus H^{0,2s-1}$$

Note that it can be defined for any $1 \leq s \leq m = \dim_{\mathbb{C}} M$. Then

$$H^{2s-1}(M, \mathbb{C}) \;=\; V_s \oplus \overline{V_s}\,,$$

and we have the projection of $H^{2s-1}(M, \mathbb{C})$ to the factor V_s. Denote by Λ_s the image of the composition

$$H^{2s-1}(M, \mathbb{Z}) \longrightarrow H^{2s-1}(M, \mathbb{C}) \longrightarrow V_s\,.$$

The s–th intermediate Jacobian of M is defined to be the complex torus

$$\mathrm{Jac}^s(M) \;:=\; \frac{V_s}{\Lambda_s}\,.$$

When $s = 1$, we get the Picard variety

$$\mathrm{Jac}^1(M) = \frac{H^{0,1}(M)}{H^1(M, \mathbb{Z})} = \mathrm{Pic}^0(M)\,.$$

Here we will be interested in the second intermediate Jacobian

$$\mathrm{Jac}^2(M) \;=\; \frac{H^{1,2}(M) \oplus H^{0,3}(M)}{H^3(M, \mathbb{Z})}\,.$$

A complex torus $T = V/\Lambda$, where Λ is a cocompact lattice in a vector space V, is called an *abelian variety* if it is a projective algebraic variety. The Kodaira embedding theorem says that T admits an embedding in a projective space if an only if there exists a Hodge form on T, meaning a closed positive form ω of type $(1,1)$ representing a rational cohomology class. The conditions which determine if such a form exists are the Riemann bilinear relations which can be formulated as follows: a class in $H^2(T, \mathbb{Z})$ is given by a bilinear form

$$Q \,:\, \Lambda \otimes_{\mathbb{Z}} \Lambda \longrightarrow \mathbb{Z}\,, \qquad Q(\lambda, \lambda') \,=\, -Q(\lambda', \lambda)\,.$$

Identifying $\Lambda_s \otimes_{\mathbb{Z}} \mathbb{C}$ with $V_s \oplus \overline{V_s}$, the Riemann bilinear relations are

$$Q(v, v') = 0, \quad v, v' \in V\,,$$
$$-\sqrt{-1}Q(v, v') > 0, 0 \neq v \in V\,.$$

Thus, for instance, given a Hodge form ω on a complex projective manifold M, the Riemann bilinear relations

$$Q \,:\, \Lambda_1 \otimes \Lambda_1 \longrightarrow \mathbb{Z}$$

given by

$$Q(\lambda, \lambda') = \int_M \omega^{n-1} \wedge \lambda \wedge \lambda'$$

produce a polarization on $\mathrm{Jac}^1(M)$ (see [GH]).

We recall the basics of a mixed Hodge structure. Let be H a finite dimensional vector space over $\mathbb{Q}$. A pure Hodge structure of weight k on H is a decomposition

$$H_{\mathbb{C}} = H \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}$$

such that $H^{q,p} = \overline{H}^{p,q}$; the bar denotes complex conjugation in $H_{\mathbb{C}}$. It has two associated filtrations, the (non-increasing) Hodge filtration F on $H_{\mathbb{C}}$

$$F^p := \bigoplus_{p' \geq p} H^{p',q} \,\subset\, H_{\mathbb{C}}\,,$$

and the (non-decreasing) weight filtration W defined over $\mathbb{Q}$

$$W_m := \bigoplus_{p+q \leq m} H^{p,q}\,.$$

A mixed Hodge structure on H consist of two filtrations: a non decreasing (weight) filtration W defined over $\mathbb{Q}$, and a non increasing (Hodge) filtration F so that F induces a Hodge filtration of weight r on each rational vector space $\mathrm{Gr}\,^W_r = W_r/W_{r-1}$.

Let M_0 be a smooth complex quasi–projective variety. The cohomology of M_0 has a mixed Hodge structure [De1, De2, De3]. The above construction of second intermediate Jacobian can be generalized to M_0 as follows:

$$\mathrm{Jac}^2(M_0) \,:=\, H^3(M_0, \mathbb{C})/(F^2H^3(M_0, \mathbb{C}) + H^3(M_0, \mathbb{Z})) \tag{23}$$

(see [Ca, p. 110]). This intermediate Jacobian is a generalized torus [Ca, p. 111].

Since the projection f in Lemma 3.7 makes $\mathcal{U}$ a vector bundle over $\mathcal{N}_\xi(r, d)$, we conclude that the corresponding homomorphism

$$f^* \,:\, H^j(\mathcal{N}_\xi(r, d), \mathbb{Z}) \longrightarrow H^j(\mathcal{U}, \mathbb{Z})$$

is an isomorphism for all j. In particular, it is an isomorphism for $j = 3$. Therefore, the following proposition holds.

Proposition 6.1.

$$\mathrm{Jac}^2(\mathcal{U}) \cong \mathrm{Jac}^2(\mathcal{N}_\xi(r,d))\,.$$

Lemma 6.2. *Let M be a smooth variety and S a closed subscheme of it of codimension k; denote $U = M \setminus S$. Then the inclusion map $U \hookrightarrow M$ induces isomorphism*

$$H^j(M,\mathbb{Z}) \cong H^j(U,\mathbb{Z})$$

for all $j < 2k-1$.

Proof. This lemma is proved in [AS, Lemma 6.1.1]. □

Proof of Theorem 1.1. Proposition 5.4 and Lemma 6.2 imply that

$$\mathrm{Jac}^2(\mathcal{M}_\xi(r,d,L) \;\cong\; \mathrm{Jac}^2(\mathcal{U})\,,$$

where $\mathcal{U}$ is the open subset in Proposition 6.1. Hence from Proposition 6.1,

$$\mathrm{Jac}^2(\mathcal{M}_\xi(r,d,L) \;\cong\; \mathrm{Jac}^2(\mathcal{N}_\xi(r,d))\,.$$

But $\mathrm{Jac}^2(\mathcal{N}_\xi(r,d)) = \mathrm{Pic}^0(X)$ [MN, p. 1201, Theorem], [NR, p. 392, Theorem 3]. Hence it only remains to provide $\mathcal{M}_\xi(r,d,L)$ with a canonical polarization, which is done in Proposition 7.1. The usual Torelli theorem then completes the proof. □

7. Reconstructing the polarization

In this section we construct a canonical polarization on $J^2(\mathcal{M}_\xi(r,d,L))$ following [Mu, Section 6] (see also [AS, Section 8]). To be precise, one shows that $H^3(\mathcal{M}_\xi(r,d,L))$ has a polarization which is natural in the sense that it can be constructed when X varies in a family.

Proposition 7.1. *Assume that if $g = 2$ then $r \geq 6$, and if $g = 3$ then $r \geq 4$. The Hodge structure $H^3(\mathcal{M}_\xi(r,d,L))$ is naturally polarized. Furthermore the isomorphism*

$$\mathrm{Jac}^2(\mathcal{M}_\xi(r,d,L)) \;\cong\; \mathrm{Jac}^2(\mathcal{N}_\xi(r,d))$$

respects the polarizations.

Proof. For brevity write $M = \mathcal{M}_\xi(r,d,L)$. We note that Proposition 5.4, Proposition 6.1 and Lemma 6.2 prove that $\mathrm{Pic}(M) = \mathbb{Z}$, thus there is a unique generator of the Picard group.

We make use of $\overline{M}$, the compactified moduli space of Hitchin pairs considered in [Ha, Sch, Si1]. Let $\mathrm{Sing}(\overline{M})$ be the singular locus of the compactified moduli space. Take a hyperplane intersection Z of M of codimension 3. By Proposition 7.2 below we have $\mathrm{codim}\,(\mathrm{Sing}(\overline{M})) \geq 4$, hence a generic such Z is smooth. As in the proof of [Mu, Proposition 6.1], we can define

$$\begin{aligned} H^3(M) \otimes H^3(M) &\longrightarrow \mathbb{Z}, \\ \beta_1 \otimes \beta_2 &\mapsto \langle \beta_1 \cup \beta_2, [Z]\rangle \end{aligned}$$

In view of Proposition 7.2, the argument in [Mu] now goes through with the obvious adaptations to prove that this is a polarization. □

Proposition 7.2. *The singular locus* $\mathrm{Sing}(\overline{M})$ *has codimension greater than or equal to* $(g-1)(r-1)$.

Proof. Write $\overline{M} = M \cup Z$ for the compactified moduli space. The singular locus of $\overline{M}$ sits inside Z, which is an orbifold submanifold of codimension 1 (Theorem 3.4 [Ha]). We can describe the singular points in $\overline{M}$ in the following way. Recall from [Ha, Sch] that $\overline{M} = (M \times \mathbb{C} - N \times \{0\})/\mathbb{C}^*$, being N the nilpotent cone defined in Section 4. Singular points correspond then to fixed points of the $\mathbb{C}^*$ action on $\mathcal{M}_\xi(r,d,L) \times \mathbb{C}$, i.e. Hitchin pairs (E,ϕ) for which there is a p–th root of the unity ζ such that $(E,\phi,0) \cong (E,\zeta_p\phi,\zeta_p \cdot 0)$. These pairs were identified by Simpson [Si1, Si3]:

Let $f : E \longrightarrow E$ be the automorphism such that $f\phi = \zeta_p\phi f$. The coefficients of the characteristic polynomial of f are holomorphic functions on X, hence constant, so the eigenvalues are constant. This gives a decomposition $E = \bigoplus_\lambda E_\lambda$ where $E_\lambda = \ker(f-\lambda)^n$. Since f is an isomorphism $\lambda \neq 0$. Now $(f-\zeta_p\lambda)^n\phi = \zeta_p^n\phi(f-\lambda)^n$ so ϕ maps the eigenspace E_λ to the eigenspace $E_{\zeta_p\lambda}$. We get then a chain of eigenvalues $\lambda, \zeta_p\lambda, \ldots, \zeta_p^{p-1}\lambda$ and, $\zeta_p^p\lambda$ becomes again an eigenvalue. It gives a decomposition of (E,ϕ) in a similar way to a Hodge bundle, except for that the indexing is by a cyclic group,

$$\begin{tikzcd}
& E_{\lambda_2} \arrow[r] & E_{\lambda_3} \arrow[dr] & \\
E_{\lambda_1} \arrow[ur] & & & E_{\lambda_4} \arrow[dll, dotted] \\
& E_{\lambda_{p-1}} \arrow[ul] & &
\end{tikzcd}$$

Such a Higgs bundle is called a *cyclotomic Hodge bundle* by Simpson and we write it as $(E = \oplus_{\lambda \in C_p}, \phi_\lambda)$, where C_p denotes the cyclic group of order p.

We can consider such pairs as forming subvarieties of M. In order to estimate their codimension, we study their space of deformations. The deformation complex (2) gives us the following deformation complex for a cyclotomic Hodge bundle $(E = \oplus_{\lambda \in C_p}, \phi_\lambda)$:

$$C^\bullet_{\lambda_k} : \bigoplus_{\lambda_k = \lambda_j - \lambda_i} \mathrm{Hom}\,(E_{\lambda_i}, E_{\lambda_j}) \longrightarrow \bigoplus_{\lambda_{k+1} = \lambda_j - \lambda_i} \mathrm{Hom}\,(E_{\lambda_i}, E_j).$$

As in Section 5, the tangent space at those points has a decomposition

$$\mathbb{H}^1(C^\bullet(E,\phi)) = \mathbb{H}^1(C^\bullet_{\lambda_k}(E,\phi)).$$

The dimension of the singular locus satisfies

$$\dim(\mathrm{Sing}(\overline{M})) \leq \dim T_{\mathrm{Sing}(\overline{M})} = \dim \mathbb{H}^1(C^\bullet_e),$$

where e is the neutral element in C_p. Therefore the codimension satisfies

$$\mathrm{codim}\,(\mathrm{Sing}(\overline{M}) \geq \dim \mathbb{H}^1(C^\bullet_{\lambda \neq e}).$$

The computations in Proposition 5.2 and Proposition 5.4 hold again for the cyclotomic Hodge bundles. This gives the estimate for the codimension of the singular locus. $\square$

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