

THE HITCHIN–KOBAYASHI CORRESPONDENCE, HIGGS PAIRS AND SURFACE GROUP REPRESENTATIONS

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ABSTRACT. We develop a complete Hitchin–Kobayashi correspondence for twisted pairs on a compact Riemann surface X . The main novelty lies in a careful study of the notion of polystability for pairs, required for having a bijective correspondence between solutions to the Hermite–Einstein equations, on one hand, and polystable pairs, on the other. Our results allow us to establish rigorously the homeomorphism between the moduli space of polystable G -Higgs bundles on X and the character variety for representations of the fundamental group of X in G . We also study in detail several interesting examples of the correspondence for particular groups and show how to significantly simplify the general stability condition in these cases.

1. INTRODUCTION

In this paper we study the Hitchin–Kobayashi correspondence for L -twisted pairs on a compact Riemann surface X . The main motivation for our study comes from non-abelian Hodge theory on X for a real semisimple Lie group G . Our results allow us to establish a one-to-one correspondence between the moduli space of G -Higgs bundles over X and the moduli space of reductive representations of the fundamental group of X in G .

The non-abelian Hodge theory correspondence has two fundamental ingredients: one ingredient is the Theorem of Corlette [7] and Donaldson [8] on the existence of harmonic metrics in flat bundles, and the other grows out of the Hitchin–Kobayashi correspondence between polystable Higgs bundles and solutions to Hitchin’s gauge theoretic equations, established by Hitchin [12] and Simpson [23, 24, 25, 26]. While the Corlette–Donaldson Theorem applies directly in our context, for the Hitchin–Kobayashi we need to work in the general setting of stable pairs treated in [1, 6]. One of the main contributions of the present paper is to establish the extension of this general correspondence to strictly polystable pairs. This is required for having a complete correspondence with solutions to the gauge theoretic equations and is essential for the application of the theory to moduli of representations of surface groups. The other main contribution lies in a careful study of the

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general stability condition in several important special cases. This leads to a simplification of the stability condition which makes it practical for applications of the theory.

We describe now briefly the content of the different sections of the paper.

In order to establish the full Hitchin–Kobayashi correspondence, in Section 2 we review the general theory of L -twisted pairs and the Hitchin–Kobayashi correspondence over a compact Riemann surface X . By an L -twisted pair over X we mean a pair (E, φ) consisting of a holomorphic $H^{\mathbb{C}}$ -principal bundle, where $H^{\mathbb{C}}$ is a complex reductive Lie group and φ is a holomorphic section of $E(B) \otimes L$, where $E(B)$ is the vector bundle associated to a complex representation $H^{\mathbb{C}} \rightarrow \mathrm{GL}(B)$ and L is a holomorphic line bundle over X . We study in full the notion of polystability and prove the correspondence between polystable pairs and solutions to the corresponding Hermite–Einstein equations for a reduction of the structure group of E to H — the maximal compact subgroup of $H^{\mathbb{C}}$. This extends the correspondence for stable pairs of [1, 6] to the strictly polystable case and solves the problem of completely characterizing the pairs which support solutions to the equations. The Hermite–Einstein equations combine the curvature term of the classical Hermite–Einstein equation for polystable vector bundles and a quadratic term on the Higgs field, which can be interpreted as a moment map (see Theorem 2.25). When the general Hermite–Einstein equation is considered for G -Higgs bundles, we call it the Hitchin equation.

In Section 3 we study non-abelian Hodge theory over a compact Riemann surface X for a general connected semisimple Lie group G . Let G be a reductive real Lie group with maximal compact subgroup $H \subset G$, let K be the canonical line bundle over X and let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be the Cartan decomposition of \mathfrak{g} . Then a G -Higgs bundle is a pair (E, φ) , consisting of a holomorphic $H^{\mathbb{C}}$ -principal bundle E over X and a holomorphic section φ of $E(\mathfrak{m}^{\mathbb{C}}) \otimes K$. Here $E(\mathfrak{m}^{\mathbb{C}})$ is the $\mathfrak{m}^{\mathbb{C}}$ -bundle associated to E via the isotropy representation $H^{\mathbb{C}} \rightarrow \mathrm{GL}(\mathfrak{m}^{\mathbb{C}})$. These objects are a particular case of the general twisted pairs introduced in Section 2. We study the deformations and the moduli spaces of G -Higgs bundles. An important result is the correspondence between the moduli space of polystable G -Higgs bundles and the moduli space of solutions to the Hitchin equations. While this is well-known when G is actually complex [12, 23, 24] or compact [16, 17], a proof for the non-compact non-complex case follows from [6] for stable G -Higgs bundles. In this paper, we prove the general case of a polystable G -Higgs bundle. The result (given by Theorem 3.23) is a consequence of the more general Hitchin–Kobayashi correspondence given in Theorem 2.25 of Section 2.11.

We then introduce the moduli space of reductive representations of the fundamental group of a compact Riemann surface X in a Lie group G . By a representation we mean a homomorphism from $\pi_1(X)$ to G , and here reductive means that the composition of the representation with the adjoint representation of G is fully reducible. When G is algebraic this is equivalent to the image of the representation of $\pi_1(X)$ in G to have reductive Zariski closure. Combining Theorem 3.23 with Corlette’s existence theorem of harmonic metrics [7], we establish in Theorem 3.32 the correspondence between this moduli space and the moduli space of polystable G -Higgs bundles when G is connected and semisimple.

In Section 4 we study how the stability condition stated in general in Section 2 simplifies for G -Higgs bundles for various groups. This includes $G = \mathrm{Sp}(2n, \mathbb{R})$ — the group of linear transformations of \mathbb{R}^{2n} which preserve the standard symplectic form — and also other groups that naturally contain $\mathrm{Sp}(2n, \mathbb{R})$, like $\mathrm{Sp}(2n, \mathbb{C})$, and $\mathrm{SL}(2n, \mathbb{C})$, as well as $\mathrm{GL}(n, \mathbb{R})$.

The notion of an L -twisted G -Higgs pair is a slight generalization of that of a G -Higgs bundle, where one allows a general line bundle L to play the role of the canonical bundle in the definition. Some (though not all) of the results of Sections 3 and 4 apply in the setting of L -twisted G -Higgs pairs at no extra cost and in these cases we choose to work in this more general setting.

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2. STABILITY OF TWISTED PAIRS AND HITCHIN-KOBAYASHI CORRESPONDENCE

In this section we introduce a general notion of polystability for pairs of the form (E, ϕ) , where E is a holomorphic principal bundle and ϕ is a section of an associated vector bundle, and we prove a Hitchin-Kobayashi correspondence for polystable pairs. There have appeared in the literature several papers [1, 6, 15, 23] with extensions of the original Hitchin-Kobayashi correspondence due to Uhlenbeck and Yau [28], obtaining different levels of generality. Lest the reader think that we have any pretension of founding a new literary genre on slight variations of the Hitchin-Kobayashi correspondence, we now briefly describe what are the new aspects which we consider, compared to the previous existing papers.

The main novelty of the present paper regarding the Hitchin-Kobayashi correspondence is the introduction and study of a general notion of polystability which is equivalent, without any additional hypothesis, to the existence of solution to the Hermite-Einstein equations corresponding to the type of pair considered. Polystability was of course well understood in the case of vector bundles and some of their generalizations as vortices, triples or Higgs bundles. However, the extensions of the Hitchin-Kobayashi correspondence to general pairs which have appeared so far deal only with stable objects (i.e., those for which the degree inequalities are always strict) satisfying a certain simplicity condition, and in this sense they are unnecessarily restricted, as the intuition obtained from the case of vector bundles suggests.

Roughly speaking, a pair (E, ϕ) is polystable if it is semistable and the structure group of E can be reduced to a smaller subgroup so as to give rise to a stable pair (this corresponds, in the vector bundle case, to the process of looking at a polystable vector bundle as a direct sum of stable vector bundles of the same slope). Our actual definition of polystability (see Subsection 2.7) is not expressed in this way, but rather in terms of reductions of the structure group from parabolic subgroups to their Levi subgroups. The existence of a reduction of the structure group leading to a stable object is proved to be a consequence of polystability in Subsection 2.10. We also prove the uniqueness of such reduction (which we call, following the usual terminology, the Jordan-Hölder reduction).

Strictly polystable vector bundles can be distinguished from stable vector bundles by the fact that their automorphism group contains elements which are not homotheties. In Subsection 2.9 we prove that something similar happens for general pairs. The Hitchin-Kobayashi correspondence for polystable pairs is proved in Subsection 2.11. Our strategy is to reduce the proof to the case of stable pairs, for which we refer to the result in [6]. Finally, we prove in Subsection 2.12 that the automorphism group of a polystable pair is reductive.

This is a consequence of two facts: first, that the group of gauge transformations which preserve a pair (E, ϕ) and the reduction of E solving the Hermite–Einstein equation is compact and, second, that the full group of automorphisms of (E, ϕ) is the complexification of the previous group (this is a general fact, which follows formally from the moment map interpretation of the equations).

We have included in this section some material on parabolic subgroups which is perhaps classical but for which we did not find any reference adapted to our point of view. These results are most of the times only sketched, but we have tried to be careful in setting the notation, so that all the notions which we are using are clearly defined.

2.1. Standard parabolic subgroups. Let H be a compact and connected Lie group and let $H^{\mathbb{C}}$ be its complexification. Parabolic subgroups of $H^{\mathbb{C}}$ can be defined in several different but equivalent ways. Here we list some of them: (1) the subgroups $P \subset H^{\mathbb{C}}$ such that the homogeneous space $H^{\mathbb{C}}/P$ is a projective variety, (2) any subgroup containing a maximal closed and connected solvable subgroup of $H^{\mathbb{C}}$ (i.e., a Borel subgroup), (3) the stabilizers of points at infinity of the visual compactification of the symmetric space $H \backslash H^{\mathbb{C}}$. Here we use a more constructive definition: we first define standard parabolic subgroups with respect to a root space decomposition, and then we define a parabolic subgroup to be any subgroup which is conjugate to a standard parabolic subgroup. The reader meeting this notion for the first time is advised to think as an example on the parabolic subgroups of $\mathrm{GL}(n, \mathbb{C})$, which are simply the stabilizers of any partial flag $0 \subset V_1 \subset \dots \subset V_r \subset \mathbb{C}^n$.

Here is some notation which will be used:

- H – a compact and connected Lie group
- $H^{\mathbb{C}}$ – the complexification of H
- $T \subset H$ – a maximal torus
- \mathfrak{h} – the Lie algebra of H
- $\mathfrak{h}^{\mathbb{C}}$ – the Lie algebra of $H^{\mathbb{C}}$
- $\mathfrak{t} \subset \mathfrak{h}$ – the Lie algebra of T
- $\mathfrak{a} \subset \mathfrak{h}^{\mathbb{C}}$ – the complexification of \mathfrak{t} , $\mathfrak{a} = \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C}$
- $\mathfrak{h}_s^{\mathbb{C}} = [\mathfrak{h}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}]$ – the semisimple part of $\mathfrak{h}^{\mathbb{C}}$
- $\mathfrak{z} \subset \mathfrak{a}$ – the center of $\mathfrak{h}^{\mathbb{C}}$
- $\mathfrak{c} \subset \mathfrak{h}_s^{\mathbb{C}}$ – the Cartan subalgebra of $\mathfrak{h}_s^{\mathbb{C}}$ defined as $\mathfrak{c} = \mathfrak{a} \cap \mathfrak{h}_s^{\mathbb{C}}$
- $\langle \cdot, \cdot \rangle$ – an invariant \mathbb{C} -bilinear pairing on $\mathfrak{h}^{\mathbb{C}}$ extending the Killing form on $\mathfrak{h}_s^{\mathbb{C}}$
- $R \subset \mathfrak{c}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{c}, \mathbb{C})$ – the roots of $\mathfrak{h}_s^{\mathbb{C}}$
- $\mathfrak{h}_{\delta} \subset \mathfrak{h}^{\mathbb{C}}$ – the root space corresponding to $\delta \in R$
- $\Delta \subset R$ – a choice of simple roots.

Using the previous notation we can write the root space decomposition of $\mathfrak{h}^{\mathbb{C}}$ as:

$$\mathfrak{h}^{\mathbb{C}} = \mathfrak{z} \oplus \mathfrak{c} \oplus \bigoplus_{\delta \in R} \mathfrak{h}_{\delta}.$$

For any $A \subset \Delta$ define R_A to be the set of roots of the form $\delta = \sum_{\beta \in \Delta} m_\beta \beta \in R$ with $m_\beta \geq 0$ for all $\beta \in A$ (so if $A = \emptyset$ then $R_A = R$). Then

$$\mathfrak{p}_A = \mathfrak{z} \oplus \mathfrak{c} \oplus \bigoplus_{\delta \in R_A} \mathfrak{h}_\delta$$

is a Lie subalgebra of $\mathfrak{h}^\mathbb{C}$. Denote by $P_A \subset H^\mathbb{C}$ the connected subgroup whose Lie algebra is \mathfrak{p}_A .

Definition 2.1. A **standard parabolic subgroup** of $H^\mathbb{C}$ is any subgroup of the form P_A , for any choice of subset $A \subset R$. A **parabolic subgroup** of $H^\mathbb{C}$ is any subgroup which is conjugate to a standard parabolic subgroup.

Define similarly R_A^0 as the set of roots $\delta = \sum_{\beta \in \Delta} m_\beta \beta$ with $m_\beta = 0$ for all $\beta \in A$. The vector space

$$(2.1) \quad \mathfrak{l}_A = \mathfrak{z} \oplus \mathfrak{c} \oplus \bigoplus_{\delta \in R_A^0} \mathfrak{h}_\delta$$

is a Lie subalgebra of \mathfrak{p}_A . Let L_A be the connected subgroup with Lie algebra \mathfrak{l}_A . Then L_A is a **Levi subgroup** of P_A , i.e., a maximal reductive subgroup of P_A . Finally,

$$(2.2) \quad \mathfrak{u}_A = \bigoplus_{\delta \in R_A \setminus R_A^0} \mathfrak{h}_\delta$$

is also a Lie subalgebra of \mathfrak{p}_A , and the connected Lie group $U_A \subset P_A$ with Lie algebra \mathfrak{u}_A is the unipotent radical of P_A . U_A is a normal subgroup of P_A and the quotient P_A/U_A is naturally isomorphic to L_A so we have

$$(2.3) \quad P_A = L_A U_A.$$

2.2. Antidominant characters of \mathfrak{p}_A . Recall that a character of a complex Lie algebra \mathfrak{g} is a complex linear map $\mathfrak{g} \rightarrow \mathbb{C}$ which factors through the quotient map $\mathfrak{g} \rightarrow \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. Here we classify the characters of parabolic subalgebras $\mathfrak{p}_A \subset \mathfrak{h}^\mathbb{C}$. We will see that all these characters come from elements of the dual of the center of the Levi subgroup $\mathfrak{l}_A \subset \mathfrak{p}_A$. Then we define antidominant characters.

Let Z be the center of $H^\mathbb{C}$, and let

$$\Gamma = \text{Ker}(\exp : \mathfrak{z} \rightarrow Z).$$

Then $\mathfrak{z}_\mathbb{R} = \Gamma \otimes_\mathbb{Z} \mathbb{R} \subset \mathfrak{z}$ is the Lie algebra of the maximal compact subgroup of Z . Let $\mathfrak{z}_\mathbb{R}^* = \text{Hom}_\mathbb{R}(\mathfrak{z}_\mathbb{R}, \mathbb{R})$ and let $\Lambda = \{\lambda \in \mathfrak{z}_\mathbb{R}^* \mid \lambda(\Gamma) \subset 2\pi i\mathbb{Z}\}$. Let $\{\lambda_\delta\}_{\delta \in \Delta} \subset \mathfrak{c}^*$ be the set of fundamental weights of $\mathfrak{h}_s^\mathbb{C}$, i.e., the duals with respect to the Killing form of the coroots $\{2\delta/\langle \delta, \delta \rangle\}_{\delta \in \Delta}$. We extend any $\lambda \in \Lambda$ to a morphism of complex Lie algebras

$$\lambda : \mathfrak{z} \oplus \mathfrak{c} \rightarrow \mathbb{C}$$

by setting $\lambda|_{\mathfrak{c}} = 0$, and similarly for any $\delta \in A$ we extend $\lambda_\delta : \mathfrak{c} \rightarrow \mathbb{C}$ to

$$\lambda_\delta : \mathfrak{z} \oplus \mathfrak{c}_A \rightarrow \mathbb{C}$$

by setting $\lambda_\delta|_{\mathfrak{z}} = 0$.

Lemma 2.2. Define $\mathfrak{z}_A = \bigcap_{\beta \in \Delta \setminus A} \text{Ker } \lambda_\beta$ if $A \neq \Delta$ and let $\mathfrak{z}_A = \mathfrak{c}$ if $A = \Delta$.

- (1) \mathfrak{z}_A is equal to the center of \mathfrak{l}_A ,
- (2) we have $(\mathfrak{p}_A/[\mathfrak{p}_A, \mathfrak{p}_A])^* \simeq \mathfrak{z}_A^*$.

Proof. Both (1) and (2) follow from the fact that for any $\delta, \delta' \in R$ we have $[h_\delta, h_{\delta'}] = h_{\delta+\delta'}$ if $\delta + \delta' \neq 0$ and $[h_\delta, h_{-\delta}] = (\text{Ker } \lambda_\delta)^\perp$ (see Theorem 2 in Chapter VI of [22]). \square

Let $\mathfrak{c}_A = \mathfrak{z}_A \cap \mathfrak{l}_A$, so that $\mathfrak{z}_A = \mathfrak{z} \oplus \mathfrak{c}_A$. By the previous lemma, the characters of \mathfrak{p}_A are in bijection with the elements in $\mathfrak{z}^* \oplus \mathfrak{c}_A^*$.

Definition 2.3. An **antidominant character** of \mathfrak{p}_A is any element of $\mathfrak{z}^* \oplus \mathfrak{c}_A^*$ of the form $\chi = z + \sum_{\delta \in A} n_\delta \lambda_\delta$, where $z \in \mathfrak{z}_\mathbb{R}^*$ and each n_δ is a nonpositive real number. If for each $\delta \in A$ we have $n_\delta < 0$ then we say that χ is **strictly antidominant**.

The restriction of the invariant form $\langle \cdot, \cdot \rangle$ to $\mathfrak{z} \oplus \mathfrak{c}_A$ is non-degenerate, so it induces an isomorphism $\mathfrak{z}^* \oplus \mathfrak{c}_A^* \simeq \mathfrak{z} \oplus \mathfrak{c}_A$. For any antidominant character χ we define $s_\chi \in \mathfrak{z} \oplus \mathfrak{c}_A \subset \mathfrak{z} \oplus \mathfrak{c}$ to be the element corresponding to χ via the previous isomorphism. One checks that s_χ belongs to \mathfrak{ih} .

2.3. Exponentiating characters of \mathfrak{p}_A to characters of P_A . A character of a complex Lie group G is a morphism of Lie groups $G \rightarrow \mathbb{C}^*$. Any character of G induces a character of \mathfrak{g} . When a character of \mathfrak{g} comes from a character of G then we say that it exponentiates. In general there are (many) characters of \mathfrak{g} which do not exponentiate, but here we prove that the set characters of \mathfrak{p}_A which exponentiate generate (as a subset of a vector space) the space of all characters of \mathfrak{p}_A . This will be used to give an algebraic definition of the degree of parabolic reductions in Subsection 2.6.

Let Z_A be the identity component of the center of L_A , and let L_A^{ss} be the connected subgroup of L_A whose Lie algebra is $[\mathfrak{l}_A, \mathfrak{l}_A]$. Then L_A^{ss} is semisimple. Define

$$Z^{ss}(L_A) := Z_A \cap L_A^{ss}.$$

The group $Z^{ss}(L_A)$ is a subgroup of the center of L_A^{ss} . The center of a semisimple group over \mathbb{C} is finite, because it coincides with the center of any of its maximal compact subgroups. Hence $Z^{ss}(L_A)$ is finite. The product map $Z_A \times L_A^{ss} \rightarrow L_A$ induces an isomorphism $L_A \simeq Z_A \times_{Z^{ss}(L_A)} L_A^{ss}$, and projection to the first factor gives a map $L_A \rightarrow Z_A/Z^{ss}(L_A)$. Composing this projection with the quotient map $P_A \rightarrow P_A/U_A \simeq L_A$ we obtain a morphism of Lie groups

$$\pi_A : P_A \rightarrow Z_A/Z^{ss}(L_A).$$

In the following lemma we use the fact that $Z^{ss}(L_A)$ is finite.

Lemma 2.4. *There exists some positive integer n (depending on the fundamental group of L_A) such that for any $\lambda \in \Lambda$ and any $\delta \in A$ the morphisms of Lie algebras $n\lambda : \mathfrak{z} \oplus \mathfrak{c}_A \rightarrow \mathbb{C}$ and $n\lambda_\delta : \mathfrak{z} \oplus \mathfrak{c}_A \rightarrow \mathbb{C}$ exponentiate to morphisms of Lie groups*

$$\exp(n\lambda) : Z_A/Z^{ss}(L_A) \rightarrow \mathbb{C}^\times, \quad \exp(n\lambda_\delta) : Z_A/Z^{ss}(L_A) \rightarrow \mathbb{C}^\times.$$

Composing the morphisms given by the previous lemma with the morphism π_A we get for any $\lambda \in \Lambda$ and $\delta \in A$ morphisms of Lie groups

$$\kappa_{n\lambda} : P_A \rightarrow \mathbb{C}^\times, \quad \kappa_{n\delta} : P_A \rightarrow \mathbb{C}^\times.$$

2.4. Recovering a parabolic subgroup from its antidominant characters.

Lemma 2.5. *Let $s \in \mathfrak{ih}$ and define the sets*

$$\begin{aligned}\mathfrak{p}_s &:= \{x \in \mathfrak{h}^\mathbb{C} \mid \text{Ad}(e^{ts})(x) \text{ is bounded as } t \rightarrow \infty\} \subset \mathfrak{h}^\mathbb{C}, \\ \mathfrak{l}_s &:= \{x \in \mathfrak{h}^\mathbb{C} \mid [x, s] = 0\} \subset \mathfrak{h}^\mathbb{C}, \\ P_s &:= \{g \in H^\mathbb{C} \mid e^{ts}ge^{-ts} \text{ is bounded as } t \rightarrow \infty\} \subset H^\mathbb{C}, \\ L_s &:= \{g \in H^\mathbb{C} \mid \text{Ad}(g)(s) = s\} \subset H^\mathbb{C}.\end{aligned}$$

The following properties hold:

- (1) *Both \mathfrak{p}_s and \mathfrak{l}_s are Lie subalgebras of $\mathfrak{h}^\mathbb{C}$ and P_s and L_s are subgroups of $H^\mathbb{C}$. Furthermore P_s and L_s are connected.*
- (2) *Let χ be an antidominant character of P_A . There are inclusions $\mathfrak{p}_A \subset \mathfrak{p}_{s_\chi}$, $\mathfrak{l}_A \subset \mathfrak{l}_{s_\chi}$, $P_A \subset P_{s_\chi}$ and $L_A \subset L_{s_\chi}$, with equality if χ is strictly antidominant.*
- (3) *For any $s \in \mathfrak{ih}$ there exists $h \in H$ and a standard parabolic subgroup P_A such that $P_s = hP_Ah^{-1}$ and $L_s = hL_Ah^{-1}$. Furthermore, there is an antidominant character χ of P_A such that $s = hs_\chi h^{-1}$.*

Proof. That $\mathfrak{l}_s, \mathfrak{p}_s$ are subalgebras and L_s, P_s are subgroups is immediate from the definitions. Let T_s be the closure of $\{e^{its} \mid t \in \mathbb{R}\}$. Then L_s is the centralizer of the torus T_s in $H^\mathbb{C}$, so by Theorem 13.2 in [3] L_s is connected. To prove that P_s is also connected, note that if g belongs to P_s , so that $e^{ts}ge^{-ts}$ is bounded as $t \rightarrow \infty$, then the limit of $\pi_s(g) := e^{ts}ge^{-ts}$ as $t \rightarrow \infty$ exists and belongs to L_s . Note by the way that the resulting map $\pi_s : P_s \rightarrow L_s$ is a morphism of Lie groups which can be identified with the projection $P_s \rightarrow P_s/U_s \simeq L_s$, where

$$U_s = \{g \in H^\mathbb{C} \mid e^{ts}ge^{-ts} \text{ converges to } 1 \text{ as } t \rightarrow \infty\} \subset P_s$$

is the unipotent radical of U_s . So if $g \in P_s$ then the map $\gamma : [0, \infty) \rightarrow H^\mathbb{C}$ defined as $\gamma(t) = e^{ts}ge^{-ts}$ extends to give a path from g to L_s , and since L_s is connected it follows that P_s is also connected. This proves (1). Let now $\chi = z + \sum_{\beta \in \Delta} n_\beta \lambda_\beta$ be an antidominant character of P_A . Let $\delta = \sum_{\beta \in \Delta} m_\beta \beta$ be a root and let $u \in \mathfrak{h}_\delta$. We have $[s_\chi, u] = \langle s_\chi, \delta \rangle u = \langle \chi, \delta \rangle u = (\sum_{\beta \in \Delta} m_\beta n_\beta \langle \beta, \beta \rangle / 2) u$. Hence $\text{Ad}(e^{ts_\chi})(u) = (\sum_{\beta \in \Delta} \exp(tn_\beta m_\beta \langle \beta, \beta \rangle / 2)) u$, so this remains bounded as $t \rightarrow \infty$ if $m_\beta \geq 0$ for any β such that $n_\beta \leq 0$. This implies that $\mathfrak{p}_A \subset \mathfrak{p}_s$ and $\mathfrak{l}_A \subset \mathfrak{l}_s$ and that the inclusions are equalities when χ is strictly dominant. The analogous statements for P_A, L_A, P_s, L_s follow from this, because the subgroups P_A, L_A, P_s, L_s are connected. Hence (2) is proved. To prove (3) take a maximal torus T_s containing $\{e^{its} \mid t \in \mathbb{R}\}$ and choose $h \in H$ such that $h^{-1}T_s h = T$ and $\text{Ad}(h^{-1})(s)$ belongs to the Weyl chamber in \mathfrak{t} corresponding to the choice of $\Delta \subset R$. Then use (2). \square

Lemma 2.6. *Let $P \subset H^\mathbb{C}$ be any parabolic subgroup, conjugate to P_A . Let χ be an antidominant character of \mathfrak{p}_A . There exists an element $s_{P,\chi} \in \mathfrak{ih}$, depending smoothly on P , which is conjugate to s_χ and such that $P \subset P_{s_{P,\chi}}$, with equality if and only if χ is strictly antidominant.*

Proof. Assume that $P = gP_Ag^{-1}$ for some $g \in H^\mathbb{C}$. From the well known equality $H^\mathbb{C}/P_A = H/(P_A \cap H) = H/(L_A \cap H)$ we deduce that there exists some $h \in H$ such that $P = hP_Ah^{-1}$. Then we set $s_{P,\chi} = hs_\chi h^{-1}$. This is well defined because h is unique up to multiplication on the right by elements of $L_A \cap H$, and these elements commute with s_χ . \square

2.5. Principal bundles and parabolic subgroups. If E is a $H^\mathbb{C}$ -principal holomorphic bundle over X and M is any set on which $H^\mathbb{C}$ acts on the left, we denote by $E(M)$ the twisted product $E \times_{H^\mathbb{C}} M$, defined as the quotient of $E \times M$ by the equivalence relation $(eh, m) \sim (e, hm)$ for any $e \in E$, $h \in H^\mathbb{C}$ and $m \in M$. The sections φ of $E(M)$ are in natural bijection with the maps $\phi : E \rightarrow M$ satisfying $\varphi(eh) = h^{-1}\varphi(e)$ for any $e \in E$ and $h \in H^\mathbb{C}$ (we call such maps antiequivariant). Furthermore, ϕ is holomorphic if and only if φ is holomorphic.

If M is a vector space (resp. complex variety) and the action of $H^\mathbb{C}$ on M is linear (resp. holomorphic) then $E(M)$ is a vector bundle (resp. holomorphic fibration). In this situation, for any complex line bundle $L \rightarrow X$ we can form a vector bundle $E(M) \otimes L$ which can be identified with $E^L(M)$, where E^L denotes the principal $H^\mathbb{C} \times \mathbb{C}^\times$ bundle $E^L = \{(e, l) \in E \times_X L \mid l \neq 0\}$ and we form the associated product by making $(h, \lambda) \in H^\mathbb{C} \times \mathbb{C}^\times$ act on $m \in M$ as λhm . Consequently, the sections of $E(M) \otimes L$ can be identified with antiequivariant maps $E^L \rightarrow M$.

Let B be a Hermitian vector space and let $\rho : H \rightarrow \mathrm{U}(B)$ be a unitary representation. The morphism ρ extends to a holomorphic representation of $H^\mathbb{C}$ in $\mathrm{GL}(B)$, which we denote also by ρ . Suppose that $P_A \subset H^\mathbb{C}$ is the parabolic subgroup corresponding to a subset $A \subset \Delta$ and let χ be an antidominant character. Define

$$B_\chi^- = \{v \in B \mid \rho(e^{ts_\chi})v \text{ remains bounded as } \mathbb{R} \ni t \rightarrow \infty\}.$$

This is a complex vector subspace of B and by (2) in Lemma 2.5 it is invariant under the action of P_A . Define also

$$B_\chi^0 = \{v \in B \mid \rho(e^{ts_\chi})v = v \text{ for any } t\} \subset B_\chi^-.$$

This is a complex subspace of B_χ^- and, using again (2) in Lemma 2.5, we deduce that B_χ^0 is invariant under the action of L_A .

Suppose that σ is a holomorphic section of $E(H^\mathbb{C}/P_A)$. Since $E(H^\mathbb{C}/P_A) \simeq E/P_A$ canonically and the quotient $E \rightarrow E/P_A$ has the structure of a P_A -principal bundle, the pullback $E_\sigma := \sigma^*E$ is a P_A -principal bundle over X , and we can identify canonically $E \simeq E_\sigma \times_{P_A} H^\mathbb{C}$ as principal $H^\mathbb{C}$ -bundles (hence, σ gives a reduction of the structure group of E to P_A). Equivalently, we can look at E_σ as a holomorphic subvariety $E_\sigma \subset E$ invariant under the action of $P_A \subset H^\mathbb{C}$ and inheriting a structure of principal bundle. It follows that $E(B) \simeq E_\sigma \times_{P_A} B$, so the vector bundle $E_\sigma \times_{P_A} B_\chi^-$ can be identified with a holomorphic subbundle

$$E(B)_{\sigma, \chi}^- \subset E(B).$$

Now suppose that σ_L is a holomorphic section of $E_\sigma(P_A/L_A)$. This section induces, exactly as before, a reduction of the structure group of E_σ from P_A to L_A . So we obtain from σ_L a principal L_A bundle E_{σ_L} and an isomorphism $E_\sigma \simeq E_{\sigma_L} \times_{L_A} P_A$. Hence $E(B) \simeq E_{\sigma_L} \times_{L_A} B$, and we can thus identify the vector bundle $E_{\sigma_L} \times_{L_A} B_\chi^0$ with a holomorphic subbundle

$$E(B)_{\sigma_L, \chi}^0 \subset E(B)_{\sigma, \chi}^-.$$

2.6. Degree of a reduction and an antidominant character. Let σ denote a reduction of the structure group of E to a standard parabolic subgroup P_A and let χ be an antidominant character of \mathfrak{p}_A . Let us write $\chi = z + \sum_{\delta \in A} n_\delta \lambda_\delta$, with $z \in \mathfrak{z}_\mathbb{R}^*$, and $z = z_1 \lambda_1 + \cdots + z_r \lambda_r$, where $\lambda_1, \dots, \lambda_r \in \Lambda$ and the z_j are real numbers. Let n be an

integer as given by Lemma 2.4. Using the characters $\kappa_{n\lambda}, \kappa_{n\delta} : P_A \rightarrow \mathbb{C}^\times$ defined in Subsection 2.3 we can construct from the principal P_A bundle E_σ line bundles $E_\sigma \times_{\kappa_{n\lambda}} \mathbb{C}$ and $E_\sigma \times_{\kappa_{n\delta}} \mathbb{C}$.

Definition 2.7. We define the **degree** of the bundle E with respect to the reduction σ and the antidominant character χ to be the real number:

$$(2.4) \quad \deg(E)(\sigma, \chi) := \frac{1}{n} \left(\sum_j z_j \deg(E_\sigma \times_{\kappa_{n\lambda_j}} \mathbb{C}) + \sum_{\delta \in A} n_\delta \deg(E_\sigma \times_{\kappa_{n\delta}} \mathbb{C}) \right).$$

This expression is independent of the choice of the λ_j 's and the integer n .

We now give another definition of the degree in terms of the curvature of connections, in the spirit of Chern–Weil theory. This definition is shorter and more natural from the point of view of proving the Hitchin–Kobayashi correspondence (but, as we said, in this paper we do not give a complete proof of it: we just reduce our general result to the one obtained in [6] for simple stable pairs; this is why the reader will not find any use of the following formula in the present paper). On the other hand, the definition in terms of Chern–Weil theory uses obviously transcendental methods, so it is not satisfying from the point of view of obtaining a polystability condition of purely algebraic nature.

Define $H_A = H \cap L_A$ and $\mathfrak{h}_A = \mathfrak{h} \cap \mathfrak{l}_A$. Then H_A is a maximal compact subgroup of L_A , so the inclusions $H_A \subset L_A$ is a homotopy equivalence. Since the inclusion $L_A \subset P_A$ is also a homotopy equivalence, given a reduction σ of the structure group of E from $H^\mathbb{C}$ to P_A one can further restrict the structure group of E to H_A in a unique way up to homotopy. Denote by E'_σ the resulting H_A principal bundle. Let $\pi_A : \mathfrak{p}_A \rightarrow \mathfrak{z} \oplus \mathfrak{c}_A$ be the differential of the projection π_A defined in Subsection 2.3. Let $\chi = z + \sum_{\delta \in A} n_\delta \lambda_\delta$ be an antidominant character. Define $\kappa_\chi = (z + \sum_{\delta} n_\delta \lambda_\delta) \circ \pi_A \in \mathfrak{p}_A^*$. Let $\mathfrak{h}_A \subset \mathfrak{l}_A \subset \mathfrak{p}_A$ be the Lie algebra of H_A . Then $\kappa_\chi(\mathfrak{h}_A) \subset \mathfrak{i}\mathbb{R}$. Choose a connection \mathbb{A} on E'_σ and denote by $F_\mathbb{A} \in \Omega^2(X, E'_\sigma \times_{\text{Ad}} \mathfrak{h}_A)$ its curvature. Then $\kappa_\chi(F_\mathbb{A})$ is a 2-form on X with values in $\mathfrak{i}\mathbb{R}$, and we have

$$\deg(E)(\sigma, \chi) := \frac{\mathbf{i}}{2\pi} \int_X \kappa_\chi(F_\mathbb{A}).$$

2.7. L -twisted pairs and stability. Let X be a closed Riemann surface and let L be a holomorphic line bundle over X . Let $H^\mathbb{C}$ be a connected complex reductive Lie group and let $\rho : H^\mathbb{C} \rightarrow \text{GL}(B)$ be a representation.

Definition 2.8. An **L -twisted pair** is a pair of the form (E, φ) , where E is a holomorphic $H^\mathbb{C}$ -principal bundle over X and φ is a holomorphic section of $E(B) \otimes L$. When it does not lead to confusion we say that (E, φ) is a pair, instead of an L -twisted pair.

Definition 2.9. Let (E, φ) be an L -twisted pair and let $\alpha \in \mathfrak{i}\mathfrak{z}_\mathbb{R} \subset \mathfrak{z}$. We say that (E, φ) is:

- **α -semistable** if: for any parabolic subgroup $P_A \subset H^\mathbb{C}$, any antidominant character χ of \mathfrak{p}_A , and any holomorphic section $\sigma \in \Gamma(E(H^\mathbb{C}/P_A))$ such that $\varphi \in H^0(E(B)_{\sigma, \chi}^- \otimes L)$, we have

$$\deg(E)(\sigma, \chi) - \langle \alpha, \chi \rangle \geq 0.$$

- **α -stable** if it is α -semistable and furthermore: for any P_A, χ and σ as above, such that $\varphi \in H^0(E(B)_{\sigma, \chi}^- \otimes L)$ and such that $A \neq \emptyset$ and $\chi \notin \mathfrak{z}_\mathbb{R}^*$, we have

$$\deg(E)(\sigma, \chi) - \langle \alpha, \chi \rangle > 0.$$

- **α -polystable** if it is α -semistable and for any P_A , χ and σ as above, such that $\varphi \in H^0(E(B)_{\sigma, \chi}^- \otimes L)$, $P_A \neq H^\mathbb{C}$ and χ is strictly antidominant, and such that

$$\deg(E)(\sigma, \chi) - \langle \alpha, \chi \rangle = 0,$$

there is a holomorphic reduction of the structure group $\sigma_L \in \Gamma(E_\sigma(P_A/L_A))$, where E_σ denotes the principal P_A -bundle obtained from the reduction σ of the structure group. Furthermore, under these hypothesis φ is required to belong to $H^0(E(B)_{\sigma_L, \chi}^0 \otimes L) \subset H^0(E(B)_{\sigma, \chi}^- \otimes L)$.

Remark 2.10. For some instances of group $H^\mathbb{C}$ and representation $H^\mathbb{C} \rightarrow \mathrm{GL}(B)$ the last condition in the definition of polystability is redundant (for example, $H^\mathbb{C} = \mathrm{GL}(n, \mathbb{C})$ with its fundamental representation on \mathbb{C}^n). This does not seem to be general fact, but we do not have any example which illustrates that the condition $\varphi \in H^0(E(B)_{\sigma_L, \chi}^0 \otimes L)$ is not a consequence of the α -semistability of (E, φ) and the existence of σ_L whenever $\deg(E)(\sigma, \chi) = \langle \alpha, \chi \rangle$ and $\varphi \in H^0(E(B)_{\sigma, \chi}^- \otimes L)$.

Remark 2.11. If we had stated the previous conditions considering reductions to arbitrary parabolic subgroups of $H^\mathbb{C}$ then we would have obtained the same definitions. Indeed, since any parabolic subgroup is (for us, by definition) conjugate to a standard parabolic subgroup, the reductions of the structure group of E to arbitrary parabolic subgroups are essentially the same as the reductions to standard parabolic subgroups.

Remark 2.12. The readers who are familiar with the stability condition for principal bundles as studied by Ramanathan [17] might find it surprising that our stability condition refers to antidominant characters of the parabolic Lie subalgebra and not only to characters of the parabolic subgroups (there are much less of the latter than of the former). The reason is that in the course of proving the Hitchin–Kobayashi correspondence one is naturally led to consider arbitrary antidominant characters of Lie subalgebras. It might be the case that the previous conditions do not vary if we only consider characters of the parabolic subgroups, but this is not at all obvious. We hope to come back to this question in the future.

2.8. The stability condition in terms of filtrations. In order to obtain a workable notion of α -(poly,semi)stability it is desirable to have a more concrete way to describe, for any holomorphic $H^\mathbb{C}$ -principal bundle E ,

- the reductions of the structure group of E to parabolic subgroups $P \subset H^\mathbb{C}$, and the (strictly or not) antidominant characters of P ,
- the subbundle $E(B)_{\sigma, \chi}^- \subset E(B)$,
- the degree $\deg(E)(\sigma, \chi)$ defined in (2.4),
- reductions to Levi factors of parabolic subgroups and the corresponding vector bundle $E(B)_{\sigma_L, \chi}^0 \subset E(B)_{\sigma, \chi}^-$.

We now discuss how to obtain in some cases such concrete descriptions, beginning with the notion of degree. In [6] the degree $\deg(E)(\sigma, \chi)$ is defined in terms of a so-called auxiliary representation (see §2.1.2 in [6]) and certain linear combinations of degrees of subbundles. The following lemma implies that definition (2.4) contains the one given in [6] as a particular case. Suppose that $\rho_W : H \rightarrow \mathrm{U}(W)$ is a representation on a Hermitian vector space, and denote the holomorphic extension $H^\mathbb{C} \rightarrow \mathrm{GL}(W)$ with the same symbol ρ_W . Let $(\mathrm{Ker} \rho_W)^\perp \subset \mathfrak{h}^\mathbb{C}$ be the orthogonal with respect to invariant pairing on $\mathfrak{h}^\mathbb{C}$ of the kernel of $\rho_W : \mathfrak{h}^\mathbb{C} \rightarrow \mathfrak{gl}(W)$, and let $\pi : \mathfrak{h}^\mathbb{C} \rightarrow (\mathrm{Ker} \rho_W)^\perp$ be the orthogonal projection.

Lemma 2.13. *Take some element $s \in \mathfrak{ih}$. Then $\rho_W(s)$ diagonalizes with real eigenvalues $\lambda_1 < \dots < \lambda_k$. Let $W_j = \text{Ker}(\lambda_j \text{Id}_W - \rho_W(s))$ and define $W_{\leq i} = \bigoplus_{j \leq i} W_j$.*

- (1) *The subgroup $P_{W,s} \subset H^\mathbb{C}$ consisting of those g such that $\rho_W(g)(W_{\leq i}) \subset W_{\leq i}$ for any i is a parabolic subgroup, which can be identified with $P_{\pi(s)}$. Let $\chi \in (\mathfrak{z} \oplus \mathfrak{c})^*$ be a character such that $s_\chi = s$. Then χ is strictly antidominant for $P_{W,s}$.*
- (2) *Suppose that for any $a, b \in (\text{Ker } \rho_W)^\perp$ we have $\langle a, b \rangle = \text{Tr } \rho_W(a) \rho_W(b)$. Let $u \in (\text{Ker } \rho_W)^\perp$ be any element, and write $\rho_W(u) = \sum \rho_W(u)_{ij}$ the decomposition in pieces $\rho_W(u)_{ij} \in \text{Hom}(W_i, W_j)$. Then*

$$(2.5) \quad \langle \chi, u \rangle = \text{Tr}(\rho_W(s) \rho_W(u)) = \lambda_k \text{Tr } \rho_W(u) + \sum_{i=1}^{k-1} (\lambda_i - \lambda_{i+1}) \text{Tr } \rho_W(u)_{ii}.$$

- (3) *Suppose that ρ_W satisfies the conditions of (2). Let E be a holomorphic $H^\mathbb{C}$ -principal bundle and let $\mathcal{W} = E(W)$ be the associated holomorphic vector bundle. Let σ be a reduction of the structure group of E to a parabolic subgroup P and let χ be an antidominant character of P . The endomorphism $\rho_W(s_\chi)$ diagonalizes with constant eigenvalues, giving rise to a decomposition $\mathcal{W} = \bigoplus_{j=1}^k \mathcal{W}_j$, where $\rho_W(s_\chi)$ restricted to \mathcal{W}_j is multiplication by $\lambda_j \in \mathbb{R}$. Suppose that $\lambda_1 < \dots < \lambda_k$. For each i the subbundle $\mathcal{W}_{\leq i} = \bigoplus_{j \leq i} \mathcal{W}_j \subset \mathcal{W}$ is holomorphic. We have:*

$$\deg(E)(\sigma, \chi) = \lambda_k \deg \mathcal{W} + \sum_{i=1}^{k-1} (\lambda_i - \lambda_{i+1}) \deg \mathcal{W}_{\leq i}.$$

Proof. The first assertion and formula (2.5) follows from easy computations. (3) follows from (2). \square

Remark 2.14. Condition (2) of the lemma is satisfied when $W = \mathfrak{h}$, endowed with the invariant metric, and $\rho_W : \mathfrak{h}^\mathbb{C} \rightarrow \text{End } W$ is the adjoint representation, since the invariant metric on \mathfrak{h} is supposed to extend the Killing pairing in the semisimple part \mathfrak{h}_s .

To clarify the other ingredients in the definition of (poly,semi)stability, we put ourselves in the situation where $H^\mathbb{C}$ is a classical group. Let $\rho : H^\mathbb{C} \rightarrow \text{GL}(N, \mathbb{C})$ be the fundamental representation. Suppose that E is an $H^\mathbb{C}$ -principal bundle, and denote by V the vector bundle associated to E and ρ . One can describe pairs (σ, χ) consisting of a reduction σ of the structure group of E to a parabolic subgroup $P \subset H^\mathbb{C}$ and an antidominant character χ of P in terms of filtrations of vector bundles

$$(2.6) \quad \mathcal{V} = (0 \subsetneq V_1 \subsetneq \dots \subsetneq V_{k-1} \subsetneq V_k = V),$$

and increasing sequences of real numbers (usually called weights)

$$(2.7) \quad \lambda_1 \leq \dots \leq \lambda_k,$$

which are arbitrary if $H^\mathbb{C} = \text{GL}(n, \mathbb{C})$, and which satisfy otherwise:

- if $H^\mathbb{C} = \text{O}(n, \mathbb{C})$ then, for any i , $V_{k-i} = V_i^\perp = \{v \in V \mid \langle v, V_i \rangle = 0\}$, where \langle, \rangle denotes the bilinear pairing given by the orthogonal structure (we implicitly define $V_0 = 0$), and $\lambda_{k-i+1} + \lambda_i = 0$.
- if $H^\mathbb{C} = \text{Sp}(2n, \mathbb{C})$ then, for any i , $V_{k-i} = V_i^\perp = \{v \in V \mid \omega(v, V_i) = 0\}$, where ω is the symplectic form on V (as before, $V_0 = 0$), and furthermore $\lambda_{k-i+1} + \lambda_i = 0$.

The resulting character χ is strictly antidominant if all the inequalities in (2.7) are strict.

Given positive integers p, q define the vector bundle $V^{p,q} = V^{\otimes p} \otimes (V^*)^{\otimes q}$. For any choice of reduction and antidominant character (σ, χ) specified by a filtration (2.6) and weights (2.7) we define

$$(V^{p,q})_{\sigma,\chi}^- = \sum_{\lambda_{i_1} + \dots + \lambda_{i_p} \leq \lambda_{j_1} + \dots + \lambda_{j_q}} V_{i_1} \otimes \dots \otimes V_{i_p} \otimes V_{j_1}^\perp \otimes \dots \otimes V_{j_q}^\perp \subset V^{p,q},$$

where $V_j^\perp = \{v \in V^* \mid \langle v, V_j \rangle = 0\}$ and \langle, \rangle is the natural pairing between V and V^* . Since $H^\mathbb{C}$ is a classical group, there is an inclusion of representations

$$B \subset (\rho^{\otimes p_1} \otimes (\rho^*)^{\otimes q_1}) \oplus \dots \oplus (\rho^{\otimes p_r} \otimes (\rho^*)^{\otimes q_r}),$$

so that the vector bundle $E(B)$ is contained in $V^{p_1, q_1} \oplus \dots \oplus V^{p_r, q_r}$. One then has

$$E(B)_{\sigma,\chi}^- = E(B) \cap ((V^{p_1, q_1})_{\sigma,\chi}^- \oplus \dots \oplus (V^{p_r, q_r})_{\sigma,\chi}^-).$$

Suppose that the invariant pairing \langle, \rangle on the Lie algebra $\mathfrak{h}^\mathbb{C}$ is defined using the fundamental representation as $\langle x, y \rangle = \text{Tr } \rho(x)\rho(y)$. This clearly satisfies the condition of (2) of Lemma 2.13, so by (3) in the same lemma we have

$$\deg(E)(\sigma, \chi) = \lambda_k \deg V + \sum_{i=1}^{k-1} (\lambda_i - \lambda_{i+1}) \deg V_i.$$

We now specify what it means to have a reduction to a Levi factor of a parabolic subgroup, as appears in the definition of polystability. Assume that (σ, χ) is a pair specified by (2.6) and (2.7), so that σ defines a reduction of the structure group of E to a parabolic subgroup $P \subset H^\mathbb{C}$, and that $\varphi \in H^0(L \otimes E(B)_{\sigma,\chi}^-)$ and $\deg(E)(\sigma, \chi) = 0$. If the pair (E, φ) is α -polystable all these assumptions imply the existence of a further reduction σ_L of the structure group of $H^\mathbb{C}$ from P to a Levi factor $L \subset P$; this is given explicitly by an isomorphism of vector bundles

$$V \simeq \text{Gr } \mathcal{V} := V_1 \oplus V_2/V_1 \oplus \dots \oplus V_k/V_{k-1}.$$

When $H^\mathbb{C} = \text{GL}(n, \mathbb{C})$ such isomorphism is arbitrary. When $H^\mathbb{C}$ is $\text{O}(n, \mathbb{C})$ (resp. $\text{Sp}(2n, \mathbb{C})$), it is also assumed that the pairing of an element of V_j/V_{j-1} with an element of V_i/V_{i-1} , using the scalar product (resp. symplectic form), is always zero unless $j + i = k + 1$. We finally describe the bundle $E(B)_{\sigma_L, \chi}^0$ in this situation. Let

$$(\text{Gr } \mathcal{V}^{p,q})_{\sigma_L, \chi}^0 = \sum_{\lambda_{i_1} + \dots + \lambda_{i_p} = \lambda_{j_1} + \dots + \lambda_{j_q}} (V_{i_1}/V_{i_1-1}) \otimes \dots \otimes (V_{i_p}/V_{i_p-1}) \otimes (V_{j_1}^\perp/V_{j_1+1}^\perp) \otimes \dots \otimes (V_{j_q}^\perp/V_{j_q+1}^\perp).$$

Then

$$E(B)_{\sigma_L, \chi}^0 = E(B) \cap ((\text{Gr } \mathcal{V}^{p_1, q_1})_{\sigma_L, \chi}^0 \oplus \dots \oplus (\text{Gr } \mathcal{V}^{p_r, q_r})_{\sigma_L, \chi}^0).$$

2.9. Infinitesimal automorphism space. For any pair (E, φ) we define the infinitesimal automorphism space of (E, φ) as

$$\text{aut}(E, \varphi) = \{s \in H^0(E(\mathfrak{h}^\mathbb{C})) \mid \rho(s)(\varphi) = 0\},$$

where we denote by $\rho : \mathfrak{h}^\mathbb{C} \rightarrow \text{End}(B)$ the morphism of Lie algebras induced by ρ . We similarly define the semisimple infinitesimal automorphism space of (E, φ) as

$$\text{aut}^{ss}(E, \varphi) = \{s \in \text{aut}(E, \varphi) \mid s(x) \text{ is semisimple for any } x \in X\}.$$

Proposition 2.15. *Suppose that (E, φ) is a α -polystable pair. Then (E, φ) is α -stable if and only if $\text{aut}^{ss}(E, \varphi) \subset H^0(E(\mathfrak{z}))$. Furthermore, if (E, φ) is α -stable then we also have $\text{aut}(E, \varphi) \subset H^0(E(\mathfrak{z}))$.*

Proof. Suppose that (E, φ) is α -polystable and that $\text{aut}^{ss}(E, \varphi) = H^0(E(\mathfrak{z}))$. We prove that (E, φ) is α -stable by contradiction. If (E, φ) were not α -stable, then there would exist a parabolic subgroup $P_A \subsetneq H^\mathbb{C}$, a holomorphic reduction $\sigma \in \Gamma(E/P_A)$, a strictly antidominant character χ such that $\deg(E)(\sigma, \chi) - \langle \alpha, \chi \rangle = 0$, and a further holomorphic reduction $\sigma_L \in \Gamma(E_\sigma/L_A)$ to the Levi L_A (here E_σ is the principal P_A bundle given by σ , satisfying $E_\sigma \times_{P_A} H^\mathbb{C} \simeq E$) such that $\varphi \in H^0(E(B)_{\sigma_L, \chi}^0 \otimes L)$. Since the adjoint action of L_A on $\mathfrak{h}^\mathbb{C}$ fixes s_χ , there is an element

$$s_{\sigma, \chi} \in H^0(E_{\sigma_L}(\mathfrak{h}^\mathbb{C})) \simeq H^0(E(\mathfrak{h}^\mathbb{C}))$$

which coincides fiberwise with s_χ . On the other hand s_χ is semisimple because it belongs to \mathfrak{ih} . The condition that $\varphi \in H^0(E(B)_{\sigma_L, \chi}^0 \otimes L)$ implies that $\rho(s_{\sigma, \chi})(\varphi) = 0$, so $s_{\sigma, \chi} \in \text{aut}^{ss}(E, \varphi)$. And the condition that $P_A \neq H^\mathbb{C}$ implies that $s_\chi \notin \mathfrak{z}$. This contradicts the assumption that $\text{aut}^{ss}(E, \varphi) = H^0(E(\mathfrak{z}))$, so (E, φ) is α -stable.

Now suppose that (E, φ) is α -stable. We want to prove that $\text{aut}(E, \varphi) = H^0(E(\mathfrak{z}))$. Let $\xi \in \text{aut}(E, \varphi)$. Since ξ is a section of $E \times_{H^\mathbb{C}} \mathfrak{h}^\mathbb{C}$, it can be viewed as an antiequivariant holomorphic map $\psi : E \rightarrow \mathfrak{h}^\mathbb{C}$. The bundle E is algebraic (to prove this, take a faithful representation $H^\mathbb{C} \rightarrow \text{GL}(n, \mathbb{C})$ and use the fact that any holomorphic vector bundle over an algebraic curve is algebraic), so by Chow's theorem ψ is algebraic. Hence ψ induces an algebraic map $\varphi : X \rightarrow \mathfrak{h}^\mathbb{C} // H^\mathbb{C}$, where $\mathfrak{h}^\mathbb{C} // H^\mathbb{C}$ denotes the affine quotient, which is an affine variety. Since X is proper, φ is constant, hence it is contained in a unique fiber $Y := \pi^{-1}(y) \subset \mathfrak{h}^\mathbb{C}$, where $\pi : \mathfrak{h}^\mathbb{C} \rightarrow \mathfrak{h}^\mathbb{C} // H^\mathbb{C}$ is the quotient map.

By a standard results on affine quotients, there is a unique closed $H^\mathbb{C}$ orbit $\mathcal{O} \subset Y$, and by a theorem of Richardson the elements in \mathcal{O} are all semisimple.

Consider the map $\sigma : Y \rightarrow \mathcal{O}$ which sends any $y \in Y$ to y_s , where $y = y_s + y_n$ is the Jordan decomposition of y (see for example [4]). We claim that this map is algebraic (note that the Jordan decomposition, when defined on the whole Lie algebra $\mathfrak{h}^\mathbb{C}$, is not even continuous). To prove the claim first consider the case $\mathfrak{h}^\mathbb{C} = \mathfrak{gl}(n, \mathbb{C})$. Then $Y \subset \mathfrak{gl}(n, \mathbb{C})$ is the set of $n \times n$ matrices with characteristic polynomial equal to some fixed polynomial, say $\prod (x - \lambda_i)^{m_i}$, with $\lambda_i \neq \lambda_j$ for $i \neq j$. By the Chinese remainder theorem there exists a polynomial $P \in \mathbb{C}[t]$ such that $P \equiv \lambda_i \pmod{(t - \lambda_i)^{m_i}}$ and $P \equiv 0 \pmod{t}$. Then the map $\sigma : Y \rightarrow \mathcal{O}$ is given by $\sigma(A) = P(A)$, which is clearly algebraic. The case of a general $\mathfrak{h}^\mathbb{C}$ can be reduced to the previous one using the adjoint representation $\text{ad} : \mathfrak{h}^\mathbb{C} \rightarrow \text{End}(\mathfrak{h}^\mathbb{C}) \simeq \mathfrak{gl}(\dim \mathfrak{h}^\mathbb{C}, \mathbb{C})$.

By construction σ is equivariant, so it induces a projection $p_E : H^0(E(Y)) \rightarrow H^0(E(\mathcal{O}))$. We define $\xi_s = p_E(\xi)$ and $\xi_n = \xi - \xi_s$. Note that the decomposition $\xi = \xi_s + \xi_n$ is simply the fiberwise Jordan decomposition of an element of the Lie algebra as the sum of a semisimple element plus a nilpotent one. We claim that both ξ_s and ξ_n belong to $\text{aut}(E, \varphi)$. To prove this we have to check that $\rho(\xi_s)(\varphi) = \rho(\xi_n)(\varphi) = 0$. But $\rho(\xi) = \rho(\xi_s) + \rho(\xi_n)$ is fiberwise the Cartan decomposition of $\rho(\xi)$, since Cartan decomposition commutes with Lie algebra representations. In addition, if $f = f_s + f_n$ is the Cartan decomposition of an endomorphism f of a finite dimensional vector space V and $v \in V$ satisfies $fv = 0$, then $f_s v = f_n v = 0$, as the reader can check putting f in Jordan form. This proves the claim.

We want to prove that $\xi_s \in H^0(E(\mathfrak{z}))$ and that $\xi_n = 0$. We will need for that the following lemma.

Lemma 2.16. *Let $s \in \mathfrak{h}^\mathbb{C}$ be a semisimple element. There exists some $h \in H^\mathbb{C}$ such that:*

- (1) *if we write $u = \text{Ad}(h^{-1})(s) = h^{-1}sh = u_r + \mathbf{i}u_i$ with $u_r, u_i \in \mathfrak{h}$, then $[u_r, u_i] = 0$;*
- (2) *there exists an element $a \in \mathfrak{h}$ such that*

$$\text{Ker ad}(s) = \text{Ad}(h)(\text{Ker ad}(u_r) \cap \text{Ker ad}(u_i)) = \text{Ad}(h) \text{Ker ad}(a).$$

Proof. Using the decomposition $\mathfrak{h}^\mathbb{C} = \mathfrak{h} \oplus \mathbf{i}\mathfrak{h}$ we define a real valued scalar product on $\mathfrak{h}^\mathbb{C}$ as follows: given $u_r + \mathbf{i}u_i, v_r + \mathbf{i}v_i \in \mathfrak{h}^\mathbb{C}$ we set $\langle u_r + \mathbf{i}u_i, v_r + \mathbf{i}v_i \rangle_\mathbb{R} := -\langle u_r, v_r \rangle - \langle u_i, v_i \rangle$. The bilinear pairing $\langle \cdot, \cdot \rangle$ restricted to \mathfrak{h} is negative definite, so the pairing $\langle \cdot, \cdot \rangle_\mathbb{R}$ is positive definite on the whole $\mathfrak{h}^\mathbb{C}$ and hence the function $\| \cdot \|^2 : \mathfrak{h}^\mathbb{C} \rightarrow \mathbb{R}$ defined by $\|s\|^2 := \langle s, s \rangle_\mathbb{R}$ is proper. Let \mathcal{O}_s be the adjoint orbit of s . Since s is semisimple, \mathcal{O}_s is a closed subset of $\mathfrak{h}^\mathbb{C}$, and hence the function $\| \cdot \|^2 : \mathcal{O}_s \rightarrow \mathbb{R}$ attains its minimum at some point $u = u_r + \mathbf{i}u_i \in \mathcal{O}_s$. That u minimizes $\| \cdot \|^2$ on its adjoint orbit means that for any $v \in \mathfrak{h}^\mathbb{C}$ we have $\langle u, [v, u] \rangle_\mathbb{R} = 0$, since we can identify $T_u \mathcal{O}_s = \{[v, u] \mid v \in \mathfrak{h}^\mathbb{C}\}$. Now we develop for any $v = v_r + \mathbf{i}v_i$, using the invariance of $\langle \cdot, \cdot \rangle$ and Jacobi rule:

$$\begin{aligned} 0 &= \langle u_r + \mathbf{i}u_i, [u_r + \mathbf{i}u_i, v_r + \mathbf{i}v_i] \rangle_\mathbb{R} \\ &= \langle u_r + \mathbf{i}u_i, ([u_r, v_r] - [u_i, v_i]) + \mathbf{i}([u_i, v_r] + [u_r, v_i]) \rangle_\mathbb{R} \\ &= -\langle u_r, [u_r, v_r] - [u_i, v_i] \rangle - \langle u_i, [u_i, v_r] + [u_r, v_i] \rangle \\ &= \langle u_r, [u_i, v_i] \rangle - \langle u_i, [u_r, v_i] \rangle \\ &= -2\langle [u_i, u_r], v_i \rangle. \end{aligned}$$

Since this holds for any choice of v , it follows that $[u_i, u_r] = 0$. So the endomorphisms $\text{ad}(u_i)$ and $\text{ad}(u_r)$ commute and hence diagonalize in the same basis with purely imaginary eigenvalues (because they respect the pairing $\langle \cdot, \cdot \rangle_\mathbb{R}$). Hence $\text{Ker ad}(u) = \text{Ker ad}(u_r + \mathbf{i}u_i) = \text{Ker}(\text{ad}(u_r) + \mathbf{i}\text{ad}(u_i)) = \text{Ker ad}(u_r) \cap \text{Ker ad}(u_i)$. Since u_r and u_i commute, they generate a torus $T_u \subset H$. Take h such that $u = \text{Ad}(h^{-1})(s)$ and choose $a \in \mathfrak{h}$ such that the closure of $\{e^{ta} \mid t \in \mathbb{R}\}$ is equal to T_u . Then $\text{Ker ad}(a) = \text{Ker ad}(u_r) \cap \text{Ker ad}(u_i)$, so the result follows. \square

We now prove that ξ_s is central. Let $u = u_r + \mathbf{i}u_i = h^{-1}y_s h$ be the element given by the previous lemma such that $[u_r, u_i] = 0$. Let $\psi_s : E \rightarrow \mathfrak{h}^\mathbb{C}$ be the antiequivariant map corresponding to $\xi_s \in H^0(E(\mathfrak{h}^\mathbb{C}))$, whose image coincides with the adjoint orbit \mathcal{O}_s . Define $E_0 = \{e \in E \mid \psi_s(e) = u\} \subset E$. Then E_0 defines a reduction of the structure group of E to the centralizer of u , which we denote by $H_0^\mathbb{C} = \{g \in H^\mathbb{C} \mid \text{Ad}(g)(u) = u\}$. Define the subgroups $P^\pm = \{g \in H^\mathbb{C} \mid e^{\pm it u_i} g e^{\mp it u_i} \text{ is bounded as } t \rightarrow \infty\} \subset H^\mathbb{C}$. By (3) in Lemma 2.5, P^\pm are parabolic subgroups and $L_{u_i} = P^+ \cap P^- = \{g \in H^\mathbb{C} \mid \text{Ad}(g)(u_i) = u_i\}$ is a common Levi subgroup of P^+ and P^- . By (1) in Lemma 2.5, $H_0^\mathbb{C}$ is a connected subgroup of $H^\mathbb{C}$, so by the same argument as in the end of the proof of Lemma 2.16 we can identify $H_0^\mathbb{C}$ with $\{g \in H^\mathbb{C} \mid \text{Ad}(g)(u_i) = u_i, \text{Ad}(g)(u_r) = u_r\}$. This implies that $H_0^\mathbb{C} \subset L_{u_i}$, hence E_0 induces a reduction σ^+ (resp. σ^-) of the structure group of E to P^+ (resp. P^-). One the other hand, if χ corresponds to $\mathbf{i}u_i$ via the isomorphism $(\mathfrak{z} \oplus \mathfrak{c})^* \simeq \mathfrak{z} \oplus \mathfrak{c}$ (so that $s_\chi = \mathbf{i}u_i$), then χ is antidominant for P^+ and $-\chi$ is antidominant for P^- .

Let $\phi : E^L \rightarrow B$ be the antiequivariant map corresponding to φ . Since $\rho(\xi_s)(\varphi) = 0$ we have $\rho(u)\phi(e) = 0$ for any $e \in E_0$. Let $v \in B$ be any element. Since u_i and u_r commute, the vectors $\rho(e^{it u_i})v$ are uniformly bounded as $t \rightarrow \infty$ if and only if the vectors

$\rho(e^{tur})\rho(e^{itu})v = \rho(e^{tu})v$ are bounded. It follows that φ belongs both to $H^0(E(B)_{\sigma^+, \chi}^- \otimes L)$ and to $H^0(E(B)_{\sigma^-, -\chi}^- \otimes L)$. Applying the α -stability condition we deduce that

$$\deg E(\sigma^+, \chi) - \langle \alpha, \chi \rangle \geq 0, \quad \text{and} \quad \deg E(\sigma^-, -\chi) - \langle \alpha, -\chi \rangle \geq 0.$$

These inequalities, together with $\deg E(\sigma^+, \chi) - \langle \alpha, \chi \rangle = -(\deg E(\sigma^-, -\chi) - \langle \alpha, -\chi \rangle)$, imply that $\deg E(\sigma, \chi) - \langle \alpha, \chi \rangle = 0$. Since we assume that (E, φ) is α -stable, such a thing can only happen if χ , and hence any element in the image of ψ_s , is central.

Finally, we prove that $\xi_n = 0$ proceeding by contradiction. Since the set of nilpotent elements $\mathfrak{h}_n^{\mathbb{C}} \subset \mathfrak{h}^{\mathbb{C}}$ contains finitely many adjoint orbits, which are locally closed in the Zariski topology, and since ξ_n is algebraic, there exists a Zariski open subset $U \subset X$ and an adjoint orbit $\mathcal{O}_n \subset \mathfrak{h}_n^{\mathbb{C}}$ such that $\xi_n(x) \in \mathcal{O}_n$ for any $x \in U$. Assume that $\xi_n(x) \neq 0$ for $x \in U$ (otherwise ξ_n vanishes identically). Consider for any $x \in U$ the weight filtration of the action of $\text{ad}(\xi_n(x))$ on $E(\mathfrak{h}^{\mathbb{C}})_x$:

$$\cdots \subset W_x^{-k} \subset W_x^{-k+1} \subset \cdots \subset W_x^{k-1} \subset W_x^k \subset \cdots,$$

which is uniquely defined by the conditions: $\text{ad}(\xi_n(x))(W_x^j) \subset W_x^{j-2}$, $\text{ad}(\xi_n(x))^{j+1}(W_x^j) = 0$ and the induced map on graded spaces $\text{Gr ad}(\xi_n(x))^j : \text{Gr } W_x^j \rightarrow \text{Gr } W_x^{-j}$ is an isomorphism. As x moves along U the spaces W_x^j give rise to an algebraic filtration of vector bundles $\cdots \subset W_U^{-k} \subset W_U^{-k+1} \subset \cdots \subset W_U^{k-1} \subset W_U^k \subset \cdots \subset E(\mathfrak{h}^{\mathbb{C}})|_U$. By the properness of the Grassmannian of subspaces of $\mathfrak{h}^{\mathbb{C}}$ these vector bundles extend to vector bundles defined on the whole X

$$(2.8) \quad \cdots \subset W^{-k} \subset W^{-k+1} \subset \cdots \subset W^{k-1} \subset W^k \subset \cdots \subset E(\mathfrak{h}^{\mathbb{C}})$$

and the induced map between graded bundles $\text{Gr ad}(\xi_n)^j : \text{Gr } W^j \rightarrow \text{Gr } W^{-j}$ is an isomorphism away from finitely many points. This implies that

$$(2.9) \quad \deg \text{Gr } W^j \leq \deg \text{Gr } W^{-j}.$$

By Jacobson–Morozov’s theorem the weight filtration (2.8) induces a reduction σ of the structure group of E to a parabolic subgroup $P \subset H^{\mathbb{C}}$ (the so-called Jacobson–Morozov’s parabolic subgroup associated to the nilpotent elements in the image of $\xi_n|_U$), and there exists an antidominant character χ of P such that $\text{ad}(s_\chi)$ preserves the weight filtration and induces on the graded piece $\text{Gr } W^j$ the map given by multiplication by j .

The subbundle $E(B)_{\sigma, \chi}^- \otimes L \subset E(B) \otimes L$ can be identified with the piece of degree 0 in the weight filtration on $E(B) \otimes L$ induced by the nilpotent endomorphism $\rho(\xi_n)$. Since $\rho(\xi_n)(\phi) = 0$, we have $\phi \in H^0(E(B)_{\sigma, \chi}^- \otimes L)$ (the kernel of a nonzero nilpotent endomorphism is included in the piece of degree zero of the weight filtration). Hence, by α -stability, $\deg(E)(\sigma, \chi) - \langle \alpha, \chi \rangle$ has to be positive. On the other hand, the character χ can be chosen to be perpendicular to \mathfrak{z} , so by (3) in Lemma 2.13 we have

$$\deg E(\sigma, \chi) - \langle \alpha, \chi \rangle = \sum_{j \in \mathbb{Z}} j \deg \text{Gr } W^j.$$

By (2.9) this is ≤ 0 , thus contradicting the stability of (E, φ) . \square

2.10. Jordan–Hölder reduction. In this subsection we associate to each α -polystable pair (E, φ) an α -stable pair. This is accomplished by picking an appropriate subgroup $H' \subset H$ (defined as the centralizer of a torus in H) and by choosing a reduction of the structure group of E to $H'^{\mathbb{C}}$. The resulting new pair is called the Jordan–Hölder reduction of (E, φ) . It is constructed using a recursive procedure in which certain choices are made,

and the main result of this subsection (see Proposition 2.20) is the proof that the resulting reduction is canonical up to isomorphism.

Let $G' \subset G$ be an inclusion of complex connected Lie subgroup with Lie algebras $\mathfrak{g}' \subset \mathfrak{g}$. Assume that the normalizer $N_G(\mathfrak{g}')$ of \mathfrak{g}' in G is equal to G' . Suppose that E is a holomorphic principal G -bundle.

Lemma 2.17. *The holomorphic reductions of the structure group of E to G' are in bijection with the holomorphic subbundles $F \subset E(\mathfrak{g})$ of Lie subalgebras satisfying this property:*

for any $x \in X$ and trivialization $E_x \simeq G$, the fiber F_x , which we identify to a subspace of \mathfrak{g} via the induced trivialization $E(\mathfrak{g})_x \simeq \mathfrak{g}$, is conjugate to \mathfrak{g}' .

Proof. Let $d = \dim \mathfrak{g}'$ and let $\text{Gr}_d(\mathfrak{g})$ denote the Grassmannian of complex d -subspaces inside \mathfrak{g} . Let $\mathcal{O}_{\mathfrak{g}'} = \{\text{Ad}(h)(\mathfrak{g}') \mid h \in G\} \subset \text{Gr}_d(\mathfrak{g})$. By assumption there is a biholomorphism $\mathcal{O}_{\mathfrak{g}'} \simeq G/G'$. Furthermore, the set of vector bundles $F \subset E(\mathfrak{g})$ satisfying the condition of the lemma is in bijection with the holomorphic sections of $E(\mathcal{O}_{\mathfrak{g}'}),$ so the result follows. \square

We now apply this principle to a particular case. Let $P \subset H^{\mathbb{C}}$ be a parabolic subgroup, let $L \subset P$ be a Levi subgroup and let $U \subset P$ be the unipotent radical. Denote $\mathfrak{u} = \text{Lie } U$, $\mathfrak{p} = \text{Lie } P$ and $\mathfrak{l} = \text{Lie } L$. The adjoint action of P on \mathfrak{p} preserves \mathfrak{u} and using the standard projection $P \rightarrow P/U \simeq L$ (see Section 2.1 and recall that P is isomorphic to P_A for some choice of A) we make P act linearly on \mathfrak{l} via the adjoint action. Hence P acts linearly on the exact sequence $0 \rightarrow \mathfrak{u} \rightarrow \mathfrak{p} \rightarrow \mathfrak{l} \rightarrow 0$. We claim that $N_P(\mathfrak{l}) = L$. To check this we identify P (up to conjugation) with some P_A , then use (2.1) and (2.2) together with the surjectivity of the exponential map $\mathfrak{u}_A \rightarrow U_A$ to deduce that no nontrivial element of U normalizes \mathfrak{l} , and finally use the decomposition $P = LU$.

Lemma 2.18. *Suppose that E_{σ} is a holomorphic principal P -bundle. The reductions of the structure group of E_{σ} from P to $L \subset P$ are in bijection with the splittings of the exact sequence of holomorphic vector bundles*

$$(2.10) \quad 0 \rightarrow E_{\sigma}(\mathfrak{u}) \rightarrow E_{\sigma}(\mathfrak{p}) \rightarrow E_{\sigma}(\mathfrak{l}) \rightarrow 0$$

given by holomorphic maps $E_{\sigma}(\mathfrak{l}) \rightarrow E_{\sigma}(\mathfrak{p})$ which are fiberwise morphisms of Lie algebras.

Proof. Since $N_P(\mathfrak{l}) = L$, we may use Lemma 2.17 with $G = P$ and $G' = L$. The subalgebras $\mathfrak{g}' \subset \mathfrak{p}$ which are conjugate to \mathfrak{p} are the same as the images of sections $\mathfrak{l} \rightarrow \mathfrak{p}$ of the exact sequence $0 \rightarrow \mathfrak{u} \rightarrow \mathfrak{p} \rightarrow \mathfrak{l} \rightarrow 0$ which are morphisms of Lie algebras. Hence the vector subbundles $F \subset E(\mathfrak{p})$ satisfying the requirements of Lemma 2.17 can be identified with the images of maps $E(\mathfrak{l}) \rightarrow E(\mathfrak{p})$ which give a section of the sequence (2.10) and which are fiberwise a morphism of Lie algebras. \square

Suppose that (E, φ) is a α -polystable pair which is not α -stable. By Proposition 2.15 there exists a semisimple non central infinitesimal automorphism $s \in \text{aut}^{ss}(E, \varphi)$. The splitting $\mathfrak{h}^{\mathbb{C}} = \mathfrak{z} \oplus \mathfrak{h}_s^{\mathbb{C}}$ (recall that $\mathfrak{h}_s^{\mathbb{C}} = [\mathfrak{h}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}]$ is the semisimple part) is invariant under the adjoint action of $H^{\mathbb{C}}$ (which is connected by assumption) hence we have $H^0(E(\mathfrak{h}^{\mathbb{C}})) = H^0(E(\mathfrak{z})) \oplus H^0(E(\mathfrak{h}_s^{\mathbb{C}}))$ so projecting to the second summand we can assume that $s \in H^0(E(\mathfrak{h}_s^{\mathbb{C}}))$.

As shown in the proof of Proposition 2.15, the image of s is contained in an adjoint orbit in $\mathfrak{h}^{\mathbb{C}}$ which contains an element $u = u_r + \mathbf{i}u_i$ such that u_r, u_i are commuting elements of \mathfrak{h} . Let $a \in \mathfrak{h}_s = [\mathfrak{h}, \mathfrak{h}]$ be an infinitesimal generator of the torus generated by u_r and u_i and let

$H_1^\mathbb{C}$ be the complexification of $H_1 := Z_H(a) = \{h \in H \mid \text{Ad}(h)(a) = a\}$. Let $\psi_s : E \rightarrow \mathfrak{h}^\mathbb{C}$ be the antiequivariant map corresponding to the section s . Then

$$E_1 = \{e \in E \mid \psi_s(e) = u\} \subset E$$

is a $H_1^\mathbb{C}$ -principal bundle, which defines a reduction of the structure group of E . We say that the pair $(E_1, H_1^\mathbb{C})$ is the **reduction of $(E, H^\mathbb{C})$ induced by s and u** .

Define $B_1 = \{v \in B \mid \rho(a)(v) = 0\}$. The restriction of ρ to H_1 preserves B_1 , so we have a subbundle $E_1(B_1) \subset E_1(B) \simeq E(B)$. Let $\phi : E^L \rightarrow B$ be the antiequivariant map inducing the section $\varphi \in H^0(E(B) \otimes L)$ (see Subsection 2.5). By the definition of the infinitesimal automorphisms, for any $(e, l) \in E_1^L$ we have $\rho(\psi_s(e))\phi(e, l) = 0$. Now $\rho(\psi_s(e)) = \rho(u_r + \mathbf{i}u_i) = \rho(u_r) + \mathbf{i}\rho(u_i)$. Since ρ restricted to H is Hermitian, $\rho(u_r)$ and $\rho(u_i)$ have purely imaginary eigenvalues, and since $[\rho(u_r), \rho(u_i)] = 0$ it follows that

$$\rho(\psi_s(e))\phi(e, l) = 0 \iff \rho(u_r)\phi(e, l) = \rho(u_i)\phi(e, l) = 0 \iff \rho(a)\phi(e, l) = 0$$

for any $(e, l) \in E^L$. This implies that $\phi(E_1^L) \subset B_1$, and consequently φ lies in the subbundle $E_1(B_1) \otimes L \subset E(B) \otimes L$. To stress this fact we rename φ with the symbol φ_1 . To sum up: assuming that (E, φ) is α -polystable but not α -stable we have obtained a subgroup $H_1 = Z_H(a) \subset H$, a H_1 -invariant subspace $B_1 \subset B$, and a new pair (E_1, φ_1) , where E_1 is a $H_1^\mathbb{C}$ principal bundle and $\varphi_1 \in H^0(E_1(B_1) \otimes L)$. We denote the Lie algebras of H_1 and its complexification by \mathfrak{h}_1 and $\mathfrak{h}_1^\mathbb{C}$.

Proposition 2.19. *The pair (E_1, φ_1) is α -polystable.*

Proof. Since H_1 is the centralizer of a and α belongs to the center of $\mathfrak{h}_1^\mathbb{C}$, we have $\alpha \in \mathfrak{h}_1^\mathbb{C}$. Hence the statement of the proposition makes sense. We first prove that (E_1, B_1) is α -semistable. Let $P_1 \subset H_1^\mathbb{C}$ be a standard parabolic subgroup. By (2) in Lemma 2.5 there is some $s \in \mathfrak{ih}_1$ (satisfying $s = s_\chi$ for an appropriate antidominant character χ of P_1) such that $P_1 = \{g \in H_1^\mathbb{C} \mid e^{ts}ge^{-ts} \text{ is bounded as } t \rightarrow \infty\}$. Since $\mathfrak{ih}_1 \subset \mathfrak{ih}$ it makes sense to define $P = \{g \in H^\mathbb{C} \mid e^{ts}ge^{-ts} \text{ is bounded as } t \rightarrow \infty\}$, which is a parabolic subgroup of $H^\mathbb{C}$, and clearly $P_1 \subset P$. Hence, any reduction σ_1 of the structure group of E_1 to P_1 , say $(E_1)_{\sigma_1} \subset E_1$, gives automatically a reduction σ of the structure group of E to P , specified by $E_\sigma = (E_1)_{\sigma_1} \times_{P_1} P \subset (E_1)_{\sigma_1} \times_{P_1} H^\mathbb{C} = E$. Furthermore, any antidominant character $\chi \in \mathfrak{ih}$ of P_1 is an antidominant character of P , and there is an equality $\deg(E_1)(\sigma_1, \chi) = \deg(E)(\sigma, \chi)$. Finally, if the section φ_1 belongs to $H^0(E_1(B_1)_{\sigma_1, \chi}^- \otimes L)$, then it also belongs to $H^0(E(B)_{\sigma, \chi}^- \otimes L)$. All this implies that (E_1, φ_1) is α -semistable.

To prove that (E_1, φ_1) is α -polystable it remains to show that if the reduction σ_1 and χ have been chosen so that $\deg(E_1)(\sigma_1, \chi) - \langle \alpha, \chi \rangle = 0$, then there is a holomorphic reduction σ_{L_1} of the structure group of $(E_1)_{\sigma_1}$ to the Levi $L_1 = \{g \in H_1^\mathbb{C} \mid \text{Ad}(g)(s) = s\}$ such that

$$(2.11) \quad \varphi_1 \in H^0(E(B_1)_{\sigma_{L_1}, \chi}^0 \otimes L).$$

Define $L = \{g \in H^\mathbb{C} \mid \text{Ad}(g)(s) = s\}$, which is a Levi subgroup of P , let $U_1 \subset P_1$ and $U \subset P$ be the unipotent radicals, and denote the corresponding Lie algebras by $\mathfrak{u}_1 = \text{Lie } U_1$, $\mathfrak{p}_1 = \text{Lie } P_1$, $\mathfrak{l}_1 = \text{Lie } L_1$, $\mathfrak{u} = \text{Lie } U$, $\mathfrak{p} = \text{Lie } P$, $\mathfrak{l} = \text{Lie } L$. By Lemma 2.18 it suffices to check that there exists a bundle morphism $w_1 : (E_1)_{\sigma_1}(\mathfrak{l}_1) \rightarrow (E_1)_{\sigma_1}(\mathfrak{p}_1)$ given fiberwise by morphisms of Lie algebras, defining a splitting of the exact sequence

$$(2.12) \quad 0 \rightarrow (E_1)_{\sigma_1}(\mathfrak{u}_1) \rightarrow (E_1)_{\sigma_1}(\mathfrak{p}_1) \rightarrow (E_1)_{\sigma_1}(\mathfrak{l}_1) \rightarrow 0.$$

Let $T \subset H$ be the closure of $\{e^{ta} \mid t \in \mathbb{R}\}$, which is a torus. Denote by $T^\vee = \text{Hom}(T, S^1)$ the group of characters of T . We have decompositions

$$\mathbf{u} = \bigoplus_{\eta \in T^\vee} \mathbf{u}_\eta, \quad \mathbf{p} = \bigoplus_{\eta \in T^\vee} \mathbf{p}_\eta, \quad \mathbf{l} = \bigoplus_{\eta \in T^\vee} \mathbf{l}_\eta,$$

and since the elements of $H_1^\mathbb{C}$ fix a , the action of $H_1^\mathbb{C}$ on \mathbf{u} , \mathbf{p} and \mathbf{l} respects the splittings above. It follows that we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & E_\sigma(\mathbf{u}) & \longrightarrow & E_\sigma(\mathbf{p}) & \longrightarrow & E_\sigma(\mathbf{l}) \longrightarrow 0 \\ & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ 0 & \longrightarrow & (E_1)_{\sigma_1}(\mathbf{u}) & \longrightarrow & (E_1)_{\sigma_1}(\mathbf{p}) & \longrightarrow & (E_1)_{\sigma_1}(\mathbf{l}) \longrightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ 0 & \longrightarrow & \bigoplus_{\eta \in T^\vee} (E_1)_{\sigma_1}(\mathbf{u}_\eta) & \longrightarrow & \bigoplus_{\eta \in T^\vee} (E_1)_{\sigma_1}(\mathbf{p}_\eta) & \longrightarrow & \bigoplus_{\eta \in T^\vee} (E_1)_{\sigma_1}(\mathbf{l}_\eta) \longrightarrow 0 \end{array}$$

Taking in the bottom row the summands corresponding to the trivial character $\eta = 0$ (the constant representation $T \rightarrow \{1\} \in S^1$) we get the exact sequence (2.12). By hypothesis the pair (E, φ) is α -polystable, so there is a section $v : E_\sigma(\mathbf{l}) \rightarrow E_\sigma(\mathbf{p})$ of the top row, given fiberwise by morphisms of Lie algebras. Using the isomorphisms and equalities in the diagram, this gives rise to a section

$$w : \bigoplus_{\eta \in T^\vee} (E_1)_{\sigma_1}(\mathbf{l}_\eta) \rightarrow \bigoplus_{\eta \in T^\vee} (E_1)_{\sigma_1}(\mathbf{p}_\eta)$$

of the bottom row. Then $w = (w_{\eta\mu})_{\eta, \mu \in T^\vee}$, where $w_{\eta\mu} : (E_1)_{\sigma_1}(\mathbf{l}_\eta) \rightarrow (E_1)_{\sigma_1}(\mathbf{p}_\mu)$, and one checks that $w_1 := w_{00}$ is fiberwise a morphism of Lie algebras and that it gives the desired splitting of the sequence (2.12). To check (2.11) we proceed as follows. First note that s_χ belongs both to the center of \mathbf{l}_1 and \mathbf{l} , hence it defines holomorphic sections $s_{\sigma_1, \chi} \in H^0((E_1)_{\sigma_1}(\mathbf{l}_1))$ and $s_{\sigma, \chi} \in H^0(E_\sigma(\mathbf{l}))$. Condition (2.11) is equivalent to

$$(2.13) \quad \rho(w_1(s_{\sigma_1, \chi}))(\varphi) = 0$$

(note that $(E_1)_{\sigma_1}(\mathbf{p}_1)$ is a subbundle of $(E_1)_{\sigma_1}(\mathbf{h}_1^\mathbb{C}) \simeq E_1(\mathbf{h}_1^\mathbb{C})$, hence it acts fiberwise on $E(B) \otimes L$). To prove this equality, we use again the hypothesis that (E, φ) is α -polystable, which implies that $\varphi \in H^0(E(B)_{\sigma_L, \chi}^- \otimes L)$, where σ_L is the reduction specified by w . This is equivalent to $\rho(w(s_{\sigma, \chi}))(\varphi) = 0$, and this implies (2.13) because $s_\chi \in \mathbf{l}_0 \subset \bigoplus_{\eta \in T^\vee} \mathbf{l}_\eta$. \square

Let (E, φ) be a α -polystable pair. Iterating the procedure described in the previous subsection as many times as possible we obtain a sequence of groups $H = H_0 \supset H_1 \supset H_2 \supset \dots$ and elements $a_j \in (\mathbf{h}_{j-1})_s = [\mathbf{h}_{j-1}, \mathbf{h}_{j-1}]$ such that $H_j = Z_{H_{j-1}}(a_j)$, vector subspaces $B = B_0 \supset B_1 \supset B_2 \supset \dots$, and α -polystable pairs $(E, \varphi) = (E_0, \varphi_0), (E_1, \varphi_1), \dots$, where E_j is a $H_j^\mathbb{C}$ -principal bundle over X and contained in E_{j-1} , and $\varphi_j \in H^0(E_j(B_j) \otimes L)$. Since $\dim H_j < \dim H_{j-1}$, this process has to eventually stop at some pair, say (E_r, φ_r) , which will necessarily be α -stable. We say that $(E_r, \varphi_r, H_r, B_r)$ is the **Jordan–Hölder** reduction of (E, φ, H, B) . To justify this terminology we need to prove that the construction is independent of the choices made in the process. Note that the elements in the sequence $\{a_0, a_1, \dots, a_l\}$ all belong to the initial Lie algebra \mathfrak{h} and they commute pairwise. Hence they generate a torus $T \subset H$, the closure of the set $\{\exp \sum t_j a_j \mid t_0, \dots, t_l \in \mathbb{R}\}$, and H_l is the centralizer in H of $T_{(E, \varphi)}$. With this in mind, the following proposition implies the uniqueness of the Jordan–Hölder reduction.

Let $H_s \subset H$ be the connected Lie subgroup whose Lie algebra is $\mathfrak{h}_s = [\mathfrak{h}, \mathfrak{h}]$.

Proposition 2.20. *Let (E, φ) be a α -polystable pair. Suppose that $T', T'' \subset H_s$ are tori, and define H' (resp. H'') to be the centralizer in H of T' (resp. T''). Let B' (resp. B'') be the fixed point set of the action of T' (resp. T'') on B , and assume that there are reductions $E' \subset E$ (resp. $E'' \subset E$) of the structure group of E to $H'^{\mathbb{C}}$ (resp. $H''^{\mathbb{C}}$). Let $\phi : E^L \rightarrow B$ the equivariant map corresponding to φ . Assume that $\phi(E'^L) \subset B' \otimes L$ and $\phi(E''^L) \subset B'' \otimes L$. Denote by $\varphi' \in H^0(E'(B') \otimes L)$ and $\varphi'' \in H^0(E''(B'') \otimes L)$ the induced sections. Finally, suppose that both (E', φ') and (E'', φ'') are α -stable. Then there is some $g \in H^{\mathbb{C}}$ such that $H'^{\mathbb{C}} = g^{-1}(H''^{\mathbb{C}})g$, $E' = E''g$, $T'^{\mathbb{C}} = g^{-1}(T''^{\mathbb{C}})g$ and $B' = \rho(g^{-1})B''$.*

Before proving Proposition 2.20 we state and prove two auxiliary lemmas.

Lemma 2.21. *Let $u', u'' \in \mathfrak{h}$ and let $s', s'' \in H^0(E(\mathfrak{h}^{\mathbb{C}}))$ be sections such that $s'(x)$ (resp. $s''(x)$) is conjugate to $\mathbf{i}u'$ (resp. $\mathbf{i}u''$) for any $x \in X$. Let $(E', H'^{\mathbb{C}})$ (resp. $(E'', H''^{\mathbb{C}})$) be the reductions of $(E, H^{\mathbb{C}})$ induced by s' and $\mathbf{i}u'$ (resp. s'' and $\mathbf{i}u''$).*

- (1) *Assume that $[s', s''] = 0$. Let $\mathfrak{h}''^{\mathbb{C}}$ be the Lie algebra of $H''^{\mathbb{C}}$. Then we can naturally identify s' with a section of $E''(\mathfrak{h}''^{\mathbb{C}})$.*
- (2) *Let \mathfrak{z}'' be the center of $\mathfrak{h}''^{\mathbb{C}}$. If $s' \in H^0(E''(\mathfrak{z}''))$ then there is some $h \in H^{\mathbb{C}}$ such that $E'' \subset E'h$ as subsets of E .*

Proof. Let $\psi', \psi'' : E \rightarrow \mathfrak{h}^{\mathbb{C}}$ be the antiequivariant maps corresponding to s', s'' . The condition $[s', s''] = 0$ implies that for any $e \in E$ the elements $\psi'(e), \psi''(e) \in \mathfrak{h}^{\mathbb{C}}$ commute. Since $E'' = (\psi'')^{-1}(\mathbf{i}u'')$, this implies that, for any $e \in E''$, $\psi'(e)$ commutes with $\mathbf{i}u''$, so $\psi'(e)$ belongs to $\mathfrak{h}''^{\mathbb{C}}$. This proves (1). We now prove (2). First observe that, being a centralizer of a semisimple element in $\mathfrak{h}^{\mathbb{C}}$, $H''^{\mathbb{C}}$ is connected (see e.g. Theorem 13.2 in [3]). Hence, the adjoint action of $H''^{\mathbb{C}}$ on $\mathfrak{h}''^{\mathbb{C}}$ fixes any element in \mathfrak{z}'' . Take some element $e \in E''$. By hypothesis, there is some $h \in H^{\mathbb{C}}$ such that $\psi'(e) = \text{Ad}(h^{-1})(\mathbf{i}u')$, so $e \in E'h$. The condition $s' \in H^0(E''(\mathfrak{z}''))$ implies that $\psi'(e) \in \mathfrak{z}''$ so, by the previous observation, for any $g \in H''^{\mathbb{C}}$ we have $\psi'(eg) = \text{Ad}(g^{-1})\text{Ad}(h^{-1})(\mathbf{i}u') = \text{Ad}(h^{-1})(\mathbf{i}u')$, hence $eg \in E'h$. It follows that $E'' \subset E'h$. \square

For any $u \in \mathfrak{h}$ we denote by $T_u \subset H$ the torus generated by u , i.e., the closure of $\{\exp tu \mid t \in \mathbb{R}\}$, and $T_u^{\mathbb{C}}$ denotes the complexification of T_u .

Lemma 2.22. *Let $u', u'' \in \mathfrak{h}_s = [\mathfrak{h}, \mathfrak{h}]$ and let $H'^{\mathbb{C}}$ (resp. $H''^{\mathbb{C}}$) be the complexification of the centralizer $Z_H(u')$ (resp. $Z_H(u'')$). If there is some $g \in H^{\mathbb{C}}$ such that $H'^{\mathbb{C}} = g^{-1}(H''^{\mathbb{C}})g$ then $T_{u'}^{\mathbb{C}} = g^{-1}T_{u''}^{\mathbb{C}}g$.*

Proof. The center of $\mathfrak{h}'^{\mathbb{C}}$ is $\mathfrak{z} \oplus \text{Lie } T_{u'}^{\mathbb{C}}$, and the sum is direct because u' is assumed to belong to \mathfrak{h}_s . Similarly, the center of $\mathfrak{h}''^{\mathbb{C}}$ is $\mathfrak{z} \oplus \text{Lie } T_{u''}^{\mathbb{C}}$. Since $H^{\mathbb{C}}$ is connected, its adjoint action on \mathfrak{z} is trivial, and hence taking the center of the Lie algebra in each side of the equality $T_{u'}^{\mathbb{C}} = g^{-1}T_{u''}^{\mathbb{C}}g$ we deduce that $\text{Lie } T_{u'}^{\mathbb{C}} = g^{-1}(\text{Lie } T_{u''}^{\mathbb{C}})g$. This implies the equality between the complexified tori. \square

We now prove Proposition 2.20.

Proof. Let $u', u'' \in \mathfrak{h}_s$ satisfy $T' = T_{u'}$ and $T'' = T_{u''}$. The existence of reductions of E to the centralizers of u' and u'' gives rise to sections $s', s'' \in \text{aut}^{ss}(E, \varphi) \subset H^0(E(\mathfrak{h}^{\mathbb{C}}))$ such that $s'(x)$ (resp. $s''(x)$) is conjugate to $\mathbf{i}s'$ (resp. $\mathbf{i}s''$) for any $x \in X$.

If $[s', s''] = 0$ then by (1) Lemma 2.21 we can view $s' \in \text{aut}^{ss}(E'', \varphi'')$ and $s'' \in \text{aut}^{ss}(E', \varphi')$. Since by assumption (E'', φ'') and (E', φ') are α -stable, by Proposition 2.15 we deduce that s' is central in the centralizer of s'' and vice-versa. By (2) in Lemma 2.21 there exist $g, h \in H^\mathbb{C}$ such that $E' \subset E''g$ and $E'' \subset E'h$. This implies that $E' \subset E''g \subset E'hg$, but $E' \subset E'hg$ clearly implies that $E' = E'hg$, which combined with the previous chain of inclusions gives $E' = E''g$. It then follows that $H^\mathbb{C} = g^{-1}(H''^\mathbb{C})g$. By Lemma 2.22 we have $T_{s'}^\mathbb{C} = g^{-1}T_{s''}^\mathbb{C}g$. Finally, since the fixed point set of $T_{s'}^\mathbb{C}$ acting on B coincides with the fixed point set of $T_{s''}^\mathbb{C}$ (and similarly for $T_{s''}^\mathbb{C}$) we have $B' = \rho(g^{-1})B''$.

Suppose now that $[s', s''] \neq 0$. There are holomorphic splittings

$$(2.14) \quad E(\mathfrak{h}^\mathbb{C}) = E_1 \oplus \cdots \oplus E_p = F_1 \oplus \cdots \oplus F_q$$

such that $\text{ad}(s')|_{E_j} = \lambda_j \text{Id}_{E_j}$ and $\text{ad}(s'')|_{F_k} = \mu_k \text{Id}_{F_k}$, where the real numbers $\lambda_1 < \cdots < \lambda_p$ (resp. $\mu_1 < \cdots < \mu_q$) are the eigenvalues of $\text{ad}(\text{is}')$ (resp. $\text{ad}(\text{is}'')$). Define for any j the subbundles $F_{\leq j} = \bigoplus_{k \leq j} F_k \subset E(\mathfrak{h}^\mathbb{C})$ and $E_{\leq j} = \bigoplus_{k \leq j} E_k \subset E(\mathfrak{h}^\mathbb{C})$. Denote by $\pi_k : E(\mathfrak{h}^\mathbb{C}) \rightarrow E_k$ the projection using the decomposition (2.14). Let $\mathcal{E}_{\leq k}$ (resp. $\mathcal{E}_k, \mathcal{F}_{\leq j}, \mathcal{F}_j$) be the sheaf of local holomorphic sections of $E_{\leq k}$ (resp. $E_k, F_{\leq j}, F_j$). Define for any j the sheaf

$$\mathcal{F}_{\leq j}^\# = \bigoplus_{k=1}^p \pi_k(\mathcal{E}_{\leq k} \cap \mathcal{F}_{\leq j}).$$

This is a subsheaf of the sheaf associated to $E(\mathfrak{h}^\mathbb{C})$, and we denote by $F_{\leq j}^\# \subset E(\mathfrak{h}^\mathbb{C})$ the subbundle obtained by taking the saturation of $\mathcal{F}_{\leq j}^\#$.

By (1) in Lemma 2.13 s'' induces a holomorphic reduction $\sigma'' \in \Gamma(E(H^\mathbb{C}/P))$ of the structure group of E to $P = P_{\text{iu}''}$.

Lemma 2.23. *The filtration $F_{\leq 1}^\# \subset \cdots \subset F_{\leq q}^\# = E(\mathfrak{h}^\mathbb{C})$ also induces a reduction $\sigma^\#$ of the structure group of E to P .*

Proof. For any $t \in \mathbb{R}$ there is a natural fiberwise action of $e^{ts'}$ on $E(H^\mathbb{C}/P)$, which allows to define $e^{ts'}\sigma'' \in \Gamma(E(H^\mathbb{C}/P))$. For the reader's convenience, we recall how this is defined. For any $x \in X$ we can identify $\sigma''(x)$ with an antiequivariant map $\xi_{\sigma''} : E_x \rightarrow H^\mathbb{C}/P$ (here $H^\mathbb{C}$ acts on the left of $H^\mathbb{C}/P$). Similarly, $s'(x)$ corresponds to a map $\psi : E_x \rightarrow \mathfrak{h}^\mathbb{C}$ which is antiequivariant and hence satisfies, for any $f \in E_x$ and $g \in H^\mathbb{C}$,

$$(2.15) \quad e^{t\psi(fg)} = g^{-1}e^{t\psi(f)}g.$$

Then $e^{ts'}\sigma''(x)$ corresponds to the antiequivariant map $\xi_{e^{ts'}\sigma''} : E_x \rightarrow H^\mathbb{C}/P$ defined as

$$\xi_{e^{ts'}\sigma''}(f) = e^{t\psi(f)}\xi_{\sigma''}(f) = \xi_{\sigma''}(fe^{-t\psi(f)}).$$

That $\xi_{e^{ts'}\sigma''}$ is antiequivariant follows from (2.15). For each x the action of $e^{ts'(x)}$ defines on the fiber $E_x(H^\mathbb{C}/P)$ a decomposition in Zariski locally closed subvarieties $\{\mathcal{C}_{x,i}\}$, the Schubert cells. Each $\mathcal{C}_{x,i}$ corresponds to a connected component $C_{x,i} \subset E_x(H^\mathbb{C}/P)$ of the fixed point set of the action of $\{e^{ts'(x)} \mid t \in \mathbb{R}\}$ on $E_x(H^\mathbb{C}/P)$, and $\mathcal{C}_{x,i}$ is the set of $z \in E_x(H^\mathbb{C}/P)$ such that $e^{ts'(x)}z$ converges to $C_{x,i}$ as $t \rightarrow \infty$. Since s' is algebraic and, for any x , $s'(x)$ is conjugate to the same element iu' , each $\mathcal{C}_i = \bigcup_{x \in X} \mathcal{C}_{x,i}$ is a Zariski locally closed subvariety of $E(H^\mathbb{C}/P)$. Since σ'' is an algebraic section of $E(H^\mathbb{C}/P)$, there is a Zariski open subset $U \subset X$ such that $\sigma''|_U$ is contained in a unique cell $\mathcal{C}_j \subset E(H^\mathbb{C}/P)$. Then for any $x \in U$ the limit $\sigma_x^\# := \lim_{t \rightarrow \infty} e^{ts'}\sigma''(x) \in C_{x,j} \subset \mathcal{C}_j$ is well defined, and the filtration $\{\mathcal{F}_{\leq j,x}^\#\}$ corresponds to $\sigma_x^\#$. As x moves along U the elements $\sigma_x^\#$ describe an

algebraic section $\sigma_U^\# \in \Gamma(U; E(H^\mathbb{C}/P))$. Finally, $F_{\leq j}^\#$ results from extending the reduction $\sigma_U^\#$ to an algebraic section $\sigma^\# \in \Gamma(E(H^\mathbb{C}/P))$, which exists and is unique thanks to the properness of the flag variety $H^\mathbb{C}/P$. \square

Let χ be the antidominant character of P corresponding to u'' , so that $s_\chi = \mathbf{i}u''$.

Lemma 2.24. *We have $\varphi \in H^0(E(B)_{\sigma^\#, \chi}^- \otimes L)$.*

Proof. Let $U \subset X$ denote, as in the preceding lemma, a nonempty Zariski open subset such that for any $x \in U$ we have $\sigma^\#(x) = \lim_{t \rightarrow \infty} e^{ts'} \sigma''(x)$. By continuity, it suffices to prove that for any $x \in U$

$$(2.16) \quad \varphi(x) \in E(B)_{\sigma^\#, \chi}^- \otimes L.$$

The vector $\varphi(x)$ corresponds to an antiequivariant map $\phi : E_x^L \rightarrow B$, whereas $\sigma^\#$ corresponds to an antiequivariant map $\xi_{\sigma^\#} : E_x \rightarrow H^\mathbb{C}/P$. Define $P_x^\# = \xi_{\sigma^\#}^{-1}(P) \subset E_x$. Then $P_x^\#$ is an orbit of the action of P on E_x on the right (which can also be obtained by identifying $E(H^\mathbb{C}/P)$ with the quotient E/P). And (2.16) is equivalent to requiring that $\phi(x)$ restricted to $(P_x^\#)^L$ is contained in B_χ^- . Define for any real t the map $\xi_{\sigma^t} : E_x \rightarrow H^\mathbb{C}/P$ as $\xi_{\sigma^t}(f) = \xi_{\sigma''}(fe^{-t\psi(f)})$, where $\psi : E_x \rightarrow \mathfrak{h}^\mathbb{C}$ is the antiequivariant map corresponding to s' . Let also P_x^t be $\xi_{\sigma^t}^{-1}(P)$. By the previous lemma, we have $\xi_{\sigma^\#} = \lim_{t \rightarrow \infty} \xi_{\sigma^t}$, so we have $P_x^\# = \lim_{t \rightarrow \infty} P_x^t$ as orbits of E_x/P . By continuity, it suffices to check that for any t the restriction of $\phi(x)$ to $(P_x^t)^L$ is contained in B_χ^- .

Since $s', s'' \in \text{aut}(E, \varphi)$, we have

$$(2.17) \quad \rho(e^{ts'})(\varphi) = \varphi$$

and we also have $\varphi \in H^0(E(B)_{\sigma'', \chi}^- \otimes L)$. Defining $P_x'' = \xi_{\sigma''}^{-1}(P)$ this implies that

$$(2.18) \quad \phi(g, l) \in B_\chi^- \quad \text{for any } g \in P_x'' \text{ and } l \in L_x.$$

Assume that $f \in P_x^t$ and $l \in L_x$. Then $\xi_{\sigma^t}(f) = \xi_{\sigma''}(fe^{-t\psi(f)}) \in P$, so $fe^{-t\psi(f)} \in P_x''$. Hence

$$\phi(f, l) = \phi(fe^{-t\psi(f)}, l) \in B_\chi^-,$$

where the equality follows from (2.17) and the inclusion follows from (2.18). This proves that $\phi(x)$ maps $(P_x^t)^L$ inside B_χ^- , so we are done. \square

Hence we can apply the α -polystability condition, which in view of Lemma 2.13 and Remark 2.14 reads

$$(2.19) \quad \deg(E)(\sigma^\#, \chi) = \mu_q \deg F_{\leq q}^\# + \sum_{j=1}^{q-1} (\mu_j - \mu_{j+1}) \deg F_{\leq j}^\# \geq 0$$

(the $\langle \alpha, \chi \rangle$ term vanishes because we assume that s'' is orthogonal to the center of \mathfrak{h}). On the other hand, since $s'' \in \text{aut}^{ss}(E, \varphi)$, the same arguments as in the proof of Proposition 2.15 imply that

$$(2.20) \quad \deg(E)(\sigma'', \chi) = \mu_q \deg F_{\leq q} + \sum_{j=1}^{q-1} (\mu_j - \mu_{j+1}) \deg F_{\leq j} = 0.$$

An easy computation shows that $\deg \mathcal{F}_{\leq j}^\sharp = \deg F_{\leq j}$, whereas in general $\deg \mathcal{F}_{\leq j}^\sharp \leq \deg F_{\leq j}^\sharp$ with equality if and only if $\mathcal{F}_{\leq j}^\sharp = (\mathcal{F}_{\leq j}^\sharp)^{\vee\vee}$, so that in general

$$\deg F_{\leq j} \leq \deg F_{\leq j}^\sharp.$$

Since $\deg F_{\leq q} = \deg \mathcal{F}_{\leq q}^\sharp = \deg F_{\leq q}^\sharp$ (because $\mathcal{F}_{\leq q}$ is equal to the sheaf associated to $E(\mathfrak{h}^\mathbb{C})$) and $\mu_j - \mu_{j+1} < 0$ for any $1 \leq j \leq q-1$, we have

$$\deg(E)(\sigma'', \chi) \geq \deg(E)(\sigma^\sharp, \chi),$$

which combined (2.19) and (2.20) yields $\deg(E)(\sigma'', \chi) = \deg(E)(\sigma^\sharp, \chi) = 0$. By the previous comments, this equality implies $\mathcal{F}_{\leq j}^\sharp = (\mathcal{F}_{\leq j}^\sharp)^{\vee\vee}$ for any j , so that $\mathcal{F}_{\leq j}^\sharp$ is the sheaf of local holomorphic sections of a subbundle $F_{\leq j}^\sharp \subset E(\mathfrak{h}^\mathbb{C})$. This has the following consequence: if we define $\mathcal{F}_l^\sharp = \bigoplus_k \pi_k(\mathcal{F}_l \cap \mathcal{E}_{\leq k})$, then \mathcal{F}_l^\sharp is also the sheaf of sections of a subbundle $F_l^\sharp \subset E(\mathfrak{h}^\mathbb{C})$ and we have $F_{\leq j}^\sharp = \bigoplus_{l \leq j} F_l^\sharp$. In particular, we obtain a decomposition $E(\mathfrak{h}^\mathbb{C}) = \bigoplus_{l \leq q} F_l^\sharp$. Let $s^\sharp = \sum_j \mu_j \text{Id}_{F_j^\sharp} \in H^0(E(\mathfrak{h}^\mathbb{C}))$. Then we have $[s', s^\sharp] = 0$ and furthermore $s^\sharp \in \text{aut}^{ss}(E, \varphi)$. These two properties imply that $s^\sharp \in \text{aut}^{ss}(E', \varphi')$, so by Proposition 2.15 s^\sharp is central in the centralizer of s' . Similarly s' is central in the centralizer of s^\sharp , so we can proceed as in the first case and deduce the statement of the theorem with s'' replaced by s^\sharp . Reversing the roles of s' and s'' we conclude the proof of Proposition 2.20. \square

2.11. Hitchin-Kobayashi correspondence. Choose a Hermitian metric h_L on the complex line bundle L , and denote by $F_L \in \Omega^2(X; \mathfrak{i}\mathbb{R})$ the curvature of the corresponding Chern connection. Suppose that $E_h \subset E$ defines a reduction of the structure group of E from $H^\mathbb{C}$ to H . Then the vector bundle $E(B) = E \times_{H^\mathbb{C}} B$ can be canonically identified with $E_h \times_H B$, and hence inherits a Hermitian structure (obtained from the Hermitian structure on B , which is preserved by H). So for any $\varphi \in H^0(E(B) \otimes L)$ it makes sense to define

$$\mu_h(\varphi) := \rho^* \left(-\frac{\mathfrak{i}}{2} \varphi \otimes \varphi^{*h, h_L} \right).$$

Here we identify $\mathfrak{i}\varphi \otimes \varphi^{*h, h_L}$ with a skew symmetric section of $\text{End}(E(B) \otimes L)^* = \text{End}(E(B))^*$, hence a section of $E_h(\mathfrak{u}(B))^*$. The map $\rho^* : E_h(\mathfrak{u}(B))^* \rightarrow E_h(\mathfrak{h})^*$ is induced by the dual of the infinitesimal action of \mathfrak{h} on B . Using the isomorphism $\mathfrak{h}^* \simeq \mathfrak{h}$ given by the non-degenerate pairing $\langle \cdot, \cdot \rangle$ we view $\mu_h(\varphi)$ as a section of $E_h(\mathfrak{h})$.

Theorem 2.25. *Let (E, φ) be a α -polystable pair. There exists a reduction h of the structure group of E from $H^\mathbb{C}$ to H , given by a subbundle $E_h \subset E$, such that*

$$(2.21) \quad \Lambda(F_h + F_L) + \mu_h(\varphi) = -\mathfrak{i}\alpha,$$

where $F_h \in \Omega^2(X; E_h(\mathfrak{h}))$ denotes the curvature of the Chern connection on E with respect to h and $\Lambda : \Omega^2(X) \rightarrow \Omega^0(X)$ is the adjoint of wedging with the volume form on X . Furthermore, if (E, φ) is α -stable then h is unique. Conversely, if (E, φ) is a pair which admits a solution to equation (2.21), then (E, φ) is α -polystable.

Proof. Suppose first of all that (E, φ) is α -stable. Then by Proposition 2.15 we have $\text{aut}^{ss}(E, \varphi) = H^0(E(\mathfrak{z}))$, so (E, φ) is simple in the sense of Definition 3.8 in [6]. Hence we can apply Theorem 4.1 of [6] to deduce the existence and uniqueness of h . (Recall that the notion of α -stability given in the present paper coincides with the one in [6] thanks to

(3) in Lemma 2.13.) If (E, φ) is α -polystable but not stable, then we consider the Jordan–Hölder reduction (E', φ', H', B') of (E, φ, H, B) . Now the pair (E', φ') is simple and we can proceed as before to get a reduction h' of the structure group of E' from $H'^{\mathbb{C}}$ to H' satisfying (2.21). But h' also defines a reduction of the structure group of E from $H^{\mathbb{C}}$ to H , by defining $E_h := E_{h'} \times_{H'} H \subset E_{h'} \times_{H'} H^{\mathbb{C}} = E$. For this choice of h , equation (2.21) still holds.

The proof of the converse is standard. One first proves that if (E, φ) admits a solution to the equations then (E, φ) is α -semistable (see for example [6]). To prove α -polystability one can use the same strategy as in the Hitchin–Kobayashi correspondence for vector bundles. Namely, assume that $h \in E(H^{\mathbb{C}}/H)$ defines a reduction of the structure group to H , in such a way that equation (2.21) is satisfied. Assume also that $P \subset H^{\mathbb{C}}$ is a parabolic subgroup, that there is a holomorphic reduction σ of the structure group of E to P , an antidominant character χ of P such that φ is contained in $E(B)_{\sigma, \chi}^- \otimes L$ and such that

$$(2.22) \quad \deg(E)(\sigma, \chi) - \langle \alpha, \chi \rangle = 0.$$

We want to prove that there is a further reduction σ_L of the structure group of E from P to L and that φ is contained in $E(B)_{\sigma_L, \chi}^0 \otimes L$.

Let $E_h \subset E$ be the principal H bundle specified by h . The reduction σ corresponds to an antiequivariant map $\xi : E \rightarrow H^{\mathbb{C}}/P$, so that $\xi(f)$ is a parabolic subgroup of $H^{\mathbb{C}}$ for each $f \in E$. Then, using the construction given in Lemma 2.6 we define an H -antiequivariant map $\psi : E_h \rightarrow \mathfrak{ih}$ by setting $\psi(f) = s_{\xi(f), \chi}$ for any $f \in E_h$. The map ψ corresponds to a section of $E_h(\mathfrak{ih})$, which we denote by

$$s_{h, \sigma, \chi} \in E_h(\mathfrak{ih}).$$

For details on the following notions the reader can consult [15]. Let \mathbb{E} be the C^∞ H -principal bundle underlying E_h , and let \mathcal{A} be the set of connections on \mathbb{E} . Each element of $A \in \mathcal{A}$ defines a holomorphic structure $\bar{\partial}_A$ on \mathbb{E} . Let also \mathcal{S} be the space of smooth sections of $\mathbb{E} \times_H B \otimes L$, and let \mathcal{G} be the gauge group of E . The space $\mathcal{A} \times \mathcal{S}$ has a natural structure of infinite dimensional symplectic manifold, with respect to which the action of \mathcal{G} is Hamiltonian and $(A, \phi) \mapsto \mu(A, \phi) := \Lambda(F_h + F_L) + \mu_h(\varphi) + \mathbf{i}\alpha$ can be identified with a moment map for this action (see Section 4 in [15]). Furthermore, $-\mathbf{i}s_{h, \sigma, \chi}$ can be identified with an element in the Lie algebra of the gauge group \mathcal{G} .

We will now apply the notions of maximal weight λ and the function λ_t (see Section 2.3 in [15]). Let $A \in \mathcal{A}$ be the element giving rise to the $\bar{\partial}$ -operator which corresponds to the holomorphic structure E . A simple computation tells that (2.22) is equivalent to the maximal weight of $-\mathbf{i}s_{h, \sigma, \chi}$ on $(\bar{\partial}_A, \varphi)$ being zero:

$$\lambda((\bar{\partial}_A, \varphi), -\mathbf{i}s_{h, \sigma, \chi}) = \lim_{t \rightarrow \infty} \lambda_t((\bar{\partial}_A, \varphi), -\mathbf{i}s_{h, \sigma, \chi}) = 0.$$

Equation (2.21) is equivalent to the vanishing of the moment map of the action of \mathcal{G} at the pair $(\bar{\partial}_A, \varphi)$. Hence we have $\lambda_0((\bar{\partial}_A, \varphi), -\mathbf{i}s_{h, \sigma, \chi}) = 0$, and since $\lambda_t((\bar{\partial}_A, \varphi), -\mathbf{i}s_{h, \sigma, \chi})$ is nondecreasing as a function of t it follows that $\lambda_t((\bar{\partial}_A, \varphi), -\mathbf{i}s_{h, \sigma, \chi}) = 0$ for any t . This implies that $e^{ts_{h, \sigma, \chi}}$ fixes the pair $(\bar{\partial}_A, \varphi)$. That $\bar{\partial}_A$ is fixed implies that $s_{h, \sigma, \chi}$ induces a holomorphic reduction σ_L of the structure group of E to L , and that φ is fixed implies that φ is contained in $E(B)_{\sigma, \chi}^- \otimes L$. \square

2.12. Automorphism groups of polystable pairs. In this section we prove that the automorphism group of an α -polystable pair is reductive. Let (E, φ) be an L -twisted pair. Let $\text{Aut}(E, \varphi)$ denote the holomorphic automorphisms of (E, φ) , i.e., the holomorphic gauge transformations $g : E \rightarrow E$ such that $\phi \circ g^L = \phi$, where $\phi : E^L \rightarrow B$ is the antiequivariant map corresponding to φ and $g^L : E \times_X L \rightarrow E \times_X L$ is the transformation acting as g in the E factor and the identity in the L factor.

The group $\text{Aut}(E, \varphi)$ carries a natural structure of Lie group with Lie algebra equal to $\text{aut}(E, \phi)$.

Lemma 2.26. *Let (E, φ) be an α -polystable pair. Then $\text{Aut}(E, \varphi)$ is a reductive Lie group.*

Proof. If (E, φ) is α -polystable, then by Theorem 2.25 there exists a reduction $h \in \Gamma(E(H^\mathbb{C}/H))$ of the structure group satisfying equation (2.21). By the arguments in the proof of Theorem 2.25 this can be interpreted as the vanishing of the moment map of the action of \mathcal{G} (the gauge group of E_h) on $\mathcal{A} \times \mathcal{S}$ at the point (A, φ) , where A is the Chern connection of E and h . It follows (see for example Proposition 1.6 in [27]) that $\text{Aut}(E, \phi)$ is the complexification of $\text{Aut}(E, \phi) \cap \mathcal{G}$. Any $g \in \text{Aut}(E, \phi) \cap \mathcal{G}$ preserves simultaneously the complex structure of E and the reduction h , hence it also preserves the Chern connection A . But the group of gauge transformations in \mathcal{G} preserving a given connection can be identified with a closed subgroup of the automorphisms of the fiber of E_h at any given point, and consequently is a compact Lie group. Hence $\text{Aut}(E, \phi) \cap \mathcal{G}$ is a compact Lie group, so by the previous argument $\text{Aut}(E, \phi)$ is reductive. \square

3. G -HIGGS BUNDLES AND THE NON-ABELIAN HODGE THEOREM

3.1. L -twisted G -Higgs pairs, G -Higgs bundles and stability. Let G be a real reductive Lie group, let $H \subset G$ be a maximal compact subgroup and let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be a Cartan decomposition, so that the Lie algebra structure on \mathfrak{g} satisfies

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}.$$

The group H acts linearly on \mathfrak{m} through the adjoint representation, and this action extends to a linear holomorphic action of $H^\mathbb{C}$ on $\mathfrak{m}^\mathbb{C} = \mathfrak{m} \otimes \mathbb{C}$. This is the **isotropy representation**:

$$(3.23) \quad \iota : H^\mathbb{C} \rightarrow \text{GL}(\mathfrak{m}^\mathbb{C}).$$

Furthermore, the Killing form on \mathfrak{g} induces on $\mathfrak{m}^\mathbb{C}$ a Hermitian structure which is preserved by the action of H .

Let X be a closed Riemann surface and let L be a holomorphic line bundle on X . Let $E(\mathfrak{m}^\mathbb{C}) = E \times_{H^\mathbb{C}} \mathfrak{m}^\mathbb{C}$ be the $\mathfrak{m}^\mathbb{C}$ -bundle associated to E via the isotropy representation. Let K be the canonical bundle of X .

Definition 3.1. An **L -twisted G -Higgs pair** on X is a pair (E, φ) , where E is a holomorphic $H^\mathbb{C}$ -principal bundle over X and φ is a holomorphic section of $E(\mathfrak{m}^\mathbb{C}) \otimes L$. A **G -Higgs bundle** on X is a K -twisted G -Higgs pair. Two L -twisted G -Higgs pairs (E, φ) and (E', φ') are **isomorphic** if there is an isomorphism $E : V \xrightarrow{\cong} E'$ such that $\varphi = f^* \varphi'$.

Remark 3.2. When G is compact $\mathfrak{m} = 0$ and hence a G -Higgs pair is simply a holomorphic principal $G^\mathbb{C}$ -bundle. When G is complex, if $U \subset G$ is a maximal compact subgroup, the Cartan decomposition of \mathfrak{g} is $\mathfrak{g} = \mathfrak{u} + i\mathfrak{u}$, where \mathfrak{u} is the Lie algebra of U . Then an L -twisted G -Higgs pair (E, φ) consists of a holomorphic G -bundle E and $\varphi \in H^0(X, E(\mathfrak{g}) \otimes L)$, where

$E(\mathfrak{g})$ is the \mathfrak{g} -bundle associated to E via the adjoint representation. These are the objects introduced originally by Hitchin [12] when $G = \mathrm{SL}(2, \mathbb{C})$ and $L = K$.

An L -twisted G -Higgs pair is thus a particular case of the general concept of an L -twisted pair introduced in Section 2. Hence α -stability, semistability and polystability are defined for any $\alpha \in \mathfrak{ih} \cap \mathfrak{z}$, where \mathfrak{z} is the center of $\mathfrak{h}^\mathbb{C}$.

3.2. Moduli spaces of G -Higgs bundles. In order to relate G -Higgs bundles to representations of the fundamental group of X (or certain central extension of the fundamental group) in G , one requires α to lie also in the center of \mathfrak{g} . Since we will be mostly concerned with G -Higgs bundles for G semisimple, this means simply $\alpha = 0$. This justifies the following terminology.

Notation 3.3. A G -Higgs bundle (E, φ) is said to be **stable** if it is 0-stable. We define **semistability** and **polystability** of G -Higgs bundles similarly.

Henceforth, we shall assume that G is connected. Then the topological classification of $H^\mathbb{C}$ -bundles E on X is given by a characteristic class $c(E) \in \pi_1(H^\mathbb{C}) = \pi_1(H) = \pi_1(G)$. For a fixed $d \in \pi_1(G)$, the **moduli space of polystable G -Higgs bundles** $\mathcal{M}_d(G)$ is by definition the set of isomorphism classes of polystable G -Higgs bundles (E, φ) such that $c(E) = d$. When G is compact, the moduli space $\mathcal{M}_d(G)$ coincides with $M_d(G^\mathbb{C})$, the moduli space of polystable $G^\mathbb{C}$ -bundles with topological invariant d .

The moduli space $\mathcal{M}_d(G)$ has the structure of a complex analytic variety. This can be seen by the standard slice method (see, e.g., Kobayashi [14]). Geometric Invariant Theory constructions are available in the literature for G real compact algebraic (Ramanathan [18]) and for G complex reductive algebraic (Simpson [25, 26]). The case of a real form of a complex reductive algebraic Lie group follows from the general constructions of Schmitt [20, 21]. We thus have the following.

Theorem 3.4. *The moduli space $\mathcal{M}_d(G)$ is a complex analytic variety, which is algebraic when G is algebraic.*

Remark 3.5. Schmitt's construction (loc. cit.) in fact applies in the more general setting of L -twisted G -Higgs pairs.

3.3. Deformation theory of G -Higgs bundles. In this section we recall some standard facts about the deformation theory of G -Higgs bundles. A convenient reference for this material is Biswas–Ramanan [2].

Definition 3.6. Let (E, φ) be a G -Higgs bundle. The *deformation complex* of (E, φ) is the following complex of sheaves:

$$(3.24) \quad C^\bullet(E, \varphi): E(\mathfrak{h}^\mathbb{C}) \xrightarrow{\mathrm{ad}(\varphi)} E(\mathfrak{m}^\mathbb{C}) \otimes K.$$

This definition makes sense because ϕ is a section of $E(\mathfrak{m}^\mathbb{C}) \otimes K$ and $[\mathfrak{m}^\mathbb{C}, \mathfrak{h}^\mathbb{C}] \subseteq \mathfrak{m}^\mathbb{C}$.

The following result generalizes the fact that the infinitesimal deformation space of a holomorphic vector bundle V is isomorphic to $H^1(\mathrm{End} V)$.

Proposition 3.7. *The space of infinitesimal deformations of a G -Higgs bundle (E, φ) is naturally isomorphic to the hypercohomology group $\mathbb{H}^1(C^\bullet(E, \varphi))$.*

For any G -Higgs bundle there is a natural long exact sequence

$$(3.25) \quad \begin{aligned} 0 \rightarrow \mathbb{H}^0(C^\bullet(E, \varphi)) &\rightarrow H^0(E(\mathfrak{h}^\mathbb{C})) \xrightarrow{\text{ad}(\varphi)} H^0(E(\mathfrak{m}^\mathbb{C}) \otimes K) \\ &\rightarrow \mathbb{H}^1(C^\bullet(E, \varphi)) \rightarrow H^1(E(\mathfrak{h}^\mathbb{C})) \xrightarrow{\text{ad}(\varphi)} H^1(E(\mathfrak{m}^\mathbb{C}) \otimes K) \rightarrow \mathbb{H}^2(C^\bullet(E, \varphi)) \rightarrow 0. \end{aligned}$$

As an immediate consequence we have the following result.

Proposition 3.8. *The infinitesimal automorphism space $\text{aut}(E, \varphi)$ defined in Section 2.9 is isomorphic to $\mathbb{H}^0(C^\bullet(E, \varphi))$.*

Let $d\iota: \mathfrak{h}^\mathbb{C} \rightarrow \text{End}(\mathfrak{m}^\mathbb{C})$ be the derivative at the identity of the complexified isotropy representation $\iota = \text{Ad}_{H^\mathbb{C}}: H^\mathbb{C} \rightarrow \text{Aut}(\mathfrak{m}^\mathbb{C})$ (cf. Section 4.1). Let $\ker d\iota \subseteq \mathfrak{h}^\mathbb{C}$ be its kernel and let $E(\ker d\iota) \subseteq E(\mathfrak{h}^\mathbb{C})$ be the corresponding subbundle. Then there is an inclusion $H^0(E(\ker d\iota)) \hookrightarrow \mathbb{H}^0(C^\bullet(E, \varphi))$.

Definition 3.9. A G -Higgs bundle (E, φ) is said to be **infinitesimally simple** if the infinitesimal automorphism space $\mathbb{H}^0(C^\bullet(E, \varphi))$ is isomorphic to $H^0(E(\ker d\iota \cap \mathfrak{z}))$.

Similarly, we have an inclusion $\ker \iota \cap Z(H^\mathbb{C}) \hookrightarrow \text{Aut}(E, \phi)$.

Definition 3.10. A G -Higgs bundle (E, φ) is said to be **simple** if $\text{Aut}(E, \varphi) = \ker \iota \cap Z(H^\mathbb{C})$, where $Z(H^\mathbb{C})$ is the center of $H^\mathbb{C}$.

As a consequence of Propositions 3.8 and 2.15 we have the following.

Proposition 3.11. *Any stable G -Higgs bundle (E, φ) with $\varphi \neq 0$ is infinitesimally simple.*

Remark 3.12. If $\ker d\iota = 0$, then (E, φ) is infinitesimally simple if and only if the vanishing $\mathbb{H}^0(C^\bullet(E, \varphi)) = 0$ holds. A particular case of this situation is when the group G is a complex semisimple group: indeed, in this case the isotropy representation is just the adjoint representation.

Next we turn to the question of the vanishing of \mathbb{H}^2 of the deformation complex. In order to deal with this question we shall use Serre duality for hypercohomology (see e.g. Theorem 3.12 in [13]), which says that there are natural isomorphisms

$$(3.26) \quad \mathbb{H}^i(C^\bullet(E, \varphi)) \cong \mathbb{H}^{2-i}(C^\bullet(E, \varphi)^* \otimes K)^*,$$

where the dual of the deformation complex (3.24) is

$$C^\bullet(E, \varphi)^*: E(\mathfrak{m}^\mathbb{C}) \otimes K^{-1} \xrightarrow{-\text{ad}(\varphi)} E(\mathfrak{h}^\mathbb{C}).$$

An important special case of this is when G is a complex group.

Proposition 3.13. *Assume that G is a complex group. Then there is a natural isomorphism*

$$\mathbb{H}^2(C^\bullet(E, \varphi)) \cong \mathbb{H}^0(C^\bullet(E, \varphi))^*.$$

Proof. This is immediate from (3.26) and the fact that the deformation complex is dual to itself, except for a sign in the map which does not influence the cohomology (cf. Remark 3.2):

$$(3.27) \quad C^\bullet(E, \varphi)^* \otimes K: E(\mathfrak{g}) \xrightarrow{-\text{ad}(\varphi)} E(\mathfrak{g}) \otimes K.$$

□

Remark 3.14. The isomorphism $\mathbb{H}^1(C^\bullet(E, \varphi)) \cong \mathbb{H}^1(C^\bullet(E, \varphi))^*$ is also important: it gives rise to the natural complex symplectic structure on the moduli space of G -Higgs bundles for complex groups G .

We have the following key observation (cf. (3.27); again we are ignoring the irrelevant change of sign in the dual complex).

Proposition 3.15. *Let G be a real group and let $G^\mathbb{C}$ be its complexification. Let (E, φ) be a G -Higgs bundle. Then there is an isomorphism of complexes:*

$$C_{G^\mathbb{C}}^\bullet(E, \varphi) \cong C_G^\bullet(E, \varphi) \oplus C_G^\bullet(E, \varphi)^* \otimes K,$$

where $C_{G^\mathbb{C}}^\bullet(E, \varphi)$ denotes the deformation complex of (E, φ) viewed as a $G^\mathbb{C}$ -Higgs bundle, and $C_G^\bullet(E, \varphi)$ denotes the deformation complex of (E, φ) viewed as a G -Higgs bundle.

Corollary 3.16. *With the same hypotheses as in the previous Proposition, there is an isomorphism*

$$\mathbb{H}^0(C_{G^\mathbb{C}}^\bullet(E, \varphi)) \cong \mathbb{H}^0(C_G^\bullet(E, \varphi)) \oplus \mathbb{H}^2(C_G^\bullet(E, \varphi))^*.$$

Proof. Immediate from the Proposition and Serre duality (3.26). \square

Proposition 3.17. *Let G be a real semisimple group and let $G^\mathbb{C}$ be its complexification. Let (E, φ) be a G -Higgs bundle which is stable viewed as a $G^\mathbb{C}$ -Higgs bundle. Then the vanishing*

$$\mathbb{H}^0(C_G^\bullet(E, \varphi)) = 0 = \mathbb{H}^2(C_G^\bullet(E, \varphi))$$

holds.

Proof. Since G is semisimple, so is $G^\mathbb{C}$. Hence, in view of Remark 3.12, the result follows at once from Corollary 3.16 and Proposition 3.11. \square

The following result on smoothness of the moduli space can be proved, for example, from the standard slice method construction referred to above.

Proposition 3.18. *Let (E, φ) be a stable G -Higgs bundle. If (E, φ) is simple and*

$$\mathbb{H}^2(C_G^\bullet(E, \varphi)) = 0,$$

then (E, φ) is a smooth point in the moduli space. In particular, if (E, φ) is a simple G -Higgs bundle which is stable as a $G^\mathbb{C}$ -Higgs bundle, then it is a smooth point in the moduli space.

Suppose now that we are in the situation of Proposition 3.18 and that a local universal family exists. Then the dimension of the component of the moduli space containing (E, φ) equals the dimension of the infinitesimal deformation space $\mathbb{H}^1(C_G^\bullet(E, \varphi))$. In view of Proposition 3.11, Remark 3.12 and Proposition 3.19, we also have $\mathbb{H}^0(C_G^\bullet(E, \varphi)) = \mathbb{H}^2(C_G^\bullet(E, \varphi)) = 0$. So we have $\mathbb{H}^1(C_G^\bullet(E, \varphi)) = -\chi(C_G^\bullet(E, \varphi))$. A remarkable fact on this equality is that, whereas the left hand side may depend on the choice of (E, φ) , the right hand side is independent of it, as we will see in the proposition below. We shall refer to $-\chi(C_G^\bullet(E, \varphi))$ as the **expected dimension** of the moduli space.

Proposition 3.19. *Let G be semisimple. Then the expected dimension of the moduli space of G -Higgs bundles is $(g - 1) \dim G^\mathbb{C}$.*

Proof. Let (E, φ) be any G -Higgs bundle. The long exact sequence (3.25) gives us

$$\chi(C_G^\bullet(E, \varphi)) - \chi(E(\mathfrak{h}^\mathbb{C})) + \chi(E(\mathfrak{m}^\mathbb{C}) \otimes K) = 0.$$

Serre duality implies that $\chi(E(\mathfrak{m}^\mathbb{C}) \otimes K) = \chi(E(\mathfrak{m}^\mathbb{C}))$ and from the Riemann–Roch formula we therefore obtain

$$-\chi(C_G^\bullet(E, \varphi)) = \deg(E(\mathfrak{m}^\mathbb{C})) + (g - 1) \operatorname{rk}(E(\mathfrak{m}^\mathbb{C})) - (\deg(E(\mathfrak{h}^\mathbb{C})) + (1 - g) \operatorname{rk}(E(\mathfrak{h}^\mathbb{C}))).$$

Any invariant pairing on $\mathfrak{g}^\mathbb{C}$ (e.g. the Killing form) induces isomorphisms $E(\mathfrak{m}^\mathbb{C}) \simeq E(\mathfrak{m}^\mathbb{C})^*$ and $E(\mathfrak{h}^\mathbb{C}) \simeq E(\mathfrak{h}^\mathbb{C})^*$. Hence $\deg(E(\mathfrak{m}^\mathbb{C})) = \deg(E(\mathfrak{h}^\mathbb{C})) = 0$, whence the result. In particular, the value of $-\chi(C_G^\bullet(E, \varphi))$ is independent of the choice of G -Higgs bundle (E, φ) . \square

Remark 3.20. Note that the actual dimension of the moduli space (if non-empty) can be smaller than the expected dimension. This happens for example when $G = \operatorname{SU}(p, q)$ with $p \neq q$ and maximal Toledo invariant (this follows from the study of $\operatorname{U}(p, q)$ -Higgs bundles in [5]) — in this case there are in fact no stable $\operatorname{SU}(p, q)$ -Higgs bundles.

3.4. G -Higgs bundles and Hitchin equations. Let G be a connected semisimple real Lie group. Let (E, φ) be a G -Higgs bundle over a compact Riemann surface X . By a slight abuse of notation, we shall denote the C^∞ -objects underlying E and φ by the same symbols. In particular, the Higgs field can be viewed as a $(1, 0)$ -form: $\varphi \in \Omega^{1,0}(E(\mathfrak{m}^\mathbb{C}))$. Let $\tau: \Omega^1(E(\mathfrak{g}^\mathbb{C})) \rightarrow \Omega^1(E(\mathfrak{g}^\mathbb{C}))$ be the compact conjugation of $\mathfrak{g}^\mathbb{C}$ combined with complex conjugation on complex 1-forms. Given a reduction h of structure group to H in the smooth $H^\mathbb{C}$ -bundle E , we denote by F_h the curvature of the unique connection compatible with h and the holomorphic structure on E .

Theorem 3.21. *There exists a reduction h of the structure group of E from $H^\mathbb{C}$ to H satisfying the Hitchin equation*

$$F_h - [\varphi, \tau(\varphi)] = 0$$

if and only if (E, φ) is polystable.

Remark 3.22. the Hitchin equation is an equation of 2-forms and makes sense without choosing metrics on X and K , whereas the general Hermite–Einstein equation (2.21) is an equation of scalars and requires a choice of metrics on X and L . Nevertheless, for any choice of metric in the conformal class of the Riemann surface structure on X (and hence on the holomorphic cotangent bundle K), the Hitchin equation is equivalent to (2.21) for this choice of metric.

Theorem 3.21 was proved by Hitchin [12] for $G = \operatorname{SL}(2, \mathbb{C})$ and Simpson [23, 24] for an arbitrary semisimple complex Lie group G . The proof for an arbitrary reductive real Lie group G when (E, φ) is stable is given in [6], and the general polystable case follows as a particular case of the more general Hitchin–Kobayashi correspondence given in Theorem 2.25.

From the point of view of moduli spaces it is convenient to fix a C^∞ principal H -bundle \mathbf{E}_H with fixed topological class $d \in \pi_1(H)$ and study the moduli space of solutions to **Hitchin’s equations** for a pair (A, φ) consisting of an H -connection A and $\varphi \in \Omega^{1,0}(X, \mathbf{E}_H(\mathfrak{m}^\mathbb{C}))$:

$$(3.28) \quad \begin{aligned} F_A - [\varphi, \tau(\varphi)] &= 0 \\ \bar{\partial}_A \varphi &= 0. \end{aligned}$$

Here d_A is the covariant derivative associated to A and $\bar{\partial}_A$ is the $(0,1)$ part of d_A , which defines a holomorphic structure on \mathbf{E}_H . The gauge group \mathcal{H} of \mathbf{E}_H acts on the space of solutions and the moduli space of solutions is

$$\mathcal{M}_d^{\text{gauge}}(G) := \{(A, \varphi) \text{ satisfying (3.28)}\} / \mathcal{H}.$$

Now, Theorem 3.21 has as a consequence the following global statement.

Theorem 3.23. *There is a homeomorphism*

$$\mathcal{M}_d(G) \cong \mathcal{M}_d^{\text{gauge}}(G)$$

To explain this correspondence we interpret the moduli space of G -Higgs bundles in terms of pairs $(\bar{\partial}_E, \varphi)$ consisting of a $\bar{\partial}$ -operator (holomorphic structure) on the $H^{\mathbb{C}}$ -bundle $\mathbf{E}_{H^{\mathbb{C}}}$ obtained from \mathbf{E}_H by the extension of structure group $H \subset H^{\mathbb{C}}$, and $\varphi \in \Omega^{1,0}(X, \mathbf{E}_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}}))$ satisfying $\bar{\partial}_E \varphi = 0$. Such pairs are in correspondence with G -Higgs bundles (E, φ) , where E is the holomorphic $H^{\mathbb{C}}$ -bundle defined by the operator $\bar{\partial}_E$ on $\mathbf{E}_{H^{\mathbb{C}}}$ and $\bar{\partial}_E \varphi = 0$ is equivalent to $\varphi \in H^0(X, E(\mathfrak{m}^{\mathbb{C}}) \otimes K)$. The moduli space of polystable G -Higgs bundles $\mathcal{M}_d(G)$ can now be identified with the orbit space

$$\{(\bar{\partial}_E, \varphi) : \bar{\partial}_E \varphi = 0, (\bar{\partial}_E, \varphi) \text{ defines a polystable } G\text{-Higgs bundle}\} / \mathcal{H}^{\mathbb{C}},$$

where $\mathcal{H}^{\mathbb{C}}$ is the gauge group of $\mathbf{E}_{H^{\mathbb{C}}}$, which is in fact the complexification of \mathcal{H} . Since there is a one-to-one correspondence between H -connections on \mathbf{E}_H and $\bar{\partial}$ -operators on $\mathbf{E}_{H^{\mathbb{C}}}$, the correspondence given in Theorem 3.23 can be interpreted by saying that in the $\mathcal{H}^{\mathbb{C}}$ -orbit of a polystable G -Higgs bundle $(\bar{\partial}_{E_0}, \varphi_0)$ we can find another Higgs bundle $(\bar{\partial}_E, \varphi)$ whose corresponding pair (d_A, φ) satisfies $F_A - [\varphi, \tau(\varphi)] = 0$, and this is unique up to H -gauge transformations.

The infinitesimal deformation space of a solution (A, φ) to Hitchin's equations can be described as the first cohomology group of a certain elliptic deformation complex. To do this, we follow Hitchin [12, § 5]. The linearized equations are:

$$\begin{aligned} d_A(\dot{A}) - [\dot{\varphi}, \tau(\varphi)] - [\varphi, \tau(\dot{\varphi})] &= 0, \\ \bar{\partial}_A \dot{\varphi} + [\dot{A}^{0,1}, \varphi] &= 0, \end{aligned}$$

for $\dot{A} \in \Omega^1(X, \mathbf{E}_H(\mathfrak{h}))$ and $\dot{\varphi} \in \Omega^{1,0}(X, \mathbf{E}_H(\mathfrak{m}^{\mathbb{C}}))$. The infinitesimal action of

$$\psi \in \text{Lie } \mathcal{H} = \Omega^0(X, \mathbf{E}_H(\mathfrak{h}))$$

is

$$(A, \phi) \mapsto (d_A \psi, [\phi, \psi]).$$

Thus the deformation theory of Hitchin's equations is governed by the (elliptic) complex

$$\begin{aligned} C^\bullet(A, \varphi) : \Omega^0(X, \mathbf{E}_H(\mathfrak{h})) &\xrightarrow{d_0} \Omega^1(X, \mathbf{E}_H(\mathfrak{h})) \oplus \Omega^{1,0}(X, \mathbf{E}_H(\mathfrak{m}^{\mathbb{C}})) \\ &\xrightarrow{d_1} \Omega^2(X, \mathbf{E}_H(\mathfrak{h})) \oplus \Omega^{1,1}(X, \mathbf{E}_H(\mathfrak{m}^{\mathbb{C}})), \end{aligned}$$

where the maps are

$$d_0(\psi) = (d_A \psi, [\varphi, \psi])$$

and

$$d_1(\psi) = (d_A(\dot{A}) - [\dot{\varphi}, \tau(\varphi)] - [\varphi, \tau(\dot{\varphi})], \bar{\partial}_A \dot{\varphi} + [\dot{A}^{0,1}, \varphi]).$$

The fact that (A, φ) is a solution to the equations, together with the gauge invariance of the equations, guarantees that $d_1 \circ d_0 = 0$. Denote by $H^i(C^\bullet(A, \varphi))$ the cohomology groups of the gauge theory deformation complex $C^\bullet(A, \varphi)$.

Let

$$\text{Aut}(A, \varphi) := \{h \in \mathcal{H} \ : \ h^*A = A, \text{ and } \iota(h)(\varphi) = \varphi\}.$$

Here $\iota : H \rightarrow \text{Aut}(\mathfrak{m})$ is the isotropy representation. Clearly $Z(H) \cap \ker \iota \subset \text{Aut}(A, \varphi)$.

Definition 3.24. Let (A, φ) be a solution of (3.28). We say that (A, φ) is **irreducible** if and only if $\text{Aut}(A, \varphi) = Z(H) \cap \ker \iota$. We say that (A, φ) is **infinitesimally irreducible** if the Lie algebra of $\text{Aut}(A, \varphi)$, which is identified with $H^0(C^\bullet(A, \varphi))$ equals $Z(\mathfrak{h}) \cap \ker d\iota$.

Proposition 3.25. Assume that $H^0(C^\bullet(A, \varphi)) = H^2(C^\bullet(A, \varphi)) = 0$ and that (A, φ) is irreducible. Then $\mathcal{M}_d^{\text{gauge}}$ is smooth at $[A, \varphi]$ and the tangent space is

$$T_{[A, \varphi]} \mathcal{M}_d^{\text{gauge}} \cong H^1(C^\bullet(A, \varphi)).$$

For a proper understanding of many aspects of the geometry of the moduli space of Higgs bundles, one needs to consider the moduli space as the gauge theory moduli space $\mathcal{M}_d^{\text{gauge}}(G)$. On the other hand, the formulation of the deformation theory in terms of hypercohomology is very convenient. Fortunately, one has the following.

Proposition 3.26. At a smooth point of the moduli space, there is a natural isomorphism of infinitesimal deformation spaces

$$H^1(C^\bullet(A, \varphi)) \cong \mathbb{H}^1(C^\bullet(E, \varphi)),$$

where the holomorphic structure on the Higgs bundle (E, φ) is given by $\bar{\partial}_A$.

As in Donaldson–Kronheimer [9, § 6.4] this can be seen by using a Dolbeault resolution to calculate $\mathbb{H}^1(C^\bullet(E, \varphi))$ and using harmonic representatives of cohomology classes, via Hodge theory. For this reason we can freely apply the complex deformation theory described in Section 3.3 to the gauge theory situation.

The following result is not essential for the present paper but we include it here for completeness. It can be deduced from the treatment of the Hitchin–Kobayashi correspondence given in Section 2.

Proposition 3.27. Under the correspondence given by Theorem 3.23, a stable G -Higgs bundle corresponds to an infinitesimally irreducible solution to Hitchin equations, while a G -Higgs bundle which is stable and simple is in correspondence with an irreducible solution.

3.5. Surface group representations. Let X be a closed oriented surface of genus g and let

$$\pi_1(X) = \langle a_1, b_1, \dots, a_g, b_g \mid \prod_{i=1}^g [a_i, b_i] = 1 \rangle$$

be its fundamental group. Let G be a connected reductive real Lie group. By a **representation** of $\pi_1(X)$ in G we understand a homomorphism $\rho : \pi_1(X) \rightarrow G$. The set of all such homomorphisms, $\text{Hom}(\pi_1(X), G)$, can be naturally identified with the subset of G^{2g} consisting of $2g$ -tuples $(A_1, B_1, \dots, A_g, B_g)$ satisfying the algebraic equation $\prod_{i=1}^g [A_i, B_i] = 1$. This shows that $\text{Hom}(\pi_1(X), G)$ is a real analytic variety, which is algebraic if G is algebraic.

The group G acts on $\text{Hom}(\pi_1(X), G)$ by conjugation:

$$(g \cdot \rho)(\gamma) = g\rho(\gamma)g^{-1}$$

for $g \in G$, $\rho \in \text{Hom}(\pi_1(X), G)$ and $\gamma \in \pi_1(X)$. If we restrict the action to the subspace $\text{Hom}^+(\pi_1(X), g)$ consisting of *reductive representations*, the orbit space is Hausdorff (see Theorem 11.4 in [19]). By a **reductive representation** we mean one that composed with the adjoint representation in the Lie algebra of G decomposes as a sum of irreducible representations. If G is algebraic this is equivalent to the Zariski closure of the image of $\pi_1(X)$ in G being a reductive group. (When G is compact every representation is reductive.) Define the *moduli space of representations* of $\pi_1(X)$ in G to be the orbit space

$$\mathcal{R}(G) = \text{Hom}^+(\pi_1(X), G)/G.$$

One has the following (see e.g. Goldman [11]).

Theorem 3.28. *The moduli space $\mathcal{R}(G)$ has the structure of a real analytic variety, which is algebraic if G is algebraic and is a complex variety if G is complex.*

Given a representation $\rho: \pi_1(X) \rightarrow G$, there is an associated flat G -bundle on X , defined as $E_\rho = \tilde{X} \times_\rho G$, where $\tilde{X} \rightarrow X$ is the universal cover and $\pi_1(X)$ acts on G via ρ . This gives in fact an identification between the set of equivalence classes of representations $\text{Hom}(\pi_1(X), G)/G$ and the set of equivalence classes of flat G -bundles, which in turn is parameterized by the cohomology set $H^1(X, G)$. We can then assign a topological invariant to a representation ρ given by the characteristic class $c(\rho) := c(E_\rho) \in \pi_1(G)$ corresponding to E_ρ . To define this, let \tilde{G} be the universal covering group of G . We have an exact sequence

$$1 \longrightarrow \pi_1(G) \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1$$

which gives rise to the (pointed sets) cohomology sequence

$$(3.29) \quad H^1(X, \tilde{G}) \longrightarrow H^1(X, G) \xrightarrow{c} H^2(X, \pi_1(G)).$$

Since $\pi_1(G)$ is abelian the orientation of X defines an isomorphism

$$H^2(X, \pi_1(G)) \cong \pi_1(G),$$

and $c(E_\rho)$ is defined as the image of E under the last map in (3.29). Thus the class $c(E_\rho)$ measures the obstruction to lifting E_ρ to a flat \tilde{G} -bundle, and hence to lifting ρ to a representation of $\pi_1(X)$ in \tilde{G} . For a fixed $d \in \pi_1(G)$, the *moduli space of reductive representations* $\mathcal{R}_d(G)$ with topological invariant d is defined as the subvariety

$$(3.30) \quad \mathcal{R}_d(G) := \{[\rho] \in \mathcal{R}(G) \mid c(\rho) = d\},$$

where as usual $[\rho]$ denotes the G -orbit $G \cdot \rho$ of $\rho \in \text{Hom}^+(\pi_1(X), G)$.

One can study deformations of a class of representations $[\rho] \in \mathcal{R}_d(G)$ by means of group cohomology (see [11]). The Lie algebra \mathfrak{g} is endowed with the structure of a $\pi_1(X)$ -module by means of the composition

$$\pi_1(X) \xrightarrow{\rho} G \xrightarrow{\text{Ad}} \text{Aut}(\mathfrak{g}).$$

Definition 3.29. Let $\rho: \pi_1(X) \rightarrow G$ be a representation of $\pi_1(X)$ in G . Let $Z_G(\rho)$ be the centralizer in G of $\rho(\pi_1(X))$. We say that ρ is **irreducible** if and only if it is reductive and $Z_G(\rho) = Z(G)$, where $Z(G)$ is the center of G . We say that ρ is an **infinitesimally irreducible** representation if it is reductive and $\text{Lie } Z_G(\rho) = \text{Lie } Z(G)$.

One has the following basic facts ([11]).

Proposition 3.30. (1) *The Zariski tangent space to $\mathcal{R}_d(G)$ at an equivalence class $[\rho]$ is isomorphic to the cohomology group $H^1(\pi_1(X), \mathfrak{g}_{\text{Ad} \circ \rho})$.*
 (2) $H^0(\pi_1(X), \mathfrak{g}_{\text{Ad} \circ \rho}) \cong \text{Lie } Z_G(\rho)$.
 (3) $H^2(\pi_1(X), \mathfrak{g}_{\text{Ad} \circ \rho}) \cong H^0(\pi_1(X), \mathfrak{g}_{\text{Ad} \circ \rho})^*$

From this one obtains the following ([11]).

Proposition 3.31. *Let G be a semisimple Lie group and let $\rho : \pi_1(X) \rightarrow G$ be irreducible. Then the equivalence class $[\rho]$ is a smooth point in $\mathcal{R}_d(G)$.*

This is simply because $Z_G(\rho) = Z(G)$ is finite and hence

$$H^0(\pi_1(X), \mathfrak{g}_{\text{Ad} \circ \rho}) = H^2(\pi_1(X), \mathfrak{g}_{\text{Ad} \circ \rho}) = 0.$$

An alternative way to study deformations of a representation is by using the corresponding flat connection. To explain this, let \mathbf{E} be a C^∞ principal G -bundle over X with fixed topological class $d \in \pi_1(G) = \pi_1(H)$. Let D be a G -connection on \mathbf{E} and let F_D be its curvature. If D is flat, i.e. $F_D = 0$, then the holonomy of D around a closed loop in X only depends on the homotopy class of the loop and thus defines a representation of $\pi_1(X)$ in G . This gives an identification¹,

$$\mathcal{R}_d(G) \cong \{\text{Reductive } G\text{-connections } D \mid F_D = 0\} / \mathcal{G},$$

where, by definition, a flat connection is reductive if the corresponding representation of $\pi_1(X)$ in G is reductive, and \mathcal{G} is the group of automorphisms of \mathbf{E} — the **gauge group**. We can now linearize the flatness condition near a flat connection D :

$$\frac{d}{dt} F(D + bt)_{t=0} = D(b)$$

for $b \in \Omega^1(X, \mathbf{E}(\mathfrak{g}))$.

Linearize the action of the gauge group $D \mapsto g \cdot D = gDg^{-1}$. For $g(t) = \exp(\psi t)$ with $\psi \in \Omega^0(X, \mathbf{E}(\mathfrak{g}))$,

$$\frac{d}{dt} (g(t) \cdot D)_{t=0} = D(\psi).$$

Thus the infinitesimal deformation space is H^1 of the complex

$$0 \rightarrow \Omega^0(X, \mathbf{E}(\mathfrak{g})) \xrightarrow{D} \Omega^1(X, \mathbf{E}(\mathfrak{g})) \xrightarrow{D} \Omega^2(X, \mathbf{E}(\mathfrak{g})) \rightarrow 0.$$

Note that $F_D = D^2 = 0$ means that this is in fact a complex.

3.6. Representations and G -Higgs bundles. We assume now that G is connected and semisimple. With the notation of the previous sections, we have the following non-abelian Hodge Theorem for representations of the fundamental group of a closed Riemann surface in a semisimple connected Lie group.

Theorem 3.32. *Let G be a connected semisimple real Lie group. There is a homeomorphism $\mathcal{R}_d(G) \cong \mathcal{M}_d(G)$. Under this homeomorphism, stable G -Higgs bundles correspond to infinitesimally irreducible representations, and stable and simple G -Higgs bundles correspond to irreducible representations.*

¹even when G is complex algebraic, this is merely a real *analytic* isomorphism, see Simpson [24, 25, 26]

Remark 3.33. On the open subvarieties defined by the smooth points of \mathcal{R}_d and \mathcal{M}_d , this correspondence is in fact an isomorphism of real analytic varieties.

Remark 3.34. There is a similar correspondence when G is reductive but not semisimple. In this case, it makes sense to consider nonzero values of the stability parameter α . The resulting Higgs bundles can be geometrically interpreted in terms of representations of the universal central extension of the fundamental group of X , and the value of α prescribes the image of a generator of the center in the representation.

The proof of Theorem 3.32 is the combination of two existence theorems for gauge-theoretic equations. To explain this, let \mathbf{E}_G be, as above, a C^∞ principal G -bundle over X with fixed topological class $d \in \pi_1(G) = \pi_1(H)$. Every G -connection D on \mathbf{E}_G decomposes uniquely as

$$D = d_A + \psi,$$

where d_A is an H -connection on \mathbf{E}_H and $\psi \in \Omega^1(X, \mathbf{E}_H(\mathfrak{m}))$. Let F_A be the curvature of d_A . We consider the following set of equations for the pair (d_A, ψ) :

$$(3.31) \quad \begin{aligned} F_A + \frac{1}{2}[\psi, \psi] &= 0 \\ d_A \psi &= 0 \\ d_A^* \psi &= 0. \end{aligned}$$

These equations are invariant under the action of \mathcal{H} , the gauge group of \mathbf{E}_H . A theorem of Corlette [7], and Donaldson [8] for $G = \mathrm{SL}(2, \mathbb{C})$, says the following.

Theorem 3.35. *There is a homeomorphism*

$$\{\text{Reductive } G\text{-connections } D \mid F_D = 0\} / \mathcal{G} \cong \{(d_A, \psi) \text{ satisfying (3.31)}\} / \mathcal{H}.$$

The first two equations in (3.31) are equivalent to the flatness of $D = d_A + \psi$, and Theorem 3.35 simply says that in the \mathcal{G} -orbit of a reductive flat G -connection D_0 we can find a flat G -connection $D = g(D_0)$ such that if we write $D = d_A + \psi$, the additional condition $d_A^* \psi = 0$ is satisfied. This can be interpreted more geometrically in terms of the reduction $h = g(h_0)$ of \mathbf{E}_G to an H -bundle obtained by the action of $g \in \mathcal{G}$ on h_0 . The equation $d_A^* \psi = 0$ is equivalent to the harmonicity of the $\pi_1(X)$ -equivariant map $\tilde{X} \rightarrow G/H$ corresponding to the new reduction of structure group h .

To complete the argument, leading to Theorem 3.32, we just need Theorem 3.21 and the following simple result.

Proposition 3.36. *The correspondence $(d_A, \varphi) \mapsto (d_A, \psi := \varphi - \tau(\varphi))$ defines a homeomorphism*

$$\{(d_A, \varphi) \text{ satisfying (3.28)}\} / \mathcal{H} \cong \{(d_A, \psi) \text{ satisfying (3.31)}\} / \mathcal{H}.$$

4. SIMPLIFIED STABILITY OF G -HIGGS BUNDLES

In this section we give concrete examples of G -Higgs bundles for various interesting cases of real reductive groups G and we show how the general stability conditions can be simplified to more workable conditions in these particular cases. Even though our main interest lies in G -Higgs bundles, we state and prove our results in the more general setting of L -twisted G -Higgs pairs, since this requires no extra work.

4.1. Jordan–Hölder reduction of G -Higgs bundles. The purpose of this subsection is to show that the Jordan–Hölder reduction (defined in Section 2.10) of a G -Higgs bundle is itself a G -Higgs bundle for a reductive subgroup $G' \subset G$.

Proposition 4.1. *Let (E, φ) be an L -twisted G -Higgs pair which is α -polystable but not α -stable. Then the Jordan–Hölder reduction of (E, φ) is an L -twisted G' -Higgs pair for some reductive subgroup $G' \subset G$.*

Proof. Recall from Section 2.10 that in the Jordan–Hölder reduction $(E', \varphi', H', (\mathfrak{m}^{\mathbb{C}})')$ of $(E, \varphi, H, \mathfrak{m}^{\mathbb{C}})$ the subgroup $H' \subset H$ is defined as the centralizer of a torus $T \subset H$ and that $(\mathfrak{m}^{\mathbb{C}})'$ is the fixed point set of T acting on $\mathfrak{m}^{\mathbb{C}}$. So it suffices to prove that the Lie algebra structure on $\mathfrak{h} \oplus \mathfrak{m}$ induces a structure of Cartan pair on $(\mathfrak{h}', (\mathfrak{m}^{\mathbb{C}})' \cap \mathfrak{m})$. The action of T on \mathfrak{h} and \mathfrak{m} induces decompositions

$$\mathfrak{h} = \bigoplus_{\eta \in T^{\vee}} \mathfrak{h}_{\eta} \quad \text{and} \quad \mathfrak{m} = \bigoplus_{\eta \in T^{\vee}} \mathfrak{m}_{\eta},$$

where T^{\vee} denotes the group of characters of T (for which we use additive notation). Then one has, as usual,

$$[\mathfrak{h}_{\eta}, \mathfrak{h}_{\mu}] \subset \mathfrak{h}_{\eta+\mu}, \quad [\mathfrak{h}_{\eta}, \mathfrak{m}_{\mu}] \subset \mathfrak{m}_{\eta+\mu}, \quad [\mathfrak{m}_{\eta}, \mathfrak{m}_{\mu}] \subset \mathfrak{h}_{\eta+\mu}$$

for any pair of characters $\eta, \mu \in T^{\vee}$. Taking $\eta = \mu = 0$ and observing that $\mathfrak{h}' = \mathfrak{h}_0$ and $(\mathfrak{m}^{\mathbb{C}})' \cap \mathfrak{m} = \mathfrak{m}_0$, it follows that

$$[\mathfrak{h}', \mathfrak{h}'] \subset \mathfrak{h}', \quad [\mathfrak{h}', (\mathfrak{m}^{\mathbb{C}})' \cap \mathfrak{m}] \subset (\mathfrak{m}^{\mathbb{C}})' \cap \mathfrak{m}, \quad [(\mathfrak{m}^{\mathbb{C}})' \cap \mathfrak{m}, (\mathfrak{m}^{\mathbb{C}})' \cap \mathfrak{m}] \subset \mathfrak{h}',$$

so that $(\mathfrak{h}', (\mathfrak{m}^{\mathbb{C}})' \cap \mathfrak{m})$ is certainly a Cartan pair. □

Remark 4.2. We can make a more precise statement: defining G' as the centralizer of T inside G we have proved that the Jordan–Hölder reduction of (E, φ) is an L -twisted G' -Higgs pair.

4.2. $\mathrm{Sp}(2n, \mathbb{C})$ -Higgs bundles. Consider now the case $G = \mathrm{Sp}(2n, \mathbb{C})$. A maximal compact subgroup of G is $H = \mathrm{Sp}(2n)$ and hence $H^{\mathbb{C}}$ coincides with $\mathrm{Sp}(2n, \mathbb{C})$. Now, if $\mathbb{W} = \mathbb{C}^{2n}$ is the fundamental representation of $\mathrm{Sp}(2n, \mathbb{C})$ and ω denotes the standard symplectic form on \mathbb{W} , the isotropy representation space is

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{sp}(\mathbb{W}) = \mathfrak{sp}(\mathbb{W}, \omega) := \{\xi \in \mathrm{End}(\mathbb{W}) \mid \omega(\xi \cdot, \cdot) + \omega(\cdot, \xi \cdot) = 0\} \subset \mathrm{End} \mathbb{W},$$

so it coincides with the adjoint representation of $\mathrm{Sp}(2n, \mathbb{C})$ on its Lie algebra. An L -twisted $\mathrm{Sp}(2n, \mathbb{C})$ -Higgs pair is thus a pair consisting of a rank $2n$ holomorphic symplectic vector bundle (W, Ω) over X (so Ω is a holomorphic section of $\Lambda^2 W^*$ whose restriction to each fiber of W is non degenerate) and a section

$$\Phi \in H^0(L \otimes \mathfrak{sp}(W)),$$

where $\mathfrak{sp}(W)$ is the vector bundle whose fiber over x is given by $\mathfrak{sp}(W_x, \Omega_x)$.

Remark 4.3. Since the center of $\mathfrak{sp}(2n, \mathbb{C})$ is trivial, $\alpha = 0$ is the only possible value for which stability of an L -twisted $\mathrm{Sp}(2n, \mathbb{C})$ -Higgs pair is defined.

Define for any filtration by holomorphic subbundles

$$\mathcal{W} = (0 = W_0 \subsetneq W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_k = W)$$

satisfying $W_{k-i} = W_i^{\perp_\Omega}$ for any i (here \perp_Ω denotes the perpendicular with respect to Ω) the set

$$\Lambda(\mathcal{W}) = \{(\lambda_1, \lambda_2, \dots, \lambda_k) \in \mathbb{R}^k \mid \lambda_i \leq \lambda_{i+1} \text{ and } \lambda_{k-i+1} + \lambda_i = 0 \text{ for any } i\}.$$

For any $\lambda \in \Lambda(\mathcal{W})$ define the following subbundle of $L \otimes \text{End } W$:

$$N(\mathcal{W}, \lambda) = L \otimes \mathfrak{sp}(W) \cap \sum_{\lambda_i \geq \lambda_j} L \otimes \text{End}(W_i, W_j).$$

Define also

$$d(\mathcal{W}, \lambda) = \sum_{j=1}^{k-1} (\lambda_j - \lambda_{j+1}) \deg W_j$$

(note that since W carries a symplectic structure we have $W \simeq W^*$ and hence $\deg W = \deg W_k = 0$).

Following again Sections 2.7 and Section 2.8, the pair $((W, \Omega), \Phi)$ is said to be

- **semistable** if for any filtration \mathcal{W} as above and any $\lambda \in \Lambda(\mathcal{W})$ such that $\Phi \in H^0(N(\mathcal{W}, \lambda))$, the following inequality holds: $d(\mathcal{W}, \lambda) \geq 0$.
- **stable** if it is semistable and furthermore, for any choice of filtration \mathcal{W} and $\lambda \in \Lambda(\mathcal{W})$ which is not identically zero (so for which there is a $j < k$ such that $\lambda_j < \lambda_{j+1}$), and such that $\Phi \in H^0(N(\mathcal{W}, \lambda))$, we have $d(\mathcal{W}, \lambda) > 0$.
- **polystable** if it is semistable and for any filtration \mathcal{W} as above and $\lambda \in \Lambda(\mathcal{W})$ satisfying $\lambda_i < \lambda_{i+1}$ for each i , $\psi \in H^0(N(\mathcal{W}, \lambda))$ and $d(\mathcal{W}, \lambda) = 0$, there is an isomorphism

$$W \simeq W_1 \oplus W_2/W_1 \oplus \dots \oplus W_k/W_{k-1}$$

such that the pairing via Ω any element of the summand W_i/W_{i-1} with an element of the summand W_j/W_{j-1} is zero unless $i + j = k + 1$; furthermore, via the isomorphism above,

$$\Phi \in H^0\left(\bigoplus_i L \otimes \text{Hom}(W_i/W_{i-1}, W_i/W_{i-1})\right).$$

We now prove an analog of Theorems 4.9 and 4.11, which implies that the definition of (semi)stability which we have given coincides with the usual one in the literature. Recall that if (W, Ω) is a symplectic vector bundle, a subbundle $W' \subset W$ is said to be isotropic if the restriction of Ω to W' is identically zero.

Theorem 4.4. *An L -twisted $\text{Sp}(2n, \mathbb{C})$ -Higgs pair $((W, \Omega), \Phi)$ is semistable if and only if for any isotropic subbundle $W' \subset W$ such that $\Phi(W') \subset L \otimes W'$ we have $\deg W' \leq 0$. Furthermore, $((W, \Omega), \Phi)$ is stable if for any nonzero and strict isotropic subbundle $0 \neq W' \subset W$ such that $\Phi(W') \subset L \otimes W'$ we have $\deg W' < 0$. Finally, $((W, \Omega), \Phi)$ is polystable if it is semistable and for any nonzero and strict isotropic subbundle $W' \subset W$ such that $\Phi(W') \subset L \otimes W'$ and $\deg W' = 0$ there is another isotropic subbundle $W'' \subset W$ such that $\Phi(W'') \subset L \otimes W''$ and $W = W' \oplus W''$.*

Proof. The proof follows the same ideas as the proofs of Theorems 4.9 and 4.11, so we just give a sketch. Take an L -twisted $\text{Sp}(2n, \mathbb{C})$ -Higgs pair $((W, \Omega), \Phi)$, and assume that for any isotropic subbundle $W' \subset W$ such that $\Phi(W') \subset L \otimes W'$ we have $\deg W' \leq 0$. We want to prove that $((W, \Omega), \Phi)$ is semistable. Choose any filtration $\mathcal{W} = (0 \subsetneq W_1 \subsetneq W_2 \subsetneq$

$\cdots \subsetneq W_k = W$) satisfying $W_{k-i} = W_i^{\perp\Omega}$ for any i . We have to understand the geometry of the convex set

$$\Lambda(\mathcal{W}, \Phi) = \{\lambda \in \Lambda(\mathcal{W}) \mid \Phi \in N(\mathcal{W}, \lambda)\} \subset \mathbb{R}^k.$$

Define for that $\mathcal{J} = \{j \mid \Phi(W_j) \subset L \otimes W_j\} = \{j_1, \dots, j_r\}$. One checks easily that if $\lambda = (\lambda_1, \dots, \lambda_k) \in \Lambda(\mathcal{W})$ then

$$(4.32) \quad \lambda \in \Lambda(\mathcal{W}, \Phi) \iff \lambda_a = \lambda_b \text{ for any } j_i \leq a \leq b \leq j_{i+1}.$$

We claim that the set of indices \mathcal{J} is symmetric:

$$(4.33) \quad j \in \mathcal{J} \iff k - j \in \mathcal{J}.$$

To check this it suffices to prove that $\Phi(W_j) \subset L \otimes W_j$ implies that $\Phi(W_j^{\perp\Omega}) \subset L \otimes W_j^{\perp\Omega}$. Suppose that this is not true, so that for some j we have $\Phi W_j \subset L \otimes W_j$ and there exists some $w \in W_j^{\perp\Omega}$ such that $\Phi w \notin L \otimes W_j^{\perp\Omega}$. Then there exists $v \in W_j$ such that $\Omega(v, \Phi w) \neq 0$. However, since $\Phi \in H^0(L \otimes \mathfrak{sp}(W))$, we must have $\Omega(v, \Phi w) = -\Omega(\Phi v, w)$, and the latter vanishes because by assumption Φv belongs to W_j . So we have reached a contradiction.

Let $\mathcal{J}' = \{j \in \mathcal{J} \mid 2j \leq k\}$ and define for any $j \in \mathcal{J}'$ the vector

$$L_j = -\sum_{c \leq j} e_c + \sum_{d \geq k-j+1} e_d,$$

where e_1, \dots, e_k is the canonical basis of \mathbb{R}^k . It follows from (4.32) and (4.33) that $\Lambda(\mathcal{W}, \Phi)$ is the positive span of the vectors $\{L_j \mid j \in \mathcal{J}'\}$. Consequently, we have

$$d(\mathcal{W}, \lambda) \geq 0 \text{ for any } \lambda \in \Lambda(\mathcal{W}, \Phi) \iff d(\mathcal{W}, L_j) \geq 0 \text{ for any } j.$$

One computes $d(\mathcal{W}, L_j) = -\deg W_{k-j} - \deg W_j$. On the other hand, since we have an exact sequence $0 \rightarrow W_{k-j} \rightarrow W^* \rightarrow W_j^* \rightarrow 0$ (the injective arrow is given by the pairing with Ω) we have $0 = \deg W^* = \deg W_{k-j} + \deg W_j^*$, so $\deg W_{k-j} = \deg W_j$ and consequently $d(\mathcal{W}, L_j) \geq 0$ is equivalent to $\deg W_j \leq 0$, which holds by assumption. Hence $((W, \Omega), \Phi)$ is semistable.

The converse statement, namely, that if $((W, \Omega), \Phi)$ is semistable then for any isotropic subbundle $W' \subset W$ such that $\Phi(W') \subset L \otimes W'$ we have $\deg W' \leq 0$ is immediate by applying the stability condition of the filtration $0 \subset W' \subset W'^{\perp\Omega} \subset W$.

Finally, the proof of the second statement on stability is very similar to case of semistability, so we omit it. The statement on polystability is also straightforward. \square

4.3. $\mathrm{SL}(n, \mathbb{C})$ -Higgs bundles. If $G = \mathrm{SL}(n, \mathbb{C})$ then the maximal compact subgroup of G is $H = \mathrm{SU}(n)$ and hence $H^{\mathbb{C}}$ coincides with $\mathrm{SL}(n, \mathbb{C})$. Now, if $\mathbb{W} = \mathbb{C}^n$ is the fundamental representation of $\mathrm{SL}(n, \mathbb{C})$, the isotropy representation space is given by the traceless endomorphisms of \mathbb{W}

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{sl}(\mathbb{W}) = \{\xi \in \mathrm{End}(\mathbb{W}) \mid \mathrm{Tr} \xi = 0\} \subset \mathrm{End} \mathbb{W},$$

so it coincides again with the adjoint representation of $\mathrm{SL}(n, \mathbb{C})$ on its Lie algebra. An L -twisted $\mathrm{SL}(n, \mathbb{C})$ -Higgs pair is thus a pair consisting of a rank n holomorphic vector bundle W over X endowed with a trivialization $\det W \simeq \mathcal{O}$ and a holomorphic section

$$\Phi \in H^0(L \otimes \mathrm{End}_0 W),$$

where $\mathrm{End}_0 W$ denotes the bundle of traceless endomorphisms of W .

Remark 4.5. Since the center of $\mathfrak{sl}(n, \mathbb{C})$ is trivial, $\alpha = 0$ is the only possible value for which stability of an L -twisted $\mathrm{SL}(n, \mathbb{C})$ -Higgs pair is defined.

Define for any filtration by holomorphic subbundles

$$\mathcal{W} = (0 = W_0 \subsetneq W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_k = W)$$

the convex set

$$\Lambda(\mathcal{W}) = \{(\lambda_1, \lambda_2, \dots, \lambda_k) \in \mathbb{R}^k \mid \lambda_i \leq \lambda_{i+1} \text{ for any } i \text{ and } \sum_i \mathrm{rk} W_i (\lambda_i - \lambda_{i+1}) = 0\}.$$

For any $\lambda \in \Lambda(\mathcal{W})$ define the following subbundle of $L \otimes \mathrm{End} W$:

$$N(\mathcal{W}, \lambda) = L \otimes \mathrm{End}_0 W \cap \sum_{\lambda_i \geq \lambda_j} L \otimes \mathrm{End}(W_i, W_j).$$

Define also

$$d(\mathcal{W}, \lambda) = \sum_{j=1}^{k-1} (\lambda_j - \lambda_{j+1}) \deg W_j$$

(since $\det W$ is trivial we have $\deg W = \deg W_k = 0$).

Following again Sections 2.7 and 2.8, (W, Φ) is said to be:

- **semistable** if for any filtration \mathcal{W} and $\lambda \in \Lambda(\mathcal{W})$ such that $\Phi \in H^0(N(\mathcal{W}, \lambda))$, we have $d(\mathcal{W}, \lambda) \geq 0$.
- **stable** if it is semistable and furthermore, for any choice of filtration \mathcal{W} and $\lambda \in \Lambda(\mathcal{W})$ which is not identically zero (so for which there is a $j < k$ such that $\lambda_j < \lambda_{j+1}$), and such that $\Phi \in H^0(N(\mathcal{W}, \lambda))$, we have $d(\mathcal{W}, \lambda) > 0$.
- **polystable** if it is semistable and for any filtration \mathcal{W} as above and $\lambda \in \Lambda(\mathcal{W})$ satisfying $\lambda_i < \lambda_{i+1}$ for each i , $\psi \in H^0(N(\mathcal{W}, \lambda))$ and $d(\mathcal{W}, \lambda) = 0$, there is an isomorphism

$$W \simeq W_1 \oplus W_2/W_1 \oplus \cdots \oplus W_k/W_{k-1}$$

with respect to which

$$\Phi \in H^0\left(\bigoplus_i L \otimes \mathrm{Hom}(W_i/W_{i-1}, W_i/W_{i-1})\right).$$

Again we have a result as Theorem 4.4 implying that the present notions of (semi)stability coincide with the usual ones.

Theorem 4.6. *An L -twisted $\mathrm{SL}(n, \mathbb{C})$ -Higgs pair (W, Φ) is semistable if and only if for any subbundle $W' \subset W$ such that $\Phi(W') \subset L \otimes W'$ we have $\deg W' \leq 0$. Furthermore, (W, Φ) is stable if for any nonzero and strict subbundle $W' \subset W$ such that $\Phi(W') \subset L \otimes W'$ we have $\deg W' < 0$. Finally, (W, Φ) is polystable if it is semistable and for each subbundle $W' \subset W$ such that $\Phi(W') \subset L \otimes W'$ and $\deg W' = 0$ there is another subbundle $W'' \subset W$ satisfying $\Phi(W'') \subset L \otimes W''$ and $W = W' \oplus W''$.*

The proof of Theorem 4.6 is very similar to that of Theorem 4.4, so we omit it.

4.4. $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles. Let $G = \mathrm{Sp}(2n, \mathbb{R})$. The maximal compact subgroup of G is $H = \mathrm{U}(n)$ and hence $H^\mathbb{C} = \mathrm{GL}(n, \mathbb{C})$. Now, if $\mathbb{V} = \mathbb{C}^n$ is the fundamental representation of $\mathrm{GL}(n, \mathbb{C})$, then the isotropy representation space is:

$$\mathfrak{m}^\mathbb{C} = S^2\mathbb{V} \oplus S^2\mathbb{V}^*.$$

An L -twisted $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs pair is thus a pair consisting of a rank n holomorphic vector bundle V over X and a section

$$\varphi = (\beta, \gamma) \in H^0(L \otimes S^2V \oplus L \otimes S^2V^*).$$

In the particular case when $L = K$, we obtain the notion of an $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle.

Notation 4.7. If W is a vector bundle and $W', W'' \subset W$ are subbundles, then $W' \otimes_S W''$ denotes the subbundle of the second symmetric power S^2W which is the image of $W' \otimes W'' \subset W \otimes W$ under the symmetrization map $W \otimes W \rightarrow S^2W$ (of course this should be defined in sheaf theoretical terms to be sure that $W' \otimes_S W''$ is indeed a subbundle, since the intersection of $W' \otimes W''$ and the kernel of the symmetrization map might change dimension from one fiber to the other). Also, we denote by $W'^\perp \subset W^*$ the kernel of the restriction map $W^* \rightarrow W'^*$.

Let α be a real number. Next we shall state the α -(semi)stability condition for an L -twisted $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs pair. In order to do this, we need to introduce some notation. For any filtration by holomorphic subbundles

$$\mathcal{V} = (0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_k = V)$$

and for any sequence of real numbers $\lambda = (\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k)$ define the subbundle²

$$(4.34) \quad N(\mathcal{V}, \lambda) = \sum_{\lambda_i + \lambda_j \leq 0} L \otimes V_i \otimes_S V_j \oplus \sum_{\lambda_i + \lambda_j \geq 0} L \otimes V_{i-1}^\perp \otimes_S V_{j-1}^\perp \subset L \otimes (S^2V \oplus S^2V^*).$$

Define also³

$$(4.35) \quad d(\mathcal{V}, \lambda, \alpha) = \lambda_k(\deg V_k - \alpha n_k) + \sum_{j=1}^{k-1} (\lambda_j - \lambda_{j+1})(\deg V_j - \alpha n_j),$$

where $n_j = \mathrm{rk} V_j$.

According to Section 2.8 (see also [6]) the α -(semi)stability condition for an L -twisted $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs pair can now be stated as follows.

Proposition 4.8. *The pair (V, φ) is α -semistable if for any filtration $\mathcal{V} = (0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_k = V)$ and for any sequence of real numbers $\lambda = (\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k)$ such that $\varphi \in H^0(N(\mathcal{V}, \lambda))$, the inequality*

$$(4.36) \quad d(\mathcal{V}, \lambda, \alpha) \geq 0$$

holds.

The pair (V, φ) is α -stable if it is α -semistable and furthermore, for any choice of \mathcal{V} and λ for which there is a $j < k$ such that $\lambda_j < \lambda_{j+1}$, whenever $\varphi \in H^0(N(\mathcal{V}, \lambda))$, we have

$$(4.37) \quad d(\mathcal{V}, \lambda, \alpha) > 0.$$

²This is the same as the bundle $L \otimes E(B)_{\sigma, \chi}^-$ of Section 2.8; we use the notation $N(\mathcal{V}, \lambda)$ for convenience.

³This expression is equal to $\deg(E)(\sigma, \chi) - \langle \alpha, \chi \rangle$ of Section 2.8.

In the particular case when $L = K$, we obtain the stability conditions for $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles by setting $\alpha = 0$ above.

It is well known that when $\varphi = 0$, the α -(semi)stability is equivalent to $\alpha = \mu(V)$ (where $\mu(V) = \deg V / \mathrm{rk} V$ is the slope of V) and V being (semi)stable. The next two theorems give a generalization of this fact for general φ , providing a much simpler α -(semi)stability condition (cf. Theorem 2.8.4.13 of Schmitt [21]).

It is important to notice that in the statement of the theorems, the inclusions in the filtration of V are not necessarily strict, in contrast to the original definition. The proofs of these theorems will be given in Subsections 4.6 and 4.7.

Theorem 4.9. *Let (V, φ) be an L -twisted $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs pair. The pair (V, φ) is α -semistable if and only if for any filtration of holomorphic subbundles $0 \subset V_1 \subset V_2 \subset V$ such that*

$$(4.38) \quad \varphi = (\beta, \gamma) \in H^0(L \otimes ((S^2 V_2 + V_1 \otimes_S V) \oplus (S^2 V_1^\perp + V_2^\perp \otimes_S V^*)))$$

we have

$$(4.39) \quad \deg V - \deg V_2 - \deg V_1 \geq \alpha(n - n_2 - n_1),$$

where $n = \mathrm{rk} V$ and $n_i = \mathrm{rk} V_i$.

Remark 4.10. The statement of the Theorem also covers the case $\varphi = 0$, as we shall now explain. If $0 = V_1 = V_2$, then the condition (4.38) is equivalent to $\beta = 0$ and the inequality (4.39) reads $\deg V \geq \alpha n$. If $V_1 = V_2 = V$, then (4.38) is equivalent to $\gamma = 0$ and the inequality (4.39) says that $\deg V \leq \alpha n$. Consequently, if $\varphi = (\beta, \gamma) = 0$, then α -semistability implies $\alpha = \deg V / \mathrm{rk} V = \mu(V)$. In this case, taking $V_1 = 0$ and $V_2 \subset V$ any subbundle, the condition (4.39) is equivalent to $\mu(V_2) \leq \mu(V)$, so V is semistable. On the other hand one can check that if V is semistable and $\alpha = \mu(V)$, then the condition (4.39) is satisfied for any filtration $0 \subset V_1 \subset V_2 \subset V$.

Theorem 4.11. *Let (V, φ) be an L -twisted $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs pair. The pair (V, φ) is α -stable if and only if the following condition is satisfied. For any filtration of holomorphic subbundles $0 \subset V_1 \subset V_2 \subset V$ such that*

$$\varphi \in H^0(L \otimes ((S^2 V_2 + V_1 \otimes_S V) \oplus (S^2 V_1^\perp + V_2^\perp \otimes_S V^*)))$$

the following holds: if at least one of the subbundles V_1 and V_2 is proper (that is, non-zero and different from V) then

$$\deg V - \deg V_2 - \deg V_1 > \alpha(n - n_2 - n_1),$$

(where $n = \mathrm{rk} V$ and $n_i = \mathrm{rk} V_i$), and in any other case

$$\deg V - \deg V_2 - \deg V_1 \geq \alpha(n - n_2 - n_1).$$

Remark 4.12. Arguing as in Remark 4.10 we deduce from the previous theorem that if $\varphi = 0$, then $(V, 0)$ is α -stable if and only if $\alpha = \deg V / \mathrm{rk} V$ and V is a stable vector bundle.

4.5. Some results on convex sets. Let W be an n dimensional vector space over \mathbb{R} . We denote the convex hull of any subset $S \subset W$ by $\text{CH}(S) \subset W$. Let h_1, h_2, \dots, h_l be elements of the dual space W^* . We assume that $l \geq n$ and that the first n elements h_1, \dots, h_n are a basis of W^* . Define for any $h \in W^*$ the set

$$\{h \leq a\} = \{v \in W \mid h(v) \leq a\} \subset W,$$

and define $\{h = a\} \subset W$ similarly.

Consider the convex subset of W

$$C = \bigcap_i \{h_i \leq 0\}$$

(here and below if no range is specified for the index then it is supposed to be the whole set $\{1, \dots, l\}$).

Remark 4.13. The fact that $\{h_1, \dots, h_l\}$ span W^* is equivalent to the condition that C does not contain any positive dimensional vector subspace of W . Indeed, if $h \in W^*$ and $Z \subset W$ is a subspace contained in $\{h \leq 0\}$, then Z is contained in $\{h = 0\}$. Consequently any vector subspace of W contained in C has to lie in $\bigcap_i \{h_i = 0\} = 0$.

Lemma 4.14. $C = \text{CH}(\partial C)$.

Proof. For any $\alpha \leq 0$ define $C_\alpha = C \cap \{h_1 + \dots + h_n = \alpha\}$. Since for any $x \in C$ we have $h_i(x) \leq 0$ and furthermore h_1, \dots, h_n is a basis of W^* , we deduce that C_α is compact. Hence $C_\alpha = \text{CH}(\partial C_\alpha)$. Now, take any $x \in C$ and set $\alpha = h_1(x) + \dots + h_n(x)$. Then $x \in C_\alpha = \text{CH}(\partial C_\alpha) \subset \text{CH}(\partial C)$. This proves the inclusion $C \subset \text{CH}(\partial C)$. The other inclusion follows from the fact that C is convex. \square

Now we have $\partial C = \bigcup_i C_i$, where $C_i = \{h_i = 0\} \cap C$. On the other hand, for any i the collection of elements h_1, \dots, h_l induce elements h'_1, \dots, h'_l on the dual of $\{h_i = 0\}$ which obviously span. Hence we may apply again the lemma to C_i and deduce that $C_i = \text{CH}(\partial C_i)$. Proceeding recursively, we deduce that C is the convex hull of the union of the sets

$$C_I = \bigcap_{i \in I} \{h_i = 0\} \cap C$$

where I runs over the collection of subsets of $\{1, \dots, l\}$ satisfying

$$(4.40) \quad |I| = n - 1 \text{ and the vectors } \{h_i \mid i \in I\} \text{ are linearly independent.}$$

Each such subset C_I is a halfline.

Lemma 4.15. Fix a basis e_1, \dots, e_n of W , and denote by e_1^*, \dots, e_n^* the dual basis. Assume that any h_i can be written either as $e_a^* - e_b^*$ or $\pm(e_a^* + e_b^*)$ for some indices a, b depending on i . Then for any I satisfying (4.40) there are disjoint subsets $P, N \subset \{1, \dots, n\}$ so that defining the element $c_I = \sum_{i \in P} e_i - \sum_{j \in N} e_j$ we have $C_I = \mathbb{R}_{\geq 0} c_I$.

Proof. Pick some I satisfying (4.40), so that $C_I = \bigcap_{i \in I} \{h_i = 0\}$ is one dimensional, and let $c_I \in W$ be an element such that $C_I = \mathbb{R}_{\geq 0} c_I$. Write $c_I = \sum \lambda_j e_j$ and take some nonzero $\lambda \in \{\lambda_1, \dots, \lambda_n\}$. Define $P_\lambda = \{j \mid \lambda_j = \lambda\}$ and $N_\lambda = \{j \mid \lambda_j = -\lambda\}$. We want to prove that for any $j \notin P_\lambda \cup N_\lambda$, $\lambda_j = 0$. Suppose the contrary. Then

$$c'_I = \sum_{j \in P_\lambda \cup N_\lambda} 2\lambda_j e_j + \sum_{j \notin P_\lambda \cup N_\lambda} \lambda_j e_j$$

does not belong to $\mathbb{R}C_I$. However, for any pair of indices a, b we clearly have

$$(e_a^* - e_b^*)c_I = 0 \implies (e_a^* - e_b^*)c'_I = 0 \quad \text{and} \quad (e_a^* + e_b^*)c_I = 0 \implies (e_a^* + e_b^*)c'_I = 0.$$

This implies by our assumption that $c'_I \in \bigcap_{i \in I} \{h_i = 0\} = C_I$, in contradiction with the fact that C_I is one dimensional. \square

4.6. Proof of Theorem 4.9. As already mentioned, when $\varphi = 0$ the pair $(V, 0)$ is α -semistable if and only if $\alpha = \mu(V)$ and V is semistable. Thus, by Remark 4.10, it suffices to consider the case $\varphi \neq 0$. Let \mathcal{V} be any filtration of V , and define

$$\Lambda(\mathcal{V}, \varphi) = \{\lambda \in \mathbb{R}^k \mid \lambda_1 \leq \dots \leq \lambda_k, \varphi \in N(\mathcal{V}, \lambda)\}.$$

The pair (V, φ) is α -semistable if for any $\lambda \in \Lambda(\mathcal{V}, \varphi)$ we have

$$d(\mathcal{V}, \lambda, \alpha) \geq 0.$$

But $d(\mathcal{V}, \lambda, \alpha)$ is clearly a linear function on λ , so to check stability it suffices to verify that $d(\mathcal{V}, \lambda, \alpha) \geq 0$ for any λ belonging to a set $\Lambda' \subset \mathbb{R}^k$ whose convex hull is $\Lambda(\mathcal{V}, \varphi)$. Define for any i, j the subbundles

$$D_{i,j} = V_i \otimes_S V_j + V_{i-1} \otimes_S V + V \otimes_S V_{j-1} \subset S^2 V$$

and

$$D_{i,j}^* = V_{i-1}^\perp \otimes_S V_{j-1}^\perp + V_i^\perp \otimes_S V^* + V^* \otimes_S V_j^\perp \subset S^2 V^*.$$

A tuple $\lambda_1 \leq \dots \leq \lambda_k$ belongs to $\Lambda(\mathcal{V}, \varphi)$ if and only if these two conditions holds:

- for any i, j such that β is contained in $H^0(L \otimes D_{i,j})$ but is not contained in the sum $H^0(L \otimes D_{i-1,j}) + H^0(L \otimes D_{i,j-1})$, we have $\lambda_i + \lambda_j \leq 0$.
- for any i, j such that γ is contained in $H^0(L \otimes D_{i,j}^*)$ but is not contained in the sum $H^0(L \otimes D_{i+1,j}^*) + H^0(L \otimes D_{i,j+1}^*)$, we have $\lambda_i + \lambda_j \geq 0$.

Hence $\Lambda(\mathcal{V}, \varphi) \subset \mathbb{R}^k$ is the intersection of halfspaces of the form $\{\lambda_i - \lambda_{i+1} \leq 0\}$ and, $\{\lambda_a + \lambda_b \leq 0\}$ (for at least one pair (a, b) , if $\beta \neq 0$) or $\{\lambda_c + \lambda_d \geq 0\}$ (for at least one pair (c, d) , if $\gamma \neq 0$). Since the only nonzero vector subspace included in the set $\Lambda = \{\lambda_1 \leq \dots \leq \lambda_k\}$ is the line generated by $(1, \dots, 1)$ and the set $\Lambda(\mathcal{V}, \varphi)$ is contained and Λ and furthermore satisfies at least one equation of the form $\lambda_a + \lambda_b \geq 0$ or $\lambda_c + \lambda_d \leq 0$, it follows that $\Lambda(\mathcal{V}, \varphi)$ does not contain any nonzero vector subspace.

So by the arguments in the previous subsection $\Lambda(\mathcal{V}, \varphi)$ is the convex hull of a collection of half lines of the form $\mathbb{R}_{\geq 0} \lambda_I$, and by Lemma 4.15 we can assume that the coordinates of λ_I are 0 and ± 1 . But if $\lambda_I \in \Lambda(\mathcal{V}, \varphi)$ we necessarily must have $c_I = (-1, \dots, -1, 0, \dots, 0, 1, \dots, 1)$, say a copies of -1 , b of 0 and $k - (a + b)$ of 1. Consider first the case when $0 < a < a + b < k$. Define now the filtration

$$\mathcal{V}' = (0 \subsetneq V_a \subsetneq V_{a+b} \subsetneq V).$$

One can easily check that

$$d(\mathcal{V}, \lambda_I, \alpha) = d(\mathcal{V}', (-1, 0, 1), \alpha) = \deg V - \deg V_a - \deg V_{a+b} - \alpha(n - n_a - n_{a+b}),$$

and that $N(\mathcal{V}, \lambda) = L \otimes ((S^2 V_{a+b} + V_a \otimes_S V) \oplus (S^2 V_a^\perp + V_{a+b}^\perp \otimes_S V^*))$.

Next we need to consider the cases where one or more of the inequalities in the condition $0 < a < a + b < k$ becomes an equality, in which case some of the inclusions in $0 \subsetneq V_a \subsetneq V_{a+b} \subsetneq V$ will not be strict. Since in the semistability condition one has to consider strict inclusions, *a priori* we should consider separately each case (so for example, if $0 < a < a + b = k$, we consider the filtration $0 \subsetneq V_a \subsetneq V$ with weights $\lambda = (-1, 0)$, and

so on). In the following table we list the possible degenerations (apart from the case $a = a + b = k = 0$, which is impossible since $k \geq 1$) and the corresponding form of the conditions $\varphi \in H^0(N(\mathcal{V}, \lambda))$ and $d(\mathcal{V}, \lambda, \alpha) \geq 0$.

Degeneration	$\varphi \in H^0(N(\mathcal{V}, \lambda))$	$d(\mathcal{V}, \lambda, c) \geq 0$
$0 = a < a + b = k$	always satisfied	always satisfied
$0 = a = a + b < k$	$\beta = 0$	$\deg V \geq \alpha n$
$0 < a = a + b = k$	$\gamma = 0$	$\deg V \leq \alpha n$
$0 < a < a + b = k$	$\gamma \in H^0(L \otimes S^2 V_a^\perp)$	$\deg V_a \leq \alpha n_a$
$0 < a = a + b < k$	$\varphi \in H^0(L \otimes (V_a \otimes V \oplus V_a^\perp \otimes V^*))$	$\deg V - 2 \deg V_a \geq \alpha(n - 2n_a)$
$0 < a < a + b < k$	$\beta \in H^0(L \otimes S^2 V_{a+b})$	$\deg V - \deg V_{a+b} \geq \alpha(n - n_{a+b})$

TABLE 4.1. Semistability conditions for degenerate filtrations

Inspecting each of these cases in turn we see that they correspond to instances of the α -semistability condition stated in the Theorem with some inclusions not being strict. More precisely, in each case the subbundle $N(\mathcal{V}, \lambda)$ turns out to coincide with $L \otimes ((S^2 V_{a+b} + V_a \otimes_S V) \oplus (S^2 V_a^\perp + V_{a+b}^\perp \otimes_S V^*))$, and the degree $d(\mathcal{V}, \lambda, \alpha)$ is equal to $\deg V - \deg V_a - \deg V_{a+b} - \alpha(n - n_a - n_{a+b})$.

4.7. Proof of Theorem 4.11. The proof is exactly like that of Theorem 4.9, except that we have to distinguish the cases in which stability implies strict inequality. We assume that $\varphi \neq 0$. Following the notation of Subsection 4.7, these are the cases in which λ contains at least two different values. If $\lambda_I = (-1, \dots, -1, 0, \dots, 0, 1, \dots, 1)$ contains a copies of -1 , b copies of 0 and $k - (a + b)$ copies of 1 , admitting that some of the numbers a , b or $k - (a + b)$ is equal to 0 , the condition that λ_I contains at least two different numbers is equivalent to asking that at least one of the bundles V_a and V_{a+b} is a proper subbundle of V (this happens in the last three rows of Table 4.1). Using the fact that $N(\mathcal{V}, c)$ is the positive span of vectors of the form λ_I (because $\varphi \neq 0$), the theorem follows.

4.8. Polystable $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles. Let α be a real number. Given a filtration \mathcal{V} of V by holomorphic strict subbundles and an increasing sequence λ of real numbers as in Section 4.4, we define $N(\mathcal{V}, \lambda)$ and $d(\mathcal{V}, \lambda, \alpha)$ by (4.34) and (4.35).

According to Section 2.8 the α -polystability condition for an L -twisted $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs pair can now be stated as follows.

Proposition 4.16. *An L -twisted $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs pair (V, φ) with $\varphi = (\beta, \gamma) \in H^0(L \otimes S^2 V \oplus L \otimes S^2 V^*)$ is α -polystable if it is semistable and for any filtration by holomorphic strict subbundles*

$$\mathcal{V} = (0 \subsetneq V_1 \subsetneq V_2 \subsetneq \dots \subsetneq V_k = V),$$

and sequence of strictly increasing real numbers $\lambda = (\lambda_1 < \dots < \lambda_k)$ such that $\varphi \in H^0(N(\mathcal{V}, \lambda))$ and $d(\mathcal{V}, \lambda, \alpha) = 0$ there is a splitting of vector bundles

$$V \simeq V_1 \oplus V_2/V_1 \oplus \dots \oplus V_k/V_{k-1}$$

with respect to which

$$\beta \in H^0\left(\bigoplus_{\lambda_i + \lambda_j = 0} L \otimes V_i/V_{i-1} \otimes_S V_j/V_{j-1}\right)$$

and

$$\gamma \in H^0\left(\bigoplus_{\lambda_i + \lambda_j = 0} L \otimes (V_i/V_{i-1})^* \otimes_S (V_j/V_{j-1})^*\right).$$

It follows from Section 2.10 that any α -polystable G -Higgs pair admits a Jordan–Hölder reduction. In order to state this result in the case of $G = \mathrm{Sp}(2n, \mathbb{R})$, we need to describe some special $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles arising from G -Higgs bundles associated to certain real subgroups $G \subseteq \mathrm{Sp}(2n, \mathbb{R})$.

The subgroup $G = \mathrm{U}(n)$. Observe that a $\mathrm{U}(n)$ -Higgs bundle is nothing but a holomorphic vector bundle V of rank n . The standard inclusion $v^{\mathrm{U}(n)}: \mathrm{U}(n) \hookrightarrow \mathrm{Sp}(2n, \mathbb{R})$ gives the correspondence

$$(4.41) \quad V \mapsto v_*^{\mathrm{U}(n)} V = (V, 0)$$

associating the $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle $v_*^{\mathrm{U}(n)} V = (V, 0)$ to the holomorphic vector bundle V .

The subgroup $G = \mathrm{U}(p, q)$. In the following we assume that $p, q \geq 1$. As is easily seen, a $\mathrm{U}(p, q)$ -Higgs bundle (cf. [5]) is given by the data $(\tilde{V}, \tilde{W}, \tilde{\varphi} = \tilde{\beta} + \tilde{\gamma})$, where \tilde{V} and \tilde{W} are holomorphic vector bundles of rank p and q , respectively, $\tilde{\beta} \in H^0(K \otimes \mathrm{Hom}(\tilde{W}, \tilde{V}))$ and $\tilde{\gamma} \in H^0(K \otimes \mathrm{Hom}(\tilde{V}, \tilde{W}))$. Let $n = p + q$. The imaginary part of the standard indefinite Hermitian metric of signature (p, q) on \mathbb{C}^n is a symplectic form, and thus there is an inclusion $v^{\mathrm{U}(p, q)}: \mathrm{U}(p, q) \hookrightarrow \mathrm{Sp}(2n, \mathbb{R})$. At the level of G -Higgs bundles, this gives rise to the correspondence

$$(4.42) \quad (\tilde{V}, \tilde{W}, \tilde{\varphi} = \tilde{\beta} + \tilde{\gamma}) \mapsto v_*^{\mathrm{U}(p, q)}(\tilde{V}, \tilde{W}, \tilde{\varphi}) = (V, \varphi = \beta + \gamma),$$

where

$$V = \tilde{V} \oplus \tilde{W}^*, \quad \beta = \begin{pmatrix} 0 & \tilde{\beta} \\ \tilde{\beta} & 0 \end{pmatrix} \quad \text{and} \quad \gamma = \begin{pmatrix} 0 & \tilde{\gamma} \\ \tilde{\gamma} & 0 \end{pmatrix}.$$

In the following we shall occasionally slightly abuse language, saying simply that $v_*^{\mathrm{U}(n)} V$ is a $\mathrm{U}(n)$ -Higgs bundle and that $v_*^{\mathrm{U}(p, q)}(\tilde{V}, \tilde{W}, \tilde{\varphi})$ is a $\mathrm{U}(p, q)$ -Higgs bundle.

Another piece of convenient notation is the following. Let (V_i, φ_i) be $\mathrm{Sp}(2n_i, \mathbb{R})$ -Higgs bundles and let $n = \sum n_i$. We can define an $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle (V, φ) by setting

$$V = \bigoplus V_i \quad \text{and} \quad \varphi = \sum \varphi_i$$

by using the canonical inclusions $H^0(K \otimes (S^2 V_i \oplus S^2 V_i^*)) \subset H^0(K \otimes (S^2 V \oplus S^2 V^*))$. We shall slightly abuse language and write $(V, \varphi) = \bigoplus (V_i, \varphi_i)$, referring to this as **the direct sum** of the (V_i, φ_i) .

With all this understood, we can state our structure theorem on polystable $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles from Section 2.10 as follows.

Proposition 4.17. *Let (V, φ) be a polystable $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle. Then there is a decomposition*

$$(V, \varphi) = (V_1, \varphi_1) \oplus \cdots \oplus (V_k, \varphi_k),$$

unique up to reordering, such that each (V_i, φ_i) is a stable G_i -Higgs bundle, where G_i is one of the following groups: $\mathrm{Sp}(2n_i, \mathbb{R})$, $\mathrm{U}(n_i)$ or $\mathrm{U}(p_i, q_i)$.

4.9. $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundles. We study now L -twisted G -Higgs pairs for $G = \mathrm{GL}(n, \mathbb{R})$. This is a case when the general L -twisted case is of particular interest since, when $L = K^2$, these are important in the study of maximal degree $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles (see [10]).

A maximal compact subgroup of $\mathrm{GL}(n, \mathbb{R})$ is $H = \mathrm{O}(n)$ and hence $H^{\mathbb{C}} = \mathrm{O}(n, \mathbb{C})$. Now, if \mathbb{W} is the standard n -dimensional complex vector space representation of $\mathrm{O}(n, \mathbb{C})$, then the isotropy representation space is:

$$\mathfrak{m}^{\mathbb{C}} = S^2 \mathbb{W}.$$

An L -twisted $\mathrm{GL}(n, \mathbb{R})$ -Higgs pair over X is thus a pair $((W, Q), \psi)$ consisting of a holomorphic $\mathrm{O}(n, \mathbb{C})$ -bundle, i.e. a rank n holomorphic vector bundle W over X equipped with a non-degenerate quadratic form Q , and a section

$$\psi \in H^0(L \otimes S^2 W).$$

Note that when $\psi = 0$ a twisted $\mathrm{GL}(n, \mathbb{R})$ -Higgs pair is simply an orthogonal bundle.

Remark 4.18. Since the center of $\mathfrak{o}(n)$ is trivial, $\alpha = 0$ is the only possible value for which stability of an L -twisted $\mathrm{GL}(n, \mathbb{R})$ -Higgs pair is defined.

In order to state the stability condition for twisted $\mathrm{GL}(n, \mathbb{R})$ -Higgs pairs, we first introduce some notation. For any filtration of vector bundles

$$\mathcal{W} = (0 = W_0 \subsetneq W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_k = W)$$

of satisfying $W_j = W_{k-j}^{\perp Q}$ (here $W_{k-j}^{\perp Q}$ denotes the orthogonal complement of W_{k-j} with respect to Q) define

$$\Lambda(\mathcal{W}) = \{(\lambda_1, \lambda_2, \dots, \lambda_k) \in \mathbb{R}^k \mid \lambda_i \leq \lambda_{i+1} \text{ and } \lambda_i + \lambda_{k-i+1} = 0 \text{ for any } i\}.$$

Define for any $\lambda \in \Lambda(\mathcal{W})$ the following bundle.

$$N(\mathcal{W}, \lambda) = \sum_{\lambda_i + \lambda_j \leq 0} L \otimes W_i \otimes_S W_j.$$

Also we define

$$d(\mathcal{W}, \lambda) = \sum_{j=1}^{k-1} (\lambda_j - \lambda_{j+1}) \deg W_j$$

(note that the quadratic form Q induces an isomorphism $W \simeq W^*$ so $\deg W = \deg W_k = 0$).

According to Section 2.8 (see also [6]) the stability conditions (for $\alpha = 0$) for an L -twisted $\mathrm{GL}(n, \mathbb{R})$ -Higgs pair can now be stated as follows.

Proposition 4.19. *an L -twisted $\mathrm{GL}(n, \mathbb{R})$ -Higgs pair (W, Q, ψ) is semistable if for all filtrations \mathcal{W} as above and all $\lambda \in \Lambda(\mathcal{W})$ such that $\psi \in H^0(N(\mathcal{W}, \lambda))$, we have $d(\mathcal{W}, \lambda) \geq 0$.*

The pair (W, ψ) is stable if it is semistable and for any choice of filtration \mathcal{W} and nonzero $\lambda \in \Lambda(\mathcal{W})$ such that $\psi \in H^0(N(\mathcal{W}, \lambda))$, we have $d(\mathcal{W}, \lambda) > 0$.

The pair (W, ψ) is polystable if it is semistable and for any filtration \mathcal{W} as above and $\lambda \in \Lambda(\mathcal{W})$ satisfying $\lambda_i < \lambda_{i+1}$ for each i , $\psi \in H^0(N(\mathcal{W}, \lambda))$ and $d(\mathcal{W}, \lambda) = 0$, there is an isomorphism

$$W \simeq W_1 \oplus W_2/W_1 \oplus \cdots \oplus W_k/W_{k-1}$$

such that pairing via Q any element of the summand W_i/W_{i-1} with an element of the summand W_j/W_{j-1} is zero unless $i + j = k + 1$; furthermore, via this isomorphism,

$$\psi \in H^0\left(\bigoplus_{\lambda_i + \lambda_j = 0} L \otimes (W_i/W_{i-1}) \otimes_S (W_j/W_{j-1})\right).$$

There is a simplification of the stability condition for orthogonal pairs analogous to Theorem 4.9 and Theorem 4.11.

Theorem 4.20. *The L -twisted $\mathrm{GL}(n, \mathbb{R})$ -Higgs pair $((W, Q), \psi)$ is semistable if and only if for any isotropic subbundle $W' \subset W$ such that $\psi \in H^0(S^2 W'^{\perp Q} \oplus W' \otimes_S W \otimes L)$ the inequality $\deg W' \leq 0$ holds. Furthermore, $((W, Q), \psi)$ is stable if it is semistable and for any isotropic strict subbundle $0 \neq W' \subset W$ such that $\psi \in H^0(S^2 W'^{\perp Q} \oplus W' \otimes_S W \otimes L)$ we have $\deg W' < 0$ holds. Finally, $((W, Q), \psi)$ is polystable if it is semistable and for any isotropic strict subbundle $0 \neq W' \subset W$ such that $\psi \in H^0(S^2 W'^{\perp Q} \oplus W' \otimes_S W \otimes L)$ and $\deg W' = 0$ there is another isotropic subbundle $W'' \subset W$ such that $\psi \in H^0(S^2 W''^{\perp Q} \oplus W'' \otimes_S W \otimes L)$ and $W = W' \oplus W''$.*

Proof. The proof is analogous to the proofs of Theorems 4.9 and 4.11. Take an L -twisted $\mathrm{GL}(n, \mathbb{R})$ -Higgs pair $((W, Q), \psi)$, and assume that for any isotropic subbundle $W' \subset W$ such that $\psi \in H^0(S^2 W'^{\perp Q} \oplus W' \otimes_S W \otimes L)$ the inequality $\deg W' \leq 0$ holds. We also assume that ψ is nonzero, for otherwise the result follows from the usual characterization of (semi)stability for $\mathrm{SO}(n, \mathbb{C})$ -principal bundles due to Ramanathan (see [17]). We want to prove that $((W, Q), \psi)$ is semistable. Choose any filtration $\mathcal{W} = (0 \subsetneq W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_k = W)$ satisfying $W_{k-i} = W_i^{\perp Q}$ for any i . Consider the convex set

$$\Lambda(\mathcal{W}, \psi) = \{\lambda \in \Lambda(\mathcal{W}) \mid \psi \in N(\mathcal{W}, \lambda)\} \subset \mathbb{R}^k.$$

Define for any i, j the subbundle

$$D_{i,j} = W_i \otimes_S W_j + W_{i-1} \otimes_S W + W \otimes_S W_{j-1} \subset S^2 W.$$

A tuple $\lambda = (\lambda_1, \dots, \lambda_k) \in \Lambda(\mathcal{W})$ belongs to $\Lambda(\mathcal{W}, \psi)$ if and only if:

for any i, j such that ψ is contained in $H^0(L \otimes D_{i,j})$ but is not contained in the sum $H^0(L \otimes D_{i-1,j}) + H^0(L \otimes D_{i,j-1})$, we have $\lambda_i + \lambda_j \leq 0$.

Hence $\Lambda(\mathcal{W}, \psi)$ is the intersection of $\Lambda(\mathcal{W})$ with the set of points in \mathbb{R}^k satisfying a collection of inequalities of the form $\lambda_a + \lambda_b \leq 0$ and $\lambda_c + \lambda_d \geq 0$ (the latter follow from the restrictions $\lambda_i + \lambda_{k-i+1} = 0$). Since $\Lambda(\mathcal{W})$ does not contain any line, a fortiori $\Lambda(\mathcal{W}, \psi)$ neither does, so (using Lemma 4.15) $\Lambda(\mathcal{W}, \psi)$ is the convex hull of a set of half lines $\{\mathbb{R}_{\geq 0} L_i \mid i \in \mathcal{I}\}$, where $L_i = (-1, \dots, -1, 0, \dots, 0, 1, \dots, 1)$ contains i copies of -1 and i copies of 1 . Consequently, we have

$$d(\mathcal{W}, \lambda) \geq 0 \text{ for any } \lambda \in \Lambda(\mathcal{W}, \psi) \iff d(\mathcal{W}, L_i) \geq 0 \text{ for any } i \in \mathcal{I}.$$

It follows from the definition that $N(\mathcal{W}, L_i) = W_i \otimes_S W + S^2 W_{k-i}$ and since $W_{k-i} = W_i^{\perp Q}$ the condition $L_i \in \Lambda(\mathcal{W}, \psi)$ can be translated into the condition

$$\psi \in H^0(S^2 W_i^{\perp Q} \oplus W_i \otimes_S W \otimes L).$$

One computes $d(\mathcal{W}, L_i) = -\deg W_{k-i} - \deg W_i$. On the other hand, since we have an exact sequence $0 \rightarrow W_{k-i} \rightarrow W^* \rightarrow W_i^* \rightarrow 0$ (the injective arrow is given by the pairing with the quadratic form Q) we have $0 = \deg W^* = \deg W_{k-i} + \deg W_i^*$, so $\deg W_{k-i} = \deg W_i$

and consequently $d(\mathcal{W}, L_i) \geq 0$ is equivalent to $\deg W_i \leq 0$, which holds by assumption. Hence $((W, Q), \psi)$ is semistable.

The converse statement, namely, that if $((W, Q), \psi)$ is semistable then for any isotropic subbundle $W' \subset W$ such that $\Phi(W') \subset L \otimes W'$ we have $\deg W' \leq 0$ is immediate by applying the stability condition of the filtration $0 \subset W' \subset W'^{\perp_Q} \subset W$.

Finally, the proof of the second statement on stability is very similar to the case of semistability, so we omit it. The statement on polystability is also straightforward. \square

Remark 4.21. The condition $\psi \in H^0(S^2 W_1^{\perp_Q} \oplus W_1 \otimes_S W \otimes L)$ is equivalent to $\tilde{\psi}(W_1) \subseteq W_1 \otimes L$, where $\tilde{\psi} = \psi \circ Q: W \rightarrow W \otimes L$.

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