REPRESENTATIONS OF SURFACE GROUPS IN REAL SYMPLECTIC GROUPS

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ABSTRACT. In this paper we study the moduli space of representations of a surface group (i.e., the fundamental group of a closed oriented surface) in the real symplectic group $\mathrm{Sp}(2n,\mathbb{R})$. The moduli space is partitioned by an integer invariant, called the Toledo invariant. This invariant is bounded by a Milnor–Wood type inequality. Our main result is a count of the number of connected components of the moduli space of maximal representations, i.e. representations with maximal Toledo invariant. Our approach uses non-abelian Hodge theory through the correspondence of the moduli space of representations with the moduli space of polystable $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundles. A key step is provided by the discovery of new discrete invariants of maximal representations. These new invariants arise from an identification, in the maximal case, of the moduli space of $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundles with a moduli space of twisted Higgs bundles for the group $\mathrm{GL}(n,\mathbb{R})$. In two appendices we develop a Hitchin–Kobayashi correspondence in the generality required for the application of Higgs bundle theory to the problem at hand. This includes a general study of the notion of polystability for G-Higgs bundles for a real reductive Lie group G.

CONTENTS

1. INTRODUCTION

In this paper we study representations of the fundamental group of a compact oriented surface X in $\mathrm{Sp}(2n, \mathbb{R})$ — the group of linear transformations of $\mathbb{R}^{2n}$ which preserve the standard symplectic form. By a representation we mean a homomorphism from $\pi_1(X)$ to $\mathrm{Sp}(2n, \mathbb{R})$. Given a representation of $\pi_1(X)$ in $\mathrm{Sp}(2n, \mathbb{R})$ there is an integer, often referred to as the *Toledo invariant*, associated to it. This integer can be obtained geometrically by considering the flat $\mathrm{Sp}(2n, \mathbb{R})$-bundle corresponding to the representation and taking a reduction of the structure group of the underlying smooth vector bundle to $\mathrm{U}(n)$ — a maximal compact subgroup of $\mathrm{Sp}(2n, \mathbb{R})$. The degree of the resulting $\mathrm{U}(n)$-bundle is the Toledo invariant (this is well defined because $\mathrm{Sp}(2n, \mathbb{R})/\mathrm{U}(n)$ is contractible, so all reductions of the structure group from $\mathrm{Sp}(2n, \mathbb{R})$ to $\mathrm{U}(n)$ are homotopic and hence define isomorphic complex vector bundles). As shown by Turaev [53] the Toledo invariant d of a representation satisfies the inequality

$$(1.1) \qquad\qquad\qquad |d| \leq n(g-1),$$

where g is the genus of the surface. When $n = 1$, one has $\mathrm{Sp}(2, \mathbb{R}) \cong \mathrm{SL}(2, \mathbb{R})$, the Toledo invariant coincides with the Euler class of the $\mathrm{SL}(2, \mathbb{R})$-bundle, and (1.1) is the classical inequality of Milnor [37] which was later generalized by Wood [55]. We shall follow custom and refer to (1.1) as as the Milnor–Wood inequality.

Given two representations, a basic question to ask is whether one can be continuously deformed into the other. Put in a more precise way, we are asking for the connected components of the space of representations

$$\mathrm{Hom}(\pi_1(X), \mathrm{Sp}(2n, \mathbb{R})).$$

As shown in [24], this space has the same number of connected components as the moduli space, or character variety,

$$\mathcal{R}(\pi_1(X), \mathrm{Sp}(2n, \mathbb{R})) = \mathrm{Hom}^+(\pi_1(X), \mathrm{Sp}(2n, \mathbb{R}))/\mathrm{Sp}(2n, \mathbb{R})$$

of reductive representations $\rho\colon \pi_1(X) \to \mathrm{Sp}(2n, \mathbb{R})$, modulo the natural equivalence given by the action of $\mathrm{Sp}(2n, \mathbb{R})$ by overall conjugation. The notation "Hom$^+$" refers to reductive representations, i.e., those whose image has reductive Zariski closure. Replacing Hom by Hom$^+$ is justified by the fact that the quotient space $\mathrm{Hom}^+(\pi_1(X), \mathrm{Sp}(2n, \mathbb{R}))/\mathrm{Sp}(2n, \mathbb{R})$ is Hausdorff, whereas $\mathrm{Hom}^+(\pi_1(X), \mathrm{Sp}(2n, \mathbb{R}))/\mathrm{Sp}(2n, \mathbb{R})$ is not Hausdorff in general (see Theorem 11.4 in [43]).

The Toledo invariant descends to the quotient so, for any d satisfying (1.1), we can define
$$\mathcal{R}_d(\pi_1(X), \mathrm{Sp}(2n, \mathbb{R})) \subset \mathcal{R}(\pi_1(X), \mathrm{Sp}(2n, \mathbb{R}))$$
to be the subspace of representations with Toledo invariant d. For ease of notation, for the remaining part of the Introduction, we shall write $\mathcal{R}_d$ for $\mathcal{R}_d(\pi_1(X), \mathrm{Sp}(2n, \mathbb{R}))$ and $\mathcal{R}$ for $\mathcal{R}(\pi_1(X), \mathrm{Sp}(2n, \mathbb{R}))$. Since the Toledo invariant varies continuously with the representation, the subspace $\mathcal{R}_d$ is a union of connected components, and our basic problem is that of counting the number of connected components of $\mathcal{R}_d$ for d satisfying (1.1). This has been done for $n = 1$ by Goldman [25, 28] and Hitchin [31], and for $n = 2$ in [29] (in the cases $d = 0$ and $|d| = 2g - 2$) and [24] (in the cases $|d| < 2g - 2$). In this paper we count the number of connected components of $\mathcal{R}_d$ for $n > 2$ when $d = 0$ and $|d| = n(g-1)$ — the maximal value allowed by the Milnor–Wood inequality. Our main result is the following (Theorem 8.7 below).

Theorem 1.1. *Let X be a compact oriented surface of genus g. Let $\mathcal{R}_d$ be the moduli space of reductive representations of $\pi_1(X)$ in $\mathrm{Sp}(2n, \mathbb{R})$ with Toledo invariant d. Let $n \geq 3$. Then*

(1) *$\mathcal{R}_0$ is non-empty and connected;*
(2) *$\mathcal{R}_{\pm n(g-1)}$ has 3.2^{2g} non-empty connected components.*

The main tool we employ to count connected components is the theory of Higgs bundles, as pioneered by Hitchin [31] for $\mathrm{SL}(2, \mathbb{R}) = \mathrm{Sp}(2, \mathbb{R})$. In the following we outline the main features of the theory which make it relevant to our problem — much more detail will be provided in the body of the paper. We fix a complex structure on X endowing it with a structure of a compact Riemann surface, which we will denote, abusing notation, also by X. An $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle over X is a triple (V, β, γ) consisting of a rank n holomorphic vector bundle V and holomorphic sections $\beta \in H^0(X, S^2 V \otimes K)$ and $\gamma \in H^0(X, S^2 V^* \otimes K)$, where K is the canonical line bundle of X. The sections β and γ are often referred to as Higgs fields. Looking at X as an algebraic curve, algebraic moduli spaces for $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle exist as a consequence of the work of Schmitt [44, 45]. Fixing $d \in \mathbb{Z}$, we denote by $\mathcal{M}_d$ the moduli space of $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundles on X with $\deg V = d$. As usual, one must introduce an appropriate stability condition (with related conditions of poly- and semistability) in order to have good moduli spaces. Thus $\mathcal{M}_d$ parametrizes isomorphism classes of polystable $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundles. A basic result of non-abelian Hodge theory, growing out of the work of Corlette [17], Donaldson [19], Hitchin [31] and Simpson [47, 48, 49, 50], is the following (Theorem 2.28 below).

Theorem 1.2. *The moduli spaces $\mathcal{R}_d$ and $\mathcal{M}_d$ are homeomorphic.*

An essential part of the proof of this Theorem follows from a Hitchin–Kobayashi correspondence between polystable $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundles and solutions to certain gauge theoretic equations, known as Hitchin's equations, for a triple (A, β, γ). Here A is a smooth unitary connection on a smooth complex vector bundle of rank n and degree d, and β and

γ are smooth counterparts to the holomorphic sections defined above. Under the correspondence, A is the Chern connection on the the holomorphic bundle V introduced above endowed with a suitable Hermitian metric which is an analogue for Higgs bundles of the Hermite–Einstein metric. In the generality required for stable $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundles, the Hitchin–Kobayashi correspondence is provided by the general theory of [10]. However, in that paper no proper attention was given to the case of polystable (non-stable) pairs in general, and we take the opportunity to fill this gap in the present paper.

Using the homeomorphism $\mathcal{R}_d \cong \mathcal{M}_d$, our problem is reduced to studying the connectedness properties of $\mathcal{M}_d$. This is done by using the Hitchin functional. This is a proper non-negative function which is defined on $\mathcal{M}_d$ using the solution to Hitchin's equations, as follows:

$$\text{(1.2)} \qquad \begin{aligned} f\colon \mathcal{M}_d &\to \mathbb{R}, \\ (A,\beta,\gamma) &\mapsto \tfrac{1}{2}\|\beta\|^2 + \tfrac{1}{2}\|\gamma\|^2. \end{aligned}$$

Here $\|\cdot\|$ is the L^2-norm obtained by using the Hermitian metric in V whose Chern connection gives a solution to Hitchin equations and integrating over X. This function arises as the moment map for the Hamiltonian circle action on the moduli space obtained by multiplying the Higgs field by an element of $\mathrm{U}(1)$. It was proved by Hitchin [31, 32] that f is proper, and this implies that f has a mimimun on each connected component of $\mathcal{M}_d$. Using this fact, our problem essentially reduces to characterizing the subvariety of minima of the Hitchin functional and studying its connectedness properties.

While we characterize the minima for every value of d satisfying the Milnor–Wood inequality (see Theorem 5.10), we only carry out the full programme for $d = 0$ and $|d| = n(g-1)$, the extreme values of d. For $d = 0$, the subvariety of minima of the Hitchin functional on $\mathcal{M}_0$ coincides with the set of Higgs bundles (V,β,γ) with $\beta = \gamma = 0$. This, in turn, can be identified with the moduli space of polystable vector bundles of rank n and degree 0. Since this moduli space is connected by the results of Narasimhan–Seshadri [39], $\mathcal{M}_0$ is connected and hence $\mathcal{R}_0$ is connected.

The analysis for the *maximal case*, $|d| = n(g-1)$, is far more involved and interesting. It turns out that in this case one of the Higgs fields β or γ for a semistable Higgs bundle (V,β,γ) becomes an isomorphism. Whether it is β or γ, actually depends on the sign of the Toledo invariant. Since the map $(V,\beta,\gamma) \mapsto (V^*,\gamma^t,\beta^t)$ defines an isomorphism $\mathcal{M}_{-d} \cong \mathcal{M}_d$, there is no loss of generality in assuming that $0 \le d \le n(g-1)$. Suppose that $d = n(g-1)$. Then $\gamma : V \to V^* \otimes K$ is an isomorphism (see Proposition 3.22). Since γ is furthermore symmetric, it equips V with a K-valued nondegenerate quadratic form. In order to have a proper quadratic bundle, we fix a square root $L_0 = K^{1/2}$ of the canonical bunle, and define $W = V^* \otimes L_0$. Then $Q := \gamma \otimes I_{L_0^{-1}} : W^* \to W$ is a symmetric isomorphism defining an orthogonal structure on W, in other words, (W,Q) is an $\mathrm{O}(n,\mathbb{C})$-holomorphic bundle. The K^2-twisted endomorphism $\psi : W \to W \otimes K^2$ defined by $\psi = (\gamma \otimes I_{K\otimes L_0}) \circ \beta \otimes I_{L_0}$ is Q-symmetric and hence (W,Q,ψ) defines what we call a K^2-twisted $\mathrm{GL}(n,\mathbb{R})$-Higgs pair, from which we can recover the original $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundle. The main result is the following (Theorem 4.4 below).

Theorem 1.3. *Let $\mathcal{M}_{\max}$ be the moduli space of polystable $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundles with $d = n(g-1)$, and let $\mathcal{M}'$ be the moduli space of polystable K^2-twisted $\mathrm{GL}(n,\mathbb{R})$-Higgs pairs. The map $(V,\beta,\gamma) \mapsto (W,Q,\psi)$ defines an isomorphism of complex algebraic varieties*

$$\mathcal{M}_{\max} \cong \mathcal{M}'.$$

We refer to this isomorphism as the *Cayley correspondence*. This name is motivated by the geometry of the bounded symmetric domain associated to the Hermitian symmetric space $\mathrm{Sp}(2n, \mathbb{R})/\mathrm{U}(n)$. The Cayley transform defines a biholomorphism between this domain and a tube type domain defined over the symmetric cone $\mathrm{GL}(n, \mathbb{R})/\mathrm{O}(n)$ — the Siegel upper half-space. In fact, there is a similar correspondence to that given in Theorem 1.3 for every semisimple Lie group G which, like $\mathrm{Sp}(2n, \mathbb{R})$, is the group of isometries of a Hermitian symmetric space of tube type (see [8] for a survey on this subject).

A key point is that the Cayley correspondence brings to the surface new topological invariants, hidden a priori, which are naturally attached to an $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle with maximal Toledo invariant. These are the first and second Stiefel-Whitney classes (w_1, w_2) of a reduction to $\mathrm{O}(n)$ of the $\mathrm{O}(n, \mathbb{C})$-bundle defined by (W, Q). It turns out that there is a connected component for each possible value of (w_1, w_2), containing K^2-twisted $\mathrm{GL}(n, \mathbb{R})$-Higgs pairs (W, Q, ψ) with $\psi = 0$. This accounts for 2.2^{2g} of the 3.2^{2g} connected components of $\mathcal{M}_{\max}$. Thus it remains to account for the 2^{2g} "extra" components. As already mentioned, the group $\mathrm{Sp}(2n, \mathbb{R})$ is the group of isometries of a Hermitian symmetric space, but it also has the property of being a split real form. In fact, up to finite coverings, it is the only Lie group with this property. In [32] Hitchin shows that for every semisimple split real Lie group G, the moduli space of reductive representations of $\pi_1(X)$ in G has a topological component which is isomorphic to $\mathbb{R}^{\dim G(2g-2)}$, and which naturally contains Teichmüller space. Indeed, when $G = \mathrm{SL}(2, \mathbb{R})$, this component can be identified with Teichmüller space, via the Riemann uniformization theorem. Since $\mathrm{Sp}(2n, \mathbb{R})$ is split, the moduli space for $\mathrm{Sp}(2n, \mathbb{R})$ must have a Hitchin component. It turns out that there are 2^{2g} isomorphic Hitchin components (this is actually true for arbitrary n). As follows from Hitchin's construction, the K^2-twisted Higgs pairs (W, Q, ψ) in the Hitchin component all have $\psi \neq 0$.

From many points of view maximal representations are the most interesting ones. They have been the object of intense study in recent years, using methods from diverse branches of geometry, and it has become clear that they enjoy very special properties. In particular, at least in many cases, maximal representations have a close relationship to geometric structures on the surface. The prototype of this philosophy is Goldman's theorem [25, 27] that the maximal representations in $\mathrm{SL}(2, \mathbb{R})$ are exactly the Fuchsian ones. In the following, we briefly mention some results of this kind.

Using bounded cohomology methods, maximal representations in general Hermitian type groups have been studied by Burger–Iozzi [11, 12] and Burger–Iozzi–Wienhard [14, 15, 16]. Among many other results, they have given a very general Milnor–Wood inequality and they have shown that maximal representations are discrete, faithful and completely reducible. One consequence of this is that the restriction to reductive representations is unnecessary in the case of the moduli space $\mathcal{R}_{\max}$ of maximal representations. Building on this work and the work of Labourie [36], Burger–Iozzi–Labourie–Wienhard [13] have shown that maximal representations in $\mathrm{Sp}(2n, \mathbb{R})$ are Anosov (in the sense of [36]). Furthermore, it has been shown that the action of the mapping class group on $\mathcal{R}_{\max}$ is proper, by Wienhard [54] (for classical simple Lie groups of Hermitian type), and by Labourie [35] (for $\mathrm{Sp}(2n, \mathbb{R})$), who also proves further geometric properties of maximal representations in $\mathrm{Sp}(2n, \mathbb{R})$.

From yet a different perspective, representations in the Teichmüller component have been studied in the work on higher Teichmüller theory of Fock–Goncharov [21], using methods

of tropical geometry. In particular, the fact that representations in the Teichmüller component for $\mathrm{Sp}(2n, \mathbb{R})$ are faithful and discrete also follows from their work

Thus, while Higgs bundle techniques are very efficient in the study of topological properties of the moduli space (like counting components), these other approaches have been more powerful in the study of special properties of individual representations. It would be very interesting indeed to gain a better understanding of the relation between these distinct methods.

We describe now briefly the content of the different sections of the paper.

In Appendix A we review the general theory of L-twisted pairs and the Hitchin–Kobayashi correspondence over a compact Riemann surface X. By an L-twisted pair over X we mean a pair (E, φ) consisting of a holomorphic $H^{\mathbb{C}}$-principal bundle, where $H^{\mathbb{C}}$ is a complex reductive Lie group and φ is a holomorphic section of $E(B) \otimes L$, where $E(B)$ is the vector bundle associated to a complex representation $H^{\mathbb{C}} \to \mathrm{GL}(B)$ and L is a holomorphic line bundle over X.

Appendix B deals with L-twisted G-Higgs pairs over a compact Riemann surface X. Let G be a reductive real Lie group with maximal compact subgroup $H \subset G$, let L be a holomorphic line bundle over X and let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be the Cartan decomposition of $\mathfrak{g}$. Then an L-twisted G-Higgs pair is a pair (E, φ), consisting of a holomorphic $H^{\mathbb{C}}$-principal bundle E over X and a holomorphic section φ of $E(\mathfrak{m}^{\mathbb{C}}) \otimes L$. Here $E(\mathfrak{m}^{\mathbb{C}})$ is the $\mathfrak{m}^{\mathbb{C}}$-bundle associated to E via the isotropy representation $H^{\mathbb{C}} \to \mathrm{GL}(\mathfrak{m}^{\mathbb{C}})$. These objects are a particular case of the general twisted pairs introduced in Appendix A. We study how the stability condition stated in general in Appendix A simplifies for L-twisted G-Higgs pairs for various groups relevant to our study. This certainly includes $G = \mathrm{Sp}(2n, \mathbb{R})$, but also other groups that naturally contain $\mathrm{Sp}(2n, \mathbb{R})$, like $\mathrm{Sp}(2n, \mathbb{C})$, and $\mathrm{SL}(2n, \mathbb{C})$, as well as $\mathrm{GL}(n, \mathbb{R})$, which is the group of isometries of the cone of the tube domain associated to $\mathrm{Sp}(2n, \mathbb{R})$.

In Section 2 we study non-abelian Hodge theory over a compact Riemann surface X for a general connected semisimple Lie group G. We introduce G-Higgs bundles over X — these are simply K-twisted G-Higgs pairs, where K is the canonical line bundle over X —, and study their deformations and their moduli spaces. An important result is the correspondence between the moduli space of polystable G-Higgs bundles and the moduli space of solutions to the Hitchin equations. While this is well-known when G is actually complex [31, 47, 48] or compact [39, 41], a proof for the non-compact non-complex case follows from [10] for stable G-Higgs bundles. In this paper, we prove the general case of a polystable G-Higgs bundle. The result is a consequence of the more general Hitchin–Kobayashi correspondence given in Theorem A.17.

We then introduce the moduli space of reductive representations of the fundamental group of a compact Riemann surface in a Lie group G and, using Corlette's existence theorem of harmonic metrics [17], we establish the correspondence between this moduli space and the moduli space of polystable G-Higgs bundles when G is connected and semisimple.

In Section 3, we specialize the non-abelian Hodge theory correspondence of Section 2 to $G = \mathrm{Sp}(2n, \mathbb{R})$ — our case of interest in this paper. Using technical results given in Appendix B, we prove basic facts about the moduli space of $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundles, including the Milnor–Wood inequality. To do this, we study and exploit the relation between the polystability of a $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundles and the $\mathrm{SL}(2n, \mathbb{C})$-Higgs bundle naturally associated to it.

In Section 4 we study the Cayley correspondence between $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundles with maximal Toledo invariant and K^2-twisted $\mathrm{GL}(n,\mathbb{R})$-Higgs pairs.

The rest of the paper is mostly devoted to the study of the connectedness properties of the moduli space of $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundles and, in particular, to prove Theorem 8.3. In Section 5 we introduce the Hitchin functional on the moduli space of $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundles and characterize its minima. We then use this and the Cayley correspondence of Section 4 to count the number of connected components of the moduli space of $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundles for $d = 0$ and $|d| = n(g-1)$. The proof of the characterization of the minima is split in two cases: the case of minima in the smooth locus of the moduli space, given in Section 6 and the case of the remaining minima, treated in Section 7.

The results of this paper have been announced in several conferences over the last four years or so, while several preliminary versions of this paper have been circulating. The main results, together with analogous results for other groups of Hermitian type have appeared in the review paper [8]. The authors apologize for having taken so long in producing this final version.

2. G-Higgs bundles and surface group representations

2.1. G-Higgs bundles. Let G be a real reductive Lie group, let $H \subset G$ be a maximal compact subgroup and let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be a Cartan decomposition, so that the Lie algebra structure on $\mathfrak{g}$ satisfies

$$[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}, \qquad [\mathfrak{h},\mathfrak{m}] \subset \mathfrak{m}, \qquad [\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}.$$

The group H acts linearly on $\mathfrak{m}$ through the adjoint representation, and this action extends to a linear holomorphic action of $H^{\mathbb{C}}$ on $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m} \otimes \mathbb{C}$ — the isotropy representation.

Let X be a compact Riemann surface and let K be its canonical line bundle.

Definition 2.1. A G-**Higgs bundle** over X is a pair (E, φ) consisting of a principal holomorphic $H^{\mathbb{C}}$-bundle E over X and a holomorphic section of $E(\mathfrak{m}^{\mathbb{C}}) \otimes K$, where $E(\mathfrak{m}^{\mathbb{C}})$ is the $\mathfrak{m}^{\mathbb{C}}$-bundle associated to E via the isotropy representation.

In other words, a G-Higgs bundle is a K-twisted G-Higgs pair in the sense of Appendix B. Thus, as for any twisted G-Higgs pair, α-stability, semistability and polystability are defined for any $\alpha \in i\mathfrak{h} \cap \mathfrak{z}$, where $\mathfrak{z}$ is the centre of $\mathfrak{h}^{\mathbb{C}}$. However, in order to relate G-Higgs bundles to representations of the fundamental group of X (or certain central extension of the fundamental group) in G, one requires α to lie also in the centre of $\mathfrak{g}$. Since we will be mostly concerned with G-Higgs bundles for G semisimple, we will take $\alpha = 0$, and we will simply talk about stability of a G-Higgs bundle, meaning 0-stability.

When G is compact $\mathfrak{m} = 0$ and hence a G-Higgs bundle is simply a holomorphic principal $G^{\mathbb{C}}$-bundle. When G is complex, if $U \subset G$ is a maximal compact subgroup, the Cartan decomposition of $\mathfrak{g}$ is $\mathfrak{g} = \mathfrak{u} + i\mathfrak{u}$, where $\mathfrak{u}$ is the Lie algebra of U. Then a G-Higgs bundle (E, φ) consists of a a holomorphic G-bundle E and $\varphi \in H^0(X, E(\mathfrak{g}) \otimes K)$, where $E(\mathfrak{g})$ is the $\mathfrak{g}$-bundle associated to E via the adjoint representation. These are the objects introduced originally by Hitchin [31] when $G = \mathrm{SL}(2,\mathbb{C})$.

Henceforth, we shall assume that G is connected. Then the topological classification of $H^{\mathbb{C}}$-bundles E on X is given by a characteristic class $c(E) \in \pi_1(H^{\mathbb{C}}) = \pi_1(H) = \pi_1(G)$. For a fixed $d \in \pi_1(G)$, the **moduli space of polystable G-Higgs bundles** $\mathcal{M}_d(G)$ is the set of isomorphism classes of polystable G-Higgs bundles (E, φ) such that $c(E) = d$. When G is compact, the moduli space $\mathcal{M}_d(G)$ coincides with $M_d(G^{\mathbb{C}})$, the moduli space of polystable $G^{\mathbb{C}}$-bundles with topological invariant d.

The moduli space $\mathcal{M}_d(G)$ has the structure of a complex analytic variety. This can be seen by the standard slice method (see, e.g., Kobayashi [34]). Geometric Invariant Theory constructions are available in the literature for G compact algebraic (Ramanathan [42]) and for G complex reductive algebraic (Simpson [49, 50]). The case of a real form of a complex reductive algebraic Lie group follows from the general constructions of Schmitt [44, 45]. We thus have the following.

Theorem 2.2. *The moduli space $\mathcal{M}_d(G)$ is a complex analytic variety, which is algebraic when G is algebraic.*

2.2. **Deformation theory of G-Higgs bundles.** In this section we recall some standard facts about the deformation theory of G-Higgs bundles. A convenenient reference for this material is Biswas–Ramanan [2].

Definition 2.3. Let (E, φ) be a G-Higgs bundle. The *deformation complex* of (E, φ) is the following complex of sheaves:

$$(2.3) \qquad C^{\bullet}(E, \varphi)\colon E(\mathfrak{h}^{\mathbb{C}}) \xrightarrow{\mathrm{ad}(\varphi)} E(\mathfrak{m}^{\mathbb{C}}) \otimes K.$$

This definition makes sense because ϕ is a section of $E(\mathfrak{m}^{\mathbb{C}}) \otimes K$ and $[\mathfrak{m}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}] \subseteq \mathfrak{m}^{\mathbb{C}}$.

The following result generalizes the fact that the infinitesimal deformation space of a holomorphic vector bundle V is isomorphic to $H^1(\mathrm{End}\, V)$.

Proposition 2.4. *The space of infinitesimal deformations of a G-Higgs bundle (E, φ) is naturally isomorphic to the hypercohomology group $\mathbb{H}^1(C^{\bullet}(E, \varphi))$.*

In particular, if (E, φ) represents a non-singular point of the moduli space $\mathcal{M}_d(G)$ then the tangent space at this point is canonically isomorphic to $\mathbb{H}^1(C^{\bullet}(E, \varphi))$.

For any G-Higgs bundle there is a natural long exact sequence

$$(2.4) \quad \begin{aligned} 0 &\to \mathbb{H}^0(C^{\bullet}(E, \varphi)) \to H^0(E(\mathfrak{h}^{\mathbb{C}})) \xrightarrow{\mathrm{ad}(\varphi)} H^0(E(\mathfrak{m}^{\mathbb{C}}) \otimes K) \\ &\to \mathbb{H}^1(C^{\bullet}(E, \varphi)) \to H^1(E(\mathfrak{h}^{\mathbb{C}})) \xrightarrow{\mathrm{ad}(\varphi)} H^1(E(\mathfrak{m}^{\mathbb{C}}) \otimes K) \to \mathbb{H}^2(C^{\bullet}(E, \varphi)) \to 0. \end{aligned}$$

As an immediate consequence we have the following result.

Proposition 2.5. *The infinitesimal automorphism space $\mathrm{aut}(E, \varphi)$ defined in Section A.7 is isomorphic to $\mathbb{H}^0(C^{\bullet}(E, \varphi))$.*

Let $d\iota\colon \mathfrak{h}^{\mathbb{C}} \to \mathrm{End}(\mathfrak{m}^{\mathbb{C}})$ be the derivative at the identity of the complexified isotropy representation $\iota = \mathrm{Ad}_{|H^{\mathbb{C}}}\colon H^{\mathbb{C}} \to \mathrm{Aut}(\mathfrak{m}^{\mathbb{C}})$ (cf. Section B.1). Let $\ker d\iota \subseteq \mathfrak{h}^{\mathbb{C}}$ be its kernel and let $E(\ker d\iota) \subseteq E(\mathfrak{h}^{\mathbb{C}})$ be the corresponding subbundle. Then there is an inclusion $H^0(E(\ker d\iota)) \hookrightarrow \mathbb{H}^0(C^{\bullet}(E, \varphi))$.

Definition 2.6. A G-Higgs bundle (E, φ) is said to be **infinitesimally simple** if the infinitesimal automorphism space $\mathbb{H}^0(C^{\bullet}(E, \varphi))$ is isomorphic to $H^0(E(\ker d\iota \cap \mathfrak{z}))$.

Similarly, we have an inclusion $\ker \iota \cap Z(H^{\mathbb{C}}) \hookrightarrow \mathrm{Aut}(E, \phi)$.

Definition 2.7. A G-Higgs bundle (E, φ) is said to be **simple** if $\mathrm{Aut}(E, \varphi) = \ker \iota \cap Z(H^{\mathbb{C}})$, where $Z(H^{\mathbb{C}})$ is the centre of $H^{\mathbb{C}}$.

As a consequence of Propositions 2.5 and A.7 we have the following.

Proposition 2.8. *Any stable G-Higgs bundle (E, φ) with $\varphi \neq 0$ is infinitesimally simple.*

Remark 2.9. If $\ker d\iota = 0$, then (E, φ) is infinitesimally simple if and only if the vanishing $\mathbb{H}^0(C^\bullet(E, \varphi)) = 0$ holds. A particular case of this situation is when the group G is a complex semisimple group: indeed, in this case the isotropy representation is just the adjoint representation.

Next we turn to the question of the vanishing of $\mathbb{H}^2$ of the deformation complex. In order to deal with this question we shall use Serre duality for hypercohomology (see e.g. Theorem 3.12 in [33]), which says that there are natural isomorphisms

$$(2.5) \qquad \mathbb{H}^i(C^\bullet(E, \varphi)) \cong \mathbb{H}^{2-i}(C^\bullet(E, \varphi)^* \otimes K)^*,$$

where the dual of the deformation complex (2.3) is

$$C^\bullet(E, \varphi)^* \colon E(\mathfrak{m}^{\mathbb{C}}) \otimes K^{-1} \xrightarrow{-\mathrm{ad}(\varphi)} E(\mathfrak{h}^{\mathbb{C}}).$$

An important special case of this is when G is a complex group.

Proposition 2.10. *Assume that G is a complex group. Then there is a natural isomorphism*

$$\mathbb{H}^2(C^\bullet(E, \varphi)) \cong \mathbb{H}^0(C^\bullet(E, \varphi))^*.$$

Proof. This is immediate from (2.5) and the fact that the the deformation complex is dual to itself, except for a sign in the map which does not influence the cohomology (cf. Section 2.1):

$$(2.6) \qquad C^\bullet(E, \varphi)^* \otimes K \colon E(\mathfrak{g}) \xrightarrow{-\mathrm{ad}(\varphi)} E(\mathfrak{g}) \otimes K.$$

$\square$

Remark 2.11. The isomorphism $\mathbb{H}^1(C^\bullet(E, \varphi)) \cong \mathbb{H}^1(C^\bullet(E, \varphi))^*$ is also important: it gives rise to the natural complex symplectic structure on the moduli space of G-Higgs bundles for complex groups G.

We have the following key observation (cf. (2.6); again we are ignoring the irrelevant change of sign in the dual complex).

Proposition 2.12. *Let G be a real group and let $G^{\mathbb{C}}$ be its complexification. Let (E, φ) be a G-Higgs bundle. Then there is an isomorphism of complexes:*

$$C^\bullet_{G^{\mathbb{C}}}(E, \varphi) \cong C^\bullet_G(E, \varphi) \oplus C^\bullet_G(E, \varphi)^* \otimes K,$$

where $C^\bullet_{G^{\mathbb{C}}}(E, \varphi)$ denotes the deformation complex of (E, φ) viewed as a $G^{\mathbb{C}}$-Higgs bundle, and $C^\bullet_G(E, \varphi)$ denotes the deformation complex of (E, φ) viewed as a G-Higgs bundle.

Corollary 2.13. *With the same hypotheses as in the previous Proposition, there is an isomorphism*

$$\mathbb{H}^0(C^\bullet_{G^{\mathbb{C}}}(E, \varphi)) \cong \mathbb{H}^0(C^\bullet_G(E, \varphi)) \oplus \mathbb{H}^2(C^\bullet_G(E, \varphi))^*.$$

Proof. Immediate from the Proposition and Serre duality (2.5).   □

Proposition 2.14. *Let G be a real semisimple group and let $G^{\mathbb{C}}$ be its complexification. Let (E, φ) be a G-Higgs bundle which is stable viewed as a $G^{\mathbb{C}}$-Higgs bundle. Then the vanishing*

$$\mathbb{H}^0(C_G^{\bullet}(E, \varphi)) = 0 = \mathbb{H}^2(C_G^{\bullet}(E, \varphi))$$

holds.

Proof. Since G is semisimple, so is $G^{\mathbb{C}}$. Hence, in view of Remark 2.9, the result follows at once from Corollary 2.13 and Proposition 2.8.   □

The following result on smoothness of the moduli space can be proved, for example, from the standard slice method construction referred to above.

Proposition 2.15. *Let (E, φ) be a stable G-Higgs bundle. If (E, φ) is simple and*

$$\mathbb{H}^2(C_G^{\bullet}(E, \varphi)) = 0,$$

then (E, φ) is a smooth point in the moduli space. In particular, if (E, φ) is a simple G-Higgs bundle which is stable as a $G^{\mathbb{C}}$-Higgs bundle, then it is a smooth point in the moduli space.

Suppose now that we are in the situation of Proposition 2.15. Then a local universal family exists (see [45]) and hence the dimension of the component of the moduli space containing (E, φ) equals the dimension of the infinitesimal deformation space $\mathbb{H}^1(C_G^{\bullet}(E, \varphi))$. In view of Proposition 2.8, Remark 2.9 and Proposition 2.16, we also have $\mathbb{H}^0(C_G^{\bullet}(E, \varphi)) = \mathbb{H}^2(C_G^{\bullet}(E, \varphi)) = 0$. So we have $\mathbb{H}^1(C_G^{\bullet}(E, \varphi)) = -\chi(C_G^{\bullet}(E, \varphi))$. A remarkable fact on this equality is that, whereas the left hand side may depend on the choice of (E, ϕ), the right hand side is independent of it, as we will see in the proposition below. We shall refer to $-\chi(C_G^{\bullet}(E, \varphi))$ as the **expected dimension** of the moduli space.

Proposition 2.16. *Let G be semisimple. Then the expected dimension of the moduli space of G-Higgs bundles is $(g - 1) \dim G^{\mathbb{C}}$.*

Proof. Let (E, φ) be any G-Higgs bundle. The long exact sequence (2.4) gives us

$$\chi(C_G^{\bullet}(E, \varphi)) - \chi(E(\mathfrak{h}^{\mathbb{C}})) + \chi(E(\mathfrak{m}^{\mathbb{C}}) \otimes K) = 0.$$

Serre duality implies that $\chi(E(\mathfrak{m}^{\mathbb{C}}) \otimes K) = \chi(E(\mathfrak{m}^{\mathbb{C}}))$ and from the Riemann–Roch formula we therefore obtain

$$-\chi(C_G^{\bullet}(E, \varphi)) = \deg(E(\mathfrak{m}^{\mathbb{C}})) + (g - 1) \operatorname{rk}(E(\mathfrak{m}^{\mathbb{C}})) - \big(\deg(E(\mathfrak{h}^{\mathbb{C}})) + (1 - g) \operatorname{rk}(E(\mathfrak{h}^{\mathbb{C}})).$$

Any invariant pairing on $\mathfrak{g}^{\mathbb{C}}$ (e.g. the Killing form) induces isomorphisms $E(\mathfrak{m}^{\mathbb{C}}) \simeq E(\mathfrak{m}^{\mathbb{C}})^*$ and $E(\mathfrak{h}^{\mathbb{C}}) \simeq E(\mathfrak{h}^{\mathbb{C}})^*$. Hence $\deg(E(\mathfrak{m}^{\mathbb{C}})) = \deg(E(\mathfrak{h}^{\mathbb{C}})) = 0$, whence the result. In particular, the value of $-\chi(C_G^{\bullet}(E, \varphi))$ is independent of the choice of G-Higgs bundle (E, φ).   □

Remark 2.17. Note that the actual dimension of the moduli space (if non-empty) can be smaller than the expected dimension. This happens for example when $G = \mathrm{SU}(p, q)$ with $p \neq q$ and maximal Toledo invariant (this follows from the study of $\mathrm{U}(p, q)$-Higgs bundles in [6]) — in this case there are in fact no stable $\mathrm{SU}(p, q)$-Higgs bundles.

2.3. G-Higgs bundles and Hitchin equations. Let G be connected semisimple real Lie group. Let (E, φ) be a G-Higgs bundle over a compact Riemann surface X. By a slight abuse of notation, we shall denote the C^∞-objects underlying E and φ by the same symbols. In particular, the Higgs field can be viewed as a $(1, 0)$-form: $\varphi \in \Omega^{1,0}(E(\mathfrak{m}^{\mathbb{C}}))$. Let $\tau \colon \Omega^1(E(\mathfrak{g}^{\mathbb{C}})) \to \Omega^1(E(\mathfrak{g}^{\mathbb{C}}))$ be the compact conjugation of $\mathfrak{g}^{\mathbb{C}}$ combined with complex conjugation on complex 1-forms. Given a reduction h of structure group to H in the smooth $H^{\mathbb{C}}$-bundle E, we denote by F_h the curvature of the unique connection compatible with h and the holomorphic structure on E.

Theorem 2.18. *There exists a reduction h of the structure group of E from $H^{\mathbb{C}}$ to H satisfying the Hitchin equation*

$$F_h - [\varphi, \tau(\varphi)] = 0$$

if and only if (E, φ) is polystable.

Theorem 2.18 was proved by Hitchin [31] for $G = \mathrm{SL}(2, \mathbb{C})$ and Simpson [47, 48] for an arbitrary semisimple complex Lie group G. The proof for an arbitrary reductive real Lie group G when (E, φ) is stable is given in [10], and the general polystable case follows as a particular case of the more general Hitchin–Kobayashi correspondence given in Theorem A.17.

From the point of view of moduli spaces it is convenient to fix a C^∞ principal H-bundle $\mathbf{E}_H$ with fixed topological class $d \in \pi_1(H)$ and study the moduli space of solutions to **Hitchin's equations** for a pair (A, φ) consisting of an H-connection A and $\varphi \in \Omega^{1,0}(X, \mathbf{E}_H(\mathfrak{m}^{\mathbb{C}}))$:

$$
(2.7) \qquad \begin{aligned} F_A - [\varphi, \tau(\varphi)] &= 0 \\ \bar{\partial}_A \varphi &= 0. \end{aligned}
$$

Here d_A is the covariant derivative associated to A and $\bar{\partial}_A$ is the $(0, 1)$ part of d_A, which defines a holomorphic structure on $\mathbf{E}_H$. The gauge group $\mathscr{H}$ of $\mathbf{E}_H$ acts on the space of solutions and the moduli space of solutions is

$$\mathcal{M}_d^{\mathrm{gauge}}(G) := \{(A, \varphi) \text{ satisfying } (2.7)\}/\mathscr{H}.$$

Now, Theorem 2.18 can be reformulated as follows.

Theorem 2.19. *There is a homeomorphism*

$$\mathcal{M}_d(G) \cong \mathcal{M}_d^{\mathrm{gauge}}(G)$$

To explain this correspondence we interpret the moduli space of G-Higgs bundles in terms of pairs $(\bar{\partial}_E, \varphi)$ consisting of a $\bar{\partial}$-operator (holomorphic structure) on the $H^{\mathbb{C}}$-bundle $\mathbf{E}_{H^{\mathbb{C}}}$ obtained from $\mathbf{E}_H$ by the extension of structure group $H \subset H^{\mathbb{C}}$, and $\varphi \in \Omega^{1,0}(X, \mathbf{E}_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}}))$ satisfying $\bar{\partial}_E \varphi = 0$. Such pairs are in correspondence with G-Higgs bundles (E, φ), where E is the holomorphic $H^{\mathbb{C}}$-bundle defined by the operator $\bar{\partial}_E$ on $\mathbf{E}_{H^{\mathbb{C}}}$ and $\bar{\partial}_E \varphi = 0$ is equivalent to $\varphi \in H^0(X, E(\mathfrak{m}^{\mathbb{C}}) \otimes K)$. The moduli space of polystable G-Higgs bundles $\mathcal{M}_d(G)$ can now be identified with the orbit space

$$\{(\bar{\partial}_E, \varphi) \ : \ \bar{\partial}_E \varphi = 0, \ (\bar{\partial}_E, \varphi) \text{ defines a polystable } G\text{-Higgs bundle}\}/\mathscr{H}^{\mathbb{C}},$$

where $\mathscr{H}^{\mathbb{C}}$ is the gauge group of $\mathbf{E}_{H^{\mathbb{C}}}$, which is in fact the complexification of $\mathscr{H}$. Since there is a one-to-one correspondence between H-connections on $\mathbf{E}_H$ and $\bar{\partial}$-operators on $\mathbf{E}_{H^{\mathbb{C}}}$, the correspondence given in Theorem 2.19 can be interpreted by saying that in the $\mathscr{H}^{\mathbb{C}}$-orbit of a polystable G-Higgs bundle $(\bar{\partial}_{E_0}, \varphi_0)$ we can find another Higgs bundle

$(\bar{\partial}_E, \varphi)$ whose corresponding pair (d_A, φ) satisfies $F_A - [\varphi, \tau(\varphi)] = 0$, and this is unique up to H-gauge transformations.

The infinitesimal deformation space of a solution (A, φ) to Hitchin's equations can be described as the first cohomology group of a certain elliptic deformation complex. To do this, we follow Hitchin [31, § 5]. The linearized equations are:

$$d_A(\dot{A}) - [\dot{\varphi}, \tau(\varphi)] - [\varphi, \tau(\dot{\varphi})] = 0,$$
$$\bar{\partial}_A \dot{\varphi} + [\dot{A}^{0,1}, \varphi] = 0,$$

for $\dot{A} \in \Omega^1(X, \mathbf{E}_H(\mathfrak{h}))$ and $\dot{\varphi} \in \Omega^{1,0}(X, \mathbf{E}_H(\mathfrak{m}^{\mathbb{C}}))$. The infinitesimal action of

$$\psi \in \mathrm{Lie}\,\mathscr{H} = \Omega^0(X, \mathbf{E}_H(\mathfrak{h}))$$

is

$$(A, \phi) \mapsto (d_A\psi, [\phi, \psi]).$$

Thus the deformation theory of Hitchin's equations is governed by the (elliptic) complex

$$C^\bullet(A, \varphi): \Omega^0(X, \mathbf{E}_H(\mathfrak{h})) \xrightarrow{d_0} \Omega^1(X, \mathbf{E}_H(\mathfrak{h})) \oplus \Omega^{1,0}(X, \mathbf{E}_H(\mathfrak{m}^{\mathbb{C}}))$$
$$\xrightarrow{d_1} \Omega^2(X, \mathbf{E}_H(\mathfrak{h})) \oplus \Omega^{1,1}(X, \mathbf{E}_H(\mathfrak{m}^{\mathbb{C}})),$$

where the maps are

$$d_0(\psi) = (d_A\psi, [\varphi, \psi])$$

and

$$d_1(\psi) = (d_A(\dot{A}) - [\dot{\varphi}, \tau(\varphi)] - [\phi, \tau(\dot{\varphi})], \bar{\partial}_A \dot{\varphi} + [\dot{A}^{0,1}, \varphi]).$$

The fact that (A, φ) is a solution to the equations, together with the gauge invariance of the equations, guarantees that $d_1 \circ d_0 = 0$. Denote by $H^i(C^\bullet(A, \varphi))$ the cohomology groups of the gauge theory deformation complex $C^\bullet(A, \varphi)$.

Let

$$\mathrm{Aut}(A, \varphi) := \{h \in \mathscr{H} \;:\; h^*A = A, \text{ and } \iota(h)(\varphi) = \varphi\}.$$

Here $\iota : H \to \mathrm{Aut}(\mathfrak{m})$ is the isotropy representation. Clearly $Z(H) \cap \ker \iota \subset \mathrm{Aut}(A, \varphi)$.

Definition 2.20. Let (A, φ) be a solution of (2.7). We say that (A, φ) is **irreducible** if and only if $\mathrm{Aut}(A, \varphi) = Z(H) \cap \ker \iota$. We say that (A, φ) is **infinitesimally irreducible** if the Lie algebra of $\mathrm{Aut}(A, \varphi)$, which is identified with $H^0(C^\bullet(A, \varphi))$ equals $Z(\mathfrak{h}) \cap \ker d\iota$.

Proposition 2.21. *Assume that $H^0(C^\bullet(A, \varphi)) = H^2(C^\bullet(A, \varphi)) = 0$ and that (A, φ) is irreducible. Then $\mathcal{M}_d^{\mathrm{gauge}}$ is smooth at $[A, \varphi]$ and the tangent space is*

$$T_{[A,\varphi]}\mathcal{M}_d^{\mathrm{gauge}} \cong H^1(C^\bullet(A, \varphi)).$$

For a proper understanding of many aspects of the geometry of the moduli space of Higgs bundles, one needs to consider the moduli space as the gauge theory moduli space $\mathcal{M}_d^{\mathrm{gauge}}(G)$. On the other hand, the formulation of the deformation theory in terms of hypercohomology is very convenient. Fortunately, one has the following.

Proposition 2.22. *At a smooth point of the moduli space, there is a natural isomorphism of infinitesimal deformation spaces*

$$H^1(C^\bullet(A, \varphi)) \cong \mathbb{H}^1(C^\bullet(E, \varphi)),$$

where the holomorphic structure on the Higgs bundle (E, φ) is given by $\bar{\partial}_A$.

As in Donaldson–Kronheimer [20, § 6.4] this can be seen by using a Dolbeault resolution to calculate $\mathbb{H}^1(C^\bullet(E, \varphi))$ and using harmonic representatives of cohomology classes, via Hodge theory. For this reason we can freely apply the complex deformation theory described in Section 2.2 to the gauge theory situation.

The following result is not essential for the present paper but we include it here for completeness. It can be deduced from the treatment of the Hitchin–Kobayashi correspondence given in Appendix A.

Proposition 2.23. *Under the correspondence given by Theorem 2.19, a stable G-Higgs bundle corresponds to an infinitesimally irreducible solution to Hitchin equations, while a G-Higgs bundle which is stable and simple is in correspondence with an irreducible solution.*

2.4. **Surface group representations.** Let X be a closed oriented surface of genus g and let

$$\pi_1(X) = \langle a_1, b_1, \ldots, a_g, b_g \mid \prod_{i=1}^{g} [a_i, b_i] = 1 \rangle$$

be its fundamental group. Let G be a connected reductive real Lie group. By a **representation** of $\pi_1(X)$ in G we understand a homomorphism $\rho \colon \pi_1(X) \to G$. The set of all such homomorphisms, $\mathrm{Hom}(\pi_1(X), G)$, can be naturally identified with the subset of G^{2g} consisting of 2g-tuples $(A_1, B_1 \ldots, A_g, B_g)$ satisfying the algebraic equation $\prod_{i=1}^{g} [A_i, B_i] = 1$. This shows that $\mathrm{Hom}(\pi_1(X), G)$ is a real analytic variety, which is algebraic if G is algebraic.

The group G acts on $\mathrm{Hom}(\pi_1(X), G)$ by conjugation:

$$(g \cdot \rho)(\gamma) = g\rho(\gamma)g^{-1}$$

for $g \in G$, $\rho \in \mathrm{Hom}(\pi_1(X), G)$ and $\gamma \in \pi_1(X)$. If we restrict the action to the subspace $\mathrm{Hom}^+(\pi_1(X), g)$ consisting of *reductive representations*, the orbit space is Hausdorff (see Theorem 11.4 in [43]). By a **reductive representation** we mean one that composed with the adjoint representation in the Lie algebra of G decomposes as a sum of irreducible representations. If G is algebraic this is equivalent to the Zariski closure of the image of $\pi_1(X)$ in G being a reductive group. (When G is compact every representation is reductive.) Define the *moduli space of representations* of $\pi_1(X)$ in G to be the orbit space

$$\mathcal{R}(G) = \mathrm{Hom}^+(\pi_1(X), G)/G.$$

One has the following (see e.g. Goldman [26]).

Theorem 2.24. *The moduli space $\mathcal{R}(G)$ has the structure of a real analytic variety, which is algebraic if G is algebraic and is a complex variety if G is complex.*

Given a representation $\rho \colon \pi_1(X) \to G$, there is an associated flat G-bundle on X, defined as $E_\rho = \widetilde{X} \times_\rho G$, where $\widetilde{X} \to X$ is the universal cover and $\pi_1(X)$ acts on G via ρ. This gives in fact an identification between the set of equivalence classes of representations $\mathrm{Hom}(\pi_1(X), G)/G$ and the set of equivalence classes of flat G-bundles, which in turn is parametrized by the cohomology set $H^1(X, G)$. We can then assign a topological invariant to a representation ρ given by the characteristic class $c(\rho) := c(E_\rho) \in \pi_1(G)$ corresponding to E_ρ. To define this, let $\widetilde{G}$ be the universal covering group of G. We have an exact sequence

$$1 \longrightarrow \pi_1(G) \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1$$

which gives rise to the (pointed sets) cohomology sequence

$$(2.8) \qquad H^1(X, \widetilde{G}) \longrightarrow H^1(X, G) \overset{c}{\longrightarrow} H^2(X, \pi_1(G)).$$

Since $\pi_1(G)$ is abelian the orientation of X defines an isomorphism

$$H^2(X, \pi_1(G)) \cong \pi_1(G),$$

and $c(E_\rho)$ is defined as the image of E under the last map in (2.8). Thus the class $c(E_\rho)$ measures the obstruction to lifting E_ρ to a flat $\widetilde{G}$-bundle, and hence to lifting ρ to a representation of $\pi_1(X)$ in $\widetilde{G}$. For a fixed $d \in \pi_1(G)$, the *moduli space of reductive representations* $\mathcal{R}_d(G)$ with topological invariant d is defined as the subvariety

$$(2.9) \qquad \mathcal{R}_d(G) := \{[\rho] \in \mathcal{R}(G) \mid c(\rho) = d\},$$

where as usual $[\rho]$ denotes the G-orbit $G \cdot \rho$ of $\rho \in \mathrm{Hom}^+(\pi_1(X), G)$.

One can study deformations of a class of representations $[\rho] \in \mathcal{R}_d(G)$ by means of group cohomology (see [26]). The Lie algebra $\mathfrak{g}$ is endowed with the structure of a $\pi_1(X)$-module by means of the composition

$$\pi_1(X) \overset{\rho}{\longrightarrow} G \overset{\mathrm{Ad}}{\longrightarrow} \mathrm{Aut}(\mathfrak{g}).$$

Definition 2.25. Let $\rho : \pi_1(X) \to G$ be a representation of $\pi_1(X)$ in G. Let $Z_G(\rho)$ be the centralizer in G of $\rho(\pi_1(X))$. We say that ρ is **irreducible** if and only if it is reductive and $Z_G(\rho) = Z(G)$, where $Z(G)$ is the centre of G. We say that ρ is an **infinitesimally irreducible** representation if it is reductive and $\mathrm{Lie}\, Z_G(\rho) = \mathrm{Lie}\, Z(G)$.

One has the following basic facts ([26]).

Proposition 2.26.   (1) *The Zariski tangent space to $\mathcal{R}_d(G)$ at an equivalence class $[\rho]$ is isomorphic to the cohomology group $H^1(\pi_1(X), \mathfrak{g}_{\mathrm{Ad}\circ\rho})$.*
   (2) $H^0(\pi_1(X), \mathfrak{g}_{\mathrm{Ad}\circ\rho}) \cong \mathrm{Lie}\, Z_G(\rho)$.
   (3) $H^2(\pi_1(X), \mathfrak{g}_{\mathrm{Ad}\circ\rho}) \cong H^0(\pi_1(X), \mathfrak{g}_{\mathrm{Ad}\circ\rho})^*$

From this one obtains the following ([26]).

Proposition 2.27. *Let G be a semisimple Lie group and let $\rho : \pi_1(X) \to G$ be irreducible. Then the equivalence class $[\rho]$ is a smooth point in $\mathcal{R}_d(G)$.*

This is simply because $Z_G(\rho) = Z(G)$ is finite and hence

$$H^0(\pi_1(X), \mathfrak{g}_{\mathrm{Ad}\circ\rho}) = H^2(\pi_1(X), \mathfrak{g}_{\mathrm{Ad}\circ\rho}) = 0.$$

An alternative way to study deformations of a representation is by using the corresponding flat connection. To explain this, let $\mathbf{E}_G$ be a C^∞ principal G-bundle over X with fixed topological class $d \in \pi_1(G) = \pi_1(H)$. Let D be a G-connection on $\mathbf{E}_G$ and let F_D be its curvature. If D is flat, i.e. $F_D = 0$, then the holonomy of D around a closed loop in X only depends on the homotopy class of the loop and thus defines a representation of $\pi_1(X)$ in G. This gives an identification[1],

$$\mathcal{R}_d(G) \cong \{\text{Reductive } G\text{-connections } D \mid F_D = 0\}/\mathscr{G},$$

---

[1]even when G is complex algebraic, this is merely a real *analytic* isomorphism, see Simpson [48, 49, 50]

where, by definition, a flat connection is reductive if the corresponding representation of $\pi_1(X)$ in G is reductive, and $\mathscr{G}$ is the group of automorphisms of $\mathbf{E}_G$ — the **gauge group**. We can now linearize the flatness condition near a flat connection D:

$$\frac{d}{dt}F(D + bt)_{t=0} = D(b)$$

for $b \in \Omega^1(X, \mathrm{Ad}(\mathbf{E}))$.

Linearize the action of the gauge group $D \mapsto g \cdot D = gDg^{-1}$. For $g(t) = \exp(\psi t)$ with $\psi \in \Omega^0(X, \mathrm{Ad}(\mathbf{E}))$,

$$\frac{d}{dt}(g(t) \cdot D)_{t=0} = D(\psi).$$

Thus the infinitesimal deformation space is H^1 of the complex

$$0 \to \Omega^0(X, \mathbf{E}(\mathfrak{g})) \xrightarrow{D} \Omega^1(X, \mathbf{E}(\mathfrak{g})) \xrightarrow{D} \Omega^2(X, \mathbf{E}(\mathfrak{g})) \to 0.$$

Note that $F_D = D^2 = 0$ means that this is in fact a complex.

2.5. **Representations and G-Higgs bundles.** We assume now that G is connected and semisimple. With the notation of the previous sections, we have the following.

Theorem 2.28. *Let G be a connected semisimple real Lie group. There is a homeomorphism $\mathcal{R}_d(G) \cong \mathcal{M}_d(G)$. Under this homeomorphism, stable G-Higgs bundles correspond to infinitesimally irreducible representations, and stable and simple G-Higgs bundles correspond to irreducible representations.*

Remark 2.29. On the open subvarieties defined by the smooth points of $\mathcal{R}_d$ and $\mathcal{M}_d$, this correspondence is in fact an isomorphism of real analytic varieties.

Remark 2.30. There is a similar correspondence when G is reductive but not semisimple. In this case, it makes sense to consider nonzero values of the stability parameter α. The resulting Higgs bundles can be geometrically interpreted in terms of representations of the universal central extension of the fundamental group of X, and the value of α prescribes the image of a generator of the center in the representation.

The proof of Theorem 2.28 is the combination of two existence theorems for gauge-theoretic equations. To explain this, let $\mathbf{E}_G$ be, as above, a C^∞ principal G-bundle over X with fixed topological class $d \in \pi_1(G) = \pi_1(H)$. Every G-connection D on $\mathbf{E}_G$ decomposes uniquely as

$$D = d_A + \psi,$$

where d_A is an H-connection on $\mathbf{E}_H$ and $\psi \in \Omega^1(X, \mathbf{E}_H(\mathfrak{m}))$. Let F_A be the curvature of d_A. We consider the following set of equations for the pair (d_A, ψ):

(2.10)
$$\begin{aligned}
F_A + \tfrac{1}{2}[\psi, \psi] &= 0 \\
d_A\psi &= 0 \\
d_A^*\psi &= 0.
\end{aligned}$$

These equations are invariant under the action of $\mathscr{H}$, the gauge group of $\mathbf{E}_H$. A theorem of Corlette [17], and Donaldson [19] for $G = \mathrm{SL}(2, \mathbb{C})$, says the following.

Theorem 2.31. *There is a homeomorphism*

$$\{Reductive\ G\text{-}connections\ D \mid F_D = 0\}/\mathscr{G} \cong \{(d_A, \psi)\ satisfying\ (2.10)\}/\mathscr{H}.$$

The first two equations in (2.10) are equivalent to the flatness of $D = d_A + \psi$, and Theorem 2.31 simply says that in the $\mathscr{G}$-orbit of a reductive flat G-connection D_0 we can find a flat G-connection $D = g(D_0)$ such that if we write $D = d_A + \psi$, the additional condition $d_A^* \psi = 0$ is satisfied. This can be interpreted more geometrically in terms of the reduction $h = g(h_0)$ of $\mathbf{E}_G$ to an H-bundle obtained by the action of $g \in \mathscr{G}$ on h_0. The equation $d_A^* \psi = 0$ is equivalent to the harmonicity of the $\pi_1(X)$-equivariant map $\widetilde{X} \to G/H$ corresponding to the new reduction of structure group h.

To complete the argument, leading to Theorem 2.28, we just need Theorem 2.18 and the following simple result.

Proposition 2.32. *The correspondence* $(d_A, \varphi) \mapsto (d_A, \psi := \varphi - \tau(\varphi))$ *defines a homeomorphism*

$$\{(d_A, \varphi) \ \text{satisfying} \ (2.7)\}/\mathscr{H} \cong \{(d_A, \psi) \ \text{satisfying} \ (2.10)\}/\mathscr{H}.$$

3. Sp$(2n, \mathbb{R})$-Higgs bundles

3.1. Sp$(2n, \mathbb{R})$-**Higgs bundles.** Let X be a compact Riemann surface. The maximal compact subgroup of Sp$(2n, \mathbb{R})$ is U(n). If $\mathbb{V} = \mathbb{C}^n$ is the fundamental representation of GL$(n, \mathbb{C})$, then the isotropy representation space is:

$$\mathfrak{m}^{\mathbb{C}} = S^2 \mathbb{V} \oplus S^2 \mathbb{V}^*.$$

An Sp$(2n, \mathbb{R})$-Higgs bundle over X is thus a triple (V, β, γ) consisting of a rank n holomorphic vector bundle V and holomorphic sections $\beta \in H^0(X, S^2 V \otimes K)$ and $\gamma \in H^0(X, S^2 V^* \otimes K)$, where K is the canonical line bundle of X.

Let (V_i, φ_i) be Sp$(2n_i, \mathbb{R})$-Higgs bundles and let $n = \sum n_i$. We can define an Sp$(2n, \mathbb{R})$-Higgs bundle (V, φ) by setting

$$V = \bigoplus V_i \quad \text{and} \quad \varphi = \sum \varphi_i$$

by using the canonical inclusions $H^0(K \otimes (S^2 V_i \oplus S^2 V_i^*)) \subset H^0(K \otimes (S^2 V \oplus S^2 V^*))$. We shall slightly abuse language and write $(V, \varphi) = \bigoplus(V_i, \varphi_i)$, referring to this as "the direct sum of the (V_i, φ_i)".

In Appendix A we introduce a very general notion of (semi-, poly-)stability for certain kind of holomorphic objects. Sp$(2n, \mathbb{R})$-Higgs bundles are instances of such kind of objects. These notions are in general rather complicated to study in concrete terms but it turns out that the general (semi-,poly-)stability conditions, when applied to Sp$(2n, \mathbb{R})$-Higgs bundles, can be sensibly simplified (this is also the case for many other kinds of objects, and it might well be true that a general simplified condition exists for all G-Higgs bundles). The simplification of the (semi-)stability conditions for Sp$(2n, \mathbb{R})$-Higgs bundles are given in Theorems B.2 and B.4 of Appendix B, with the parameter value $\alpha = 0$ (recall from Section 2.1 that we are fixing this value, since it is the one relevant for the study of representations of surface groups). Before giving a precise statement we introduce some notation. If W is a vector bundle and $W', W'' \subset W$ are subbundles, then $W' \otimes_S W''$ denotes the subbundle of the second symmetric power $S^2 W$ which is the image of $W' \otimes W'' \subset W \otimes W$ under the symmetrization map $W \otimes W \to S^2 W$ (of course this should be defined in sheaf theoretical terms to be sure that $W' \otimes_S W''$ is indeed a subbundle, since the intersection of $W' \otimes W''$ and the kernel of the symmetrization map might change dimension from one fiber

to the other). Also, we denote by $W'^{\perp} \subset W^*$ the kernel of the restriction map $W^* \to W'^*$. Now the simplified (semi-)stability condition is given by the following proposition.

Proposition 3.1. *An* $\mathrm{Sp}(2n, \mathbb{R})$*-Higgs bundle* (V, ϕ) *is stable if, for any filtration of subbundles*

$$0 \subset V_1 \subset V_2 \subset V$$

such that

(3.11) $\qquad \beta \in H^0(K \otimes (S^2 V_2 + V_1 \otimes_S V)), \quad \gamma \in H^0(K \otimes (S^2 V_1^{\perp} + V_2^{\perp} \otimes_S V^*)),$

the following holds: if at least one of the subbundles V_1 *and* V_2 *is proper, then the inequality*

(3.12) $$\deg(V) - \deg(V_1) - \deg(V_2) > 0$$

holds and, in any other case,

(3.13) $$\deg(V) - \deg(V_1) - \deg(V_2) \geq 0.$$

The condition for (V, φ) *to be semistable is obtained by omitting the strict inequality (3.12).*

The following observation will be useful many times below.

Remark 3.2. If $0 \subset V_1 \subset V_2 \subset V$ is a filtration of vector bundles then for any $\beta \in H^0(K \otimes S^2 V)$ and $\gamma \in H^0(K \otimes S^2 V^*)$ the condition $\beta \in H^0(K \otimes (S^2 V_2 + V_1 \otimes_S V))$ is equivalent to $\beta V_2^{\perp} \subset K \otimes V_1$ and $\beta V_1^{\perp} \subset K \otimes V_2$, and similarly $\gamma \in H^0(K \otimes (S^2 V_1^{\perp} + V_2^{\perp} \otimes_S V^*))$ is equivalent to $\gamma V_1 \subset K \otimes V_2^{\perp}$ and $\gamma V_2 \subset K \otimes V_1^{\perp}$, where $V_i^{\perp}$ is the kernel of the projection $V^* \to V_i^*$ and we view β and γ as symmetric maps $\beta : V^* \to K \otimes V$ and $\gamma : V \to K \otimes V^*$. Thus, if we use a local basis of V adapted to the filtration $0 \subseteq V_1 \subseteq V_2 \subseteq V$ and the dual basis of V^*, then the matrix of γ is of the form

$$\begin{pmatrix} 0 & 0 & * \\ 0 & * & * \\ * & * & * \end{pmatrix},$$

while the matrix of β has the form

$$\begin{pmatrix} * & * & * \\ * & * & 0 \\ * & 0 & 0 \end{pmatrix}.$$

The deformation complex (2.3) for an $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle $(V, \varphi = \beta + \gamma)$ is

(3.14) $\qquad C^{\bullet}(V, \varphi) \colon \mathrm{End}(V) \xrightarrow{\mathrm{ad}(\varphi)} S^2 V \otimes K \oplus S^2 V^* \otimes K$
$$\psi \mapsto (-\beta \psi^t - \psi \beta, \gamma \psi + \psi^t \gamma)$$

Proposition 3.3. *An* $\mathrm{Sp}(2n, \mathbb{R})$*-Higgs bundle* (V, φ) *is infinitesimally simple if and only if* $\mathbb{H}^0(C^{\bullet}(V, \varphi)) = 0$. *Equivalently,* (V, φ) *is infinitesimally simple if and only if there is a non-zero* $\psi \in H^0(\mathrm{End}(V))$ *such that*

$$\mathrm{ad}(\varphi)(\psi) = (-\beta \psi^t - \psi \beta, \gamma \psi + \psi^t \gamma) = (0, 0).$$

Proof. For $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundles one has that $\ker(d\iota) = 0$. Thus the first statement is immediate from Definition 2.6. The equivalent statement now follows from the long exact sequence (2.4), recalling that in this case the deformation complex (2.3) is given by (3.14). $\qquad \square$

Proposition 3.4. *An* $\mathrm{Sp}(2n, \mathbb{R})$*-Higgs bundle* (V, φ) *is simple if and only if* $\mathrm{Aut}(V, \varphi) = \{\pm \mathrm{Id}\}$.

Proof. Since $\lambda \in \mathbb{C}^* = Z(H^{\mathbb{C}})$ acts on the isotropy representation $\mathfrak{m}^{\mathbb{C}} = S^2\mathbb{V} \oplus S^2\mathbb{V}^*$ by $(\beta, \gamma) \mapsto (\lambda^2\beta, \lambda^{-2}\gamma)$ we have $\ker \iota \cap Z(H^{\mathbb{C}}) = \{\pm 1\}$, so the statement follows directly from Definition 2.7. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.5. Contrary to the case of vector bundles, stability of a $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle does not imply that it is simple. To give an example of this phenomenon, take two different square roots, M_1 and M_2, of K. Define $V = M_1 \oplus M_2$, then $S^2V \otimes K = \mathscr{O} \oplus M_1^{-1}M_2^{-1} \oplus \mathscr{O}$. Let $\gamma = (1, 0, 1)$, $\beta = 0$ and set $\varphi = (\beta, \gamma)$. Then (V, φ) is not simple. However, we shall show that (V, φ) is stable. Since V has rank 2, in any filtration $0 \subset V_1 \subset V_2 \subset V$ some inclusion is in fact an equality. Hence we have to verify the semistability condition for the cases listed in Table B.1 and the stability condition (with strict inequality) for the cases listed in the last three rows of the same Table. This is easy, using the fact that γ is non-degenerate (note that for any proper $V_1 \subset V$ this means that γ cannot belong to $H^0(S^2V_1^{\perp})$). The phenomenon described by this example will be described in a systematic way in Theorem 3.12 below.

3.2. Stable and non-simple $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundles.

The goal of this section is to obtain a complete understanding of how a stable $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle can fail to be simple. The main result is Theorem 3.12.

For this, we need to describe some special $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundles arising from G-Higgs bundles associated to certain real subgroups $G \subseteq \mathrm{Sp}(2n, \mathbb{R})$.

The subgroup $G = \mathrm{U}(n)$. Observe that a $\mathrm{U}(n)$-Higgs bundle is nothing but a holomorphic vector bundle V of rank n. The standard inclusion $\upsilon^{\mathrm{U}(n)} \colon \mathrm{U}(n) \hookrightarrow \mathrm{Sp}(2n, \mathbb{R})$ gives the correspondence

$$(3.15) \qquad\qquad\qquad V \mapsto \upsilon_*^{\mathrm{U}(n)}V = (V, 0)$$

associating the $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle $\upsilon_*^{\mathrm{U}(n)}V = (V, 0)$ to the holomorphic vector bundle V.

Remark 3.6. Note that $(V, 0)$ is never simple as an $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle, since its automorphism group contains the non-zero scalars $\mathbb{C}^*$.

The subgroup $G = \mathrm{U}(p, q)$. In the following we assume that $p, q \geq 1$. As is easily seen, a $\mathrm{U}(p, q)$-Higgs bundle (cf. [6]) is given by the data $(\tilde{V}, \tilde{W}, \tilde{\varphi} = \tilde{\beta} + \tilde{\gamma})$, where $\tilde{V}$ and $\tilde{W}$ are holomorphic vector bundles of rank p and q, respectively, $\tilde{\beta} \in H^0(K \otimes \mathrm{Hom}(\tilde{W}, \tilde{V}))$ and $\tilde{\gamma} \in H^0(K \otimes \mathrm{Hom}(\tilde{V}, \tilde{W}))$. Let $n = p + q$. The imaginary part of the standard indefinite Hermitian metric of signature (p, q) on $\mathbb{C}^n$ is a symplectic form, and thus there is an inclusion $\upsilon^{\mathrm{U}(p,q)} \colon \mathrm{U}(p, q) \hookrightarrow \mathrm{Sp}(2n, \mathbb{R})$. At the level of G-Higgs bundles, this gives rise to the correspondence

$$(3.16) \qquad (\tilde{V}, \tilde{W}, \tilde{\varphi} = \tilde{\beta} + \tilde{\gamma}) \mapsto \upsilon_*^{\mathrm{U}(p,q)}(\tilde{V}, \tilde{W}, \tilde{\varphi}) = (V, \varphi = \beta + \gamma),$$

where

$$V = \tilde{V} \oplus \tilde{W}^*, \quad \beta = \begin{pmatrix} 0 & \tilde{\beta} \\ \tilde{\beta} & 0 \end{pmatrix} \quad \text{and} \quad \gamma = \begin{pmatrix} 0 & \tilde{\gamma} \\ \tilde{\gamma} & 0 \end{pmatrix}.$$

Remark 3.7. Again, we note that the $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle $v_*^{\mathrm{U}(p,q)}(\tilde{V}, \tilde{W}, \tilde{\varphi})$ is not simple, since it has the automorphism $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

We shall need a few lemmas for the proof of Theorem 3.12.

Lemma 3.8. *Let (V, φ) be an $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle and assume that there is a non-trivial splitting $(V, \varphi) = (V_a \oplus V_b, \varphi_a + \varphi_b)$ such that $\varphi_\nu \in H^0(K \otimes (S^2 V_\nu \oplus S^2 V_\nu^*))$ for $\nu = a, b$. Assume that the $\mathrm{Sp}(2n_a, \mathbb{R})$-Higgs bundle (V_a, φ_a) is not stable. Then (V, φ) is not stable.*

Proof. Since (V_a, φ_a) is not stable there is a filtration $0 \subset V_{a1} \subset V_{a2} \subset V_a$ such that
$$\beta \in H^0(K \otimes (S^2 V_{a2} + V_{a1} \otimes_S V)), \quad \gamma \in H^0(K \otimes (S^2 V_{a1}^\perp + V_{a2}^\perp \otimes_S V^*))$$
and
$$(3.17) \qquad \deg(V_a) - \deg(V_{a1}) - \deg(V_{a1}) \le 0.$$
Consider the filtration $0 \subset V_1 \subset V_2 \subset V$ obtained by setting
$$V_1 = V_{a1}, \quad V_2 = V_{a2} \oplus V_b.$$
Using Remark 3.2 one readily sees that this filtration satisfies the conditions (3.11). Since
$$\deg(V) - \deg(V_1) - \deg(V_2) = \deg(V_a) - \deg(V_{a1}) - \deg(V_{a1}),$$
it follows from (3.17) that (V, φ) is not stable. $\qquad\square$

Lemma 3.9. *Let (V, φ) be an $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle and assume that there is a non-trivial splitting $V = V_a \oplus V_b$ such that $\varphi \in H^0(K \otimes (S^2 V_a \oplus S^2 V_a^*))$. In other words, $(V, \varphi) = (V_a \oplus V_b, \varphi_a + 0)$ with $(V_b, 0) = v_*^{\mathrm{U}(n_b)} V_b$. Then (V, φ) is not stable.*

Proof. It is immediate from Lemma 3.8 and Remark B.5 that V_b is a stable vector bundle with $\deg(V_b) = 0$. Hence
$$\deg(V) = \deg(V_a).$$
Consider the filtration $0 \subset V_1 \subset V_2 \subset V$ obtained by setting $V_1 = 0$ and $V_2 = V_a$. As before this filtration satisfies (3.11). Therefore the calculation
$$\deg(V) - \deg(V_1) - \deg(V_2) = \deg(V) - \deg(V_a) = 0$$
shows that (V, φ) is not stable. $\qquad\square$

Lemma 3.10. *Let $(V, \varphi) = v_*^{\mathrm{U}(p,q)}(V_a, V_b^*, \tilde{\varphi})$ be an $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle arising from a $\mathrm{U}(p, q)$-Higgs bundle $(V_a, V_b^*, \tilde{\varphi})$ with $p, q \ge 1$. Then (V, φ) is not stable.*

Proof. The $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle (V, ϕ) is given by
$$V = V_a \oplus V_b, \quad \beta = \begin{pmatrix} 0 & \tilde{\beta} \\ \tilde{\beta} & 0 \end{pmatrix} \quad \text{and} \quad \gamma = \begin{pmatrix} 0 & \tilde{\gamma} \\ \tilde{\gamma} & 0 \end{pmatrix}.$$
Let $V_1 = V_2 = V_a$ and consider the filtration $0 \subset V_1 \subset V_2 \subset V$. Again this filtration satisfies the conditions (3.11). Thus, if (V, φ) is stable, we have from (3.12)
$$\deg(V) - 2\deg(V_a) < 0.$$
Similarly, considering $V_1 = V_2 = V_b$, we obtain
$$\deg(V) - 2\deg(V_b) < 0,$$

so we conclude that
$$\deg(V) = \deg(V_a) + \deg(V_b) < \deg(V),$$
which is absurd.

$\square$

Lemma 3.11. *Let $(\tilde{V}, \tilde{\varphi})$ be an $\mathrm{Sp}(2\tilde{n}, \mathbb{R})$-Higgs bundle. Then the $\mathrm{Sp}(4\tilde{n}, \mathbb{R})$-Higgs bundle $(\tilde{V} \oplus \tilde{V}, \tilde{\varphi} + \tilde{\varphi})$ is not stable.*

Proof. Consider the automorphism $f = \frac{1}{\sqrt{2\mathbf{i}}} \left(\begin{smallmatrix} 1 & \mathbf{i} \\ \mathbf{i} & 1 \end{smallmatrix}\right)$ of $V = \tilde{V} \oplus \tilde{V}$. Write $\beta = \left(\begin{smallmatrix} \tilde{\beta} & 0 \\ 0 & \tilde{\beta} \end{smallmatrix}\right)$ and $\gamma = \left(\begin{smallmatrix} \tilde{\gamma} & 0 \\ 0 & \tilde{\gamma} \end{smallmatrix}\right)$. Then we have that
$$(V, \varphi) \cong (\tilde{V} \oplus \tilde{V}, f \cdot \beta + f \cdot \gamma),$$
where
$$f \cdot \beta = f\beta f^t = \begin{pmatrix} 0 & \tilde{\beta} \\ \tilde{\beta} & 0 \end{pmatrix} \quad \text{and} \quad f \cdot \gamma = (f^t)^{-1}\gamma f^{-1} = \begin{pmatrix} 0 & \tilde{\gamma} \\ \tilde{\gamma} & 0 \end{pmatrix}.$$
We shall see that $(\tilde{V} \oplus \tilde{V}, f \cdot \beta + f \cdot \gamma)$ is not stable. To this end, consider the filtration $0 \subset V_1 \subset V_2 \subset \tilde{V} \oplus \tilde{V}$ obtained by setting $V_1 = V_2 = \tilde{V}$. This satisfies (3.11). But, on the other hand,
$$\deg(\tilde{V} \oplus \tilde{V}) - \deg(V_1) - \deg(V_2) = 0$$
so $(\tilde{V} \oplus \tilde{V}, f \cdot \beta + f \cdot \gamma)$ is not stable. $\square$

Theorem 3.12. *Let (V, φ) be a stable $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle. If (V, φ) is not simple, then one of the following alternatives occurs:*

(1) *The vanishing $\varphi = 0$ holds and V is a stable vector bundle of degree zero. In this case, $\mathrm{Aut}(V, \varphi) \cong \mathbb{C}^*$.*

(2) *There is a decomposition, unique up to reordering, $(V, \varphi) = (\bigoplus_{i=1}^k V_i, \sum_{i=1}^k \varphi_i)$ with $\phi_i = \beta_i + \gamma_i \in H^0(K \otimes (S^2 V_i \oplus S^2 V_i^*))$, such that each (V_i, ϕ_i) is a stable and simple $\mathrm{Sp}(2n_i, \mathbb{R})$-Higgs bundle. Furthermore, each $\varphi_i \neq 0$ and $(V_i, \varphi_i) \ncong (V_j, \varphi_j)$ for $i \neq j$. The automorphism group of (V, φ) is*
$$\mathrm{Aut}(V, \varphi) \cong \mathrm{Aut}(V_1, \varphi_1) \times \cdots \times \mathrm{Aut}(V_k, \varphi_k) \cong (\mathbb{Z}/2)^k.$$

Recall that an example of the second situation was described in Remark 3.5.

Proof. First of all, we note that if $\varphi = 0$ then it is immediate from Remark B.5 that alternative (1) occurs.

Next, consider the case $\varphi \neq 0$. Since (V, φ) is not simple, there is an automorphism $\sigma \in \mathrm{Aut}(V, \varphi) \setminus \{\pm 1\}$. If σ were a multiple of the identity, say $\sigma = \lambda \,\mathrm{Id}$ with $\lambda \in \mathbb{C}^*$, then it would act on $\varphi = \beta + \gamma$ by $\beta \mapsto \lambda^2 \beta$ and $\gamma \mapsto \lambda^{-2}\gamma$. Since $\varphi \neq 0$ this would force σ to be equal to 1 or -1, in contradiction with our choice. Hence σ is not a multiple of the identity. We know from Lemma A.18 that $\mathrm{Aut}(V, \varphi)$ is reductive. This implies that σ may be chosen in such a way that there is a splitting $V = \bigoplus V_i$ in eigenbundles for σ such that the action of σ on V_i is given by multiplication by some $\sigma_i \in \mathbb{C}^*$. It follows that the action of σ on $S^2 V \oplus S^2 V^*$ is given by

(3.18) $\sigma = \sigma_i \sigma_j \colon V_i \otimes V_j \to V_i \otimes V_j$ and $\sigma = \sigma_i^{-1} \sigma_j^{-j} \colon V_i^* \otimes V_j^* \to V_i^* \otimes V_j^*.$

If we denote by $\varphi_{ij} = \beta_{ij} + \gamma_{ij}$ the component of φ in $H^0(K \otimes (V_i \otimes V_j \oplus V_i^* \otimes V_j^*))$ (symmetrizing the tensor product if $i = j$), then

$$(3.19) \qquad\qquad\qquad \sigma_i \sigma_j \neq 1 \implies \varphi_{ij} = 0.$$

Suppose that $\varphi_{i_0 j_0} \neq 0$ for some $i_0 \neq j_0$. From (3.19) we conclude that $\sigma_{i_0} \sigma_{j_0} = 1$. But then $\sigma_i \sigma_{j_0} \neq 1$ for $i \neq i_0$ and $\sigma_{i_0} \sigma_j \neq 1$ for $j \neq j_0$. Hence, again by (3.19), $\varphi_{ij} = 0 = \varphi_{i_0 j}$ if $i \neq i_0$ or $j \neq j_0$. Thus $(V_{i_0}, V_{j_0}^*, \varphi_{i_0 j_0})$ is a $\mathrm{U}(p, q)$-Higgs Bundle and we have a non-trivial decomposition $(V, \varphi) = (V_a \oplus V_b, \varphi_a + \varphi_b)$ with $(V_a, \varphi_a) = \upsilon_*^{\mathrm{U}(p,q)}(V_{i_0}, V_{j_0}^*, \varphi_{i_0 j_0})$. By Lemma 3.10 the $\mathrm{Sp}(2n_a, \mathbb{R})$-Higgs bundle (V_a, φ_a) is not stable so, by Lemma 3.8, (V, φ) is not stable. This contradiction shows that $\varphi_{ij} = 0$ for $i \neq 0$.

It follows that $\varphi = \sum \varphi_i$ with $\phi_i \in H^0(K \otimes (S^2 V_i \oplus S^2 V_i^*))$. By Lemma 3.8 each of the summands (V_i, φ_i) is a stable $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle and by Lemma 3.9 each φ_i must be non-zero. Also, from (3.18), $\sigma \cdot \beta_i = \sigma_i^2 \beta_i$ and $\sigma \cdot \gamma_i = \sigma_i^{-2} \gamma_i$ so we conclude that the only possible eigenvalues of σ are 1 and -1. Thus the decomposition $(V, \varphi) = \bigoplus(V_i, \varphi_i)$ has in fact only two summands and, more importantly, $\sigma^2 = 1$. This means that all non-trivial elements of $\mathrm{Aut}(V, \varphi)$ have order two and therefore $\mathrm{Aut}(V, \varphi)$ is abelian.

Now, the summands (V_i, φ_i) may not be simple but, applying the preceding argument inductively to each of the (V_i, φ_i), we eventually obtain a decomposition $(V, \varphi) = (\bigoplus V_i, \sum \varphi_i)$ where each (V_i, φ_i) is stable and simple, and $\varphi_i \neq 0$. Since $\mathrm{Aut}(V, \varphi)$ is abelian, the successive decompositions of V in eigenspaces can in fact be carried out simultaneously for all $\sigma \in \mathrm{Aut}(V, \varphi) \setminus \{\pm 1\}$. From this the uniqueness of the decomposition and the statement about the automorphism group of (V, φ) are immediate.

Finally, Lemma 3.9 and Lemma 3.11 together imply that the (V_i, φ_i) are mutually non-isomorphic. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

3.3. $\mathrm{Sp}(2n, \mathbb{R})$-, $\mathrm{Sp}(2n, \mathbb{C})$- and $\mathrm{SL}(2n, \mathbb{C})$-Higgs bundles: stability conditions. An $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle can be viewed as a Higgs bundle for the larger complex groups $\mathrm{Sp}(2n, \mathbb{C})$ and $\mathrm{SL}(2n, \mathbb{C})$. The goal of this section is to understand the relation between the various corresponding stability notions. The main results are Theorems 3.13 and 3.14 below.

We have seen in Section B.8 that an $\mathrm{SL}(m, \mathbb{C})$-Higgs bundle is a pair (W, Φ) where W is a rank m holomorphic vector bundle on the Riemann surface X and $\Phi \in H^0(K \otimes \mathrm{End}(W))$. As was shown in Theorem B.10, (W, Φ) is stable if for any subbundle $W' \subset W$ such that $\Phi(W') \subset K \otimes W'$ we have $\deg W' < 0$ (and similarly for semistability).

We have also seen, in Section B.7, that an $\mathrm{Sp}(2n, \mathbb{C})$-Higgs bundle is given by $((W, \Omega), \Phi)$, where (W, Ω) is a rank $2n$ holomorphic symplectic vector bundle (i.e., Ω is a holomorphic symplectic form on W) and $\Phi \in H^0(K \otimes \mathrm{End}(W))$ is symplectic, i.e.,

$$(3.20) \qquad\qquad\qquad \Omega(\Phi u, v) + \Omega(u, \Phi v) = 0$$

for local holomorphic sections u and v of W. Recall from Theorem B.9 that $((W, \Omega), \Phi)$ is stable if and only if for any isotropic subbundle $W' \subset W$ such that $\Phi(W') \subset K \otimes W'$ we have $\deg W' < 0$ (and similarly for semistability).

Given an $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle (V, φ) with $\varphi = (\beta, \gamma) \in H^0(K \otimes (S^2 V \oplus S^2 V^*))$ one can associate to it an $\mathrm{Sp}(2n, \mathbb{C})$-Higgs bundle $((W, \Omega), \Phi)$ given by

$$(3.21) \qquad W = V \oplus V^*, \quad \Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \quad \text{and} \quad \Omega\big((v, \xi), (w, \eta)\big) = \xi(w) - \eta(v),$$

for local holomorphic sections v, w of V and ξ, η of V^* (i.e. Ω is the canonical symplectic structure on $V \oplus V^*$).

Since $\mathrm{Sp}(2n, \mathbb{C}) \subset \mathrm{SL}(2n, \mathbb{C})$, every $\mathrm{Sp}(2n, \mathbb{C})$-Higgs bundle $((W, \Omega), \Phi)$ gives rise to an $\mathrm{SL}(2n, \mathbb{C})$-Higgs bundle (W, Φ). If $((W, \Omega), \Phi)$ is obtained from an $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle (V, φ) we denote the associated $\mathrm{SL}(2n, \mathbb{C})$-Higgs bundle by

$$H(V, \varphi) = (W, \Phi) = (V \oplus V^*, \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}).$$

Theorem 3.13. *Let $(V, \varphi = (\beta, \gamma))$ be an $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle and let $(W, \Phi) = H(V, \varphi)$ be the corresponding $\mathrm{SL}(2n, \mathbb{C})$-Higgs bundle. Then*

(1) *if (W, Φ) is stable then (V, φ) is stable;*
(2) *if (V, φ) is stable and simple then (W, Φ) is stable unless there is an isomorphism $f : V \xrightarrow{\cong} V^*$ such that $\beta f = f^{-1}\gamma$, in which case (W, Φ) is polystable;*
(3) *(W, Φ) is semistable if and only if (V, φ) is semistable.*
(4) *(W, Φ) is polystable if and only if (V, φ) is polystable;*

In particular, if $\deg V \neq 0$ then (W, Φ) is stable if and only if (V, φ) is stable.

For the statement of the following Theorem, recall from Section B.9 that a $\mathrm{GL}(n, \mathbb{R})$-Higgs bundle is given by $((W, Q), \psi)$, where (W, Q) is rank n orthogonal bundle and $\psi \in H^0(K \otimes S^2 W)$. The stability condition for $\mathrm{GL}(n, \mathbb{R})$-Higgs bundles is given in Theorem B.11.

Theorem 3.14. *Let (V, φ) be a stable and simple $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle. Then (V, φ) is stable as an $\mathrm{Sp}(2n, \mathbb{C})$-Higgs bundle, unless there is a symmetric isomorphism $f : V \xrightarrow{\cong} V^*$ such that $\beta f = f^{-1}\gamma$. Moreover, if such an f exists, let $\psi = \beta = f^{-1}\gamma f^{-1} \in H^0(K \otimes S^2 V)$. Then the $\mathrm{GL}(n, \mathbb{R})$-Higgs bundle $((V, f), \psi)$ is stable, even as a $\mathrm{GL}(n, \mathbb{C})$-Higgs bundle.*

The proof of Theorem 3.13 is given below in Section 3.4 and the proof of Theorem 3.14 is given below in Section 3.5.

The following observation is not essential for our main line of argument. We include it since it might be of independent interest.

Remark 3.15. Suppose we are in Case (2) of Theorem 3.13. Decompose $f = f_s + f_a : V \xrightarrow{\cong} V$ in its symmetric and anti-symmetric parts, given by $f_s = \frac{1}{2}(f + f^t)$ and $f_a = \frac{1}{2}(f - f^t)$. Let $V_a = \ker(f_s)$ and $V_s = \ker(f_a)$. There is then a decomposition $V = V_a \oplus V_s$ and f decomposes as

$$f = \begin{pmatrix} f_s & 0 \\ 0 & f_a \end{pmatrix} : V_s \oplus V_a \to V_s^* \oplus V_a^*.$$

Write $\gamma_{sa} : V_a \to V_s^* \otimes K$ for the component of γ in $H^0(K \otimes V_a^* \otimes V_s^*)$ and similarly for the other mixed components of β and γ. Since f intertwines β and γ, one has that $\gamma_{as} = f_a \beta_{as} f_s$. Hence

$$\gamma_{sa} = \gamma_{as}^t = f_s^t \beta_{as}^t f_a^t = -f_s \beta_{sa} f_a = -\gamma_{sa}.$$

It follows that $\gamma_{sa} = 0$ and similarly for the other mixed terms. Thus there is a decomposition $(V, \varphi) = (V_s \oplus V_a, \varphi_s + \varphi_a)$. If (V, φ) is simple then one of the summands must be trivial. The case when $(V, \varphi) = (V_s, \varphi_s)$ is the one covered in Theorem 3.14. In the other case, when $(V, \varphi) = (V_a, \varphi_a)$, the antisymmetric map f defines a symplectic form on V. If we let $\psi = \beta f = f^{-1}\gamma$, one easily checks that ψ is symplectic (cf. (3.20)). Thus,

in this case, (V, ϕ) comes in fact from an $\mathrm{Sp}(n, \mathbb{C})$-Higgs bundle $((V, f), \psi)$. This is a stable $\mathrm{Sp}(n, \mathbb{C})$-Higgs bundle, since (V, ψ) is a stable $\mathrm{GL}(n, \mathbb{C})$-Higgs bundle (cf. the proof of Theorem 3.14 below).

3.4. **Proof of Theorem 3.13.** The proof of the theorem is split into several lemmas. We begin with the following lemma which proves that Higgs bundle stability of $H(V, \varphi)$ implies stability of (V, φ).

Lemma 3.16. *Let $(V, \varphi = (\beta, \gamma))$ be an $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle, and let*

$$\Phi = \left(\begin{array}{cc} 0 & \beta \\ \gamma & 0 \end{array} \right) : V \oplus V^* \to K \otimes (V \oplus V^*).$$

The pair (V, φ) is semistable if and only if for any pair of subbundles $A \subset V$ and $B \subset V^*$ satisfying $B^\perp \subset A$, $A^\perp \subset B$ and $\Phi(A \oplus B) \subset K \otimes (A \oplus B)$ we have $\deg(A \oplus B) \leq 0$.

The pair (V, φ) is stable if and only if it is semistable and for any pair of subbundles $A \subset V$ and $B \subset V^*$, at least one of which is proper, and satisfying $B^\perp \subset A$ (equivalently, $A^\perp \subset B$) and $\Phi(A \oplus B) \subset K \otimes (A \oplus B)$, the inequality $\deg(A \oplus B) < 0$ holds.

Proof. Suppose that $A \subset V$ and $B \subset V^*$ satisfy the conditions of the lemma. Then setting $V_2 := A$ and $V_1 := B^\perp$ we obtain a filtration $0 \subset V_1 \subset V_2 \subset V$ which, thanks to Remark 3.2, satisfies (3.11).

Conversely, given a filtration $0 \subset V_1 \subset V_2 \subset V$ for which (3.11) holds, we get subbundles $A := V_2 \subset V$ and $B := V_1^\perp \subset V^*$ satisfying the conditions of the lemma. Finally, we have

$$\deg(A \oplus B) = \deg(V_1^\perp \oplus V_2) = \deg V_1 + \deg V_2 - \deg V,$$

so the lemma follows from Theorem B.4. (For the case of stability, note that at least one of V_1 and V_2 is a proper subbundle of V if and only if at least one of $A \subset V$ and $B \subset V^*$ is a proper subbundle.) $\qquad\square$

Remark 3.17. In the proof we have used the following formula: if $F \subset E$ is an inclusion of vector bundles, then $\deg F^\perp = \deg F - \deg E$. To check this, observe that there is an exact sequence $0 \to F^\perp \to E^* \to F^* \to 0$, and apply the additivity of the degree w.r.t. exact sequences together with $\deg E^* = -\deg E$ and $\deg F^* = -\deg F$.

The following lemma resumes the proof of equivalence between Higgs bundle stability and stability when V is not isomorphic to V^*.

Lemma 3.18. *Suppose that (V, φ) is semistable, and define $\Phi \colon V \oplus V^* \to K \otimes (V \oplus V^*)$ as previously. Then any subbundle $0 \neq W' \subsetneq V \oplus V^*$ such that $\Phi(W') \subset K \otimes W'$ satisfies $\deg W' \leq 0$. Furthermore, if (V, φ) is stable and simple, one can get equality only if there is an isomorphism $f \colon V \to V^*$ such that $\beta f = f^{-1}\gamma$, and in this case $(W, \Phi) = H(V, \varphi)$ is polystable.*

Proof. Fix a subbundle $W' \subset V \oplus V^*$ satisfying $\Phi(W') \subset K \otimes W'$. We prove the lemma in various steps.

1. Denote by $p \colon V \oplus V^* \to V$ and $q \colon V \oplus V^* \to V^*$ the projections, and define subsheaves $A = p(W')$ and $B = q(W')$. It follows from $\Phi W' \subset K \otimes W'$ that $\beta B \subset K \otimes A$ and $\gamma A \subset K \otimes B$ (for example, using that $\Phi p = q\Phi$ and $\Phi q = p\Phi$). Since both β and γ are

symmetric we deduce that $\beta A^\perp \subset K \otimes B^\perp$ and $\gamma B^\perp \subset K \otimes A^\perp$ as well. It follows from this that if we define subsheaves

$$A_0 = A + B^\perp \subset V \qquad \text{and} \qquad B_0 = B + A^\perp \subset V^*$$

then we have $B_0^\perp \subset A_0$, $A_0^\perp \subset B_0$ and $\Phi(A_0 \oplus B_0) \subset K \otimes (A_0 \oplus B_0)$.

We can apply Lemma 3.16 also to subsheaves by replacing any subsheaf of V or V^* by its saturation, which is now a subbundle of degree not less than that of the subsheaf. Hence we deduce that

$$(3.22) \qquad \deg A_0 + \deg B_0 = \deg(A + B^\perp) + \deg(B + A^\perp) \le 0,$$

and equality holds if and only if $A + B^\perp = V$ and $B + A^\perp = V^*$.

Now we compute (using repeatedly the formula in Remark 3.17)

$$\begin{aligned}
\deg(A + B^\perp) &= \deg A + \deg B^\perp - \deg(A \cap B^\perp) \\
&= \deg A + \deg B - \deg V^* - \deg((A^\perp + B)^\perp) \\
&= \deg A + \deg B - \deg V^* - \deg(A^\perp + B) + \deg V^* \\
&= \deg A + \deg B - \deg(A^\perp + B).
\end{aligned}$$

Consequently $\deg A + \deg B = \deg(A + B^\perp) + \deg(A^\perp + B)$, so (3.22) implies that

$$(3.23) \qquad \deg A + \deg B \le 0,$$

with equality if and only if $A + B^\perp = V$ and $B + A^\perp = V^*$.

2. Let now $A' = W' \cap V$ and $B' = W' \cap V^*$. Using again that $\Phi(W') \subset K \otimes W'$ we prove that $\beta B' \subset K \otimes A'$ and $\gamma A' \subset K \otimes B'$. Now, the same reasoning as above (considering $(A' + B'^\perp) \oplus (B' + A'^\perp)$ and so on) proves that

$$(3.24) \qquad \deg A' + \deg B' \le 0,$$

with equality if and only if $A' + B'^\perp = V$ and $A'^\perp + B' = V^*$.

3. Observe that there are exact sequences of sheaves

$$0 \to B' \to W' \to A \to 0 \qquad \text{and} \qquad 0 \to A' \to W' \to B \to 0,$$

from which we obtain the formulae

$$\deg W' = \deg A + \deg B' \qquad \text{and} \qquad \deg W' = \deg B + \deg A'.$$

Adding up and using (3.23) together with (3.24) we obtain the desired inequality

$$\deg W' \le 0.$$

4. Finally we consider the case when (V, φ) is stable and simple. Suppose that $\deg W' = 0$. Then we have equality both in (3.23) and in (3.24). Hence, $A + B^\perp = V$, $A^\perp + B = V^*$, $A' + B'^\perp = V$ and $A'^\perp + B' = V^*$. But $A^\perp + B = (A \cap B^\perp)^\perp$ and $A'^\perp + B' = (A' \cap B'^\perp)^\perp$, so we deduce that

$$A \oplus B^\perp = V \qquad \text{and} \qquad A' \oplus B'^\perp = V.$$

If one of these decompositions were nontrivial then V would not be simple, in contradiction with our assumptions. Consequently we must have $A = V$, $B^\perp = 0$ (because $W' \ne 0$) and similarly $A' = 0$, $B'^\perp = V^*$ (because $W' \ne V \oplus V^*$). This implies that the projections $p : W' \to A$ and $q : W' \to B$ induce isomorphisms $u : W' \simeq V$ and $v : W' \simeq V^*$. Finally, defining $f := v \circ u^{-1} : V \to V^*$ we find an isomorphism which satisfies $\beta f = f^{-1}\gamma$ because $\Phi W' \subset K \otimes W'$.

To prove that in this case $(W, \Phi) = H(V, \varphi)$ is strictly polystable just observe that $W' = \{(u, fu) \mid u \in V\}$ and define $W'' = \{(u, -fu) \mid u \in V\}$. It is then straightforward to check that $V \oplus V^* = W' \oplus W''$, that $\Phi W' \subset K \otimes W'$ and that $\Phi W'' \subset K \otimes W''$. Finally note that the Higgs bundle (W', Φ) is stable: any Φ-invariant subbundle $W_0 \subset W'$ is also a Φ-invariant subbundle of $(V \oplus V^*, \Phi)$. Hence, if $\deg W_0 = 0$ the argument of the previous paragraph shows that W_0 has to have the same rank as V, so $W_0 = W'$. Analogously, one sees that (W'', Φ) is a stable Higgs bundle. $\square$

Lemma 3.19. *An* $\mathrm{Sp}(2n, \mathbb{R})$*-Higgs bundle* (V, φ) *is semistable if and only if* $H(V, \varphi)$ *is semistable.*

Proof. Both Lemmas 3.16 and 3.18 are valid if we substitute all strict inequalities by inequalities (and of course remove the last part in the statement of Lemma 3.18). Combining these two modified lemmas we get the desired result. $\square$

Lemma 3.20. *An* $\mathrm{Sp}(2n, \mathbb{R})$*-Higgs bundle* $(V, \varphi = (\beta, \gamma))$ *is polystable if and only if* $H(V, \varphi)$ *is polystable.*

Proof. If (V, φ) is polystable then Lemmas 3.16 and 3.18 imply that $H(V, \varphi)$ is polystable.

Now assume that $(W, \Phi) = H(V, \varphi)$ is polystable, so that $W = \bigoplus_{i=1}^{N} W_i$ with $\Phi W_i \subset K \otimes W_i$ and every $(W_i, \Phi|_{W_i})$ is stable with $\deg W_i = 0$.

1. We claim that for any subbundle $U \subset W$ satisfying $\deg U = 0$ and $\Phi(U) \subset K \otimes U$ there is an isomorphism $\psi : W \to W$ which commutes with Φ and a set $I \subset \{1, \ldots, N\}$ such that $U = \psi(\bigoplus_{i \in I} W_i)$. To prove the claim we use induction on N (the case $N = 1$ being obvious). Let $W_{\geq 2} = \bigoplus_{i \geq 2} W_i$ and denote by $p_{\geq 2} : W \to W_{\geq 2}$ the projection. Then we have an exact sequence

$$0 \to W_1 \cap U \to U \to p_{\geq 2}(U) \to 0.$$

Since both $W_1 \cap U$ and $p_{\geq 2}(U)$ are invariant under Φ, by polystability their degrees must be ≤ 0. And since according to the exact sequence above the sum of their degrees must be 0, the only possibility is that

$$\deg W_1 \cap U = 0 \qquad \text{and} \qquad \deg p_{\geq 2}(U) = 0.$$

Now we apply the induction hypothesis to the inclusion $p_{\geq 2}(U) \subset W_{\geq 2}$ and deduce that there is an isomorphism $\psi_2 : W_{\geq 2} \to W_{\geq 2}$ commuting with Φ and a subset $I_2 \subset \{2, \ldots, N\}$ such that

$$p_{\geq 2}(U) = \psi_2(\bigoplus_{i \in I_2} W_i).$$

Since $\deg W_1 \cap U = 0$ and W_1 is stable, only two things can happen. Either $W_1 \cap U = W_1$ or $W_1 \cap U = 0$. In the first case we have

$$U = W_1 \oplus \bigoplus_{i \in I_2} \psi(W_i),$$

so putting $I = \{1\} \cap I_2$ and $\psi = \mathrm{diag}(1, \psi_2)$ the claim is proved. If instead $W_1 \cap U = 0$ then there is a map $\xi : p_{\geq 2}(U) \to W_1$ such that

$$U = \{(\xi(v), v) \in W_1 \oplus p_{\geq 2}\}.$$

Since U is Φ-invariant we deduce that ξ must commute with Φ. If we now extend ξ to $W_{\geq 2}$ by defining $\xi(\psi_2(W_j)) = 0$ for any $j \in \{2, \ldots, N\} \setminus I_2$ then the claim is proved by setting $I = I_2$ and

$$\psi = \begin{pmatrix} 1 & \xi \circ \psi_2 \\ 0 & \psi_2 \end{pmatrix}.$$

2. Define for any $W' \subset W$ the subsheaves $R(W') = p(W') \oplus q(W')$ (recall that $p : W \to V$ and $q : W \to V^*$ are the projections) and $r(W') = (W' \cap V) \oplus (W' \cap V^*)$. Reasoning as in the first step of the proof of Lemma 3.18 we deduce that if W' is Φ-invariant then both $R(W')$ and $r(W')$ are Φ invariant, so in particular we must have $\deg R(W') \leq 0$ and $\deg r(W') \leq 0$. In case $\deg W' = 0$ these inequalities imply $\deg R(W') = \deg r(W') = 0$ (using the exact sequences $0 \to W' \cap V^* \to W' \to p(W') \to 0$ and $0 \to W' \cap V \to W' \to q(W') \to 0$).

Assume that there is some summand in $\{W_1, \ldots, W_N\}$, say W_1, such that $0 \neq r(W_1)$ or $R(W_1) \neq W$. Suppose, for example, that $W' := R(W_1) \neq W$ (the other case is similar). Let $A = p(W_1)$ and $B = q(W_1)$, so that $W' = A \oplus B$. By the observation above and the claim proved in **1** we know that there is an isomorphism $\psi : W \to W$ which commutes with Φ and such that, if we substitute $\{W_i\}_{1 \leq i \leq N}$ by $\{\psi(W_i)\}_{1 \leq i \leq N}$ and we reorder the summands if necessary, then we may write $W' = W_1 \oplus \cdots \oplus W_k$ for some $k < N$. Now let $W'' = W_{k+1} \oplus \cdots \oplus W_N$. We clearly have $W = W' \oplus W''$, so the inclusion of $W'' \subset W = V \oplus V^*$ composed with the projection $V \oplus V^* \to V/A \oplus V^*/B = W/W'$ induces an isomorphism. Consequently we have $V = A \oplus W'' \cap V$. Let us rename for convenience $V_1 := A$ and $V_2 := W'' \cap V$. Then, using the fact that each W_i is Φ-invariant we deduce that we can split both β and γ as

$$\beta = (\beta_1, \beta_2) \in H^0(K \otimes S^2 V_1) \oplus H^0(K \otimes S^2 V_2),$$
$$\gamma = (\gamma_1, \gamma_2) \in H^0(K \otimes S^2 V_1^*) \oplus H^0(K \otimes S^2 V_2^*).$$

Hence, if we put $\varphi_i = (\beta_i, \gamma_i)$ for $i = 1, 2$ then we may write

$$(V, \varphi) = (V_1, \varphi_1) \oplus (V_2, \varphi_2).$$

3. Our strategy is now to apply recursively the process described in **2**. Observe that if $N \geq 3$ then for at least one i we have $R(W_i) \neq W$, because there must be a summand whose rank is strictly less that the rank of V. Hence the projection of this summand to V is not exhaustive.

Consequently, we can apply the process and split V in smaller and smaller pieces, until we arrive at a decomposition

$$(V, \varphi) = (V_1, \varphi_1) \oplus \cdots \oplus (V_j, \varphi_j)$$

such that we can not apply **2** to any $H(V_i, \varphi_i)$ For each (V_i, φ_i) there are two possibilities. Either $H(V_i, \varphi_i)$ is stable, in which case (V_i, φ_i) is stable (by Lemma 3.16), or $H(V_i, \varphi_i)$ splits in two stable Higgs bundles $W_i' \oplus W_i''$ which satisfy:

$$R(W_i') = R(W_i'') = W \qquad \text{and} \qquad r(W_i') = r(W_i'') = 0.$$

But in this case it is easy to check that (V_i, φ_i) is also stable.

By the preceeding lemma, (V, φ) is semistable. Suppose it is not stable. Then there is a filtration $0 \subset V_1 \subset V_2 \subset V$ such that $\Phi(V_2 \oplus V_1^\perp) \subset K \otimes (V_2 \oplus V_1^\perp)$ and $W' := V_2 \oplus V_1^\perp = 0$ has degree $\deg W' = 0$.

Define $W_{\geq 2} = \bigoplus_{i \geq 2} W_i$, and let $p_2 : W \to W_{\geq 2}$ denote the projection. We have an exact sequence

$$0 \to W' \cap W_1 \to W' \to p_2(W') \to 0.$$

It is easy to check that $\Phi(W' \cap W_1) \subset K \otimes (W' \cap W_1)$ and that $\Phi(p_2(W')) \subset K \otimes p_2(W')$. Since both W_1 and and $W_{\geq 2}$ are polystable, we must have $\deg W' \cap W_1 \leq 0$ and $\deg p_2(W') \leq 0$. Finally, since $\deg W' = 0$, the exact sequence above implies that $\deg W' \cap W_1 = 0$ and $\deg p_2(W') = 0$. Now W_1 is stable, so $W' \cap W_1$ can only be either 0 or W_1. Reasoning inductively with $p_2(W') \subset W_{\geq 2}$ in place of $W' \subset W$ we deduce that there must be some $I \subset \{1, \ldots, k\}$ such that

$$W' = \bigoplus_{i \in I} W_i.$$

Since each $(W_i, \Phi|_{W_i})$ is stable, it is easy to check (for example using induction on N) that one must have $\deg V_2 \oplus V_1^\perp = W_j$ for some j. This easily implies that $V_2 = V \cap W_j$ and if we define

$$V' = \bigoplus_{i \neq j} p(W_j)$$

then $V = V' \oplus V_2$. Applying the same process to V' and V_2 we arrive at the conclusion that (V, φ) is polystable. $\qquad \square$

3.5. **Proof of Theorem 3.14.** An $\mathrm{Sp}(2n, \mathbb{C})$-Higgs bundle $((W, \Omega), \Phi)$ is stable if the $\mathrm{SL}(2n, \mathbb{C})$-Higgs bundle (W, Φ) is stable. Thus the result is immediate from Theorem 3.13, unless we are in Case (2) of that Theorem. In that case, we have seen in the last paragraph of the proof of Lemma 3.18 that

(1) There is an isomorphism f as stated, except for the symmetry condition.
(2) There is an isomorphism $V \oplus V^* = W' \oplus W''$, where $W' = \{(u, f(u)) \mid u \in V\}$ and $W'' = \{(u, -f(u)) \mid u \in V\}$, and W' and W'' are Φ-invariant subbundles of W.
(3) The $\mathrm{SL}(2n, \mathbb{C})$-Higgs bundle (W, Φ) is strictly polystable, decomposing as the direct sum of stable $\mathrm{GL}(n, \mathbb{C})$-Higgs bundles:

(3.25) $$(W, \Phi) = (W', \Phi') \oplus (W'', \Phi'').$$

Note also that $(W', \Phi') \simeq (W'', \Phi'')$.

Now, from Theorem B.9 we have that for the $\mathrm{Sp}(2n, \mathbb{C})$-Higgs bundle $((W, \Omega), \Phi)$ to be strictly semistable, it must have an isotropic Φ-invariant subbundle of degree zero. But the decomposition (3.25) shows that the only degree zero Φ-invariant subbundles are W' and W''. The subbundle W' is isotropic if and only if, for all local sections u, v of V, we have

$$\Omega((u, f(u), (v, f(v))) = 0 \iff \langle u, f(v) \rangle = \langle v, f(u) \rangle,$$

that is, if and only if f is symmetric. Analogously, W'' is isotropic if and only if f is symmetric. The first part of the conclusion follows.

For the second part, consider the $\mathrm{GL}(n, \mathbb{R})$-Higgs bundle $((V, f), \beta f)$. This is stable as a $\mathrm{GL}(n, \mathbb{C})$-Higgs bundle because $(V, \beta f) \simeq (W', \Phi')$, which is stable. Thus, in particular, $((V, f), \beta f)$ is stable as a $\mathrm{GL}(n, \mathbb{R})$-Higgs bundle (see Theorem B.11). $\qquad \square$

3.6. **Milnor–Wood inequality and moduli space of** $\mathrm{Sp}(2n,\mathbb{R})$**-Higgs bundles.** The topological invariant attached to an $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundle (V,β,γ) is an element in the fundamental group of $\mathrm{U}(n)$. Since $\pi_1(U(n)) \cong \mathbb{Z}$, this is an integer, which coincides with the degree of V.

We have the following Higgs bundle incarnation of the Milnor–Wood inequality (1.1) (see [29, 6]).

Proposition 3.21. *Let* (V,β,γ) *be a semistable* $\mathrm{Sp}(2n,\mathbb{R})$*-Higgs bundle and let* $d = \deg(V)$*. Then*

$$\tag{3.26} d \leq \mathrm{rank}(\gamma)(g-1)$$
$$\tag{3.27} -d \leq \mathrm{rank}(\beta)(g-1),$$

This is proved by first using the equivalence between the semistability of (V,β,γ) and the $\mathrm{SL}(2n,\mathbb{C})$-Higgs bundle (W,Φ) associated to it, and then applying the semistability numerical criterion to special Higgs subbundles defined by the kernel and image of Φ.

As a consequence of Proposition 3.21 we have the following.

Proposition 3.22. *Let* (V,β,γ) *be a semistable* $\mathrm{Sp}(2n,\mathbb{R})$*-Higgs bundle and let* $d = \deg(V)$*. Then*

$$|d| \leq n(g-1).$$

Furthermore,

    (1) $d = n(g-1)$ *holds if and only if* $\gamma\colon V \to V^* \otimes K$ *is an isomorphism;*
    (2) $d = -n(g-1)$ *holds if and only if* $\beta\colon V^* \to V \otimes K$ *is an isomorphism.*

Recall from our general discussion in Section 2 that $\mathcal{M}_d(\mathrm{Sp}(2n,\mathbb{R}))$ denotes the moduli space of $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundles (V,β,γ) with $\deg(V) = d$. For brevity we shall henceforth write simply $\mathcal{M}_d$ for this moduli space.

Combining Theorem 2.2 with Proposition 2.16 we have the following.

Proposition 3.23. *The moduli space* $\mathcal{M}_d$ *is a complex algebraic variety. Its expected dimension is* $(g-1)(2n^2+n)$*.*

One has the following immediate duality result.

Proposition 3.24. *The map* $(V,\beta,\gamma) \mapsto (V^*, \gamma^t, \beta^t)$ *gives an isomorphism* $\mathcal{M}_d \cong \mathcal{M}_{-d}$*.*

As a corollary of Proposition 3.22, we obtain the following.

Proposition 3.25. *The moduli space* $\mathcal{M}_d$ *is empty unless*

$$|d| \leq n(g-1).$$

3.7. **Smoothness and polystability of** $\mathrm{Sp}(2n,\mathbb{R})$**-Higgs bundles.** We study now the smoothness properties of the moduli space. As a corollary of Proposition 2.15 and Theorem 3.14 we have the following.

Proposition 3.26. *Let* (V,φ) *be an* $\mathrm{Sp}(2n,\mathbb{R})$*-Higgs bundle which is stable and simple and assume that there is no symmetric isomorphism* $f\colon V \xrightarrow{\cong} V^*$ *intertwining* β *and* γ*. Then* (V,φ) *represents a smooth point of the moduli space of polystable* $\mathrm{Sp}(2n,\mathbb{R})$*-Higgs bundles.*

So, a stable $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundle (V,φ) in $\mathcal{M}_d$ with $d \neq 0$ can only fail to be a smooth point of the moduli space if it is not simple — this gives rise to an orbifold-type singularity — or if, in spite of being simple, there is an isomorphism $V \simeq V^*$ intertwining β and γ. Of course, this can only happen if $d = \deg V = 0$. Generally, if (V,φ) is polystable, but not stable it is also a singular point of $\mathcal{M}_d$.

We shall need the following analogue of Proposition 3.26 for $\mathrm{U}(n)$-, $\mathrm{U}(p,q)$- and $\mathrm{GL}(n,\mathbb{R})$-Higgs bundles

Proposition 3.27.   (1) *A stable* $\mathrm{U}(n)$-*Higgs bundle represents a smooth point in the moduli space of* $\mathrm{U}(n)$-*Higgs bundles.*
   (2) *A stable* $\mathrm{U}(p,q)$-*Higgs bundle represents a smooth point of the moduli space of* $\mathrm{U}(p,q)$-*Higgs bundles.*
   (3) *A* $\mathrm{GL}(n,\mathbb{R})$-*Higgs bundle which is stable as a* $\mathrm{GL}(n,\mathbb{C})$-*Higgs bundle represents a smooth point in the moduli space of* $\mathrm{GL}(n,\mathbb{R})$-*Higgs bundles.*

Proof. (1) A stable $\mathrm{U}(n)$-Higgs bundles is nothing but a stable vector bundle, so this is classical.

(2) A stable $\mathrm{U}(p,q)$-Higgs bundle is also stable as $\mathrm{GL}(p+q,\mathbb{C})$-Higgs bundle (see [6]). Thus the result follows from Proposition 2.15 and the fact that a $\mathrm{GL}(p+q,\mathbb{C})$-Higgs bundle is simple.

(3) This holds by the same argument as in (2). $\square$

It will be convenient to make the following definition for $\mathrm{GL}(n,\mathbb{R})$-Higgs bundles, analogous to the way we associate $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundles to vector bundles and $\mathrm{U}(p,q)$-Higgs bundles in (3.15) and (3.16), respectively (cf. Theorem 3.14). Given a $\mathrm{GL}(n,\mathbb{R})$-Higgs bundle $((W,Q),\psi)$, let $f \colon W \to W^*$ be the symmetric isomorphism associated to Ω. Define an associated $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundle

$$(3.28) \qquad\qquad (V,\varphi) = \upsilon_*^{\mathrm{GL}(n,\mathbb{R})}((W,Q),\psi)$$

by setting

$$V = W, \quad \beta = \psi \quad \text{and} \quad \gamma = f\psi f.$$

Since no confusion is likely to occur, in the following we shall slightly abuse language, saying simply that $\upsilon_*^{\mathrm{GL}(n,\mathbb{R})}((W,Q),\psi)$ is a $\mathrm{GL}(n,\mathbb{R})$-Higgs bundle. Similarly we shall say that $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundles (V,φ) obtained from the constructions (3.15) and (3.16) are $\mathrm{U}(n)$-Higgs bundles and $\mathrm{U}(p,q)$-Higgs bundles, respectively. With this understood, we can state our structure theorem on polystable $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundles from Section B.6 as follows.

Proposition 3.28. *Let* (V,φ) *be a polystable* $\mathrm{Sp}(2n,\mathbb{R})$-*Higgs bundle. Then there is a decomposition*

$$(V,\varphi) = (V_1,\varphi_1) \oplus \cdots \oplus (V_k,\varphi_k),$$

unique up to reordering, such that each (V_i,φ_i) *is a stable* G_i-*Higgs bundle, where* G_i *is one of the following groups:* $\mathrm{Sp}(2n_i,\mathbb{R})$, $\mathrm{U}(n_i)$ *or* $\mathrm{U}(p_i,q_i)$.

Theorem 3.29. *Let* (V,φ) *be a polystable* $\mathrm{Sp}(2n,\mathbb{R})$-*Higgs bundle. Then there is a decomposition* $(V,\varphi) = (V_1,\varphi_1) \oplus \cdots \oplus (V_k,\varphi_k)$, *unique up to reordering, such that each of the* $\mathrm{Sp}(2n_i,\mathbb{R})$-*Higgs bundles* (V_i,φ_i) *is one of the following:*

   (1) *A stable and simple* $\mathrm{Sp}(2n_i,\mathbb{R})$-*Higgs bundle.*

(2) *A stable* $\mathrm{U}(p_i, q_i)$*-Higgs bundle with* $n_i = p_i + q_i$.
(3) *A stable* $\mathrm{U}(n_i)$*-Higgs bundle.*
(4) *A* $\mathrm{GL}(n_i, \mathbb{R})$*-Higgs bundle which is stable as a* $\mathrm{GL}(n_i, \mathbb{C})$*-Higgs bundle.*

Each (V_i, φ_i) *is a smooth point in the moduli space of* G_i*-Higgs bundles, where* G_i *is the corresponding real group* $\mathrm{Sp}(2n_i, \mathbb{R})$, $\mathrm{U}(p_i, q_i)$, $\mathrm{U}(n_i)$ *or* $\mathrm{GL}(n_i, \mathbb{R})$.

Proof. This follows from Propositions 3.26, 3.27 and 3.28 and Theorems 3.12 and 3.14   □

Remark 3.30. The existence of the decomposition of a polystable $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle (V, φ) given in Proposition 3.28 can also be seen in a more down to earth way, as we now briefly outline. Let (V, φ) be a polystable $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle and let $(W, \Phi) = H(V, \varphi)$ be the corresponding $\mathrm{SL}(2n, \mathbb{C})$-Higgs bundle. By Theorems 3.13 and B.10 we have that

$$(3.29) \qquad\qquad (W, \Phi) = \bigoplus (W_i, \Phi_i),$$

where (W_i, Φ_i) are stable $\mathrm{GL}(n_i, \mathbb{C})$-Higgs bundles. We can control the shape of the summands (W_i, Φ_i) by considering the subbundles $A \oplus B$ described in Lemma 3.16. By considering a maximal destabilizing $W' = A \oplus B \subseteq E$ and analyzing the induced stable quotient $W'' = (V/A) \oplus V^*/B$ with the induced Higgs field, one sees that (W_i, Φ_i) is in fact isomorphic to $H(V_i, \varphi_i)$, where (V_i, φ_i) is of one of the three types $\mathrm{U}(n_i)$, $\mathrm{Sp}(2n_i, \mathbb{R})$, and $\mathrm{U}(p_i, q_i)$. The different types correspond to whether $(V/A)^*$ and V^*/B are isomorphic or not.

4. MAXIMAL DEGREE $\mathrm{Sp}(2n, \mathbb{R})$-HIGGS BUNDLES AND THE CAYLEY CORRESPONDENCE

4.1. Cayley correspondence. In this section we shall describe the $\mathrm{Sp}(2n, \mathbb{R})$ moduli space for the extreme value $|d| = n(g - 1)$. In fact, for the rest of this section we shall assume that $d = n(g - 1)$. This involves no loss of generality, since, by Proposition 3.24, $(V, \varphi) \mapsto (V^*, \varphi^t)$ gives an isomorphism between the $\mathrm{Sp}(2n, \mathbb{R})$ moduli spaces for d and $-d$. The main result is Theorem 4.4, which we refer to as the *Cayley correspondence*. This is stated as Theorem 1.3 in the Introduction, where the reason for the name is also explained.

When γ is an isomorphism, the stability condition for $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundles, given by Theorem B.4 (with $\alpha = 0$), simplifies further. Here is a key observation:

Proposition 4.1. *Let* (V, γ, β) *be an* $\mathrm{Sp}(2n, \mathbb{R})$*-Higgs bundle and assume that* $\gamma \colon V \to V^* \otimes K$ *is an isomorphism. If* $0 \subseteq V_1 \subseteq V_2 \subseteq V$ *is a filtration such that* $\gamma \in H^0(K \otimes (S^2 V_1^\perp + V_2^\perp \otimes_S V^*))$, *then* $V_2 = V_1^{\perp \gamma}$.

Proof. This follows from the interpretation of the condition on γ given in Remark 3.2.   □

Proposition 4.2. *Let* (V, β, γ) *be an* $\mathrm{Sp}(2n, \mathbb{R})$*-Higgs bundle and assume that* $\gamma \colon V \to V^* \otimes L$ *is an isomorphism. Let* $\tilde{\beta} = (\beta \otimes 1) \circ \gamma \colon V \to V \otimes L^2$. *Then* (V, β, γ) *is stable if and only if for any* $V_1 \subset V$ *such that* $V_1 \subseteq V_1^{\perp \gamma}$ *(i.e.,* V_1 *is isotropic with respect to* γ*) and* $\tilde{\beta}(V_1) \subseteq V_1 \otimes L^2$, *the condition*

$$\mu(V_1) < g - 1$$

is satisfied.

Proof. Note that $\tilde{\beta}$ is symmetric with respect to γ (viewed as an K-valued quadratic form on V). From Remark 3.2 one sees that $\beta \in H^0(K \otimes (S^2 V_2 + V_1 \otimes_S V))$ if and only if $\tilde{\beta}$ preserves the filtration $0 \subseteq V_1 \subseteq V_2 \subseteq V$. But from Lemma 4.1 we have $V_2 = V_1^{\perp \gamma}$.

Hence $\tilde{\beta}$ preserves V_1 if and only if it preserves V_2 (here one uses that $\tilde{\beta}$ is symmetric with respect to γ). Given this correspondence between the subobjects, one can easily translate the stability condition. $\square$

Let (V, β, γ) be an $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle with $d = n(g-1)$ such that $\gamma \in H^0(K \otimes S^2 V^*)$ is an isomorphism. Let $L_0 = K^{1/2}$ be a fixed square root of K, and define $W = V^* \otimes L_0$. Then $Q := \gamma \otimes I_{L_0^{-1}} : W^* \to W$ is a symmetric isomorphism defining an orthogonal structure on W, in other words, (W, Q) is an $\mathrm{O}(n, \mathbb{C})$-holomorphic bundle. The K^2-twisted endomorphism $\psi : W \to W \otimes K^2$ defined by $\psi = (\gamma \otimes I_{K \otimes L_0}) \circ \beta \otimes I_{L_0}$ is Q-symmetric and hence (W, Q, ψ) defines a K^2-twisted $\mathrm{GL}(n, \mathbb{R})$-Higgs pair, from which we can recover the original $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle.

Theorem 4.3. *Let (V, β, γ) be a $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle with $d = n(g-1)$ such that γ is an isomorphism. Let (W, Q, ψ) be the corresponding K^2-twisted $\mathrm{GL}(n, \mathbb{R})$-Higgs pair. Then (V, β, γ) is semistable (resp. stable, polystable) if and only if (W, Q, ψ) is semistable (resp. stable, polystable).*

Proof. This follows from the simplified stability conditions given in Theorem B.11 and Proposition 4.2, using the translation $W_1 = V_1^* \otimes L_0$. Similarly for semistability and polystability. $\square$

Theorem 4.4. *Let $\mathcal{M}_{\max}$ be the moduli space of polystable $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundles with $d = n(g-1)$ and let $\mathcal{M}'$ be the moduli space of polystable K^2-twisted $\mathrm{GL}(n, \mathbb{R})$-Higgs pairs. The map $(V, \beta, \gamma) \mapsto (W, Q, \psi)$ defines an isomorphism of complex algebraic varieties*

$$\mathcal{M}_{\max} \cong \mathcal{M}'.$$

Proof. Let (V, β, γ) be a semistable $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle with $d = n(g-1)$. By Proposition 3.22, γ is an isomorphism and hence the map $(V, \beta, \gamma) \mapsto (W, Q, \psi)$ is well defined. The result follows now from Theorem 4.3 and the existence of local universal families (see [45]). $\square$

4.2. **Invariants of $\mathrm{GL}(n, \mathbb{R})$-Higgs pairs.** To a K^2-twisted $\mathrm{GL}(n, \mathbb{R})$-Higgs pair (W, Q, ψ) one can attach topological invariants corresponding to the first and second Stiefel-Whitney classes of a reduction to $\mathrm{O}(n)$ of the $\mathrm{O}(n, \mathbb{C})$ bundle defined by (W, Q). The first class $w_1 \in H^1(X, \mathbb{Z}_2) \cong \mathbb{Z}_2^{2g}$ measures the obstruction for the $\mathrm{O}(n)$-bundle to have an orientation, i.e. to the existence of a reduction to a $\mathrm{SO}(n)$ bundle, while the second one $w_2 \in H^2(X, \mathbb{Z}_2) \cong \mathbb{Z}_2$ measures the obstruction to lifting the $\mathrm{O}(n)$-bundle to a $\mathrm{Pin}(n)$-bundle, where

$$1 \to \mathbb{Z}_2 \to \mathrm{Pin}(n) \to \mathrm{O}(n) \to 1.$$

If we define

$$\mathcal{M}'_{w_1, w_2} := \{(W, Q, \psi) \in \mathcal{M}' \text{ such that } w_1(W) = w_1 \text{ and } w_2(W) = w_2\},$$

we have that

$$(4.30) \qquad \mathcal{M}' = \bigcup_{w_1, w_2} \mathcal{M}'_{w_1, w_2}.$$

We thus have, via the isomorphism given by Theorem 4.4, that the moduli space $\mathcal{M}_{\max}$ is partitioned in disjoint closed subvarieties corresponding to fixing (w_1, w_2).

5. THE HITCHIN FUNCTIONAL

5.1. The Hitchin functional. In order to define this functional, we consider the moduli space of $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundles (V, φ) from the gauge theory point of view, i.e., we use the identification of $\mathcal{M}_d$ with the moduli space $\mathcal{M}_d^{\mathrm{gauge}}$ of solutions (A, φ) to the Hitchin equations given by Theorem 2.19. There is an action of S^1 on $\mathcal{M}_d$ via multiplication of φ by scalars: $(A, \varphi) \mapsto (A, e^{i\theta}\varphi)$. Restricted to the smooth locus $\mathcal{M}_d^s$ this action is hamiltonian with symplectic moment map $-f$, where the *Hitchin functional* f is defined by

$$(5.31) \qquad \begin{aligned} & f \colon \mathcal{M}_d \to \mathbb{R}, \\ & (A, \varphi) \mapsto \tfrac{1}{2}\|\varphi\|^2 = \tfrac{1}{2}\|\beta\|^2 + \tfrac{1}{2}\|\gamma\|^2. \end{aligned}$$

Here $\|\cdot\|$ is the L^2-norm obtained by using the Hermitian metric in V and integrating over X. The function f is well defined on the whole moduli space (not just on the smooth locus). It was proved by Hitchin [31, 32] that f is proper and therefore it has a minimum on every closed subspace of $\mathcal{M} = \bigcup_d \mathcal{M}_d$. Thus we have the following result.

Proposition 5.1. *Let $\mathcal{M}' \subseteq \mathcal{M}$ be any closed subspace and let $\mathcal{N}' \subseteq \mathcal{M}'$ be the subspace of local minima of f on $\mathcal{M}'$. If $\mathcal{N}'$ is connected then so is $\mathcal{M}'$.* $\qquad\square$

The following observation was also made by Hitchin [32].

Proposition 5.2. *The Hitchin functional is additive with respect to direct sum of $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundles, in other words,*

$$f\Big(\bigoplus (V_i, \varphi_i)\Big) = \sum f(V_i, \varphi_i).$$

Let (V, φ) represent a smooth point of $\mathcal{M}_d$. Then the moment map condition shows that the critical points of f are exactly the fixed points of the circle action. These can be identified as follows (cf. [31, 32, 48]).

Proposition 5.3. *An $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle (V, φ) represents a fixed point of the circle action on $\mathcal{M}_d$ if and only if it is a* complex variation of Hodge structure *(also called a* Hodge bundle*): this means that there is a decomposition in holomorphic subbundles*

$$V = \bigoplus F_i$$

for real indices, or weights, *i such that, attributing weight $-i$ to F_i^*, $\varphi = (\beta, \gamma)$ has weight one with respect to this decomposition; more explicitly this means that*

$$\gamma \colon F_i \to F_{-i-1}^* \otimes K \qquad and \qquad \beta \colon F_i^* \to F_{-i+1} \otimes K.$$

$\qquad\square$

The decomposition $V = \bigoplus F_i$ of Proposition 5.3 gives rise to corresponding decompositions

$$(5.32) \qquad \operatorname{End}(V)_k = \bigoplus_{i-j=k} F_i \otimes F_j^*,$$

$$(5.33) \qquad (S^2 V \otimes K)_{k+1} = \bigoplus_{\substack{i+j=k+1 \\ i<j}} F_i \otimes F_j \otimes K \oplus S^2 F_{\frac{k+1}{2}} \otimes K,$$

$$(5.34) \qquad (S^2 V^* \otimes K)_{k+1} = \bigoplus_{\substack{-i-j=k+1 \\ i<j}} F_i^* \otimes F_j^* \otimes K \oplus S^2 F^*_{-\frac{k+1}{2}} \otimes K.$$

The map $\operatorname{ad}(\varphi)$ in the deformation complex (2.3) has weight 1 with respect to these decompositions, so that we can define complexes

$$(5.35) \qquad C_k^\bullet(V, \varphi) \colon \operatorname{End}(V)_k \xrightarrow{\operatorname{ad}(\varphi)} (S^2 V \otimes K \oplus S^2 V^* \otimes K)_{k+1},$$

for any k. The deformation complex (2.3) decomposes accordingly as

$$C^\bullet(V, \varphi) = \bigoplus C_k^\bullet(V, \varphi).$$

We shall also need the positive weight subcomplex

$$(5.36) \qquad C_-^\bullet(V, \varphi) = \bigoplus_{k>0} C_k^\bullet(V, \varphi).$$

It can be shown (see, e.g., [23, §3.2]) that $\mathbb{H}^1(C_k^\bullet(V, \varphi))$ is the weight $-k$-subspace of $\mathbb{H}^1(C^\bullet(V, \varphi))$ for the infinitesimal circle action. Thus $\mathbb{H}^1(C_-^\bullet(V, \varphi))$ is the positive weight space for the infinitesimal circle action.

Proposition 5.4. *Let (V, φ) be a polystable $\operatorname{Sp}(2n, \mathbb{R})$-Higgs bundle whose isomorphism class is fixed under the circle action.*

(1) *Assume that (V, φ) is simple and stable as an $\operatorname{Sp}(2n, \mathbb{C})$-Higgs bundle. Then (V, φ) represents a local minimum of f if and only if $\mathbb{H}^1(C_-^\bullet(V, \varphi)) = 0$.*

(2) *Suppose that there is a family (V_t, φ_t) of polystable $\operatorname{Sp}(2n, \mathbb{R})$-Higgs bundles, parametrized by t in the open unit disk D, deforming (V, φ) (i.e., such that $(V_0, \varphi_0) = (V, \varphi)$) and that the corresponding infinitesimal deformation is a non-zero element of $\mathbb{H}^1(C_-^\bullet(V, \varphi))$. Then (V, φ) is not a local minimum of f on $\mathcal{M}_d$.*

Proof. (1) From Proposition 2.15, when the hypotheses are satisfied, (V, φ) represents a smooth point of the moduli space. Then one can use the moment map condition on f to show that $\mathbb{H}^1(C_k^\bullet(V, \varphi))$ is the eigenvalue $-k$ subspace of the Hessian of f (cf. [23, §3.2]; this goes back to Frankel [22], at least). This proves (1).

(2) Take a corresponding family of solutions to Hitchin's equations. One can then prove that the second variation of f along this family is negative in certain directions (see Hitchin [32, § 8]). $\qquad \square$

5.2. **A cohomological criterion for minima.** The following result was proved in [6, Proposition 4.14[2] and Remark 4.16]. It is the key to obtaining the characterization of the minima of the Hitchin functional f.

___
[2]a corrected proof can be found in [9, Lemma 3.11]

Proposition 5.5. *Let (V, φ) be a polystable $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle whose isomorphism class is fixed under the circle action. Then for any k we have $\chi(C_k^\bullet(V, \varphi)) \leq 0$ and equality holds if and only if*

$$\mathrm{ad}(\varphi) \colon \mathrm{End}(V)_k \to (S^2 V \otimes K \oplus S^2 V^* \otimes K)_{k+1}$$

is an isomorphism.

Corollary 5.6. *Let (V, φ) be a simple $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle which is stable as an $\mathrm{Sp}(2n, \mathbb{C})$-Higgs bundle. If (V, φ) is fixed under the circle action then it represents a local minimum of f if and only if the map*

$$\mathrm{ad}(\varphi) \colon \mathrm{End}(V)_k \to (S^2 V \otimes K \oplus S^2 V^* \otimes K)_{k+1}$$

is an isomorphism for all $k > 0$.

Proof. We have the vanishing $\mathbb{H}^0(C_k^\bullet(V, \varphi)) = \mathbb{H}^2(C_k^\bullet(V, \varphi)) = 0$ for all $k > 0$ from Proposition 2.14. Hence $\dim \mathbb{H}^1(C_-^\bullet(V, \varphi)) = -\chi(C_-^\bullet(V, \varphi))$. Now the result is immediate from Proposition 5.5 and (1) of Proposition 5.4. $\qquad\square$

5.3. **Minima of the Hitchin functional.** In order to describe the minima, it is convenient to define the following subspaces of $\mathcal{M}_d$.

Definition 5.7. For each d, define the following subspace of $\mathcal{M}_d$.

$$\mathcal{N}_d = \{(V, \beta, \gamma) \in \mathcal{M}_d \mid \beta = 0 \text{ or } \gamma = 0\}.$$

Remark 5.8. It is easy to see that polystability of (V, φ) implies that, in fact,

$$\begin{aligned}
\mathcal{N}_d &= \{(V, \beta, \gamma) \mid \beta = 0\} & \text{for } d > 0, \\
\mathcal{N}_d &= \{(V, \beta, \gamma) \mid \gamma = 0\} & \text{for } d < 0, \\
\mathcal{N}_d &= \{(V, \beta, \gamma) \mid \beta = \gamma = 0\} & \text{for } d = 0.
\end{aligned}$$

Note, in particular, that for $d = 0$ the vanishing of one of the secions β or γ implies the vanishing of the other one.

Proposition 5.9. *Let (V, φ) be a polystable $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle with $\beta = 0$ or $\gamma = 0$. Then (V, φ) represents the absolute minimum of f on $\mathcal{M}_d$. Thus $\mathcal{N}_d$ is contained in the subspace of local minima of f on $\mathcal{M}_d$.*

Proof. This can be seen in a way similar to the proof of [6, Proposition 4.5]. $\qquad\square$

Theorem 5.10. *Let (V, β, γ) be a polystable $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle and assume that $n \geq 3$. Then (V, β, γ) represents a minimum of the Hitchin functional if and only if one of the following situations occurs:*

(1) *(V, β, γ) belongs to $\mathcal{N}_d$.*
(2) *The degree $d = -n(g-1)$ with $n = 2q + 1$ odd, and there exists a square root L of K such that the bundle V is of the form*

$$V = \bigoplus_{\lambda=-q}^{q} L^{-1} K^{-2\lambda}.$$

With respect to this decomposition of V and the corresponding decomposition of V^*, the maps β and γ are of the form:

$$\beta = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & 0 \\ 0 & 1 & \cdot^{\cdot^{\cdot}} & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad \gamma = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & 1 \\ 0 & 0 & \cdot^{\cdot^{\cdot}} & \vdots \\ 0 & 1 & \cdots & 0 \end{pmatrix}$$

where, in the matrix for β, we denote by 1 the canonical section of

$$\mathrm{Hom}((L^{-1}K^{-2\lambda})^*, L^{-1}K^{2\lambda}) \otimes K \cong \mathcal{O}$$

and analogously for γ.

(3) *The degree $d = -n(g-1)$ with $n = 2q+2$ even, and there exists a square root L of K such that the bundle V is of the form*

$$V = \bigoplus_{\lambda=-q}^{q+1} LK^{-2\lambda}.$$

With respect to this decomposition of V and the corresponding decomposition of V^*, the maps β and γ are of the form given above.

(4) *The degree $d = n(g-1)$ and the dual $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundle $(V', \beta', \gamma') = (V^*, \gamma^t, \beta^t)$ is of the form given in (2) or (3) above.*

Definition 5.11. If (V, β, γ) is a minimum which does not belong to $\mathcal{N}_d$ we say that it is a **quiver type** minimum.

Remark 5.12. The cases $n = 1$ and $n = 2$ are special and were treated in [31] and [29], respectively (cf. (1) of Corollary 6.6 and Remark 6.7).

Proof of Theorem 5.10. This proof relies on the results of Sections 6 and 7 below.

Consider first the case of simple $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundles (V, φ) which are stable as $\mathrm{Sp}(2n, \mathbb{C})$-Higgs bundles. In this case, the analysis of the minima is based on Corollary 5.6 and is carried out in Section 6 below. The main result is Theorem 6.8, which says that Theorem 5.10 holds for such (V, φ).

Next, consider a polystable $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle (V, φ) which is not simple and stable as an $\mathrm{Sp}(2n, \mathbb{C})$-Higgs bundle. Then the decomposition $(V, \varphi) = \bigoplus(V_i, \varphi_i)$ given in the structure Theorem 3.29 is non-trivial. The main result of Section 7, Proposition 7.1, says that if such a (V, φ) is a local minimum then it belongs to $\mathcal{N}_d$, i.e., $\beta = 0$ or $\gamma = 0$. This concludes the proof. $\square$

6. Minima in the smooth locus of the moduli space

In this section we consider simple $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundles (V, ϕ) which are stable as $\mathrm{Sp}(2n, \mathbb{C})$-Higgs bundles. Thus, by Proposition 2.15, they belong to the smooth locus of the moduli space $\mathcal{M}_d$. In Theorem 6.8 below we prove that the statement of Theorem 5.10 holds in this case.

Our results are based on a careful analysis of the structure of $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundles (V, ϕ) satisfying the criterion of Corollary 5.6.

6.1. **Hodge bundles.** In this subsection we give a description of simple $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundles, which are complex variations of Hodge structure (cf. Proposition 5.3).

Lemma 6.1. *Let (V, β, γ) be an $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle and suppose there are splittings $V = V_a \oplus V_b$ and $V^* = V_c^* \oplus V_d^*$, at least one of which is non-trivial, and satisfying*

$$\beta(V_c^*) \subset V_a \otimes K, \qquad \gamma(V_a) \subset V_c \otimes K, \qquad \beta(V_d^*) \subset V_b \otimes K, \qquad \gamma(V_b) \subset V_d \otimes K.$$

Then (V, β, γ) is not simple.

Proof. Suppose the splitting $V = V_a \oplus V_b$ is non-trivial and, with respect to this decomposition, let

$$\sigma = (1, -1) \in \mathrm{Aut}(V)$$

Then, clearly, σ fixes $\varphi = (\beta, \gamma)$ and hence $\sigma \in \mathrm{Aut}(V, \varphi)$. Therefore (V, φ) is not simple (cf. Proposition 3.4). An analogous argument works when the splitting $V^* = V_c^* \oplus V_d^*$ is non-trivial. $\qquad\square$

Now assume that the $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle $(V, \varphi) = (V, \beta, \gamma)$ is a Hodge bundle, i.e., $V = \bigoplus F_i$, $\beta \colon F_i^* \to F_{-i+1} \otimes K$ and $\gamma \colon F_i \to F_{-i-1}^* \otimes K$, as descibed in Proposition 5.3, and let

$$\mathbf{F}_i = F_i \oplus F_{-i}^*$$

be the weight i subspace of $V \oplus V^*$. Then φ has components

$$\varphi_i \colon \mathbf{F}_i \to \mathbf{F}_{i+1} \otimes K.$$

Let i_0 and i_1 be the smallest and largest weights corresponding to non-zero weight spaces, respectively:

$$i_0 = \min\{i \mid \mathbf{F}_i \neq 0\},$$
$$i_1 = \max\{i \mid \mathbf{F}_i \neq 0\}.$$

Lemma 6.2. *If $(V, \varphi) = (V, \beta, \gamma)$ is simple and a complex variation of Hodge structure, then*

(1) *any weight i for which $\mathbf{F}_i \neq 0$ differs from i_0 by an integer,*
(2) *the map $\varphi_i \colon \mathbf{F}_i \to \mathbf{F}_{i+1} \otimes K$ is non-vanishing for $i = i_0, i_0 + 1, \ldots, i_1 - 1$, and*
(3) *only one of the bundles in the decomposition $\mathbf{F}_i = F_i \oplus F_{-i}^*$ is non-zero for $i = i_0, i_0 + 1, \ldots, i_1 - 1$.*

Proof. (1) Let

$$E' = \bigoplus_{i - n \in \mathbb{Z}} \mathbf{F}_i \qquad \text{and} \qquad E'' = \bigoplus_{i - n \notin \mathbb{Z}} \mathbf{F}_i$$

and define splittings $V = V_a \oplus V_b$ and $V^* = V_c^* \oplus V_d^*$ by the requirement that

$$E' = V_a \oplus V_c^* \qquad \text{and} \qquad E'' = V_b \oplus V_d^*.$$

Clearly these splittings satisfy the conditions of Lemma 6.1 and hence, since (V, β, γ) is simple, we conclude that $E'' = 0$.

(2) If $\varphi_j = 0$ for some j, we can let

$$E' = \bigoplus_{i_0 \leq i \leq j} \mathbf{F}_i \qquad \text{and} \qquad E'' = \bigoplus_{j+1 \leq i \leq i_1} \mathbf{F}_i,$$

and we obtain a contradiction with simplicity of (V, β, γ), using Lemma 6.1, as in the proof of (1).

(3) We can organize the bundles F_i and F_i^* and the maps between them in the diagram

(6.37)

$$
\begin{array}{ccccccc}
F_{i_0} & \xrightarrow{\gamma} & F_{i_0+1} & \xrightarrow{\gamma} & \cdots & , \\
& \nearrow \searrow & & \nearrow \searrow & \nearrow \\
F_{-i_0}^* & \xrightarrow{\beta} & F_{-i_0-1}^* & \xrightarrow{\beta} & \cdots
\end{array}
$$

where the maps are twisted by K, i.e., $F_i \xrightarrow{\gamma} F_{-i-1}^*$ should be interpreted to mean that we have a map $F_i \xrightarrow{\gamma} F_{-i-1}^* \otimes K$, and analogously for β. From this it is clear that, if we define

$$
\begin{aligned}
V_a &= \bigoplus F_{i_0+2i} & V_b &= \bigoplus F_{i_0+1+2i} \\
V_c^* &= \bigoplus F_{-i_0-1+2i}^* & V_d^* &= \bigoplus F_{-i_0+2i}^*,
\end{aligned}
$$

and let $E' = V_a \oplus V_c^*$ and $E'' = V_b \oplus V_d^*$ then, again using Lemma 6.1, simplicity implies that $E' = 0$ or $E'' = 0$. Since the bundles $\mathbf{F}_i$ are just the direct sums of the bundles appearing in each column in the diagram (6.37) above, this concludes the proof. $\qquad\square$

Proposition 6.3. *If $(V, \varphi) = (\bigoplus F_i, \beta, \gamma)$ is simple and a Hodge bundle, we have either the diagram*

(6.38)

$$
\begin{array}{ccccccc}
F_{i_1} & & F_{i_1-2} & & & F_{i_0+1} \\
\nwarrow \beta & \searrow \gamma & \nwarrow \beta & & \cdots & \nwarrow \beta \\
& F_{1-i_1}^* & & F_{3-i_1}^* & & F_{-i_0}^*
\end{array}
$$

or the diagram

(6.39)

$$
\begin{array}{ccccccc}
& F_{i_1-1} & & F_{i_1-3} & & & F_{i_0} \\
\nearrow \gamma & \nwarrow \beta & \nearrow \gamma & & \cdots & & \searrow \gamma \\
F_{-i_1}^* & & F_{2-i_1}^* & & & F_{-i_0+1}^*,
\end{array}
$$

where each of the maps is non-zero. Here the maps are twisted by K, i.e., $F_i \xrightarrow{\gamma} F_{-i-1}^*$ should be interpreted to mean that we have a map $F_i \xrightarrow{\gamma} F_{-i-1}^* \otimes K$, and analogously for β.

Proof. Immediate from Lemma 6.2. $\qquad\square$

Remark 6.4. Recall from Proposition 3.24 that, for each d, there is an isomorphism $\mathcal{M}_d \xrightarrow{\cong} \mathcal{M}_{-d}$, given by the duality $(V, \beta, \gamma) \mapsto (V^*, \gamma^t, \beta^t)$. Under this duality the situations (6.38) and (6.39) correspond (in fact, as we shall see, the former situation corresponds to $d < 0$, whereas the latter corresponds to $d > 0$). Henceforth we shall assume, for definiteness, that we are in the situation (6.38) of Proposition 6.3.

6.2. **Application of the criterion for minima.** In this section we combine the description of the complex variations of Hodge structure given in Proposition 6.3 and the criterion given in Corollary 5.6 to determine the minima of the Hitchin functional corresponding to simple $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundles which are stable as $\mathrm{Sp}(2n, \mathbb{C})$-Higgs bundles.

Let
$$m = i_0 + 1 \qquad \text{and} \qquad M = i_1$$
be the smallest and largest weight, respectively, appearing in the decomposition $V = \bigoplus F_i$. Then we can write

(6.40)
$$V = \bigoplus_{\lambda=0}^{p} F_{M-2\lambda},$$

where
$$p = (M - m)/2, \qquad M = p + \frac{1}{2} \qquad \text{and} \qquad m = -p + \frac{1}{2}.$$
Note also that $m = 1 - M$.

Theorem 6.5. *Let (V, β, γ) be a Hodge bundle of the type described in (6.38) of Proposition 6.3. Assume that $\beta \neq 0$ and $\gamma \neq 0$. Then the map*
$$\mathrm{ad}(\varphi) \colon \mathrm{End}(V)_k \to (S^2 V \otimes K \oplus S^2 V^* \otimes K)_{k+1}$$
is an isomorphism for all $k > 0$ if and only if the following holds:

(i) *For any $0 \leq \lambda \leq p$ the rank of $F_{M-2\lambda}$ is 1;*
(ii) *for any $0 \leq \lambda \leq p - 1$ the piece of β in*
$$F_{M-2\lambda} \otimes F_{m+2\lambda} \otimes K \subset S^2 V \otimes K$$
never vanishes;
(iii) *for any $1 \leq \lambda \leq p - 1$ the piece of γ in*
$$F^*_{M-2\lambda} \otimes F^*_{m+2\lambda-2} \otimes K \subset S^2 V^* \otimes K$$
never vanishes.

An analogous statement holds for Hodge bundles of the type described in (6.39) of Proposition 6.3.

Proof. The assumption $\beta \neq 0$ and $\gamma \neq 0$ means that in the decomposition (6.40) we have $p \geq 1$. If we take the piece in degree $k = 2p$ of the map $\mathrm{ad}(\varphi)$, we get
$$A := \mathrm{ad}(\varphi)_{2p} : F_M \otimes F^*_m \to S^2 F_M \otimes K,$$
which by assumption is an isomorphism. Computing the ranks $r_i = \mathrm{rk}(F_i)$, we deduce
$$r_M r_m = \frac{r_M(r_M + 1)}{2}.$$
To prove that $r_M = r_m = 1$, we assume the contrary and show that this leads to a contradiction. If $r_M > 1$ then by the formula above we must have $r_m < r_M$. Let b be the piece of β in $F_M \otimes F_m \otimes K \subset (S^2 V \otimes K)_{2p}$. Then the map A sends any $e \in F_M \otimes F^*_m$ to
$$A(e) = eb + be^*.$$
The first summand denotes the composition of maps
$$F^*_M \xrightarrow{b} F_m \xrightarrow{e} F_M$$

and the second summand
$$F_M^* \xrightarrow{e^*} F_m^* \xrightarrow{b} F_M.$$
Take a basis $u_1, \ldots, u_{r_M}$ of F_M whose first r_m elements are a basis of $b(F_m^*)$, and take on F_M^* the dual basis. If we write the matrices of eb and be^* with respect to these basis, one readily checks that the $(r_M - r_m) \times (r_M - r_m)$ block in the bottom left of both matrices vanishes. Consequently, an element in $S^2 F_M$ represented by a symmetric matrix whose entry at the bottom left corner is nonzero cannot belong to the image of A. Hence A is not an isomorphism, in contradiction to our assumption, so we deduce that
$$r_M = r_m = 1.$$
One also deduces that the section $b \in H^0(F_M \otimes F_m \otimes K)$ never vanishes. This proves statements (i) and (ii) when $\lambda = 0$ or p.

Observation. The following observation will be useful: if $e \in F_i \otimes F_j^* \subset \mathrm{End}(V)$, then any nonzero piece of $\mathrm{ad}(\varphi)(e)$ in the decomposition (5.33) belongs to a summand of the form $F_i \otimes F_u \otimes K$, and any nonzero piece in (5.34) belongs to a summand of the form $F_j^* \otimes F_v^* \otimes K$ (in both cases the symmetrization should be understood if the two indices coincide). This follows from the fact that $\mathrm{ad}(\varphi)(e)$ is the sum of compositions of e with another map (either on the right and on the left). Hence each summand in $\mathrm{ad}(\varphi)(e)$ must share with e at least the domain or the target.

Now let us take any $k = 2p - 2\lambda \geq 1$, such that $\lambda \geq 1$, so that $1 \leq \lambda \leq p - 1$. Then we have
$$(6.41) \qquad \mathrm{End}(V)_{2p-2\lambda} = F_M \otimes F_{m+2\lambda}^* \oplus F_{M-2} \otimes F_{m+2\lambda-2}^* \oplus \cdots \oplus F_{M-2\lambda} \otimes F_m^*.$$
We claim that there is no nonzero block in $(S^2 V^* \otimes K)_{2p-2\lambda+1}$ of the form $F_{m+2\lambda}^* \otimes F_v^* \otimes K$. Indeed, for that one should take $v = -(2p - 2\lambda + 1) - (m + 2\lambda) = -M - 1$, but $F_{-M-1} = 0$, because $-M - 1 < m$. On the other hand, $(S^2 V^* \otimes K)_{2p-2\lambda+1}$ contains the block $F_M \otimes F_{M-2\lambda} \otimes K$ and no other block involving F_M. Hence we must have
$$\mathrm{ad}(\varphi)_k(F_M \otimes F_{m+2\lambda}^*) \subset F_M \otimes F_{M-2\lambda} \otimes K.$$
Taking ranks and using the fact that $\mathrm{ad}(\varphi)_k$ is injective, we deduce that
$$r_{m+2\lambda} \leq r_{M-2\lambda}.$$
Since $1 \leq \lambda \leq p - 1 \Longleftrightarrow 1 \leq p - \lambda \leq p - 1$, we automatically deduce that
$$r_{m+2p-2\lambda} \leq r_{M-2p+2\lambda}.$$
But $m + 2p = M$, so we conclude that
$$(6.42) \qquad\qquad\qquad r_{m+2\lambda} = r_{M-2\lambda}.$$

Let us distinguish two possibilities.

Case (1). Suppose that $\lambda = 2l + 1$ is odd. Then we have
$$S^2 F_{m+\lambda-1}^* \otimes K \subset (S^2 V^* \otimes K)_{2p-2\lambda+1},$$
and the observation above implies that
$$\mathrm{ad}(\varphi)_{2p-2\lambda}^{-1}(S^2 F_{m+\lambda-1}^* \otimes K) \subset F_{M-\lambda-1} \otimes F_{m+\lambda-1}^*.$$
The argument given above for $\lambda = 0$ proves now that the piece of γ in
$$F_{M-\lambda-1}^* \otimes F_{m+\lambda-1}^* \otimes K$$
never vanishes.

Case (2). Suppose that $\lambda = 2l$ is even. Then we have
$$S^2 F_{M-\lambda} \otimes K \subset (S^2 V \otimes K)_{2p-2\lambda+1},$$
and the observation above implies that
$$\mathrm{ad}(\varphi)_{2p-2\lambda}^{-1}(S^2 F_{M-\lambda} \otimes K) \subset F_{M-\lambda} \otimes F_{m+\lambda}^*.$$
The argument given above for $\lambda = 0$ proves now that the piece of β in
$$F_{M-\lambda} \otimes F_{m+\lambda} \otimes K$$
never vanishes.

These arguments prove statements (ii) and (iii).

We are now going to prove that for any $1 \le \lambda \le p/2$ the ranks $r_{M-2\lambda} = r_{m+2\lambda} = 1$ using induction. Fix such a λ and assume that for any $0 \le l < \lambda$ we have $r_{M-2l} = r_{m+2l} = 1$ (when $l = 0$ we already know this is true). Since $2p - 2\lambda \ge 1$ we must have

(6.43) $\mathrm{rk}\,\mathrm{End}(V)_{2p-2\lambda} = \mathrm{rk}(S^2 V \otimes K \oplus S^2 V^* \otimes K)_{2p-2\lambda+1}.$

Using induction we can compute the left hand side:
$$\mathrm{rk}\,\mathrm{End}(V)_{2p-2\lambda} = r_M r_{m+2\lambda} + r_{M-2} r_{m+2\lambda-2} + \cdots + r_{M-2\lambda+2} r_{m+2} + r_{M-2\lambda} r_m$$
$$= r_{m+2\lambda} + r_{M-2\lambda} + (\lambda - 1).$$
We now distinguish again two cases.

Case (1). Suppose that $\lambda = 2l + 1$ is odd. Then we compute
$$\mathrm{rk}(S^2 V)_{2p-2\lambda+1} = r_M r_{M-2\lambda} + r_{M-2} r_{M-2\lambda+2} + \cdots + r_{M-\lambda+1} r_{M-\lambda-1}$$
$$= r_{M-2\lambda} + l$$
and
$$\mathrm{rk}(S^2 V^*)_{2p-2\lambda+1} = r_m r_{m+2\lambda-2} + r_{m+2} r_{m+2\lambda-4} + \cdots + r_{m+\lambda-3} r_{m+\lambda+1}$$
$$+ \binom{r_{m+\lambda-1} + 1}{2} = l + 1.$$
Comparing the two computations it follows from (6.43) that
$$r_{m+2\lambda} = 1,$$
and using (6.42) we deduce that
$$r_{M-2\lambda} = 1.$$

Case (2). Now suppose that $\lambda = 2l$ is even. Then we have
$$\mathrm{rk}(S^2 V)_{2p-2\lambda+1} = r_M r_{M-2\lambda} + r_{M-2} r_{M-2\lambda+2} + \cdots + r_{M-\lambda+2} r_{M-\lambda-2}$$
$$+ \binom{r_{M-\lambda}}{2} = r_{M-2\lambda} + l$$
and
$$\mathrm{rk}(S^2 V^*)_{2p-2\lambda+1} = r_m r_{m+2\lambda-2} + r_{m+2} r_{m+2\lambda-4} + \cdots + r_{m+\lambda-2} r_{m+\lambda}$$
$$= l.$$
Comparing again the two computations we deduce that
$$r_{m+2\lambda} = r_{M-2\lambda} = 1.$$
This finishes the proof of statement (i) and thus the proof of the Theorem. $\square$

Corollary 6.6. *Let (V, β, γ) be a Hodge bundle of the type described in* (6.38) *of Proposition 6.3. Assume that $\beta \neq 0$ and $\gamma \neq 0$ and that the map*

$$\mathrm{ad}(\varphi)\colon \mathrm{End}(V)_k \to (S^2 V \otimes K \oplus S^2 V^* \otimes K)_{k+1}$$

is an isomorphism for all $k > 0$. Then the following holds.

(1) *If $n = 2$ then $F_{\frac{3}{2}} \otimes F_{-\frac{1}{2}} \otimes K \cong \mathbb{C}$.*

(2) *If $n = 2q + 1 \geq 3$ is odd then $\beta\colon F^*_{\frac{1}{2}-2\lambda} \xrightarrow{\cong} F_{\frac{1}{2}+2\lambda}K$ for any integer $-q \leq \lambda \leq q$. In particular, there exists a square root L of K such that for any integer $-q \leq \lambda \leq q$ we have*

$$F_{M-2(q-\lambda)} \cong F_{m+2(\lambda+q)} \cong F_{\frac{1}{2}+2\lambda} \cong L^{-1} \otimes K^{-2\lambda},$$

and the bundle V is of the form

$$V = \bigoplus_{\lambda=-q}^{q} L^{-1} K^{-2\lambda}.$$

(3) *If $n = 2q + 2 \geq 4$ then $\gamma\colon F_{-\frac{1}{2}} \xrightarrow{\cong} F^*_{-\frac{1}{2}}K$ and $\beta\colon F^*_{-\frac{1}{2}-2\lambda} \xrightarrow{\cong} F_{-\frac{1}{2}+2\lambda}K$ for any integer $-q \leq \lambda \leq q+1$. In particular, there exists a square root L of K such that for any integer $-q \leq \lambda \leq q+1$ we have*

$$F_{-\frac{1}{2}+2\lambda} \cong L \otimes K^{-2\lambda} \cong F_{M-2(q+1-\lambda)} \cong F_{m+2(\lambda+q)},$$

and the bundle V is of the form

$$V = \bigoplus_{\lambda=-q}^{q+1} L K^{-2\lambda}.$$

(4) *For any $n \geq 2$, the degree of V is $\deg V = n(1-g)$.*

(5) *For any $n \geq 2$, an $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle of the form described in (1)–(3) above is stable as an $\mathrm{SL}(2n, \mathbb{C})$-Higgs bundle, and thus also as an $\mathrm{Sp}(2n, \mathbb{C})$-Higgs bundle.*

Analogous statements hold for Hodge bundles of the type described in (6.39) *of Proposition 6.3. In particular, in this case the degree of V is $\deg V = n(g-1)$ (cf. Remark 6.4).*

Remark 6.7. In the case $n = 1$ it is not possible for (V, φ) to be a Hodge bundle with $\beta \neq 0$ and $\gamma \neq 0$.

Proof of Corollary 6.6. First we observe that, since the F_i are all line bundles, we have $n = p + 1$, $M = p + \frac{1}{2}$ and $m = -p + \frac{1}{2}$.

(1) In this case we have $n = 2$, $p = 1$, $M = 3/2$, $m = -1/2$. Then, taking $\lambda = 0$ in (ii) of Theorem 6.5 we get $F_{\frac{3}{2}} \otimes F_{-\frac{1}{2}} \otimes K \cong \mathbb{C}$.

(2) In this case we have $n = p + 1 = 2q + 1$ so that $M = 2q + 1/2$ and $m = -2q + 1/2$. Hence, using (ii) and (iii) of Theorem 6.5, we can describe the structure of the maps β and

γ in the following diagram:

,
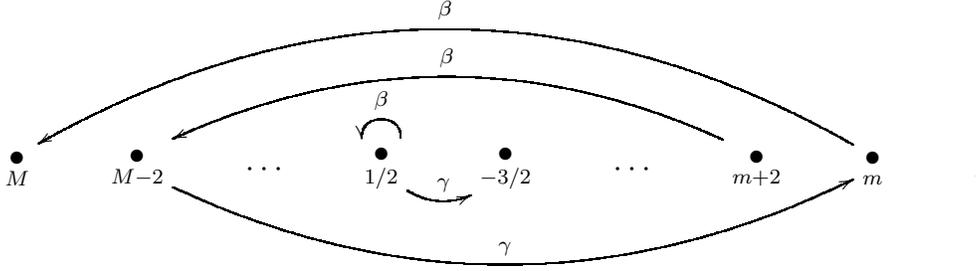
where an arrow $\begin{smallmatrix}\bullet\\i\end{smallmatrix} \xrightarrow{\beta} \begin{smallmatrix}\bullet\\j\end{smallmatrix}$ means that there is an isomorphism $\beta\colon F_i^* \to F_j \otimes K$ (and thus $j = -i + 1$); similarly, an arrow $\begin{smallmatrix}\bullet\\i\end{smallmatrix} \xrightarrow{\gamma} \begin{smallmatrix}\bullet\\j\end{smallmatrix}$ means that there is an isomorphism $\gamma\colon F_i \to F_j^* \otimes K$. In particular, we see that the isomorphism $\beta\colon F_{\frac{1}{2}}^* \xrightarrow{\cong} F_{\frac{1}{2}} \otimes K$ means that $F_{\frac{1}{2}} \cong L^{-1}$ for a square root L of K. This proves the case $\lambda = 0$ of (2). Now repeated application of (ii) and (iii) of Theorem 6.5 proves the general case. Note that this argument can be phrased as saying that the graph above is connected and its only closed loop is the one at $1/2$: thus the remaining F_i are uniquely determined by $F_{\frac{1}{2}}$.

(3) In this case we have $n = p + 1 = 2q + 2$ so that $M = 2q + 3/2$ and $m = -2q - 1/2$ and, as above, we have a diagram

.
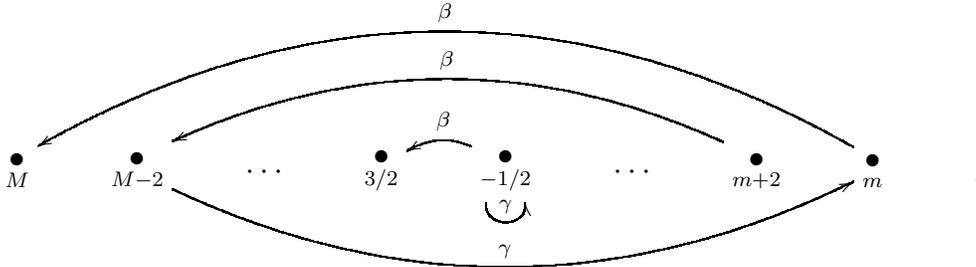
The argument is now analogous to the previous case.

(4) Easy from the formulas for V given in (2) and (3).

(5) Let (V, φ) be of the kind described in (1)–(3), and consider the associated $\mathrm{SL}(2n, \mathbb{C})$-Higgs bundle $(V \oplus V^*, \Phi) = H(V, \varphi)$. The Φ-invariant subbundles of $V \oplus V^*$ are of the form $\bigoplus_{i \geq i_0}(F_i \oplus F_{-i}^*)$. From the given description, it is easy to check that such a subbundle, when proper and non-zero, has degree strictly negative. $\square$

Finally, we use the analysis carried out so far to determine the minima of the Hitchin functional on the locus of the moduli space corresponding to simple $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundles which are stable as $\mathrm{Sp}(2n, \mathbb{C})$-Higgs bundles.

Theorem 6.8. *Let (V, β, γ) be a simple $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle which is stable as an $\mathrm{Sp}(2n, \mathbb{C})$-Higgs bundle.*

(1) *If $|d| < n(g-1)$ then (V, β, γ) represents a minimum of the Hitchin functional if and only if it belongs to $\mathcal{N}_d$.*

(2) *If $|d| = n(g-1)$ and $n \geq 3$ then (V, β, γ) represents a minimum of the Hitchin functional if and only if one of the following situations occurs:*

(i) *the $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle (V, β, γ) belongs to $\mathcal{N}_d$;*

(ii) *the* $\mathrm{Sp}(2n,\mathbb{R})$-*Higgs bundle* (V,β,γ) *is of the type described in (2) or (3) of Corollary 6.6. In this case we say that* (V,β,γ) *is a* quiver type *minimum.*

Proof. If (V,β,γ) belongs to $\mathcal{N}_d$ then we know from Proposition 5.9 that it represents a minimum. And, if (V,β,γ) is of the type described in (2) or (3) of Corollary 6.6, then Corollary 5.6 and Theorem 6.5 show that it represents a minimum.

On the other hand, if (V,β,γ) is a minimum which does not belong to $\mathcal{N}_d$, then Corollary 5.6, Theorem 6.5 and Corollary 6.6 show that it is of the type described in (2) or (3) of Corollary 6.6. □

7. Minima on the entire moduli space

7.1. **Main result and strategy of proof.** In Section 6 we characterized the minima of the Hitchin functional on the locus of $\mathcal{M}_d$ corresponding to simple $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundles (V,φ) which are stable as $\mathrm{Sp}(2n,\mathbb{C})$-Higgs bundles. In this section we provide the remaining results required to extend this characterization to the whole moduli space, thus completing the proof of Theorem 5.10. As explained in the proof of that Theorem, what is required is to rule out certain type of potential minima of the Hitchin functional. In each case this is done by using (2) of Proposition 5.4. The main result of this Section is the following.

Proposition 7.1. *Let* $(V,\varphi = \beta + \gamma)$ *be a polystable* $\mathrm{Sp}(2n,\mathbb{R})$-*Higgs bundle and assume that the decomposition* $(V,\varphi) = (V_1,\varphi_1) \oplus \cdots \oplus (V_k,\varphi_k)$ *of Theorem 3.29 is non-trivial. If* (V,φ) *is a local minimum of the Hitchin functional then either* $\beta = 0$ *or* $\gamma = 0$.

Proof. The starting point is the structure Theorem 3.29. Recall that this describes a polystable $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundle as a direct sum

$$(7.44) \qquad\qquad (V,\varphi) = \bigoplus(V_i,\varphi_i),$$

where each $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundle (V_i,φ_i) comes from a G_i-Higgs bundle which is a smooth point in its respective moduli space. If (V,φ) is a minimum, then Proposition 5.2 implies that each (V_i,φ_i) is a minimum on the corresponding moduli space of G_i-Higgs bundles. Consider each of the possible G_i's in turn.

The case $G_i = \mathrm{Sp}(2n_i,\mathbb{R})$. This is the case covered by Theorem 6.8. (Except for the case $n_i = 2$, which will require special attention.)

The case $G_i = \mathrm{U}(n_i)$. In this case $\varphi_i = 0$ for any G_i-Higgs bundle, as we have already seen.

The case $G_i = \mathrm{U}(p_i,q_i)$. In this case, the minima of the Hitchin functional were determined in [6]. There it is shown that a $\mathrm{U}(p_i,q_i)$-Higgs bundle $(\tilde{V}_i,\tilde{W}_i,\tilde{\beta}+\tilde{\gamma})$ is a minimum if and only if $\tilde{\beta} = 0$ or $\tilde{\gamma} = 0$. Hence $(V_i,\varphi_i) = v_*^{\mathrm{U}(p_i,q_i)}(\tilde{V}_i,\tilde{W}_i,\tilde{\beta}+\tilde{\gamma})$ (cf. (3.16)) is a minimum if and only if $\beta_i = 0$ or $\gamma_i = 0$

The case $G_i = \mathrm{GL}(n_i,\mathbb{R})$. The moduli space of such Higgs bundles was studied in [7]. Using the results of that paper we show in Lemma 7.8 below that a $\mathrm{Sp}(2n_i,\mathbb{R})$-Higgs bundle (V_i,φ_i) coming from a $\mathrm{GL}(n_i,\mathbb{R})$-Higgs bundle is a minimum if and only if $\varphi_i = 0$.

A quiver type minimum (V,φ) is simple and stable as a $\mathrm{Sp}(2n,\mathbb{C})$-Higgs bundle by (5) of Corollary 6.6. Thus, to conclude the proof of the Proposition, it remains to show that if (V,φ) is a minimum and the decomposition (7.44) is non-trivial, then it belongs to $\mathcal{N}_d$,

i.e., $\beta = 0$ or $\gamma = 0$. By the above analysis of the minima coming from G_i-Higgs bundles, it therefore suffices to show that (V, φ) is not a minimum when the decomposition (7.44) falls in one of the following cases:

(1) There is a (V_i, φ_i) in $\mathcal{N}_{d_i}$ with $\beta_i \neq 0$ and a (V_j, φ_j) in $\mathcal{N}_{d_j}$ with $\gamma_j \neq 0$.
(2) There is a (V_i, φ_i) which is a quiver type minimum and a (V_j, φ_j) which lies in $\mathcal{N}_{d_i}$.
(3) There are (distinct) (V_i, φ_i) and (V_j, φ_j) which are quiver type minima.

In order to accomodate the possibility $n_i = 2$, the quiver type minima must here be understood to include all minima with $\beta \neq 0$ and $\gamma \neq 0$ (cf. (1) of Corollary 6.6). The case $n_i = 1$ is included since such minima must have $\beta = 0$ or $\gamma = 0$ (cf. Remark 6.7).

Note that, by Proposition 5.2, in fact it suffices to consider the case when $k = 2$ in (7.44). With this in mind, the results of Lemmas 7.2, 7.4 and 7.6 below conclude the proof.   □

7.2. Deforming a sum of minima in $\mathcal{N}_d$.

Lemma 7.2. *Let (V, φ) be a polystable* $\mathrm{Sp}(2n, \mathbb{R})$*-Higgs bundle which decomposes as a direct sum* $(V, \varphi) = (V', \varphi') \oplus (V'', \varphi'')$ *with* $\varphi' = (\beta', \gamma')$ *and* $\varphi'' = (\beta'', \gamma'')$. *Suppose that* $\beta' = 0$, $\gamma' \neq 0$, $\beta'' \neq 0$ *and* $\gamma'' = 0$. *Suppose additionally that* (V', φ') *and* (V'', φ'') *are stable* $\mathrm{Sp}(2n, \mathbb{R})$*-Higgs bundles or stable* $\mathrm{U}(p, q)$*-Higgs bundles. Then* (V, φ) *is not a minimum of* f *on* $\mathcal{M}_d$. *The same is true if* $\beta' \neq 0$, $\gamma' = 0$, $\beta'' = 0$ *and* $\gamma'' \neq 0$.

Proof. We prove the Lemma by applying the criterion in (2) of Proposition 5.4. As a first step, we identify the complex $C_-^\bullet$ defined in (5.36), and for that we need to know the weights of each piece V', V''. Recall that the weight of φ', φ'' is always 1.

(1) Since $\gamma': V' \to V'^* K$, the weight on V'^* is $1 + \lambda' = -\lambda'$, where λ' is the weight on V'. Thus $\lambda' = -1/2$.
(2) Similarly, the weight on V'' is $\lambda'' = 1/2$.

From this it follows immediately that the complex $C_-^\bullet$ is given by

$$C_-^\bullet: \ \mathrm{Hom}(V', V'') \to 0,$$

so that

$$\mathbb{H}^1(C_-^\bullet) = H^1(\mathrm{Hom}(V', V'')).$$

Recall from Remark 5.8 that $d' = \deg(V') \geq 0$ and $d'' \leq 0$ so, by Riemann–Roch,

$$H^1(\mathrm{Hom}(V', V'')) \neq 0.$$

This proves that $C_-^\bullet$ has nonzero first hypercohomology. To finish the argument we need to integrate any element of $\mathbb{H}^1(C_-^\bullet)$ to a deformation of (V, φ) through polystable $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundles.

Chose any[3] nonzero element $a \in H^1(\mathrm{Hom}(V', V''))$. Denote by D the open unit disk. Define $\mathbb{V}' = D \times V'$ and $\mathbb{V}'' = D \times V''$, which we view as vector bundles over $X \times D$. We denote by $\gamma'_D : \mathbb{V}' \to \mathbb{V}'^* \otimes K$ (here K denotes the pullback to $X \times D$) the extension of γ' which is constant on the D direction, and we define similarly $\beta''_D : \mathbb{V}''^* \to \mathbb{V}'' \otimes K$. Take the extension

$$0 \to \mathbb{V}'' \to \mathbb{V} \to \mathbb{V}' \to 0$$

---

[3]when one of (V', φ') and (V'', φ'') is a $\mathrm{U}(p, q)$-Higgs bundle, this choice is not completely arbitrary, cf. the proof of Lemma 7.3 below.

classified by

$$a \otimes 1 \in H^1(\mathrm{Hom}(\mathbb{V}', \mathbb{V}'')) = H^1(X; \mathrm{Hom}(V', V'')) \otimes H^0(D; \mathbb{C}).$$

The restriction of this to $X \times \{t\}$ is the extension

(7.45)
$$0 \to V'' \to V_t \to V' \to 0$$

classified by $ta \in H^1(\mathrm{Hom}(V', V''))$. Define $\gamma_D : \mathbb{V} \to \mathbb{V}^* \otimes K$ as the composition

$$\mathbb{V} \longrightarrow \mathbb{V}' \xrightarrow{\gamma'_D} \mathbb{V}'^* \otimes K \to \mathbb{V}^* \otimes K,$$

where the first arrow comes from the exact sequence defining $\mathbb{V}$ and the third one comes from dualising the same exact sequence and tensoring by the pullback of K. Similarly, define $\beta_D : \mathbb{V}^* \to \mathbb{V} \otimes K$ as the composition

$$\mathbb{V}^* \longrightarrow \mathbb{V}''^* \xrightarrow{\beta''_D} \mathbb{V}'' \otimes K \to \mathbb{V} \otimes K.$$

The resulting triple $(\mathbb{V}, \beta_D, \gamma_D)$ is a family of symplectic Higgs bundles parametrized by the disk, whose restriction to the origin coincides with (V, φ), and which integrates the element a in the deformation complex.

It remains to show that each member of the family $(\mathbb{V}, \beta_D, \gamma_D)$ is a polystable $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle. This is done in Lemma 7.3 below. We have thus proved that (V, φ) is not a local minimum. $\qquad\square$

Lemma 7.3. *The* $\mathrm{Sp}(2n, \mathbb{R})$*-Higgs bundle* $(V_t, \varphi_t = \beta_t + \gamma_t)$ *on* X*, obtained by restricting to* $X \times \{t\}$ *the family* $(\mathbb{V}, \beta_D, \gamma_D)$ *constructed in the proof of Lemma 7.2, is polystable.*

Proof. It will be convenient to use the stability condition for $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundles as given in Lemma 3.16. Thus, if (V_t, φ_t) is not stable, there are subbundles $A \subset V_t$ and $B \subset V_t^*$ such that $\gamma_t(A) \subset B \otimes K$ and $\beta_t(B) \subset A \otimes K$, and with $\deg(A \oplus B) = 0$. Since X is a Riemann surface, the kernel of the restriction to A of the sheaf map $V_t \to V''$ is locally free and corresponds to a subbundle $A' \subset A$. The quotient $A'' := A/A'$ then gives a subbundle $A'' \subset V''$ so that we have a commutative diagram with exact rows and columns:

(7.46)
$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A'' & \longrightarrow & A & \longrightarrow & A' & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V'' & \longrightarrow & V_t & \longrightarrow & V' & \longrightarrow & 0.
\end{array}
$$

Similarly, we obtain subbundles $B'' \subset V''^*$ and $B' \subset V'^*$ and a diagram:

(7.47)
$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & B' & \longleftarrow & B & \longleftarrow & B'' & \longleftarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & V'^* & \longleftarrow & V_t & \longleftarrow & V''^* & \longleftarrow & 0.
\end{array}
$$

One easily checks that $B'^{,\perp} \subset A'$ and $B''^{,\perp} \subset A''$. By definition of γ_t, the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & V' & \longrightarrow & V_t & \longrightarrow & V'' & \longrightarrow & 0 \\
& & & & & & & & \\
& & \Big\downarrow \gamma' & & \Big\downarrow \gamma_t & & & & \\
& & & & & & & & \\
0 & \longleftarrow & V'^* & \longleftarrow & V_t & \longleftarrow & V''^* & \longleftarrow & 0.
\end{array}
$$

commutes. Thus, since $\gamma_t(A) \subset B \otimes K$, we have that $\gamma'(A') \subset B' \otimes K$. Similarly, $\beta''(B'') \subset A'' \otimes K$. It follows that the pair of subbundles $A' \subset V'$ and $B' \subset V'^*$ destabilizes (V', φ') and that the pair of subbundles $A'' \subset V''$ and $B'' \subset V''^*$ destabilizes (V'', φ'').

Consider now the case in which both (V', φ') and (V'', φ'') are stable $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundles. Then we must have $A' \oplus B' = V' \oplus V'^*$ or $A' \oplus B' = 0$ and similarly for $A'' \oplus B''$. The only case in which the original destabilizing subbundle $A \oplus B \subset V_t \oplus V_t^*$ is non-trivial is when $A' \oplus B' = V' \oplus V'^*$ and $A'' \oplus B'' = 0$ (or vice-versa). But, in this case, $V' \cong A' \cong A$ and hence (7.46) shows that the non-trivial extension (7.45) splits, which is a contradiction. Hence there is no non-trivial destabilizing pair of subbundles of (V_t, φ_t), which is therefore stable.

It remains to deal with case in which one, or both, of (V', φ') and (V'', φ'') are stable $\mathrm{U}(p, q)$-Higgs bundles. The remaining cases being similar, for definiteness we consider the case in which (V'', φ'') is a stable $\mathrm{Sp}(2n'', \mathbb{R})$-Higgs bundle and (V', φ') is a stable $\mathrm{U}(n_1', n_2')$-Higgs bundle, i.e.,

$$
V' = V_1' \oplus V_2', \qquad \varphi' = \gamma' \in H^0(V_1' \otimes V_2' \otimes K).
$$

In addition to the cases considered above, we now also need to consider the case when $A' \oplus B'$ is non-trivial, say $A' \oplus B' = V_1' \oplus V_2'^*$. There are now two possibilities for $A'' \oplus B''$: either it is zero or it equals $V'' \oplus V''^*$; we leave the first (simpler) case to the reader and consider the second one. In this case, the element

$$
a = a_1 + a_2 \in H^1(\mathrm{Hom}(V', V'') = H^1(\mathrm{Hom}(V_1', V'')) \oplus H^1(\mathrm{Hom}(V_2', V''))
$$

chosen in the proof of Lemma 7.2 above must be taken such that both a_1 and a_2 are non-zero (this is possible by Riemann–Roch). Thus, for $i = 1, 2$ we have a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & V'' & \longrightarrow & V_{t_i} & \longrightarrow & V_i' & \longrightarrow & 0 \\
& & \Big\| & & \Big\downarrow & & \Big\downarrow & & \\
0 & \longrightarrow & V'' & \longrightarrow & V_t & \longrightarrow & V_1' \oplus V_2' & \longrightarrow & 0
\end{array}
$$

of non-trivial extensions, where the two vertical maps on the right are inclusions. This, together with (7.47) for $B' = V_2'^*$ and $B'' = V''^*$, gives rise to the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & V_2'^* & \longrightarrow & B & \longrightarrow & V''^* & \longrightarrow & 0 \\
& & \Big\downarrow & & \Big\downarrow & & \Big\| & & \\
0 & \longrightarrow & V_1'^* \oplus V_2'^* & \longrightarrow & V_t^* & \longrightarrow & V''^* & \longrightarrow & 0 \\
& & \Big\downarrow & & \Big\downarrow & & \Big\| & & \\
0 & \longrightarrow & V_2'^* & \longrightarrow & V_{t_2}^* & \longrightarrow & V''^* & \longrightarrow & 0.
\end{array}
$$

The composites of the vertical maps on the left and on the right are isomorphisms. Hence the composite of the middle vertical maps is also an isomorphism and this provides a

splitting of the extension

$$0 \to V_1'^* \to V_t^* \to V_{t_2}^* \to 0.$$

Denote the splitting maps in the dual split extension by

$$i \colon V_1' \to V_t \qquad \text{and} \qquad p \colon V_t \to V_{t_2}.$$

We now have a diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & V'' & \longrightarrow & V_{t_1} & \longrightarrow & V_1' & \longrightarrow & 0 \\
& & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V'' & \longrightarrow & V_t & \longrightarrow & V_1' \oplus V_2' & \longrightarrow & 0 \\
& & \| & & \downarrow{\scriptstyle p} & & \downarrow & & \\
0 & \longrightarrow & V'' & \longrightarrow & V_{t_2} & \longrightarrow & V_2' & \longrightarrow & 0,
\end{array}
$$

where the vertical maps on the right are the natural inclusion and projection, respectively. Using the existence of the splitting map $i \colon V_1' \to V_t$ and the inclusion $V_{t_2} \to V_t$ one readily sees that this diagram commutes. This finally gives us the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & V_t/V_{t_1} & \overset{\cong}{\longrightarrow} & V_2' & \longrightarrow & 0 \\
& & \| & & \downarrow & & \| & & \\
0 & \longrightarrow & V'' & \longrightarrow & V_{t_2} & \longrightarrow & V_2' & \longrightarrow & 0,
\end{array}
$$

which shows that the sequence at the bottom is split, a contradiction. $\qquad\square$

7.3. Deforming a sum of a quiver type minimum and a minimum in $\mathcal{N}_d$.

Lemma 7.4. *Let (V, φ) be a polystable $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle which decomposes as a direct sum $(V, \varphi) = (V', \varphi') \oplus (V'', \varphi'')$ with $\varphi' = (\beta', \gamma')$ and $\varphi'' = (\beta'', \gamma'')$. Suppose that*

(1) *(V', φ') is a quiver type minimum,*
(2) *(V'', φ'') is a minimum with $\beta'' = 0$ or $\gamma'' = 0$ which is a stable G''-Higgs bundle for G'' one of the following groups: $\mathrm{Sp}(2n'', \mathbb{R})$, $\mathrm{U}(p'', q'')$, $\mathrm{U}(n'')$ or $\mathrm{GL}(n'', \mathbb{R})$.*

Then (V, φ) is not a minimum of f on $\mathcal{M}_d$.

Proof. Consider for definiteness the case in which (V', φ') is a quiver type minimum with $\deg(V') = n'(1 - g)$ and (V'', φ'') has $\gamma'' = 0$ and $\beta'' \neq 0$. The case in which $\beta'' = 0$ and $\gamma'' \neq 0$ can be treated along the same lines as the present case, so we will not give the details. The case in which (V', φ') is a quiver type minimum with $\deg(V') = n'(g - 1)$ is obtained by symmetry. Note that some degenerate cases can occur, namely:

(1) (V', φ') is a quiver type minimum with $\mathrm{rk}(V') = 2$ (cf. (1) of Corollary 6.6).
(2) (V'', φ'') has $\beta'' = \gamma'' = 0$.

With respect to Case (1), all we need for the arguments below is that $\beta \colon F_{\frac{3}{2}}^* \overset{\cong}{\longrightarrow} F_{-\frac{1}{2}} \otimes K$ is an isomorphism, which is guaranteed by (1) of Corollary 6.6. In what concerns Case (2), slight modifications are required in the arguments given below; we leave these to the reader.

With these introductory remarks out of the way, Corollary 6.6 tells us that V' decomposes as a direct sum of line bundles $V' = F_m \oplus \cdots \oplus F_M$ and that restricting β' we get

an isomorphism

$$\beta' : F_m^* \xrightarrow{\simeq} F_M \otimes K.$$

Our first task is to identify nonzero elements in the first hypercohomology of $C_-^\bullet$. A good place to look for them is in the hypercohomology of the piece of highest weight in the deformation complex, which is

(7.48) $$V''^* \otimes F_M \oplus V'' \otimes F_m^* \to V'' \otimes F_M \otimes K.$$

This morphism cannot be an isomorphism, because the ranks do not match. Thus Proposition 5.5 implies that $\mathbb{H}^1$ of this complex is non-vanishing.

In the hypercohomology long exact sequence (cf. (2.4)) of the complex (7.48), the map

$$H^0(V''^* \otimes F_M \oplus V'' \otimes F_m^*) = H^0(V''^* \otimes F_M) \oplus H^0(V'' \otimes F_m^*) \to H^0(V'' \otimes F_M \otimes K)$$

is always onto because the map $f : H^0(V'' \otimes F_m^*) \to H^0(V'' \otimes F_M \otimes K)$ is induced by tensoring $\beta' : F_m^* \to F_M \otimes K$ (which is an isomorphism) with the identity map $V'' \to V''$, so f is also an isomorphism. Hence the image of $H^0(V'' \otimes F_M \otimes K) \to \mathbb{H}^1$ is zero, and this by exactness implies that $\mathbb{H}^1 \to H^1(V''^* \otimes F_M \oplus V'' \otimes F_m^*)$ is injective. We now want to characterize the image of this inclusion. Tensoring the Higgs fields β'' and β' with the indentity on F_M and V'' respectively, we get maps

$$\beta'' : V''^* \otimes F_M \to V'' \otimes F_M \otimes K,$$

and

$$\beta' : V'' \otimes F_m^* \xrightarrow{\simeq} V'' \otimes F_M \otimes K.$$

Now the map ζ in the long exact sequence

$$\mathbb{H}^1 \to H^1(V''^* \otimes F_M \oplus V'' \otimes F_m^*) \xrightarrow{\zeta} H^1(V'' \otimes F_M \otimes K) \to \mathbb{H}^2$$

can be intepreted as follows: given elements $(\delta, \epsilon) \in H^1(V''^* \otimes F_M) \oplus H^1(V'' \otimes F_m^*)$,

$$\zeta(\delta, \epsilon) = -\beta''(\delta) - \beta'(\epsilon) \in H^1(V'' \otimes F_M \otimes K).$$

Hence we may take a nonzero pair (δ, η) satisfying $\beta''(\delta) + \beta'(\epsilon) = 0$ and corresponding to a nonzero element in the hypercohomology of the complex (7.48). We next prove that the deformation along (δ, η) is unobstructed, by giving an explicit construction of a family of Higgs bundles (V_t, β_t, γ_t) parameterized by $t \in \mathbb{C}$ and restricting to $(V' \oplus V'', \varphi' + \varphi'')$ at $t = 0$.

Pick Dolbeault representatives $a_\delta \in \Omega^{0,1}(V''^* \otimes F_M)$ and $a_\epsilon \in \Omega^{0,1}(F_m^* \otimes V'')$ of δ and ϵ. We are going to construct a pair (W_t, ν_t) satisfying the following.

- There is a C^∞ isomorphism of vector bundles $W_t \simeq F_M \oplus V'' \oplus F_m$ with respect to which the $\bar\partial$ operator of W_t can be written as

$$\bar\partial_{W_t} = \begin{pmatrix} \bar\partial_{F_M} & ta_\delta & t^2\gamma \\ 0 & \bar\partial_{V''} & ta_\epsilon \\ 0 & 0 & \bar\partial_{F_m} \end{pmatrix} = \bar\partial_0 + ta_1 + t^2 a_2,$$

  where $\gamma \in \Omega^{0,1}(F_m^* \otimes F_M)$ will be specified later,
- ν_t is a holomorphic section of $H^0(S^2 W_t \otimes K)$ of the form

$$\nu_t = \beta' + \beta'' + t\nu_1.$$

Now the condition $\bar{\partial}_{W_t}\nu_t = 0$ translates into

$$\bar{\partial}_0(\beta' + \beta'') = 0,$$
$$\bar{\partial}_1\nu_1 + a_1(\beta' + \beta'') = 0,$$
$$a_1\nu_1 + a_2(\beta' + \beta'') = 0.$$

The first equation is automatically satisfied. As for the second equation note that

$$a_1(\beta' + \beta'') = \beta''(a_\delta) + \beta'(a_\epsilon) \in \Omega^{1,1}(V'' \otimes_S F_M).$$

Since by hypothesis the Dolbeault cohomology class represented by $\beta''(a_\delta) + \beta'(a_\epsilon)$ is equal to zero, we may chose a value of $\nu_1 \in \Omega^{0,1}(V'' \otimes_S F_M)$ solving the second equation. It remains to consider the third equation. Note that $a_2\beta'' = 0$ and that $a_2\beta' = \gamma(\beta') \in \Omega^{1,1}(F_M \otimes F_M)$. Since β' is an isomorphism, for any $\eta \in \Omega^{1,1}(F_M \otimes F_M)$ there exist some γ such that $\gamma(\beta') = \eta$. Taking $\eta = -a_1\nu_1$, we obtain a value of γ solving the third equation above.

It follows from the construction that there are short exact sequences of holomorphic bundles

$$0 \to F_M \to W_t \to Z_t \to 0, \qquad\qquad 0 \to V'' \to Z_t \to F_m \to 0.$$

Dualizing both sequences we have inclusions $F_m^* \to Z_t^*$ and $Z_t^* \to W_t^*$ which can be composed to get an inclusion

$$(7.49) \qquad\qquad\qquad\qquad\qquad F_m^* \to W_t^*.$$

Now let

$$V_t = W_t \oplus \bigoplus_{m < \lambda < M} F_\lambda.$$

To finish the construction of the family of Higgs bundles we have to define holomorphic maps

$$\beta_t : V_t^* \to V_t \otimes K, \qquad\qquad \gamma_t : V_t \to V_t^* \otimes K$$

defining sections in $H^0(S^2V_t \otimes K)$ and $H^0(S^2V_t^* \otimes K)$ respectively. The following conditions are in fact satisfied by a unique choice of maps (β_t, γ_t):

- the restriction of β_t to W_t is equal to ν_t,
- the restriction of β_t to $\bigoplus_{m < \lambda < M} F_\lambda$ is equal to β',
- the restriction of γ_t to W_t is equal to 0,
- the restriction of γ_t to $F_M \subset V_t$ is 0,
- the restriction of γ_t to $F_{M-2} \subset V_t$ is the composition of $\gamma' : F_{M-2} \to F_m^* \otimes K$ with the inclusion (7.49) tensored by the identity on K,
- the restriction of γ_t to $\bigoplus_{m < \lambda < M-2} F_\lambda$ is equal to γ'.

The proof of the lemma is completed by using Lemma 7.5. $\square$

Lemma 7.5. *The* $\mathrm{Sp}(2n, \mathbb{R})$-*Higgs bundle* (V_t, φ_t), *obtained by restricting the family constructed in the proof of Lemma 7.4 to* $X \times \{t\}$, *is polystable.*

Proof. Analogous to the proof of Lemma 7.3. $\square$

7.4. **Deforming a sum of two quiver type minima.**

Lemma 7.6. *Let (V, φ) be a polystable $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle which decomposes as a direct sum $(V, \varphi) = (V', \varphi') \oplus (V'', \varphi'')$ with $\varphi' = (\beta', \gamma')$ and $\varphi'' = (\beta'', \gamma'')$. Suppose that both (V', φ') and (V'', φ'') are quiver type minima. Then (V, φ) is not a minimum of f on $\mathcal{M}_d$.*

Proof. Suppose we have two minima which are quiver pairs (minimal degree)

$$V' = F'_{m'} \oplus \cdots \oplus F'_{M'} = \bigoplus F'_\lambda \qquad \text{and} \qquad V'' = F''_{m''} \oplus \cdots \oplus F''_{M''} = \bigoplus F''_\mu.$$

All morphisms $\beta', \beta'', \gamma', \gamma''$ are isomorphisms. We want to deform $V' \oplus V''$.

The same ideas as before tell us (looking at the negative deformation complex) that we should look at the piece of the exact sequence of maximal weight, which is

$$C^\bullet : F'^*_{m'} \otimes F''_{M''} \oplus F''^*_{m''} \otimes F'_{M'} \to F'_{M'} \otimes F''_{M''} \otimes K.$$

Define $V''_0 := F''_{m''} \oplus F''_{M''}$. The restriction of the β'' to V''_0 defines an isomorphism

$$\beta''_0 : V''^*_0 \to V''_0 \otimes K,$$

so we can apply exactly the same construction as before, replacing V'' by V''_0, and obtain a deformation $W_{t\delta, t\epsilon}$ of the bundle

$$F'_{m'} \oplus F'_{M'} \oplus V''_0 = F'_{m'} \oplus F'_{M'} \oplus F''_{m''} \oplus F''_{M''}.$$

A very important point, however, is that now the extension classes of the bundles W_δ and W_ϵ are more restricted, since they belong respectively to the groups $H^1(F''^*_{m''} \otimes F'_{M'})$ and $H^1(F'^*_{m'} \otimes F''_{M''})$. In particular, to define $W_{t\epsilon}$ the line bundle $F'_{m'}$ only *merges* with $F''_{M''}$, and not with $F''_{m''}$. This implies that there is a map

$$(7.50) \qquad\qquad\qquad\qquad W_{t\epsilon} \to F''_{m''}$$

which deforms the projection $V''_0 \to F''_{m''}$.

We leave all the remaining F'_λ and F''_μ untouched. There are only two maps which have to be deformed (apart from the β's which are internal in $W_{\delta,\epsilon}$). These are

$$\gamma' : F'_{m'} \to F'^*_{M'-2} \otimes K \qquad \text{and} \qquad \gamma'' : F''_{m''} \to F''^*_{M''-2} \otimes K.$$

The first one can be deformed to a map

$$\gamma'_{\delta,\epsilon} : W_{t\delta, t\epsilon} \to F'^*_{M'-2} \otimes K$$

exactly as in the previous section. As for γ'', we combine the projection $W_{t\delta, t\epsilon} \to W_{t\epsilon}$ with the map in (7.50) and with γ'' to obtain the desired deformation

$$W_{t\delta, t\epsilon} \to F''^*_{M''-2} \otimes K.$$

Lemma 7.7 below completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Lemma 7.7. *The $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle (V_t, φ_t), obtained by restricting the family constructed in the proof of Lemma 7.6 to $X \times \{t\}$, is polystable.*

Proof. Analogous to the proof of Lemma 7.3. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

7.5. GL$(n, \mathbb{R})$-Higgs bundles. In this section, we will assume that

$$(V, \varphi) = \upsilon_*^{\mathrm{GL}(n,\mathbb{C})}((W, Q), \psi)$$

is an Sp$(2n, \mathbb{R})$-Higgs bundle associated to a GL$(n, \mathbb{R})$-Higgs bundle $((W, Q), \psi)$. Recall that $d = \deg(V) = 0$ in this case.

Lemma 7.8. *Let (V, φ) be the Sp$(2n, \mathbb{R})$-Higgs bundle associated to a GL$(n, \mathbb{R})$-Higgs bundle $((W, Q), \psi)$ as in (3.28). If (V, φ) is a minimum of f on $\mathcal{M}_0$ then $\varphi = 0$.*

Proof. In [7] it is shown that there are two types of minima on the moduli space GL$(n, \mathbb{R})$-Higgs bundles $((W, Q), \psi)$. The first type has $\psi = 0$. The second type corresponds to the minimum on the Hitchin–Teichmüller component and has non-vanishing Higgs field. They are of the form:

$$W = F_{-m} \oplus \cdots \oplus F_m$$

for line bundles F_i, indexed by integers for $n = 2m+1$ odd and half-integers for $n = 2m+1$ even. More precisely, $F_i \cong K^{-i}$ such that, in particular, $F_i \cong F_{-i}^*$. With respect to this decomposition of W,

$$Q = \begin{pmatrix} 0 & \cdots & \cdots & 0 & 1 \\ \vdots & & & \cdot^{\cdot^{\cdot}} & 0 \\ \vdots & & 1 & & \vdots \\ 0 & \cdot^{\cdot^{\cdot}} & & & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix} \quad \text{and} \quad \psi = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

We shall apply the criterion in (2) of Proposition 5.4 to show that $\upsilon_*^{\mathrm{GL}(n,\mathbb{C})}((W, Q), \psi)$ is not a minimum of the Hitchin functional for such $((W, Q), \psi)$.

Recall that $V = W$, $\beta = \psi f^{-1}$ and $\gamma = f\psi$, where $f: V \to V^*$ is the symmetric isomorphism associated to Q. Hence the components of β and γ are the canonical sections

$$\beta: F_i^* \to F_{-i+1} \otimes K \quad \text{and} \quad \gamma: F_i \to F_{-i-1}^* \otimes K.$$

Since φ has weight one, the weight of F_i is i (cf. Proposition 5.3). It follows that the highest weight piece of the complex $C_-^\bullet$ defined in (5.36) is

$$C_{2m}^\bullet: \mathrm{Hom}(F_{-m}, F_m) \to 0.$$

Hence

$$\mathbb{H}^1(C_{2m}^\bullet) = H^1(\mathrm{Hom}(F_{-m}, F_m)) = H^1(K^{-2m}),$$

which is non-vanishing. Take a non-zero $a \in H^1(\mathrm{Hom}(F_{-m}, F_m))$. Let D be the open unit disk and let $\mathbb{F}_j$ be the pull-back of F_j to $X \times D$. Let

(7.51) $$0 \to \mathbb{F}_m \to \mathbb{W}_a \to \mathbb{F}_{-m} \to 0$$

be the extension with class

$$a \otimes 1 \in H^1(\mathrm{Hom}(\mathbb{F}_{-m}, \mathbb{F}_m)) \cong H^1(X; \mathrm{Hom}(F_{-m}, F_m)) \otimes H^0(D; \mathbb{C}).$$

Then $\mathbb{V}_a = \mathbb{W}_a \oplus \bigoplus_{i < m} \mathbb{F}_i$ is a family deforming V which is tangent to a at $t = 0 \in D$. To obtain the required deformation of (V, φ) it thus remains to define the Higgs field $\varphi_D \in H^0(S^2 \mathbb{V}_a \otimes K)$ deforming φ. The only pieces of φ which do not automatically lift are the ones involving F_{-m} and F_m, i.e., $\beta \in H^0(\mathrm{Hom}(F_{-m+1}^*, F_m) \otimes K)$ and $\gamma \in$

$H^0(\mathrm{Hom}(F_{-m}, F_{m-1}^*) \otimes K)$. In order to lift β, clearly we should define β_D to be the composition

$$\mathbb{F}^*_{-m+1} \xrightarrow{\beta} \mathbb{F}_m \to \mathbb{W}_a,$$

where the last map is induced from the injection in (7.51). A similar construction gives the lift γ_D of γ. We have thus constrcuted a family $(\mathbb{V}_a, \beta_D, \gamma_D)$ which is tangent to $a \in H^1(C_{2m}^\bullet(V, \varphi))$ for $t = 0 \in D$. Hence Lemma 7.9 below completes the proof. $\qquad \square$

Lemma 7.9. *The* $\mathrm{Sp}(2n, \mathbb{R})$*-Higgs bundle* (V_t, φ_t)*, obtained by restricting* $(\mathbb{V}_a, \beta_D, \gamma_D)$ *constructed in the proof of Lemma 7.8 above to* $X \times \{t\}$*, is polystable.*

Proof. Analogous to the proof of Lemma 7.3. $\qquad \square$

8. Counting components: main results

8.1. **Connected components of** $\mathcal{M}_d$ **for** $d = 0$ **and** $|d| = n(g-1)$**.** With the description of the minima of the Hitchin functional given in Theorem 5.10 at our disposal we are now in a position to complete the count of connected components of the moduli space in the situation of $d = 0$ and $|d| = n(g - 1)$.

Proposition 8.1. *The quiver type minima belong to a Hitchin–Teichmüller component of the moduli space. In particular, they are stable and simple and correspond to smooth points of the moduli space.*

Proof. This is immediate from the description of the $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundles of the Hitchin–Teichmüller component given in [32]. $\qquad \square$

Proposition 8.2. *Assume that* $d = -n(g - 1)$ *and let* (V, β, γ) *be a quiver type minimum for the Hitchin functional. Let* L_0 *be a fixed square root of the canonical bundle, giving rise to the Cayley correspondence isomorphism* $\mathcal{M}_{-n(g-1)} \xrightarrow{\cong} \mathcal{M}'$ *of Theorem 4.4, via* $V \mapsto W \otimes L_0$*. Then the following holds.*

(1) *The second Stiefel–Whitney class* $w_2(W) \in H^2(X, \mathbb{Z}_2)$ *vanishes.*
(2) *If* n *is odd, the first Stiefel–Whitney class* $w_1(W)$ *corresponds to the two-torsion point* $L^{-1}L_0$ *in the Jacobian of* X *under the standard identification* $J_2 \cong H^1(X, \mathbb{Z}_2)$*.*
(3) *If* n *is even, the first Stiefel–Whitney class* $w_1(W) \in H^1(X, \mathbb{Z}_2)$ *vanishes.*

Proof. Easy (similar to the arguments given in [32] for $G = \mathrm{SL}(n, \mathbb{R})$). $\qquad \square$

Theorem 8.3. *Let* X *be a compact oriented surface of genus* g*. Let* $\mathcal{M}_d$ *be the moduli space of polystable* $\mathrm{Sp}(2n, \mathbb{R})$*-Higgs bundles of degree* d*. Let* $n \geq 3$*. Then*

(1) $\mathcal{M}_0$ *is non-empty and connected;*
(2) $\mathcal{M}_{\pm n(g-1)}$ *has* 3.2^{2g} *non-empty connected components.*

Proof. (1) When $d = 0$, we have from Theorem 5.10 that the subspace of minima of the Hitchin functional on $\mathcal{M}_0$ is $\mathcal{N}_0$. It is immediate from Theorem 3.13 that $\mathcal{N}_0$ is isomorphic to the moduli space of poly-stable vector bundles of degree zero. This moduli space is well known to be non-empty and connected and hence $\mathcal{M}_0$ is non-empty and connected.

(2) For definiteness assume that $d = -n(g-1)$. The decomposition (4.30) given by the Cayley correspondence gives a decomposition

$$(8.52) \qquad \mathcal{M}_{-n(g-1)} = \bigcup_{w_1, w_2} \mathcal{M}_{w_1, w_2},$$

where $\mathcal{M}_{w_1, w_2}$ corresponds to $\mathcal{M}'_{w_1, w_2}$ under the Cayley correspondence.

For each possible value of (w_1, w_2), there may be one or more corresponding Hitchin–Teichmüller components contained in $\mathcal{M}_{w_1, w_2}$ (cf. Proposition 8.2); denote by $\tilde{\mathcal{M}}_{w_1, w_2}$ the complement to these. Since minima in $\mathcal{N}_{-n(g-1)}$ (i.e. with $\gamma = 0$) clearly do not belong to Hitchin–Teichmüller components, we see that the subspace of minima of $\tilde{\mathcal{M}}_{w_1, w_2}$ consists of those (V, β, γ) which have $\gamma = 0$. Thus, under the Cayley correspondence, this subspace of minima is identified with the moduli space of poly-stable $O(n, \mathbb{C})$-bundles with the given Stiefel–Whitney classes (w_1, w_2). The moduli space of principal bundles for a connected group and fixed topological type is known to be connected by Ramanathan [41, Proposition 4.2]. However, since $O(n, \mathbb{C})$ is not connected the result of Ramanathan cannot be applied directly. But, all that is required for his argument is that semistability is an open condition and thus, in fact the moduli space in question is connected (cf. [40]). It follows that the subspace of minima on $\tilde{\mathcal{M}}_{w_1, w_2}$ is connected and, hence, this space itself is connected by Proposition 5.1. Additionally, each $\tilde{\mathcal{M}}_{w_1, w_2}$ is non-empty (see, e.g., [40]). Therefore, there is one connected component $\tilde{\mathcal{M}}_{w_1, w_2}$ for each of the 2^{2g+1} possible values of (w_1, w_2). Adding to this the 2^{2g} Hitchin–Teicmüller components gives a total of 3.2^{2g} connected components, as stated.

This accounts for all the connected components of $\mathcal{M}_{-n(g-1)}$ since there are no other minima of the Hitchin functional. $\qquad\square$

8.2. **Representations and $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundles.** Let $\mathcal{R} := \mathcal{R}(\mathrm{Sp}(2n, \mathbb{R}))$ be the moduli space of reductive representations of $\pi_1(X)$ in $\mathrm{Sp}(2n, \mathbb{R})$. Since $\mathrm{U}(n) \subset \mathrm{Sp}(2n, \mathbb{R})$ is a maximal compact subgroup, we have

$$\pi_1(\mathrm{Sp}(2n, \mathbb{R})) \cong \pi_1(\mathrm{U}(n)) \cong \mathbb{Z},$$

and the topological invariant attached to a representation $\rho \in \mathcal{R}$ is hence an element $d = d(\rho) \in \mathbb{Z}$. This integer is called the **Toledo invariant** and coincides with the first Chern class of a reduction to a $\mathrm{U}(n)$-bundle of the flat $\mathrm{Sp}(2n, \mathbb{R})$-bundle associated to ρ.

Fixing the invariant $d \in \mathbb{Z}$ we consider, as in (2.9),

$$\mathcal{R}_d := \{\rho \in \mathcal{R} \quad \text{such that} \quad d(\rho) = d\}.$$

Proposition 8.4. *The transformation $\rho \mapsto (\rho^t)^{-1}$ in $\mathcal{R}$ induces an isomorphism of the moduli spaces $\mathcal{R}_d$ and $\mathcal{R}_{-d}$.*

As shown in Turaev [53] (cf. also Domic–Toledo [18], the Toledo invariant d of a representation satisfies the Milnor–Wood type inequality

$$(8.53) \qquad |d| \leq n(g-1).$$

As a consequence we have the following.

Proposition 8.5. *The moduli space $\mathcal{R}_d$ is empty unless*

$$|d| \leq n(g-1).$$

As a special case of Theorem 2.28 we have the following.

Proposition 8.6. *The moduli spaces $\mathcal{R}_d$ and $\mathcal{M}_d$ are homeomorphic.*

From Proposition 8.6 and Theorem 8.3 we have the main result of this paper regarding the connectedness properties of $\mathcal{R}$ given by the following.

Theorem 8.7. *Let X be a compact oriented surface of genus g. Let $\mathcal{R}_d$ be the moduli space of reductive representations of $\pi_1(X)$ in $\mathrm{Sp}(2n, \mathbb{R})$. Let $n \geq 3$. Then*

(1) *$\mathcal{R}_0$ is non-empty and connected;*
(2) *$\mathcal{R}_{\pm n(g-1)}$ has 3.2^{2g} non-empty connected components.*

APPENDIX A. STABILITY OF TWISTED PAIRS AND HITCHIN–KOBAYASHI CORRESPONDENCE

A.1. **Standard parabolic subgroups.** Here we set up some notations about parabolic subgroups , which will be used when stating a general notion of (poly, semi)-stability (see Chapter IV in [4] for more details). First some basic notation.

$$H - \text{a compact and connected Lie group}$$
$$H^{\mathbb{C}} - \text{the complexification of } H$$
$$\mathfrak{h} - \text{the Lie algebra of } H$$
$$\mathfrak{h}^{\mathbb{C}} - \text{the Lie algebra of } H^{\mathbb{C}}$$
$$\mathfrak{h}_s^{\mathbb{C}} = [\mathfrak{h}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}] - \text{the semisimple part of } \mathfrak{h}^{\mathbb{C}}$$
$$\mathfrak{z} \subset \mathfrak{a} - \text{the center of } \mathfrak{h}^{\mathbb{C}}$$
$$T \subset H - \text{a maximal torus}$$
$$\mathfrak{t} \subset \mathfrak{h} - \text{the Lie algebra of } T$$
$$\mathfrak{a} \subset \mathfrak{h}^{\mathbb{C}} - \text{the complexification of } \mathfrak{t}$$
$$\langle \cdot, \cdot \rangle - \text{an invariant } \mathbb{C}\text{-bilinear pairing on } \mathfrak{h}^{\mathbb{C}} \text{ extending the Killing form on } \mathfrak{h}_s^{\mathbb{C}}$$
$$R \subset \mathfrak{c}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{c}, \mathbb{C}) - \text{the roots of } \mathfrak{h}_s^{\mathbb{C}}$$
$$\mathfrak{h}_\delta \subset \mathfrak{h}^{\mathbb{C}} - \text{the root space corresponding to } \delta \in R$$
$$\Delta \subset R - \text{a choice of simple roots.}$$

Defining $\mathfrak{c} = \mathfrak{a} \cap \mathfrak{h}_s^{\mathbb{C}}$ we have $\mathfrak{a} = \mathfrak{z} \oplus \mathfrak{c}$ and moreover

$$\mathfrak{h}^{\mathbb{C}} = \mathfrak{z} \oplus \mathfrak{c} \oplus \bigoplus_{\delta \in R} \mathfrak{h}_\delta.$$

For any $A \subset \Delta$ define R_A to be the set of roots $\delta = \sum_{\beta \in \Delta} m_\beta \beta \in R$ with $m_\beta \geq 0$ for all $\beta \in A$ (so if $A = \emptyset$ then $R_A = R$). Then

$$\mathfrak{p}_A = \mathfrak{z} \oplus \mathfrak{c} \oplus \bigoplus_{\delta \in R_A} \mathfrak{h}_\delta$$

is a Lie subalgebra of $\mathfrak{h}^{\mathbb{C}}$. Denote by $P_A \subset H^{\mathbb{C}}$ the connected subgroup whose Lie algebra is $\mathfrak{p}_A$. The group P_A is a parabolic subgroup of $H^{\mathbb{C}}$, and any parabolic subgroup of $H^{\mathbb{C}}$ is

conjugate to P_A for some A. Define similarly R_A^0 as the set of roots $\delta = \sum_{\beta \in \Delta} m_\beta \beta$ with $m_\beta = 0$ for all $\beta \in A$. The vector space

$$(A.54) \qquad \qquad \mathfrak{l}_A = \mathfrak{z} \oplus \mathfrak{c} \oplus \bigoplus_{\delta \in R_A^0} \mathfrak{h}_\delta$$

is a Lie subalgebra of $\mathfrak{p}_A$. Let L_A be the connected subgroup with Lie algebra $\mathfrak{l}_A$. Then L_A is a Levi subgroup of P_A, i.e., a maximal reductive subgroup of P_A. Finally,

$$(A.55) \qquad \qquad \mathfrak{u}_A = \bigoplus_{\delta \in R_A \setminus R_A^0} \mathfrak{h}_\delta$$

is also a Lie subalgebra of $\mathfrak{p}_A$, and the connected Lie group $U_A \subset P_A$ with Lie algebra $\mathfrak{u}_A$ is the unipotent radical of P_A. U_A is a normal subgroup of P_A and the quotient P_A/U_A is naturally isomorphic to L_A so we have

$$(A.56) \qquad \qquad P_A = L_A U_A.$$

A.2. **Characters of parabolic subgroups.** Let Z be the center of $H^{\mathbb{C}}$, and let $\Gamma = \mathrm{Ker}(\exp : \mathfrak{z} \to Z)$. Then $\mathfrak{z}_{\mathbb{R}} = \Gamma \otimes_{\mathbb{Z}} \mathbb{R} \subset \mathfrak{z}$ is the Lie algebra of the maximal compact subgroup of Z. Let $\mathfrak{z}_{\mathbb{R}}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{z}_{\mathbb{R}}, \mathbf{i}\mathbb{R})$ and let $\Lambda = \{\lambda \in \mathfrak{z}_{\mathbb{R}}^* \mid \lambda(\Gamma) \subset 2\pi\mathbf{i}\mathbb{Z}\}$. Let $\{\lambda_\delta\}_{\delta \in \Delta} \subset \mathfrak{c}^*$ be the set of fundamental weights of $\mathfrak{h}_s^{\mathbb{C}}$, i.e., the duals with respect to the Killing form of the coroots $\{2\delta/\langle \delta, \delta \rangle\}_{\delta \in \Delta}$. We extend any $\lambda \in \Lambda$ to a morphism of complex Lie algebras $\lambda : \mathfrak{z} \oplus \mathfrak{c}_A \to \mathbb{C}$ by setting $\lambda|_{\mathfrak{z}} = 0$, and similarly for any $\delta \in A$ we extend $\lambda_\delta : \mathfrak{c}_A \to \mathbb{C}$ to $\lambda_\delta : \mathfrak{z} \oplus \mathfrak{c}_A \to \mathbb{C}$ by setting $\lambda_\delta|_{\mathfrak{z}} = 0$.

Let Z_A° be the identity component of the center of L_A, and let L_A^{ss} be the maximal connected semisimple Lie subgroup of L_A (i.e., the connected subgroup whose Lie algebra is $[\mathfrak{l}_A, \mathfrak{l}_A]$). Define

$$Z^{oss}(L_A) := Z_A^\circ \cap L_A^{ss}.$$

The product map $Z_A^\circ \times L_A^{ss} \to L_A$ induces an isomorphism $L_A \simeq Z_A^\circ \times_{Z^{oss}(L_A)} L_A^{ss}$, and projection to the first factor gives a map $L_A \to Z_A^\circ/Z^{oss}(L_A)$. Composing this projection with the quotient map $P_A \to P_A/U_A \simeq L_A$ we obtain a morphism of Lie groups

$$\pi_A : P_A \to Z_A^\circ/Z^{oss}(L_A).$$

Let $\mathfrak{z}_A \subset \mathfrak{l}_A$ be the Lie algebra of Z_A°. Let $\mathfrak{c}_A = \bigcap_{\beta \in \Delta \setminus A} \mathrm{Ker}\,\lambda_\beta$ if $A \neq \Delta$ and let $\mathfrak{c}_A = \mathfrak{c}$ if $A = \Delta$. Then we have

$$\mathfrak{z}_A = \mathfrak{z} \oplus \mathfrak{c}_A.$$

This follows from the fact that for any $\delta, \delta' \in R$ we have $[h_\delta, h_{\delta'}] = h_{\delta+\delta'}$ if $\delta + \delta' \neq 0$ and $[h_\delta, h_{-\delta}] = (\mathrm{Ker}\,\lambda_\delta)^\perp$ (see Theorem 2 in Chapter VI of [46]).

Lemma A.1. *There exists some positive integer n (depending on the fundamental group of L_A) such that for any $\lambda \in \Lambda$ and any $\delta \in A$ the morphisms of Lie algebras $n\lambda : \mathfrak{z} \oplus \mathfrak{c}_A \to \mathbb{C}$ and $n\lambda_\delta : \mathfrak{z} \oplus \mathfrak{c}_A \to \mathbb{C}$ exponentiate to morphisms of Lie groups*

$$\exp(n\lambda) : Z_A^\circ/Z^{oss}(L_A) \to \mathbb{C}^\times, \qquad \qquad \exp(n\lambda_\delta) : Z_A^\circ/Z^{oss}(L_A) \to \mathbb{C}^\times.$$

Composing the morphisms given by the previous lemma with the morphism P_A we get for any $\lambda \in \Lambda$ and $\delta \in A$ morphisms of Lie groups

$$\kappa_{n\lambda} : P_A \to \mathbb{C}^\times, \qquad \qquad \kappa_{n\delta} : P_A \to \mathbb{C}^\times.$$

A.3. **Antidominant characters of $\mathfrak{p}_A$.** An **antidominant character** of $\mathfrak{p}_A$ is any element of $\mathfrak{z}^* \oplus \mathfrak{c}_A^*$ of the form $\chi = z + \sum_{\delta \in A} n_\delta \lambda_\delta$, where $z \in \mathfrak{z}_\mathbb{R}^*$ and each n_δ is a nonpositive real number. If for each $\delta \in A$ we have $n_\delta < 0$ then we say that χ is **strictly antidominant**. The restriction of the invariant form $\langle , \rangle$ to $\mathfrak{z} \oplus \mathfrak{c}_A$ is nondegenerate, so it induces an isomorphism $\mathfrak{z}^* \oplus \mathfrak{c}_A^* \simeq \mathfrak{z} \oplus \mathfrak{c}_A$. For any antidominant character χ we define $s_\chi \in \mathfrak{z} \oplus \mathfrak{c}_A \subset \mathfrak{z} \oplus \mathfrak{c}$ to be the element corresponding to χ via the previous isomorphism. One checks that s_χ belongs to $i\mathfrak{h}$. The following lemma implies that one can recover from s_χ the parabolic subgroup P_A and all related objects.

Lemma A.2. *Let $s \in i\mathfrak{h}$ and define the sets*

$$\mathfrak{p}_s := \{x \in \mathfrak{h}^\mathbb{C} \mid \mathrm{Ad}(e^{ts})(x) \text{ is bounded as } t \to \infty\} \subset \mathfrak{h}^\mathbb{C},$$

$$\mathfrak{l}_s := \{x \in \mathfrak{h}^\mathbb{C} \mid [x, s] = 0 \} \subset \mathfrak{h}^\mathbb{C},$$

$$P_s := \{g \in H^\mathbb{C} \mid e^{ts} g e^{-ts} \text{ is bounded as } t \to \infty\} \subset H^\mathbb{C},$$

$$L_s := \{g \in H^\mathbb{C} \mid \mathrm{Ad}(g)(s) = s \} \subset H^\mathbb{C}.$$

The following properties hold:

(1) *Both $\mathfrak{p}_s$ and $\mathfrak{l}_s$ are Lie subalgebras of $\mathfrak{h}^\mathbb{C}$ and P_s and L_s are subgroups of $H^\mathbb{C}$. Furthermore P_s and L_s are connected.*

(2) *Let χ be an antidominant character of P_A. There are inclusions $\mathfrak{p}_A \subset \mathfrak{p}_{s_\chi}$, $\mathfrak{l}_A \subset \mathfrak{l}_{s_\chi}$, $P_A \subset P_{s_\chi}$ and $L_A \subset L_{s_\chi}$, with equality if χ is strictly antidominant.*

(3) *For any $s \in i\mathfrak{h}$ there exists $h \in H$ and a standard parabolic subgroup P_A such that $P_s = h P_A h^{-1}$ and $L_s = h L_A h^{-1}$. Furthermore, there is an antidominant character χ of P_A such that $s = h s_\chi h^{-1}$.*

Proof. That $\mathfrak{l}_s, \mathfrak{p}_s$ are subalgebras and L_s, P_s are subgroups is immediate from the definitions. Let T_s be the closure of $\{e^{its} \mid t \in \mathbb{R}\}$. Then L_s is the centralizer of the torus T_s in $H^\mathbb{C}$, so by Theorem 13.2 in [3] is connected. To prove that P_s is also connected, note that if g belongs to P_s, so that $e^{ts} g e^{-ts}$ is bounded as $t \to \infty$, then the limit of $\pi_s(g) := e^{ts} g e^{-ts}$ as $t \to \infty$ exists and belongs to L_s. Note by the way that the resulting map $\pi_s : P_s \to L_s$ is a morphism of Lie groups which can be identified with the projection $P_s \to P_s/U_s \simeq L_s$, where

$$U_s = \{g \in H^\mathbb{C} \mid e^{ts} g e^{-ts} \text{ converges to } 1 \text{ as } t \to \infty\} \subset P_s$$

is the unipotent radical of U_s. So if $g \in P_s$ then the map $\gamma : [0, \infty) \to H^\mathbb{C}$ defined as $\gamma(t) = e^{ts} g e^{-ts}$ extends to give a path from g to L_s, and since L_s is connected it follows that P_s is also connected. This proves (1). Let now $\chi = z + \sum_{\beta \in \Delta} n_\beta \lambda_\beta$ be an antidominant character of P_A. Let $\delta = \sum_{\beta \in \Delta} m_\beta \beta$ be a root and let $u \in \mathfrak{h}_\delta$. We have $[s_\chi, u] = \langle s_\chi, \delta \rangle u = \langle \chi, \delta \rangle u = (\sum_{\beta \in \Delta} m_\beta n_\beta \langle \beta, \beta \rangle /2) u$. Hence $\mathrm{Ad}(e^{ts_\chi})(u) = (\sum_{\beta \in \Delta} \exp(t n_\beta m_\beta \langle \beta, \beta \rangle /2)) u$, so this remains bounded as $t \to \infty$ if $m_\beta \geq 0$ for any β such that $n_\beta \leq 0$. This implies that $\mathfrak{p}_A \subset \mathfrak{p}_s$ and $\mathfrak{l}_A \subset \mathfrak{l}_s$ and that the inclusions are equalities when χ is strictly dominant. The analogous statements for P_A, L_A, P_s, L_s follow from this, because the subgroups P_A, L_A, P_s, L_s are connected. Hence (2) is proved. To prove (3) take a maximal torus T_s containing $\{e^{its} \mid t \in \mathbb{R}\}$ and choose $h \in H$ such that $h^{-1} T_s h = T$ and $\mathrm{Ad}(h^{-1})(s)$ belongs to the Weyl chamber in $\mathfrak{t}$ corresponding to the choice of $\Delta \subset R$. Then use (2). $\square$

Lemma A.3. *Let $P \subset H^\mathbb{C}$ be any parabolic subgroup, conjugate to P_A. Let χ be an antidominant character of $\mathfrak{p}_A$. There exists an element $s_{P,\chi} \in i\mathfrak{h}$, depending smoothly on P, which is conjugate to s_χ and such that $P \subset P_{s_{P,\chi}}$, with equality if and only if χ is strictly antidominant.*

Proof. Assume that $P = gP_Ag^{-1}$ for some $g \in H^{\mathbb{C}}$. From the well known equality $H^{\mathbb{C}}/P_A = H/(P_A \cap H) = H/(L_A \cap H)$ we deduce that there exists some $h \in H$ such that $P = hP_Ah^{-1}$. Then we set $s_{P,\chi} = hs_\chi h^{-1}$. This is well defined because h is unique up to multiplication on the right by elements of $L_A \cap H$, and these elements commute with s_χ. □

A.4. **Principal bundles and parabolic subgroups.** If E is a $H^{\mathbb{C}}$-principal holomorphic bundle over X and M is any set on which $H^{\mathbb{C}}$ acts on the left, we denote by $E(M)$ the twisted product $E \times_{H^{\mathbb{C}}} M$, defined as the quotient of $E \times M$ by the equivalence relation $(eh, m) \sim (e, hm)$ for any $e \in E$, $h \in H^{\mathbb{C}}$ and $m \in M$. The sections φ of $E(M)$ are in natural bijection with the maps $\phi : E \to M$ satisfying $\varphi(eh) = h^{-1}\varphi(e)$ for any $e \in E$ and $h \in H^{\mathbb{C}}$ (we call such maps antiequivariant). Furthermore, ϕ is holomorphic if and only if φ is holomorphic.

If M is a vector space (resp. complex variety) and the action of $H^{\mathbb{C}}$ on M is linear (resp. holomorphic) then $E(M)$ is a vector bundle (resp. holomorphic fibration). In this situation, for any complex line bundle $L \to X$ we can form a vector bundle $E(M) \otimes L$ which can be identified with $E^L(M)$, where E^L denotes the principal $H^{\mathbb{C}} \times \mathbb{C}^\times$ bundle $E^L = \{(e, l) \in E \times_X L \mid l \neq 0\}$ and we form the associated product by making $(h, \lambda) \in H^{\mathbb{C}} \times \mathbb{C}^\times$ act on $m \in M$ as λhm. Consequently, the sections of $E(M) \otimes L$ can be identified with antiequivariant maps $E^L \to M$.

Let B be a Hermitian vector space and let $\rho : H \to U(B)$ be a unitary representation. The morphism ρ extends to a holomorphic representation of $H^{\mathbb{C}}$ in $GL(B)$, which we denote also by ρ. Suppose that $P_A \subset H^{\mathbb{C}}$ is the parabolic subgroup corresponding to a subset $A \subset \Delta$ and let χ be an antidominant character. Define

$$B_\chi^- = \{v \in B \mid \rho(e^{ts_\chi})v \text{ remains bounded as } \mathbb{R} \ni t \to \infty\}.$$

This is a complex subspace of B and by (2) in Lemma A.2 it is invariant under the action of P_A. Define also

$$B_\chi^0 = \{v \in B \mid \rho(e^{ts_\chi})v = v \text{ for any } t \} \subset B_\chi^-.$$

This is a complex subspace of B_χ^- and, using again (2) in Lemma A.2, we deduce that B_χ^0 is invariant under the action of L_A.

Suppose that σ is a holomorphic section of $E(H^{\mathbb{C}}/P_A)$. Since $E(H^{\mathbb{C}}/P_A) \simeq E/P_A$ canonically and the quotient $E \to E/P_A$ has the structure of a P_A-principal bundle, the pullback $E_\sigma := \sigma^*E$ is a P_A-principal bundle over X, and we can identify canonically $E \simeq E_\sigma \times_{P_A} H^{\mathbb{C}}$ as principal $H^{\mathbb{C}}$-bundles (hence, σ gives a reduction of the structure group of E to P_A). Equivalently, we can look at E_σ as a holomorphic subvariety $E_\sigma \subset E$ invariant under the action of $P_A \subset H^{\mathbb{C}}$ and inheriting a structure of principal bundle. It follows that $E(B) \simeq E_\sigma \times_{P_A} B$, so the vector bundle $E_\sigma \times_{P_A} B_\chi^-$ can be identified with a holomorphic subbundle

$$E(B)_{\sigma,\chi}^- \subset E(B).$$

Now suppose that σ_L is a holomorphic section of $E_\sigma(P_A/L_A)$. This section induces, exactly as before, a reduction of the structure group of E_σ from P_A to L_A. So we obtain from σ_L a principal L_A bundle E_{σ_L} and an isomorphism $E_\sigma \simeq E_{\sigma_L} \times_{L_A} P_A$. Hence $E(B) \simeq E_{\sigma_L} \times_{L_A} B$, and we can thus identify the vector bundle $E_{\sigma_L} \times_{L_A} B_\chi^0$ with a holomorphic subbundle

$$E(B)_{\sigma_L,\chi}^0 \subset E(B)_{\sigma,\chi}^-.$$

Let us write $\chi = z + \sum_{\delta \in A} n_\delta \lambda_\delta$, with $z \in \mathfrak{z}_\mathbb{R}^*$, and $z = z_1 \lambda_1 + \cdots + z_r \lambda_r$, where $\lambda_1, \ldots, \lambda_r \in \Lambda$ and the z_j are real numbers. Let n be an integer as given by Lemma A.1. Using the characters $\kappa_{n\lambda}, \kappa_{n\delta} : P_A \to \mathbb{C}^\times$ defined in Subsection A.2 we can construct from the principal P_A bundle E_σ line bundles $E_\sigma \times_{\kappa_{n\lambda}} \mathbb{C}$ and $E_\sigma \times_{\kappa_{n\delta}} \mathbb{C}$. We define the **degree** of the bundle E with respect to the reduction σ and the antidominant character χ to be the real number:

$$(\text{A.57}) \qquad \deg(E)(\sigma, \chi) := \frac{1}{n} \left(\sum_j z_j \deg(E_\sigma \times_{\kappa_{n\lambda_j}} \mathbb{C}) + \sum_{\delta \in A} n_\delta \deg(E_\sigma \times_{\kappa_{n\delta}} \mathbb{C}) \right).$$

This expression is independent of the choice of the λ_j's and the integer n.

Although this will play no role in our results, we now stop to give another definition of the degree in terms of the curvature of connections, in the spirit of Chern–Weil theory. Define $H_A = H \cap L_A$ and $\mathfrak{h}_A = \mathfrak{h} \cap \mathfrak{l}_A$. Then H_A is a maximal compact subgroup of L_A, so the inclusions $H_A \subset L_A$ is a homotopy equivalence. Since the inclusion $L_A \subset P_A$ is also a homotopy equivalence, given a reduction σ of the structure group of E from $H^\mathbb{C}$ to P_A one can further restrict the structure group of E to H_A in a unique way up to homotopy. Denote by E_σ' the resulting H_A principal bundle. Let $\pi_A : \mathfrak{p}_A \to \mathfrak{z} \oplus \mathfrak{c}_A$ be the differential of the projection π_A defined in Subsection A.2. Let $\chi = z + \sum_{\delta \in A} n_\delta \lambda_\delta$ be an antidominant character. Define $\kappa_\chi = (z + \sum_\delta n_\delta \lambda_\delta) \circ \pi_A \in \mathfrak{p}_A^*$. Let $\mathfrak{h}_A \subset \mathfrak{l}_A \subset \mathfrak{p}_A$ be the Lie algebra of H_A. Then $\kappa_\chi(\mathfrak{h}_A) \subset \mathbf{i}\mathbb{R}$. Choose a connection $\mathbb{A}$ on E_σ' and denote by $F_\mathbb{A} \in \Omega^2(X, E_\sigma' \times_{\text{Ad}} \mathfrak{h}_A)$ its curvature. Then $\kappa_\chi(F_\mathbb{A})$ is a 2-form on X with values in $\mathbf{i}\mathbb{R}$, and we have

$$\deg(E)(\sigma, \chi) := \frac{\mathbf{i}}{2\pi} \int_X \kappa_\chi(F_\mathbb{A}).$$

A.5. **Stability of L-twisted pairs.** Let L be a holomorphic line bundle over X. We define an L-twisted pair to be a pair of the form (E, φ), where E is a holomorphic $H^\mathbb{C}$-principal bundle over X and φ is a holomorphic section of $E(B) \otimes L$. When it does not lead to confusion we say that (E, φ) is a pair, instead of a L-twisted pairs.

Let $\alpha \in \mathbf{i}\mathfrak{z}_\mathbb{R} \subset \mathfrak{z}$. We say that (E, φ) is:

- α-**semistable** if: for any parabolic subgroup $P_A \subset H^\mathbb{C}$, any antidominant character χ for P_A, and any holomorphic section $\sigma \in \Gamma(E(H^\mathbb{C}/P_A))$ such that $\varphi \in H^0(E(B)_{\sigma,\chi}^- \otimes L)$, we have

$$\deg(E)(\sigma, \chi) - \langle \alpha, \chi \rangle \geq 0.$$

- α-**stable** if it is α-semistable and furthermore: for any P_A, χ and σ as above, such that $\varphi \in H^0(E(B)_{\sigma,\chi}^- \otimes L)$ and such that $A \neq \emptyset$ and $\chi \notin \mathfrak{z}_\mathbb{R}^*$, we have

$$\deg(E)(\sigma, \chi) - \langle \alpha, \chi \rangle > 0.$$

- α-**polystable** if it is α-semistable and for any P_A, χ and σ as above, such that $\varphi \in H^0(E(B)_{\sigma,\chi}^- \otimes L)$, $P_A \neq H^\mathbb{C}$ and χ is strictly antidominant, and such that

$$\deg(E)(\sigma, \chi) - \langle \alpha, \chi \rangle = 0,$$

  there is a holomorphic reduction of the structure group $\sigma_L \in \Gamma(E_\sigma(P_A/L_A))$, where E_σ denotes the principal P_A-bundle obtained from the reduction σ of the structure group. Furthermore, under these hypothesis φ is required to belong to $H^0(E(B)_{\sigma_L,\chi}^0 \otimes L) \subset H^0(E(B)_{\sigma,\chi}^- \otimes L)$.

Remark A.4. For some instances of group $H^{\mathbb{C}}$ and representation $H^{\mathbb{C}} \to GL(B)$ the last condition in the definition of polystability is redundant (for example, $H^{\mathbb{C}} = GL(n, \mathbb{C})$ with its fundamental representation on $\mathbb{C}^n$). This does not seem to be general fact, but we do not have any example which illustrates that the condition $\varphi \in \mathrm{H}^0(E(B)^0_{\sigma_L, \chi} \otimes L)$ is not a consequence of the α-semistability of (E, φ) and the existence of σ_L whenever $\deg(E)(\sigma, \chi) = \langle \alpha, \chi \rangle$ and $\varphi \in H^0(E(B)^-_{\sigma, \chi} \otimes L)$.

A.6. **The stability condition in terms of filtrations.** In order to obtain a workable notion of α-(poly,semi)stability it is desirable to have a more concrete way to describe, for any holomorphic $H^{\mathbb{C}}$-principal bundle E,

- the reductions of the structure group of E to parabolic subgroups $P \subset H^{\mathbb{C}}$, and the (strictly or not) antidominant characters of P,
- the subbundle $E(B)^-_{\sigma, \chi} \subset E(B)$,
- the degree $\deg(E)(\sigma, \chi)$ defined in (A.57),
- reductions to Levi factors of parabolic subgroups and the corresponding vector bundle $E(B)^0_{\sigma_L, \chi} \subset E(B)^-_{\sigma, \chi}$.

We now discuss how to obtain in some cases such concrete descriptions, beginning with the notion of degree. In [10] the degree $\deg(E)(\sigma, \chi)$ is defined in terms of a so-called auxiliary representation (see §2.1.2 in [10]) and certain linear combinations of degrees of subbundles. The following lemma implies that definition (A.57) contains the one given in [10] as a particular case. Suppose that $\rho_W : H \to \mathrm{U}(W)$ is a representation on a Hermitian vector space, and denote the holomorphic extension $H^{\mathbb{C}} \to \mathrm{GL}(W)$ with the same symbol ρ_W. Let $(\mathrm{Ker}\, \rho_W)^\perp \subset \mathfrak{h}^{\mathbb{C}}$ be the orthogonal with respect to invariant pairing on $\mathfrak{h}^{\mathbb{C}}$ of the kernel of $\rho_W : \mathfrak{h}^{\mathbb{C}} \to \mathfrak{gl}(W)$, and let $\pi : \mathfrak{h}^{\mathbb{C}} \to (\mathrm{Ker}\, \rho_W)^\perp$ be the orthogonal projection.

Lemma A.5. *Take some element $s \in i\mathfrak{h}$. Then $\rho_W(s)$ diagonalizes with real eigenvalues $\lambda_1 < \cdots < \lambda_k$. Let $W_j = \mathrm{Ker}(\lambda_j \, \mathrm{Id}_W - \rho_W(s))$ and define $W_{\leq i} = \bigoplus_{j \leq i} W_j$.*

(1) *The subgroup $P_{W,s} \subset H^{\mathbb{C}}$ consisting of those g such that $\rho_W(g)(W_{\leq i}) \subset W_{\leq i}$ for any i is a parabolic subgroup, which can be identified with $P_{\pi(s)}$. Let $\chi \in (\mathfrak{z} \oplus \mathfrak{c})^*$ be a character such that $s_\chi = s$. Then χ is strictly antidominant for $P_{W,s}$.*

(2) *Suppose that for any $a, b \in (\mathrm{Ker}\, \rho_W)^\perp$ we have $\langle a, b \rangle = \mathrm{Tr}\, \rho_W(a)\rho_W(b)$. Let $u \in (\mathrm{Ker}\, \rho_W)^\perp$ be any element, and write $\rho_W(u) = \sum \rho_W(u)_{ij}$ the decomposition in pieces $\rho_W(u)_{ij} \in \mathrm{Hom}(W_i, W_j)$. Then*

$$(\mathrm{A}.58) \qquad \langle \chi, u \rangle = \mathrm{Tr}(\rho_W(s)\rho_W(u)) = \lambda_k \,\mathrm{Tr}\, \rho_W(u) + \sum_{i=1}^{k-1} (\lambda_i - \lambda_{i+1}) \,\mathrm{Tr}\, \rho_W(u)_{ii}.$$

(3) *Suppose that ρ_W satisfies the conditions of (2). Let E be a holomorphic $H^{\mathbb{C}}$-principal bundle and let $\mathcal{W} = E(W)$ be the associated holomorphic vector bundle. Let σ be a reduction of the structure group of E to a parabolic subgroup P and an let χ be an antidominant character of P. The endomorphism $\rho_W(s_\chi)$ diagonalizes with constant eigenvalues, giving rise to a decomposition $\mathcal{W} = \bigoplus_{j=1}^k \mathcal{W}_j$, where $\rho_W(s_\chi)$ restricted to $\mathcal{W}_j$ is multiplication by $\lambda_j \in \mathbb{R}$. Suppose that $\lambda_1 < \cdots < \lambda_k$. For each i the subbundle $\mathcal{W}_{\leq i} = \bigoplus_{j \leq i} \mathcal{W}_j \subset \mathcal{W}$ is holomorphic. We have:*

$$\deg(E)(\sigma, \chi) = \lambda_k \deg \mathcal{W} + \sum_{i=1}^{k-1} (\lambda_i - \lambda_{i+1}) \deg \mathcal{W}_{\leq i}.$$

Proof. The first assertion and formula (A.58) follows from easy computations. (3) follows from (2).   □

Remark A.6. Condition (2) of the lemma is satisfied when $W = \mathfrak{h}$, endowed with the invariant metric, and $\rho_W : \mathfrak{h}^{\mathbb{C}} \to \operatorname{End} W$ is the adjoint representation, since the invariant metric on $\mathfrak{h}$ is supposed to extend the Killing pairing in the semisimple part $\mathfrak{h}_s$.

To clarify the other ingredients in the definition of (poly,semi)stability, we put ourselves in the situation where $H^{\mathbb{C}}$ is a classical group, so that there is the so-called fundamental representation $\rho : H^{\mathbb{C}} \to \operatorname{GL}(N, \mathbb{C})$ with N depending on $H^{\mathbb{C}}$. Suppose that E is an $H^{\mathbb{C}}$-principal bundle, and denote by V the vector bundle associated to E and ρ. One can describe pairs (σ, χ) consisting of a reduction σ of the structure group of E to a parabolic subgroup $P \subset H^{\mathbb{C}}$ and an antidominant character χ of P in terms of filtrations of vector bundles

$$(A.59) \qquad\qquad \mathscr{V} = (0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{k-1} \subsetneq V_k = V),$$

and increasing sequences of real numbers (usually called weights)

$$(A.60) \qquad\qquad \lambda_1 \leq \cdots \leq \lambda_k,$$

which are arbitrary if $H^{\mathbb{C}} = \operatorname{GL}(n, \mathbb{C})$, and which satisfy otherwise:

- if $H^{\mathbb{C}} = \operatorname{O}(n, \mathbb{C})$ then, for any i, $V_{k-i} = V_i^{\perp} = \{v \in V \mid \langle v, V_i \rangle = 0\}$, where $\langle, \rangle$ denotes the bilinear pairing given by the orthogonal structure (we implicitly define $V_0 = 0$), and $\lambda_{k-i+1} + \lambda_i = 0$.
- if $H^{\mathbb{C}} = \operatorname{Sp}(2n, \mathbb{C})$ then, for any i, $V_{k-i} = V_i^{\perp} = \{v \in V \mid \omega(v, V_i) = 0\}$, where ω is the symplectic form on V (as before, $V_0 = 0$), and furthermore $\lambda_{k-i+1} + \lambda_i = 0$.

The resulting character χ is strictly antidominant if all the inequalities in (A.60) are strict.

Given positive integers p, q define the vector bundle $V^{p,q} = V^{\otimes p} \otimes (V^*)^{\otimes q}$. For any choice of reduction and antidominant character (σ, χ) specified by a filtration (A.59) and weights (A.60) we define

$$(V^{p,q})^-_{\sigma,\chi} = \sum_{\lambda_{i_1} + \cdots + \lambda_{i_p} \leq \lambda_{j_1} + \cdots + \lambda_{j_q}} V_{i_1} \otimes \cdots \otimes V_{i_p} \otimes V_{j_1}^{\perp} \otimes \cdots \otimes V_{j_q}^{\perp} \subset V^{p,q},$$

where $V_j^{\perp} = \{v \in V^* \mid \langle v, V_j \rangle = 0\}$ and $\langle, \rangle$ is the natural pairing between V and V^*. Since $H^{\mathbb{C}}$ is a classical group, there is an inclusion of representations

$$B \subset (\rho^{\otimes p_1} \otimes (\rho^*)^{\otimes q_1}) \oplus \cdots \oplus (\rho^{\otimes p_r} \otimes (\rho^*)^{\otimes q_r}),$$

so that the vector bundle $E(B)$ is contained in $V^{p_1, q_1} \oplus \cdots \oplus V^{p_r, q_r}$. One then has

$$E(B)^-_{\sigma,\chi} = E(B) \cap ((V^{p_1,q_1})^-_{\sigma,\chi} \oplus \cdots \oplus (V^{p_r,q_r})^-_{\sigma,\chi}).$$

Suppose that the invariant pairing $\langle, \rangle$ on the Lie algebra $\mathfrak{h}^{\mathbb{C}}$ is defined using the fundamental representation as $\langle x, y \rangle = \operatorname{Tr} \rho(x)\rho(y)$. This clearly satisfies the condition of (2) of Lemma A.5, so by (3) in the same lemma we have

$$\deg(E)(\sigma, \chi) = \lambda_k \deg V + \sum_{i=1}^{k-1} (\lambda_i - \lambda_{i+1}) \deg V_i.$$

We now specify what it means to have a reduction to a Levi factor of a parabolic subgroup, as appears in the definition of polystability. Assume that (σ, χ) is a pair specified by (A.59) and (A.60), so that σ defines a reduction of the structure group of E to a

parabolic subgroup $P \subset H^{\mathbb{C}}$, and that $\varphi \in \mathrm{H}^0(L \otimes E(B)_{\sigma,\chi}^-)$ and $\deg(E)(\sigma, \chi) = 0$. If the pair (E, φ) is α-polystable all these assumptions imply the existence of a further reduction σ_L of the structure group of $H^{\mathbb{C}}$ from P to a Levi factor $L \subset P$; this is given explicitly by an isomorphism of vector bundles

$$V \simeq \mathrm{Gr}\,\mathscr{V} := V_1 \oplus V_2/V_1 \oplus \cdots \oplus V_k/V_{k-1}.$$

When $H^{\mathbb{C}} = \mathrm{GL}(n, \mathbb{C})$ such isomorphism is arbitrary. When $H^{\mathbb{C}}$ is $\mathrm{O}(n, \mathbb{C})$ (resp. $\mathrm{Sp}(2n, \mathbb{C})$), it is also assumed that the pairing of an element of V_j/V_{j-1} with an element of V_i/V_{i-1}, using the scalar product (resp. symplectic form), is always zero unless $j + i = k + 1$. We finally describe the bundle $E(B)_{\sigma_L,\chi}^0$ in this situation. Let

$$(\mathrm{Gr}\,\mathscr{V}^{p,q})_{\sigma_L,\chi}^0 = \sum_{\lambda_{i_1}+\cdots+\lambda_{i_p}=\lambda_{j_1}+\cdots+\lambda_{j_q}} (V_{i_1}/V_{i_1-1})\otimes\cdots\otimes(V_{i_p}/V_{i_p-1})\otimes(V_{j_1}^\perp/V_{j_1+1}^\perp)\otimes\cdots\otimes(V_{j_q}^\perp/V_{j_q+1}^\perp).$$

Then

$$E(B)_{\sigma_L,\chi}^0 = E(B) \cap ((\mathrm{Gr}\,\mathscr{V}^{p_1,q_1})_{\sigma_L,\chi}^0 \oplus \cdots \oplus (\mathrm{Gr}\,\mathscr{V}^{p_r,q_r})_{\sigma_L,\chi}^0).$$

A.7. **Infinitesimal automorphism space.** For any pair (E, φ) we define the infinitesimal automorphism space of (E, φ) as

$$\mathrm{aut}(E, \varphi) = \{s \in \mathrm{H}^0(E(\mathfrak{h}^{\mathbb{C}})) \mid \rho(s)(\varphi) = 0\},$$

where we denote by $\rho : \mathfrak{h}^{\mathbb{C}} \to \mathrm{End}(B)$ the morphism of Lie algebras induced by ρ. We similarly define the semisimple infinitesimal automorphism space of (E, φ) as

$$\mathrm{aut}^{ss}(E, \varphi) = \{s \in \mathrm{aut}(E, \varphi) \mid s(x) \text{ is semisimple for any } x \in X\}.$$

Proposition A.7. *Suppose that (E, φ) is a α-polystable pair. Then (E, φ) is α-stable if and only if $\mathrm{aut}^{ss}(E, \varphi) \subset \mathrm{H}^0(E(\mathfrak{z}))$. Furthermore, if (E, φ) is α-stable then we also have $\mathrm{aut}(E, \varphi) \subset \mathrm{H}^0(E(\mathfrak{z}))$.*

Proof. Suppose that (E, φ) is α-polystable and that $\mathrm{aut}^{ss}(E, \varphi) = \mathrm{H}^0(E(\mathfrak{z}))$. We prove that (E, φ) is α-stable by contradiction. If (E, φ) were not α-stable, then there would exist a parabolic subgroup $P_A \subsetneq H^{\mathbb{C}}$, a holomorphic reduction $\sigma \in \Gamma(E/P_A)$, a strictly antidominant character χ such that $\deg(E)(\sigma, \chi) - \langle \alpha, \chi \rangle = 0$, and a further holomorphic reduction $\sigma_L \in \Gamma(E_\sigma/L_A)$ to the Levi L_A (here E_σ is the principal P_A bundle given by σ, satisfying $E_\sigma \times_{P_A} H^{\mathbb{C}} \simeq E$) such that $\varphi \in \mathrm{H}^0(E(B)_{\sigma_L,\chi}^0 \otimes L)$. Since the adjoint action of L_A on $\mathfrak{h}^{\mathbb{C}}$ fixes s_χ, there is an element

$$s_{\sigma,\chi} \in \mathrm{H}^0(E_{\sigma_L}(\mathfrak{h}^{\mathbb{C}})) \simeq \mathrm{H}^0(E(\mathfrak{h}^{\mathbb{C}}))$$

which coincides fiberwise with s_χ. On the other hand s_χ is semisimple because it belongs to $i\mathfrak{h}$. The condition that $\varphi \in \mathrm{H}^0(E(B)_{\sigma_L,\chi}^0 \otimes L)$ implies that $\rho(s_{\sigma,\chi})(\varphi) = 0$, so $s_{\sigma,\chi} \in \mathrm{aut}^{ss}(E, \varphi)$. And the condition that $P_A \neq H^{\mathbb{C}}$ implies that $s_\chi \notin \mathfrak{z}$. This contradicts the assumption that $\mathrm{aut}^{ss}(E, \varphi) = \mathrm{H}^0(E(\mathfrak{z}))$, so (E, φ) is α-stable.

Now suppose that (E, φ) is α-stable. We want to prove that $\mathrm{aut}(E, \varphi) = \mathrm{H}^0(E(\mathfrak{z}))$. Let $\xi \in \mathrm{aut}(E, \varphi)$. Since ξ is a section of $E \times_{H^{\mathbb{C}}} \mathfrak{h}^{\mathbb{C}}$, it can be viewed as an antiequivariant holomorphic map $\psi : E \to \mathfrak{h}^{\mathbb{C}}$. The bundle E is algebraic (to prove this, take a faithful representation $H^{\mathbb{C}} \to \mathrm{GL}(n, \mathbb{C})$ and use the fact that any holomorphic vector bundle over an algebraic curve is algebraic), so by Chow's theorem ψ is algebraic. Hence ψ induces an algebraic map $\varphi : X \to \mathfrak{h}^{\mathbb{C}}/\!/H^{\mathbb{C}}$, where $\mathfrak{h}^{\mathbb{C}}/\!/H^{\mathbb{C}}$ denotes the affine quotient, which is

an affine variety. Since X is proper, φ is constant, hence it is contained in a unique fiber $Y := \pi^{-1}(y) \subset \mathfrak{h}^{\mathbb{C}}$, where $\pi : \mathfrak{h}^{\mathbb{C}} \to \mathfrak{h}^{\mathbb{C}} /\!/ H^{\mathbb{C}}$ is the quotient map.

By a standard results on affine quotients, there is a unique closed $H^{\mathbb{C}}$ orbit $\mathscr{O} \subset Y$, and by a theorem of Richardson the elements in $\mathscr{O}$ are all semisimple. Consider the map $\sigma : Y \to \mathscr{O}$ which sends any $y \in Y$ to y_s, where $y = y_s + y_n$ is the Jordan decomposition of y (see for example [4]). We claim that this map is algebraic (note that the Jordan decomposition, when defined on the whole Lie algebra $\mathfrak{h}^{\mathbb{C}}$, is not even continuous). To prove the claim first consider the case $\mathfrak{h}^{\mathbb{C}} = \mathfrak{gl}(n, \mathbb{C})$. Then $Y \subset \mathfrak{gl}(n, \mathbb{C})$ is the set of $n \times n$ matrices with characteristic polynomial equal to some fixed polynomial, say $\prod (x - \lambda_i)^{m_i}$, with $\lambda_i \neq \lambda_j$ for $i \neq j$. By the Chinese remainder theorem there exists a polynomial $P \in \mathbb{C}[t]$ such that $P \equiv \lambda_i \mod (t - \lambda_i)^{m_i}$ and $P \equiv 0 \mod t$. Then the map $\sigma : Y \to \mathscr{O}$ is given by $\sigma(A) = P(A)$, which is clearly algebraic. The case of a general $\mathfrak{h}^{\mathbb{C}}$ can be reduced to the previous one using the adjoint representation $\mathrm{ad} : \mathfrak{h}^{\mathbb{C}} \to \mathrm{End}(\mathfrak{h}^{\mathbb{C}}) \simeq \mathfrak{gl}(\dim \mathfrak{h}^{\mathbb{C}}, \mathbb{C})$.

By construction σ is equivariant, so it induces a projection $p_E : \mathrm{H}^0(E(Y)) \to \mathrm{H}^0(E(\mathscr{O}))$. We define $\xi_s = p_E(\xi)$ and $\xi_n = \xi - \xi_s$. Note that the decomposition $\xi = \xi_s + \xi_n$ is simply the fiberwise Jordan decomposition of an element of the Lie algebra as the sum of a semisimple element plus a nilpotent one. We claim that both ξ_s and ξ_n belong to $\mathrm{aut}(E, \phi)$. To prove this we have to check that $\rho(\xi_s)(\phi) = \rho(\xi_n)(\phi) = 0$. But $\rho(\xi) = \rho(\xi_s) + \rho(\xi_n)$ is fiberwise the Cartan decomposition of $\rho(\xi)$, since Cartan decomposition commutes with Lie algebra representations. In addition, if $f = f_s + f_n$ is the Cartan decomposition of an endomorphism f of a finite dimensional vector space V and $v \in V$ satisfies $fv = 0$, then $f_s v = f_n v = 0$, as the reader can check putting f in Jordan form. This proves the claim.

We want to prove that $\xi_s \in \mathrm{H}^0(E(\mathfrak{z}))$ and that $\xi_n = 0$. We will need for that the following lemma.

Lemma A.8. *Let $s \in \mathfrak{h}^{\mathbb{C}}$ be a semisimple element. There exists some $h \in H^{\mathbb{C}}$ such that:*

(1) *if we write $u = \mathrm{Ad}(h^{-1})(s) = h^{-1}sh = u_r + \mathbf{i}u_i$ with $u_r, u_i \in \mathfrak{h}$, then $[u_r, u_i] = 0$;*
(2) *there exists an element $a \in \mathfrak{h}$ such that*

$$\mathrm{Ker}\, \mathrm{ad}(s) = \mathrm{Ad}(h)(\mathrm{Ker}\, \mathrm{ad}(u_r) \cap \mathrm{Ker}\, \mathrm{ad}(u_i)) = \mathrm{Ad}(h)\, \mathrm{Ker}\, \mathrm{ad}(a).$$

Proof. Using the decomposition $\mathfrak{h}^{\mathbb{C}} = \mathfrak{h} \oplus \mathbf{i}\mathfrak{h}$ we define a real valued scalar product on $\mathfrak{h}^{\mathbb{C}}$ as follows: given $u_r + \mathbf{i}u_i, v_r + \mathbf{i}v_i \in \mathfrak{h}^{\mathbb{C}}$ we set $\langle u_r + \mathbf{i}u_i, v_r + \mathbf{i}v_i \rangle_{\mathbb{R}} := -\langle u_r, v_r \rangle - \langle u_i, v_i \rangle$. The bilinear pairing $\langle, \rangle$ restricted to $\mathfrak{h}$ is negative definite, so the pairing $\langle, \rangle_{\mathbb{R}}$ is positive definite on the whole $\mathfrak{h}^{\mathbb{C}}$ and hence the function $\| \cdot \|^2 : \mathfrak{h}^{\mathbb{C}} \to \mathbb{R}$ defined by $\|s\|^2 := \langle s, s \rangle_{\mathbb{R}}$ is proper. Let $\mathscr{O}_s$ be the adjoint orbit of s. Since s is semisimple, $\mathscr{O}_s$ is a closed subset of $\mathfrak{h}^{\mathbb{C}}$, and hence the function $\| \cdot \|^2 : \mathscr{O}_s \to \mathbb{R}$ attains its minimum at some point $u = u_r + \mathbf{i}u_i \in \mathscr{O}_s$. That u minimizes $\| \cdot \|^2$ on its adjoint orbit means that for any $v \in \mathfrak{h}^{\mathbb{C}}$ we have $\langle u, [v, u] \rangle_{\mathbb{R}} = 0$, since we can identify $T_u\mathscr{O}_s = \{[v, u] \mid v \in \mathfrak{h}^{\mathbb{C}}\}$. Now we develop for any $v = v_r + \mathbf{i}v_i$, using the biinvariance of $\langle, \rangle$ and Jacobi rule:

$$\begin{aligned}
0 &= \langle u_r + \mathbf{i}u_i, [u_r + \mathbf{i}u_i, v_r + \mathbf{i}v_i] \rangle_{\mathbb{R}} \\
&= \langle u_r + \mathbf{i}u_i, ([u_r, v_r] - [u_i, v_i]) + \mathbf{i}([u_i, v_r] + [u_r, v_i]) \rangle_{\mathbb{R}} \\
&= -\langle u_r, [u_r, v_r] - [u_i, v_i] \rangle - \langle u_i, [u_i, v_r] + [u_r, v_i] \rangle \\
&= \langle u_r, [u_i, v_i] \rangle - \langle u_i, [u_r, v_i] \rangle \\
&= -2\langle [u_i, u_r], v_i \rangle.
\end{aligned}$$

Since this holds for any choice of v, it follows that $[u_i, u_r] = 0$. So the endomorphisms $\mathrm{ad}(u_i)$ and $\mathrm{ad}(u_r)$ commute and hence diagonalize in the same basis with purely imaginary eigenvalues (because they respect the pairing $\langle \cdot, \cdot \rangle_{\mathbb{R}}$). Hence $\mathrm{Ker}\,\mathrm{ad}(u) = \mathrm{Ker}\,\mathrm{ad}(u_r + \mathbf{i}u_i) = \mathrm{Ker}(\mathrm{ad}(u_r) + \mathbf{i}\mathrm{ad}(u_i)) = \mathrm{Ker}\,\mathrm{ad}(u_r) \cap \mathrm{Ker}\,\mathrm{ad}(u_i)$. Since u_r and u_i commute, they generate a torus $T_u \subset H$. Take h such that $u = \mathrm{Ad}(h^{-1})(s)$ and choose $a \in \mathfrak{h}$ such that the closure of $\{e^{ta} \mid t \in \mathbb{R}\}$ is equal to T_u. Then $\mathrm{Ker}\,\mathrm{ad}(a) = \mathrm{Ker}\,\mathrm{ad}(u_r) \cap \mathrm{Ker}\,\mathrm{ad}(u_i)$, so the result follows. $\qquad\square$

We now prove that ξ_s is central. Let $u = u_r + \mathbf{i}u_i = h^{-1}y_s h$ be the element given by the previous lemma such that $[u_r, u_i] = 0$. Let $\psi_s : E \to \mathfrak{h}^{\mathbb{C}}$ be the antiequivariant map corresponding to $\xi_s \in \mathrm{H}^0(E(\mathfrak{h}^{\mathbb{C}}))$, whose image coincides with the adjoint orbit $\mathscr{O}_s$. Define $E_0 = \{e \in E \mid \psi_s(e) = u\} \subset E$. Then E_0 defines a reduction of the structure group of E to the centralizer of u, which we denote by $H_0^{\mathbb{C}} = \{g \in H^{\mathbb{C}} \mid \mathrm{Ad}(g)(u) = u\}$. Define the subgroups $P^{\pm} = \{g \in H^{\mathbb{C}} \mid e^{\pm \mathbf{i}tu_i} g e^{\mp \mathbf{i}tu_i} \text{ is bounded as } t \to \infty \} \subset H^{\mathbb{C}}$. By (3) in Lemma A.2, $P^{\pm}$ are parabolic subgroups and $L_{u_i} = P^+ \cap P^- = \{g \in H^{\mathbb{C}} \mid \mathrm{Ad}(g)(u_i) = u_i\}$ is a common Levi subgroup of P^+ and P^-. By (1) in Lemma A.2, $H_0^{\mathbb{C}}$ is a connected subgroup of $H^{\mathbb{C}}$, so by the same argument as in the end of the proof of Lemma A.8 we can identify $H_0^{\mathbb{C}}$ with $\{g \in H^{\mathbb{C}} \mid \mathrm{Ad}(g)(u_i) = u_i,\ \mathrm{Ad}(g)(u_r) = u_r\}$. This implies that $H_0^{\mathbb{C}} \subset L_{u_i}$, hence E_0 induces a reduction σ^+ (resp. σ^-) of the structure group of E to P^+ (resp. P^-). One the other hand, if χ corresponds to $\mathbf{i}u_i$ via the isomorphism $(\mathfrak{z} \oplus \mathfrak{c})^* \simeq \mathfrak{z} \oplus \mathfrak{c}$ (so that $s_{\chi} = \mathbf{i}u_i$), then χ is antidominant for P^+ and $-\chi$ is antidominant for P^-.

Let $\phi : E^L \to B$ be the antiequivariant map corresponding to φ. Since $\rho(\xi_s)(\varphi) = 0$ we have $\rho(u)\phi(e) = 0$ for any $e \in E_0$. Let $v \in B$ be any element. Since u_i and u_r commute, the vectors $\rho(e^{\mathbf{i}tu_i})v$ are uniformly bounded as $t \to \infty$ if and only if the vectors $\rho(e^{tu_r})\rho(e^{\mathbf{i}tu_i})v = \rho(e^{tu})v$ are bounded. It follows that φ belongs both to $\mathrm{H}^0(E(B)^-_{\sigma^+, \chi} \otimes L)$ and to $\mathrm{H}^0(E(B)^-_{\sigma^-, -\chi} \otimes L)$. Applying the α-stability condition we deduce that

$$\deg E(\sigma^+, \chi) - \langle \alpha, \chi \rangle \geq 0, \quad \text{and} \quad \deg E(\sigma^-, -\chi) - \langle \alpha, -\chi \rangle \geq 0.$$

These inequalities, together with $\deg E(\sigma^+, \chi) - \langle \alpha, \chi \rangle = -(\deg E(\sigma^-, -\chi) - \langle \alpha, -\chi \rangle)$, imply that $\deg E(\sigma, \chi) - \langle \alpha, \chi \rangle = 0$. Since we assume that (E, φ) is α-stable, such a thing can only happen if χ, and hence any element in the image of ψ_s, is central.

Finally, we prove that $\xi_n = 0$ proceeding by contradiction. Since the set of nilpotent elements $\mathfrak{h}_n^{\mathbb{C}} \subset \mathfrak{h}^{\mathbb{C}}$ contains finitely many adjoint orbits, which are locally closed in the Zariski topology, and since ξ_n is algebraic, there exists a Zariski open subset $U \subset X$ and an adjoint orbit $\mathscr{O}_n \subset \mathfrak{h}_n^{\mathbb{C}}$ such that $\xi_n(x) \in \mathscr{O}_n$ for any $x \in U$. Assume that $\xi_n(x) \neq 0$ for $x \in U$ (otherwise ξ_n vanishes identically). Consider for any $x \in U$ the weight filtration of the action of $\mathrm{ad}(\xi_n(x))$ on $E(\mathfrak{h}^{\mathbb{C}})_x$:

$$\cdots \subset W_x^{-k} \subset W_x^{-k+1} \subset \cdots \subset W_x^{k-1} \subset W_x^k \subset \dots,$$

which is uniquely defined by the conditions: $\mathrm{ad}(\xi_n(x))(W_x^j) \subset W_x^{j-2}$, $\mathrm{ad}(\xi_n(x))^{j+1}(W_x^j) = 0$ and the induced map on graded spaces $\mathrm{Gr}\,\mathrm{ad}(\xi_n(x))^j : \mathrm{Gr}\,W_x^j \to \mathrm{Gr}\,W_x^{-j}$ is an isomorphism. As x moves along U the spaces W_x^j give rise to an algebraic filtration of vector bundles $\cdots \subset W_U^{-k} \subset W_U^{-k+1} \subset \cdots \subset W_U^{k-1} \subset W_U^k \subset \cdots \subset E(\mathfrak{h}^{\mathbb{C}})|_U$. By the properness of the Grassmannian of subspaces of $\mathfrak{h}^{\mathbb{C}}$ these vector bundles extend to vector bundles defined on the whole X

$$(A.61) \qquad \cdots \subset W^{-k} \subset W^{-k+1} \subset \cdots \subset W^{k-1} \subset W^k \subset \cdots \subset E(\mathfrak{h}^{\mathbb{C}})$$

and the induced map between graded bundles $\operatorname{Gr}\operatorname{ad}(\xi_n)^j : \operatorname{Gr}W^j \to \operatorname{Gr}W^{-j}$ is an isomorphism away from finitely many points. This implies that

$$(A.62) \qquad\qquad\qquad \deg\operatorname{Gr}W^j \leq \deg\operatorname{Gr}W^{-j}.$$

By Jacobson–Morozov's theorem the weight filtration (A.61) induces a reduction σ of the structure group of E to a parabolic subgroup $P \subset H^{\mathbb{C}}$ (the so-called Jacobson–Morozov's parabolic subgroup associated to the nilpotent elements in the image of $\xi_n|_U$), and there exists an antidominant character χ of P such that $\operatorname{ad}(s_\chi)$ preserves the weight filtration and induces on the graded piece $\operatorname{Gr}W^j$ the map given by multiplication by j.

The subbundle $E(B)^-_{\sigma,\chi} \otimes L \subset E(B) \otimes L$ can be identified with the piece of degree 0 in the weight filtration on $E(B) \otimes L$ induced by the nilpotent endomorphism $\rho(\xi_n)$. Since $\rho(\xi_n)(\phi) = 0$, we have $\phi \in \operatorname{H}^0(E(B)^-_{\sigma,\chi} \otimes L)$ (the kernel of a nonzero nilpotent endomorphism is included in the piece of degree zero of the weight filtration). Hence, by α-stability, $\deg(E)(\sigma,\chi) - \langle \alpha, \chi \rangle$ has to be positive. On the other hand, the character χ can be chosen to be perpendicular to $\mathfrak{z}$, so by (3) in Lemma A.5 we have

$$\deg E(\sigma,\chi) - \langle \alpha, \chi \rangle = \sum_{j \in \mathbb{Z}} j \deg\operatorname{Gr}W^j.$$

By (A.62) this is ≤ 0, thus contradicting the stability of (E,φ). $\qquad\square$

A.8. **Jordan–Hölder reduction.** In this subsection we associate to each α-polystable pair (E,φ) an α-stable pair. This is accomplished by picking an appropriate subgroup $H' \subset H$ (defined as the centralizer of a torus in H) and by choosing a reduction of the structure group of E to $H'^{\mathbb{C}}$. The resulting new pair is called the Jordan–Hölder reduction of (E,φ). It is constructed using a recursive procedure in which certain choices are made, and the main result of this subsection (see Proposition A.12) is the proof that the resulting reduction is canonical up to isomorphism.

Let $G' \subset G$ be an inclusion of complex connected Lie subgroup with Lie algebras $\mathfrak{g}' \subset \mathfrak{g}$. Assume that the normalizer $N_G(\mathfrak{g}')$ of $\mathfrak{g}'$ in G is equal to G'. Suppose that E is a holomorphic principal G-bundle.

Lemma A.9. *The holomorphic reductions of the structure group of E to G' are in bijection with the holomorphic subbundles $F \subset E(\mathfrak{g})$ of Lie subalgebras satisfying this property:*

> *for any $x \in X$ and trivialization $E_x \simeq G$, the fiber F_x, which we identify to a subspace of $\mathfrak{g}$ via the induced trivialization $E(\mathfrak{g})_x \simeq \mathfrak{g}$, is conjugate to $\mathfrak{g}'$.*

Proof. Let $d = \dim\mathfrak{g}'$ and let $\operatorname{Gr}_d(\mathfrak{g})$ denote the Grassmannian of complex d-subspaces inside $\mathfrak{g}$. Let $\mathscr{O}_{\mathfrak{g}'} = \{\operatorname{Ad}(h)(\mathfrak{g}') \mid h \in G\} \subset \operatorname{Gr}_d(\mathfrak{g})$. By assumption there is a biholomorphism $\mathscr{O}_{\mathfrak{g}'} \simeq G/G'$. Furthermore, the set of vector bundles $F \subset E(\mathfrak{g})$ satisfying the condition of the lemma is in bijection with the holomorphic sections of $E(\mathscr{O}_{\mathfrak{g}'})$, so the result follows. $\quad\square$

We now apply this principle to a particular case. Let $P \subset H^{\mathbb{C}}$ be a parabolic subgroup, let $L \subset P$ be a Levi subgroup and let $U \subset P$ be the unipotent radical. Denote $\mathfrak{u} = \operatorname{Lie}U$, $\mathfrak{p} = \operatorname{Lie}P$ and $\mathfrak{l} = \operatorname{Lie}L$. The adjoint action of P on $\mathfrak{p}$ preserves $\mathfrak{u}$ and using the standard projection $P \to P/U \simeq L$ (see Section A.1 and recall that P is isomorphic to P_A for some choice of A) we make P act linearly on $\mathfrak{l}$ via the adjoint action. Hence P acts linearly on the exact sequence $0 \to \mathfrak{u} \to \mathfrak{p} \to \mathfrak{l} \to 0$. We claim that $N_P(\mathfrak{l}) = L$. To check this we identify P (up to conjugation) with some P_A, then use (A.54) and (A.55) together with

the surjectivity of the exponential map $\mathfrak{u}_A \to U_A$ to deduce that no nontrivial element of U normalizes $\mathfrak{l}$, and finally use the decomposition $P = LU$.

Lemma A.10. *Suppose that E_σ is a holomorphic principal P-bundle. The reductions of the structure group of E_σ from P to $L \subset P$ are in bijection with the splittings of the exact sequence of holomorphic vector bundles*

$$(A.63) \qquad\qquad 0 \to E_\sigma(\mathfrak{u}) \to E_\sigma(\mathfrak{p}) \to E_\sigma(\mathfrak{l}) \to 0$$

given by holomorphic maps $E_\sigma(\mathfrak{l}) \to E_\sigma(\mathfrak{p})$ which are fiberwise morphisms of Lie algebras.

Proof. Since $N_P(\mathfrak{l}) = L$, we may use Lemma A.9 with $G = P$ and $G' = L$. The subalgebras $\mathfrak{g}' \subset \mathfrak{p}$ which are conjugate to $\mathfrak{p}$ are the same as the images of sections $\mathfrak{l} \to \mathfrak{p}$ of the exact sequence $0 \to \mathfrak{u} \to \mathfrak{p} \to \mathfrak{l} \to 0$ which are morphisms of Lie algebras. Hence the vector subbundles $F \subset E(\mathfrak{p})$ satisfying the requirements of Lemma A.9 can be identified with the images of maps $E(\mathfrak{l}) \to E(\mathfrak{p})$ which give a section of the sequence (A.63) and which are fiberwise a morphism of Lie algebras. $\qquad\square$

Suppose that (E, φ) is a α-polystable pair which is not α-stable. By Proposition A.7 there exists a semisimple non central infinitesimal automorphism $s \in \mathrm{aut}^{ss}(E, \varphi)$. The splitting $\mathfrak{h}^{\mathbb{C}} = \mathfrak{z} \oplus \mathfrak{h}_s^{\mathbb{C}}$ (recall that $\mathfrak{h}_s^{\mathbb{C}} = [\mathfrak{h}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}]$ is the semisimple part) is invariant under the adjoint action of $H^{\mathbb{C}}$ (which is connected by assumption) hence we have $\mathrm{H}^0(E(\mathfrak{h}^{\mathbb{C}})) = \mathrm{H}^0(E(\mathfrak{z})) \oplus \mathrm{H}^0(E(\mathfrak{h}_s^{\mathbb{C}}))$ so projecting to the second summand we can assume that $s \in \mathrm{H}^0(E(\mathfrak{h}_s^{\mathbb{C}}))$.

As shown in the proof of Proposition A.7, the image of s is contained in an adjoint orbit in $\mathfrak{h}^{\mathbb{C}}$ which contains an element $u = u_r + \mathbf{i}u_i$ such that u_r, u_i are commuting elements of $\mathfrak{h}$. Let $a \in \mathfrak{h}_s = [\mathfrak{h}, \mathfrak{h}]$ be an infinitesimal generator of the torus generated by u_r and u_i and let $H_1^{\mathbb{C}}$ be the complexification of $H_1 := Z_H(a) = \{h \in H \mid \mathrm{Ad}(h)(a) = a\}$. Let $\psi_s : E \to \mathfrak{h}^{\mathbb{C}}$ be the antiequivariant map corresponding to the section s. Then

$$E_1 = \{e \in E \mid \psi_s(e) = u\} \subset E$$

is a $H_1^{\mathbb{C}}$-principal bundle, which defines a reduction of the structure group of E. We say that the pair $(E_1, H_1^{\mathbb{C}})$ is the **reduction of $(E, H^{\mathbb{C}})$ induced by s and u**.

Define $B_1 = \{v \in B \mid \rho(a)(v) = 0\}$. The restriction of ρ to H_1 preserves B_1, so we have a subbundle $E_1(B_1) \subset E_1(B) \simeq E(B)$. Let $\phi : E^L \to B$ be the antiequivariant map inducing the section $\varphi \in \mathrm{H}^0(E(B) \otimes L)$ (see Subsection A.4). By the definition of the infinitesimal automorphisms, for any $(e, l) \in E_1^L$ we have $\rho(\psi_s(e))\phi(e, l) = 0$. Now $\rho(\psi_s(e)) = \rho(u_r + \mathbf{i}u_i) = \rho(u_r) + \mathbf{i}\rho(u_i)$. Since ρ restricted to H is Hermitian, $\rho(u_r)$ and $\rho(u_i)$ have purely imaginary eigenvalues, and since $[\rho(u_r), \rho(u_i)] = 0$ it follows that

$$\rho(\psi_s(e))\phi(e, l) = 0 \quad\Longleftrightarrow\quad \rho(u_r)\phi(e, l) = \rho(u_i)\phi(e, l) = 0 \quad\Longleftrightarrow\quad \rho(a)\phi(e, l) = 0$$

for any $(e, l) \in E^L$. This implies that $\phi(E_1^L) \subset B_1$, and consequently φ lies in the subbundle $E_1(B_1) \otimes L \subset E(B) \otimes L$. To stress this fact we rename φ with the symbol φ_1. To sum up: assuming that (E, φ) is α-polystable but not α-stable we have obtained a subgroup $H_1 = Z_H(a) \subset H$, a H_1-invariant subspace $B_1 \subset B$, and a new pair (E_1, φ_1), where E_1 is a $H_1^{\mathbb{C}}$ principal bundle and $\varphi_1 \in \mathrm{H}^0(E_1(B_1) \otimes L)$. We denote the Lie algebras of H_1 and its complexification by $\mathfrak{h}_1$ and $\mathfrak{h}_1^{\mathbb{C}}$.

Proposition A.11. *The pair (E_1, B_1) is α-polystable.*

Proof. Since H_1 is the centralizer of a and α belongs to the center of $\mathfrak{h}^{\mathbb{C}}$, we have $\alpha \in \mathfrak{h}_1^{\mathbb{C}}$. Hence the statement of the proposition makes sense. We first prove that (E_1, B_1) is α-semistable. Let $P_1 \subset H_1^{\mathbb{C}}$ be a standard parabolic subgroup. By (2) in Lemma A.2 there is some $s \in i\mathfrak{h}_1$ (satisfying $s = s_\chi$ for an appropriate antidominant character χ of P_1) such that $P_1 = \{g \in H_1^{\mathbb{C}} \mid e^{ts}ge^{-ts}$ is bounded as $t \to \infty \}$. Since $i\mathfrak{h}_1 \subset i\mathfrak{h}$ it makes sense to define $P = \{g \in H^{\mathbb{C}} \mid e^{ts}ge^{-ts}$ is bounded as $t \to \infty \}$, which is a parabolic subgroup of $H^{\mathbb{C}}$, and clearly $P_1 \subset P$. Hence, any reduction σ_1 of the structure group of E_1 to P_1, say $(E_1)_{\sigma_1} \subset E_1$, gives automatically a reduction σ of the structure group of E to P, specified by $E_\sigma = (E_1)_{\sigma_1} \times_{P_1} P \subset (E_1)_{\sigma_1} \times_{P_1} H^{\mathbb{C}} = E$. Furthermore, any antidominant character $\chi \in i\mathfrak{h}$ of P_1 is an antidominant character of P, and there is an equality $\deg(E_1)(\sigma_1, \chi) = \deg(E)(\sigma, \chi)$. Finally, if the section φ_1 belongs to $\mathrm{H}^0(E_1(B_1)_{\sigma_1,\chi}^- \otimes L)$, then it also belongs to $\mathrm{H}^0(E(B)_{\sigma,\chi}^- \otimes L)$. All this implies that (E_1, φ_1) is α-semistable.

To prove that (E_1, φ_1) is α-polystable it remains to show that if the reduction σ_1 and χ have been chosen so that $\deg(E_1)(\sigma_1, \chi) - \langle \alpha, \chi \rangle = 0$, then there is a holomorphic reduction σ_{L_1} of the structure group of $(E_1)_{\sigma_1}$ to the Levi $L_1 = \{g \in H_1^{\mathbb{C}} \mid \mathrm{Ad}(g)(s) = s\}$ such that

$$(A.64) \qquad\qquad \varphi_1 \in \mathrm{H}^0(E(B_1)_{\sigma_{L_1},\chi}^0 \otimes L).$$

Define $L = \{g \in H^{\mathbb{C}} \mid \mathrm{Ad}(g)(s) = s\}$, which is a Levi subgroup of P, let $U_1 \subset P_1$ and $U \subset P$ be the unipotent radicals, and denote the corresponding Lie algebras by $\mathfrak{u}_1 = \mathrm{Lie}\, U_1$, $\mathfrak{p}_1 = \mathrm{Lie}\, P_1$, $\mathfrak{l}_1 = \mathrm{Lie}\, L_1$, $\mathfrak{u} = \mathrm{Lie}\, U$, $\mathfrak{p} = \mathrm{Lie}\, P$, $\mathfrak{l} = \mathrm{Lie}\, L$. By Lemma A.10 it suffices to check that there exists a bundle morphism $w_1 : (E_1)_{\sigma_1}(\mathfrak{l}_1) \to (E_1)_{\sigma_1}(\mathfrak{p}_1)$ given fiberwise by morphisms of Lie algebras, defining a splitting of the exact sequence

$$(A.65) \qquad\qquad 0 \to (E_1)_{\sigma_1}(\mathfrak{u}_1) \to (E_1)_{\sigma_1}(\mathfrak{p}_1) \to (E_1)_{\sigma_1}(\mathfrak{l}_1) \to 0.$$

Let $T \subset H$ be the closure of $\{e^{ta} \mid t \in \mathbb{R}\}$, which is a torus. Denote by $T^\vee = \mathrm{Hom}(T, S^1)$ the group of characters of T. We have decompositions

$$\mathfrak{u} = \bigoplus_{\eta \in T^\vee} \mathfrak{u}_\eta, \qquad \mathfrak{p} = \bigoplus_{\eta \in T^\vee} \mathfrak{p}_\eta, \qquad \mathfrak{l} = \bigoplus_{\eta \in T^\vee} \mathfrak{l}_\eta,$$

and since the elements of $H_1^{\mathbb{C}}$ fix a, the action of $H_1^{\mathbb{C}}$ on $\mathfrak{u}$, $\mathfrak{p}$ and $\mathfrak{l}$ respects the splittings above. It follows that we have a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \to & E_\sigma(\mathfrak{u}) & \to & E_\sigma(\mathfrak{p}) & \to & E_\sigma(\mathfrak{l}) & \to & 0 \\
& & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \\
0 & \to & (E_1)\sigma_1(\mathfrak{u}) & \to & (E_1)_{\sigma_1}(\mathfrak{p}) & \to & (E_1)_{\sigma_1}(\mathfrak{l}) & \to & 0 \\
& & \| & & \| & & \| & & \\
0 & \to & \bigoplus_{\eta \in T^\vee} (E_1)\sigma_1(\mathfrak{u}_\eta) & \to & \bigoplus_{\eta \in T^\vee} (E_1)_{\sigma_1}(\mathfrak{p}_\eta) & \to & \bigoplus_{\eta \in T^\vee} (E_1)_{\sigma_1}(\mathfrak{l}_\eta) & \to & 0
\end{array}
$$

Taking in the bottom row the summands corresponding to the trivial character $\eta = 0$ (the constant representation $T \to \{1\} \in S^1$) we get the exact sequence (A.65). By hypothesis the pair (E, φ) is α-polystable, so there is a section $v : E_\sigma(\mathfrak{l}) \to E_\sigma(\mathfrak{p})$ of the top row, given fiberwise by morphisms of Lie algebras. Using the isomorphisms and equalities in the diagram, this gives rise to a section

$$w : \bigoplus_{\eta \in T^\vee} (E_1)_{\sigma_1}(\mathfrak{l}_\eta) \to \bigoplus_{\eta \in T^\vee} (E_1)_{\sigma_1}(\mathfrak{p}_\eta)$$

of the bottom row. Then $w = (w_{\eta\mu})_{\eta,\mu \in T^\vee}$, where $w_{\eta\mu} : (E_1)_{\sigma_1}(\mathfrak{l}_\eta) \to (E_1)_{\sigma_1}(\mathfrak{p}_\mu)$, and one checks that $w_1 := w_{00}$ is fiberwise a morphism of Lie algebras and that it gives the desired splitting of the sequence (A.65). To check (A.64) we proceed as follows. First note that s_χ belongs both to the center of $\mathfrak{l}_1$ and $\mathfrak{l}$, hence it defines holomorphic sections $s_{\sigma_1,\chi} \in \mathrm{H}^0((E_1)_{\sigma_1}(\mathfrak{l}_1))$ and $s_{\sigma,\chi} \in \mathrm{H}^0(E_\sigma(\mathfrak{l}))$. Condition (A.64) is equivalent to

$$(\text{A.66}) \qquad\qquad\qquad \rho(w_1(s_{\sigma_1,\chi}))(\varphi) = 0$$

(note that $(E_1)_{\sigma_1}(\mathfrak{p}_1)$ is a subbundle of $(E_1)_{\sigma_1}(\mathfrak{h}_1^{\mathbb{C}}) \simeq E_1(\mathfrak{h}_1^{\mathbb{C}})$, hence it acts fiberwise on $E(B) \otimes L$). To prove this equality, we use again the hypothesis that (E, φ) is α-polystable, which implies that $\varphi \in \mathrm{H}^0(E(B)_{\sigma_L,\chi}^- \otimes L)$, where σ_L is the reduction specified by w. This is equivalent to $\rho(w(s_{\sigma,\chi}))(\varphi) = 0$, and this implies (A.66) because $s_\chi \in \mathfrak{l}_0 \subset \bigoplus_{\eta \in T^\vee} \mathfrak{l}_\eta$. $\square$

Let (E, φ) be a α-polystable pair. Iterating the procedure described in the previous subsection as many times as possible we obtain a sequence of groups $H = H_0 \supset H_1 \supset H_2 \supset \dots$ and elements $a_j \in (\mathfrak{h}_{j-1})_s = [\mathfrak{h}_{j-1}, \mathfrak{h}_{j-1}]$ such that $H_j = Z_{H_{j-1}}(a_j)$, vector subspaces $B = B_0 \supset B_1 \supset B_2 \supset \dots$, and α-polystable pairs $(E, \varphi) = (E_0, \varphi_0)$, $(E_1, \varphi_1), \dots$, where E_j is a $H_j^{\mathbb{C}}$-principal bundle over X and contained in E_{j-1}, and $\varphi_j \in \mathrm{H}^0(E_j(B_j) \otimes L)$. Since $\dim H_j < \dim H_{j-1}$, this process has to eventually stop at some pair, say (E_r, φ_r), which will necessarily be α-stable. We say that $(E_r, \varphi_r, H_r, B_r)$ is the **Jordan–Hölder** reduction of (E, φ, H, B). To justify this terminology we need to prove that the construction is independent of the choices made in the process. Note that the elements in the sequence $\{a_0, a_1, \dots, a_l\}$ all belong to the initial Lie algebra $\mathfrak{h}$ and they commute pairwise. Hence they generate a torus $T \subset H$, the closure of the set $\{\exp \sum t_j a_j \mid t_0, \dots, t_l \in \mathbb{R}\}$, and H_l is the centralizer in H of $T_{(E,\varphi)}$. With this in mind, the following proposition implies the uniqueness of the Jordan–Hölder reduction.

Let $H_s \subset H$ be the connected Lie subgroup whose Lie algebra is $\mathfrak{h}_s = [\mathfrak{h}, \mathfrak{h}]$.

Proposition A.12. *Let (E, φ) be a α-polystable pair. Suppose that $T', T'' \subset H_s$ are tori, and define H' (resp. H'') to be the centralizer in H of T' (resp. T''). Let B' (resp. B'') be the fixed point set of the action of T' (resp. T'') on B, and assume that there are reductions $E' \subset E$ (resp. $E'' \subset E$) of the structure group of E to $H'^{\mathbb{C}}$ (resp. $H''^{\mathbb{C}}$). Let $\phi : E^L \to B$ the equivariant map corresponding to φ. Assume that $\phi(E'^L) \subset B' \otimes L$ and $\phi(E''^L) \subset B'' \otimes L$. Denote by $\varphi' \in \mathrm{H}^0(E'(B') \otimes L)$ and $\varphi'' \in \mathrm{H}^0(E''(B'') \otimes L)$ the induced sections. Finally, suppose that both (E', φ') and (E'', φ'') are α-stable. Then there is some $g \in H^{\mathbb{C}}$ such that $H'^{\mathbb{C}} = g^{-1}(H''^{\mathbb{C}})g$, $E' = E''g$, $T'^{\mathbb{C}} = g^{-1}(T''^{\mathbb{C}})g$ and $B' = \rho(g^{-1})B''$.*

Before proving Proposition A.12 we state and prove two auxiliary lemmas.

Lemma A.13. *Let $u', u'' \in \mathfrak{h}$ and let $s', s'' \in \mathrm{H}^0(E(\mathfrak{h}^{\mathbb{C}}))$ be sections such that $s'(x)$ (resp. $s''(x)$) is conjugate to $\mathbf{i}u'$ (resp. $\mathbf{i}u''$) for any $x \in X$. Let $(E', H'^{\mathbb{C}})$ (resp. $(E'', H''^{\mathbb{C}})$) be the reductions of $(E, H^{\mathbb{C}})$ induced by s' and $\mathbf{i}u'$ (resp. s'' and $\mathbf{i}u''$).*

   (1) *Assume that $[s', s''] = 0$. Let $\mathfrak{h}''^{\mathbb{C}}$ be the Lie algebra of $H''^{\mathbb{C}}$. Then we can naturally identify s' with a section of $E''(\mathfrak{h}''^{\mathbb{C}})$.*
   (2) *Let $\mathfrak{z}''$ be the center of $\mathfrak{h}''^{\mathbb{C}}$. If $s' \in \mathrm{H}^0(E''(\mathfrak{z}''))$ then there is some $h \in H^{\mathbb{C}}$ such that $E'' \subset E'h$ as subsets of E.*

Proof. Let $\psi', \psi'' : E \to \mathfrak{h}^{\mathbb{C}}$ be the antiequivariant maps corresponding to s', s''. The condition $[s', s''] = 0$ implies that for any $e \in E$ the elements $\psi'(e), \psi''(e) \in \mathfrak{h}^{\mathbb{C}}$ commute. Since $E'' = (\psi'')^{-1}(\mathbf{i}u'')$, this implies that, for any $e \in E''$, $\psi'(e)$ commutes with $\mathbf{i}u''$, so

$\psi'(e)$ belongs to $\mathfrak{h}''^{\mathbb{C}}$. This proves (1). We now prove (2). First observe that, being a centralizer of a semisimple element in $\mathfrak{h}^{\mathbb{C}}$, $H''^{\mathbb{C}}$ is connected (see e.g. Theorem 13.2 in [3]). Hence, the adjoint action of $H''^{\mathbb{C}}$ on $\mathfrak{h}''^{\mathbb{C}}$ fixes any element in $\mathfrak{z}''$. Take some element $e \in E''$. By hypothesis, there is some $h \in H^{\mathbb{C}}$ such that $\psi'(e) = \mathrm{Ad}(h^{-1})(\mathbf{i}u')$, so $e \in E'h$. The condition $s' \in \mathrm{H}^0(E''(\mathfrak{z}''))$ implies that $\psi'(e) \in \mathfrak{z}''$ so, by the previous observation, for any $g \in H''^{\mathbb{C}}$ we have $\psi'(eg) = \mathrm{Ad}(g^{-1})\mathrm{Ad}(h^{-1})(\mathbf{i}u') = \mathrm{Ad}(h^{-1})(\mathbf{i}u')$, hence $eg \in E'h$. It follows that $E'' \subset E'h$. $\qquad\square$

For any $u \in \mathfrak{h}$ we denote by $T_u \subset H$ the torus generated by u, i.e., the closure of $\{\exp tu \mid t \in \mathbb{R}\}$, and $T_u^{\mathbb{C}}$ denotes the complexification of T_u.

Lemma A.14. *Let $u', u'' \in \mathfrak{h}_s = [\mathfrak{h}, \mathfrak{h}]$ and let $H'^{\mathbb{C}}$ (resp. $H''^{\mathbb{C}}$) be the complexification of the centralizer $Z_H(u')$ (resp. $Z_H(u'')$). If there is some $g \in H^{\mathbb{C}}$ such that $H'^{\mathbb{C}} = g^{-1}(H''^{\mathbb{C}})g$ then $T_{u'}^{\mathbb{C}} = g^{-1}T_{u''}^{\mathbb{C}}g$.*

Proof. The center of $\mathfrak{h}'^{\mathbb{C}}$ is $\mathfrak{z} \oplus \mathrm{Lie}\, T_{u'}^{\mathbb{C}}$, and the sum is direct because u' is assumed to belong to $\mathfrak{h}_s$. Similarly, the center of $\mathfrak{h}''^{\mathbb{C}}$ is $\mathfrak{z} \oplus \mathrm{Lie}\, T_{u''}^{\mathbb{C}}$. Since $H^{\mathbb{C}}$ is connected, its adjoint action on $\mathfrak{z}$ is trivial, and hence taking the center of the Lie algebra in each side of the equality $T_{u'}^{\mathbb{C}} = g^{-1}T_{u''}^{\mathbb{C}}g$ we deduce that $\mathrm{Lie}\, T_{u'}^{\mathbb{C}} = g^{-1}(\mathrm{Lie}\, T_{u''}^{\mathbb{C}})g$. This implies the equality between the complexified tori. $\qquad\square$

We now prove Proposition A.12.

Proof. Let $u', u'' \in \mathfrak{h}_s$ satisfy $T' = T_{u'}$ and $T'' = T_{u''}$. The existence of reductions of E to the centralizers of u' and u'' gives rise to sections $s', s'' \in \mathrm{aut}^{ss}(E, \varphi) \subset \mathrm{H}^0(E(\mathfrak{h}^{\mathbb{C}}))$ such that $s'(x)$ (resp. $s''(x)$) is conjugate to $\mathbf{i}s'$ (resp. $\mathbf{i}s''$) for any $x \in X$.

If $[s', s''] = 0$ then by (1) Lemma A.13 we can view $s' \in \mathrm{aut}^{ss}(E'', \varphi'')$ and $s'' \in \mathrm{aut}^{ss}(E', \varphi')$. Since by assumption (E'', φ'') and (E', φ') are α-stable, by Proposition A.7 we deduce that s' is central in the centralizer of s'' and vice-versa. By (2) in Lemma A.13 there exist $g, h \in H^{\mathbb{C}}$ such that $E' \subset E''g$ and $E'' \subset E'h$. This implies that $E' \subset E''g \subset E'hg$, but $E' \subset E'hg$ clearly implies that $E' = E'hg$, which combined with the previous chain of inclusions gives $E' = E''g$. It then follows that $H'^{\mathbb{C}} = g^{-1}(H''^{\mathbb{C}})g$. By Lemma A.14 we have $T_{s'}^{\mathbb{C}} = g^{-1}T_{s''}^{\mathbb{C}}g$. Finally, since the fixed point set of $T_{s'}^{\mathbb{C}}$ acting on B coincides with the fixed point set of $T_{s'}$ (and similarly for $T_{s''}^{\mathbb{C}}$) we have $B' = \rho(g^{-1})B''$.

Suppose now that $[s', s''] \neq 0$. There are holomorphic splittings

(A.67) $$E(\mathfrak{h}^{\mathbb{C}}) = E_1 \oplus \cdots \oplus E_p = F_1 \oplus \cdots \oplus F_q$$

such that $\mathrm{ad}(s')|_{E_j} = \lambda_j \, \mathrm{Id}_{E_j}$ and $\mathrm{ad}(s'')|_{F_k} = \mu_k \, \mathrm{Id}_{F_k}$, where the real numbers $\lambda_1 < \cdots < \lambda_p$ (resp. $\mu_1 < \cdots < \mu_q$) are the eigenvalues of $\mathrm{ad}(\mathbf{i}s')$ (resp. $\mathrm{ad}(\mathbf{i}s'')$). Define for any j the subbundles $F_{\leq j} = \bigoplus_{k \leq j} F_k \subset E(\mathfrak{h}^{\mathbb{C}})$ and $E_{\leq j} = \bigoplus_{k \leq j} E_k \subset E(\mathfrak{h}^{\mathbb{C}})$. Denote by $\pi_k : E(\mathfrak{h}^{\mathbb{C}}) \to E_k$ the projection using the decomposition (A.67). Let $\mathscr{E}_{\leq k}$ (resp. $\mathscr{E}_k$, $\mathscr{F}_{\leq j}$, $\mathscr{F}_j$) be the sheaf of local holomorphic sections of $E_{\leq k}$ (resp. E_k, $F_{\leq j}$, F_j). Define for any j the sheaf

$$\mathscr{F}_{\leq j}^{\sharp} = \bigoplus_{k=1}^{p} \pi_k(\mathscr{E}_{\leq k} \cap \mathscr{F}_{\leq j}).$$

This is a subsheaf of the sheaf associated to $E(\mathfrak{h}^{\mathbb{C}})$, and we denote by $F_{\leq j}^{\sharp} \subset E(\mathfrak{h}^{\mathbb{C}})$ the subbundle obtained by taking the saturation of $\mathscr{F}_{\leq j}^{\sharp}$.

By (1) in Lemma A.5 s'' induces a holomorphic reduction $\sigma'' \in \Gamma(E(H^{\mathbb{C}}/P))$ of the structure group of E to $P = P_{\mathbf{i}u''}$.

Lemma A.15. *The filtration $F^{\sharp}_{\leq 1} \subset \cdots \subset F^{\sharp}_{\leq q} = E(\mathfrak{h}^{\mathbb{C}})$ also induces a reduction $\sigma^{\sharp}$ of the structure group of E to P.*

Proof. For any $t \in \mathbb{R}$ there is a natural fiberwise action of $e^{ts'}$ on $E(H^{\mathbb{C}}/P)$, which allows to define $e^{ts'}\sigma'' \in \Gamma(E(H^{\mathbb{C}}/P))$. For the reader's convenience, we recall how this is defined. For any $x \in X$ we can identify $\sigma''(x)$ with an antiequivariant map $\xi_{\sigma''} : E_x \to H^{\mathbb{C}}/P$ (here $H^{\mathbb{C}}$ acts on the left of $H^{\mathbb{C}}/P$). Similarly, $s'(x)$ corresponds to a map $\psi : E_x \to \mathfrak{h}^{\mathbb{C}}$ which is antiequivariant and hence satisfies, for any $f \in E_x$ and $g \in H^{\mathbb{C}}$,

$$(A.68) \qquad e^{t\psi(fg)} = g^{-1}e^{t\psi(f)}g.$$

Then $e^{ts'}\sigma''(x)$ corresponds to the antiequivariant map $\xi_{e^{ts'}\sigma''} : E_x \to H^{\mathbb{C}}/P$ defined as

$$\xi_{e^{ts'}\sigma''}(f) = e^{t\psi(f)}\xi_{\sigma''}(f) = \xi_{\sigma''}(fe^{-t\psi(f)}).$$

That $\xi_{e^{ts'}\sigma''}$ is antiequivariant follows from (A.68). For each x the action of $e^{ts'(x)}$ defines on the fiber $E_x(H^{\mathbb{C}}/P)$ a decomposition in Zariski locally closed subvarieties $\{\mathscr{C}_{x,i}\}$, the Schubert cells. Each $\mathscr{C}_{x,i}$ corresponds to a connected component $C_{x,i} \subset E_x(H^{\mathbb{C}}/P)$ of the fixed point set of the action of $\{e^{ts'(x)} \mid t \in \mathbb{R}\}$ on $E_x(H^{\mathbb{C}}/P)$, and $\mathscr{C}_{x,i}$ is the set of $z \in E_x(H^{\mathbb{C}}/P)$ such that $e^{ts'(x)}z$ converges to $C_{x,i}$ as $t \to \infty$. Since s' is algebraic and, for any x, $s'(x)$ is conjugate to the same element $\mathbf{i}u'$, each $\mathscr{C}_i = \bigcup_{x \in X} \mathscr{C}_{x,i}$ is a Zariski locally closed subvariety of $E(H^{\mathbb{C}}/P)$. Since σ'' is an algebraic section of $E(H^{\mathbb{C}}/P)$, there is a Zariski open subset $U \subset X$ such that $\sigma''|_U$ is contained in a unique cell $\mathscr{C}_j \subset E(H^{\mathbb{C}}/P)$. Then for any $x \in U$ the limit $\sigma^{\sharp}_x := \lim_{t \to \infty} e^{ts'}\sigma''(x) \in C_{x,j} \subset \mathscr{C}_j$ is well defined, and the filtration $\{\mathscr{F}^{\sharp}_{\leq j,x}\}$ corresponds to $\sigma^{\sharp}_x$. As x moves along U the elements $\sigma^{\sharp}_x$ describe an algebraic section $\sigma^{\sharp}_U \in \Gamma(U; E(H^{\mathbb{C}}/P))$. Finally, $F^{\sharp}_{\leq j}$ results from extending the reduction $\sigma^{\sharp}_U$ to an algebraic section $\sigma^{\sharp} \in \Gamma(E(H^{\mathbb{C}}/P))$, which exists and is unique thanks to the properness of the flag variety $H^{\mathbb{C}}/P$. $\qquad\square$

Let χ be the antidominant character of P corresponding to u'', so that $s_\chi = \mathbf{i}u''$.

Lemma A.16. *We have $\varphi \in \mathrm{H}^0(E(B)^-_{\sigma^{\sharp},\chi} \otimes L)$.*

Proof. Let $U \subset X$ denote, as in the preceeding lemma, a nonempty Zariski open subset such that for any $x \in U$ we have $\sigma^{\sharp}(x) = \lim_{t \to \infty} e^{ts'}\sigma''(x)$. By continuity, it suffices to prove that for any $x \in U$

$$(A.69) \qquad \varphi(x) \in E(B)^-_{\sigma^{\sharp},\chi} \otimes L.$$

The vector $\varphi(x)$ corresponds to an antiequivariant map $\phi : E^L_x \to B$, whereas $\sigma^{\sharp}$ corresponds to an antiequivariant map $\xi_{\sigma^{\sharp}} : E_x \to H^{\mathbb{C}}/P$. Define $P^{\sharp}_x = \xi^{-1}_{\sigma^{\sharp}}(P) \subset E_x$. Then $P^{\sharp}_x$ is an orbit of the action of P on E_x on the right (which can also be obtained by identifying $E(H^{\mathbb{C}}/P)$ with the quotient E/P). And (A.69) is equivalent to requiring that $\phi(x)$ restricted to $(P^{\sharp}_x)^L$ is contained in B^-_χ. Define for any real t the map $\xi_{\sigma^t} : E_x \to H^{\mathbb{C}}/P$ as $\xi_{\sigma^t}(f) = \xi_{\sigma''}(fe^{-t\psi(f)})$, where $\psi : E_x \to \mathfrak{h}^{\mathbb{C}}$ is the antiequivariant map corresponding to s'. Let also P^t_x be $\xi^{-1}_{\sigma^t}(P)$. By the previous lemma, we have $\xi_{\sigma^{\sharp}} = \lim_{t \to \infty} \xi_{\sigma^t}$, so we have $P^{\sharp}_x = \lim_{t \to \infty} P^t_x$ as orbits of E_x/P. By continuity, it suffices to check that for any t the restriction of $\phi(x)$ to $(P^t_x)^L$ is contained in B^-_χ.

Since $s', s'' \in \mathrm{aut}(E, \varphi)$, we have

$$(A.70) \qquad\qquad \rho(e^{ts'})(\varphi) = \varphi$$

and we also have $\varphi \in \mathrm{H}^0(E(B)^-_{\sigma'', \chi} \otimes L)$. Defining $P''_x = \xi^{-1}_{\sigma''}(P)$ this implies that

$$(A.71) \qquad\qquad \phi(g, l) \in B^-_\chi \qquad \text{for any } g \in P''_x \text{ and } l \in L_x.$$

Assume that $f \in P^t_x$ and $l \in L_x$. Then $\xi_{\sigma^t}(f) = \xi_{\sigma''}(fe^{-t\psi(f)}) \in P$, so $fe^{-t\psi(f)} \in P''_x$. Hence

$$\phi(f, l) = \phi(fe^{-t\psi(f)}, l) \in B^-_\chi,$$

where the equality follows from (A.70) and the inclusion follows from (A.71). This proves that $\phi(x)$ maps $(P^t_x)^L$ inside B^-_χ, so we are done. $\qquad\square$

Hence we can apply the α-polystability condition, which in view of Lemma A.5 and Remark A.6 reads

$$(A.72) \qquad \deg(E)(\sigma^\sharp, \chi) = \mu_q \deg F^\sharp_{\leq q} + \sum_{j=1}^{q-1} (\mu_j - \mu_{j+1}) \deg F^\sharp_{\leq j} \geq 0$$

(the $\langle \alpha, \chi \rangle$ term vanishes because we assume that s'' is orthogonal to the center of $\mathfrak{h}$). On the other hand, since $s'' \in \mathrm{aut}^{ss}(E, \varphi)$, the same arguments as in the proof of Proposition A.7 imply that

$$(A.73) \qquad \deg(E)(\sigma'', \chi) = \mu_q \deg F_{\leq q} + \sum_{j=1}^{q-1} (\mu_j - \mu_{j+1}) \deg F_{\leq j} = 0.$$

An easy computation shows that $\deg \mathscr{F}^\sharp_{\leq j} = \deg F_{\leq j}$, whereas in general $\deg \mathscr{F}^\sharp_{\leq j} \leq \deg F^\sharp_{\leq j}$ with equality if and only if $\mathscr{F}^\sharp_{\leq j} = (\mathscr{F}^\sharp_{\leq j})^{\vee\vee}$, so that in general

$$\deg F_{\leq j} \leq \deg F^\sharp_{\leq j}.$$

Since $\deg F_{\leq q} = \deg \mathscr{F}^\sharp_{\leq q} = \deg F^\sharp_{\leq q}$ (because $\mathscr{F}_{\leq q}$ is equal to the sheaf associated to $E(\mathfrak{h}^\mathbb{C})$) and $\mu_j - \mu_{j+1} < 0$ for any $1 \leq j \leq q - 1$, we have

$$\deg(E)(\sigma'', \chi) \geq \deg(E)(\sigma^\sharp, \chi),$$

which combined (A.72) and (A.73) yields $\deg(E)(\sigma'', \chi) = \deg(E)(\sigma^\sharp, \chi) = 0$. By the previous comments, this equality implies $\mathscr{F}^\sharp_{\leq j} = (\mathscr{F}^\sharp_{\leq j})^{\vee\vee}$ for any j, so that $\mathscr{F}^\sharp_{\leq j}$ is the sheaf of local holomorphic sections of a subbundle $F^\sharp_{\leq j} \subset E(\mathfrak{h}^\mathbb{C})$. This has the following consequence: if we define $\mathscr{F}^\sharp_l = \bigoplus_k \pi_k(\mathscr{F}_l \cap \mathscr{E}_{\leq k})$, then $\mathscr{F}^\sharp_l$ is also the sheaf of sections of a subbundle $F^\sharp_l \subset E(\mathfrak{h}^\mathbb{C})$ and we have $F^\sharp_{\leq j} = \bigoplus_{l \leq j} F^\sharp_l$. In particular, we obtain a decomposition $E(\mathfrak{h}^\mathbb{C}) = \bigoplus_{l \leq q} F^\sharp_l$. Let $s^\sharp = \sum_j \mu_j \, \mathrm{Id}_{F^\sharp_j} \in \mathrm{H}^0(E(\mathfrak{h}^\mathbb{C}))$. Then we have $[s', s^\sharp] = 0$ and furthermore $s^\sharp \in \mathrm{aut}^{ss}(E, \varphi)$. These two properties imply that $s^\sharp \in \mathrm{aut}^{ss}(E', \varphi')$, so by Proposition A.7 $s^\sharp$ is central in the centralizer of s'. Similarly s' is central in the centralizer of $s^\sharp$, so we can proceed as in the first case and deduce the statement of the theorem with s'' replaced by $s^\sharp$. Reversing the roles of s' and s'' we obtain the theorem. $\qquad\square$

A.9. **Hitchin-Kobayashi correspondence.** Choose a Hermitian metric h_L, on the complex line bundle L, and denote by $F_L \in \Omega^2(X; \mathbf{i}\mathbb{R})$ the curvature of the corresponding Chern connection. Suppose that $E_h \subset E$ defines a reduction of the structure group of E from $H^{\mathbb{C}}$ to H. Then the vector bundle $E(B) = E \times_{H^{\mathbb{C}}} B$ can be canonically identified with $E_h \times_H B$, and hence inherits a Hermitian structure (obtained from the Hermitian structure on B, which is preserved by H). So for any $\varphi \in \mathrm{H}^0(E(B) \otimes L)$ it makes sense to define

$$\mu_h(\varphi) := \rho^* \left(-\frac{\mathbf{i}}{2} \varphi \otimes \varphi^{*h,h_L} \right).$$

Here we identify $\mathbf{i}\varphi \otimes \varphi^{*h,h_L}$ with a skew symmetric section of $\mathrm{End}(E(B) \otimes L)^* = \mathrm{End}(E(B))^*$, hence a section of $E_h(\mathfrak{u}(B))^*$. The map $\rho^* : E_h(\mathfrak{u}(B))^* \to E_h(\mathfrak{h})^*$ is induced by the dual of the infinitesimal action of $\mathfrak{h}$ on B. Using the isomorphism $\mathfrak{h}^* \simeq \mathfrak{h}$ given by the nondegenerate pairing $\langle \cdot, \cdot \rangle$ we view $\mu_h(\varphi)$ as a section of $E_h(\mathfrak{h})$.

Theorem A.17. . *Let (E, φ) be a α-polystable pair. There exists a reduction h of the structure group of E from $H^{\mathbb{C}}$ to H, given by a subbundle $E_h \subset E$, such that*

$$\text{(A.74)} \qquad\qquad \Lambda(F_h + F_L) + \mu_h(\varphi) = -\mathbf{i}\alpha,$$

where $F_h \in \Omega^2(X; E_h(\mathfrak{h}))$ denotes the curvature of the Chern connection on E with respect to h and $\Lambda : \Omega^2(X) \to \Omega^0(X)$ is the adjoint of wedging with the volume form on X. Furthermore, if (E, φ) is α-stable then h is unique. Conversely, if (E, φ) is a pair which admits a solution to equation (A.74), then (E, φ) is α-polystable.

Proof. Suppose first of all that (E, φ) is α-stable. Then by Proposition A.7 we have $\mathrm{aut}^{ss}(E, \varphi) = \mathrm{H}^0(E(\mathfrak{z}))$, so (E, φ) is simple in the sense of Definition 3.8 in [10]. Hence we can apply Theorem 4.1 of [10] to deduce the existence and uniqueness of h. (Recall that the notion of α-stability given in the present paper coincides with the one in [10] thanks to (3) in Lemma A.5.) If (E, φ) is α-polystable but not stable, then we consider the Jordan–Hölder reduction (E', φ', H', B') of (E, φ, H, B). Now the pair (E', φ') is simple and we can proceed as before to get a reduction h' of the structure group of E' from $H'^{\mathbb{C}}$ to H' satisfying (A.74). But h' also defines a reduction of the structure group of E from $H^{\mathbb{C}}$ to H, by defining $E_h := E_{h'} \times_{H'} H \subset E_{h'} \times_{H'} H^{\mathbb{C}} = E$. For this choice of h, equation (A.74) still holds.

The proof of the converse is standard. One first proves that if (E, φ) admits a solution to the equations then (E, φ) is α-semistable (see for example [10]). To prove α-polystability one can use the same strategy as in the Hitchin–Kobayashi correspondence for vector bundles. Namely, assume that $h \in E(H^{\mathbb{C}}/H)$ defines a reduction of the structure group to H, in such a way that equation (A.74) is satisfied. Assume also that $P \subset H^{\mathbb{C}}$ is a parabolic subgroup, that there is a holomorphic reduction σ of the structure group of E to P, an antidominant character χ of P such that φ is contained in $E(B)^-_{\sigma,\chi} \otimes L$ and such that

$$\text{(A.75)} \qquad\qquad \deg(E)(\sigma, \chi) - \langle \alpha, \chi \rangle = 0.$$

We want to prove that there is a further reduction σ_L of the structrue group of E from P to L and that φ is contained in $E(B)^0_{\sigma_L,\chi} \otimes L$.

Let $E_h \subset E$ be the principal H bundle specified by h. The reduction σ corresponds to an antiequivariant map $\xi : E \to H^{\mathbb{C}}/P$, so that $\xi(f)$ is a parabolic subgroup of $H^{\mathbb{C}}$ for each $f \in E$. Then, using the construction given in Lemma A.3 we define an H-antiequivariant

map $\psi : E_h \to \mathbf{i}\mathfrak{h}$ by setting $\psi(f) = s_{\xi(f),\chi}$ for any $f \in E_h$. The map ψ corresponds to a section of $E_h(\mathbf{i}\mathfrak{h})$, which we denote by

$$s_{h,\sigma,\chi} \in E_h(\mathbf{i}\mathfrak{h}).$$

For details on the following notions the reader can consult [38]. Let $\mathbb{E}$ be the C^∞ H-principal bundle underlying E_h, and let $\mathscr{A}$ be the set of connections on $\mathbb{E}$. Each element of $A \in \mathscr{A}$ defines a holomorphic structure $\overline{\partial}_A$ on $\mathbb{E}$. Let also $\mathscr{S}$ be the space of smooth sections of $\mathbb{E} \times_H B \otimes L$, and let $\mathscr{G}$ be the gauge group of E. The space $\mathscr{A} \times \mathscr{S}$ has a natural structure of infinite dimensional symplectic manifold, with respect to which the action of $\mathscr{G}$ is Hamiltonian and $(A, \phi) \mapsto \mu(A, \phi) := \Lambda(F_h + F_L) + \mu_h(\varphi) + \mathbf{i}\alpha$ can be identified with a moment map for this action (see Section 4 in [38]). Furthermore, $-\mathbf{i}s_{h,\sigma,\chi}$ can be identified with an element in the Lie algebra of the gauge group $\mathscr{G}$.

We will now apply the notions of maximal weight λ and the function λ_t (see Section 2.3 in [38]). Let $A \in \mathscr{A}$ be the element giving rise to the $\overline{\partial}$-operator which corresponds to the holomorphic structure E. A simple computation tells that (A.75) is equivalent to the maximal weight of $-\mathbf{i}s_{h,\sigma,\chi}$ on $(\overline{\partial}_A, \varphi)$ being zero:

$$\lambda((\overline{\partial}_A, \varphi), -\mathbf{i}s_{h,\sigma,\chi}) = \lim_{t \to \infty} \lambda_t((\overline{\partial}_A, \varphi), -\mathbf{i}s_{h,\sigma,\chi}) = 0.$$

Equation (A.74) is equivalent to the vanishing of the moment map of the action of $\mathscr{G}$ at the pair $(\overline{\partial}_A, \phi)$. Hence we have $\lambda_0((\overline{\partial}_A, \varphi), -\mathbf{i}s_{h,\sigma,\chi}) = 0$, and since $\lambda_t((\overline{\partial}_A, \varphi), -\mathbf{i}s_{h,\sigma,\chi})$ is nondecreasing as a function of t it follows that $\lambda_t((\overline{\partial}_A, \varphi), -\mathbf{i}s_{h,\sigma,\chi}) = 0$ for any t. This implies that $e^{ts_{h,\sigma,\chi}}$ fixes the pair $(\overline{\partial}_A, \varphi)$. That $\overline{\partial}_A$ is fixed implies that $s_{h,\sigma,\chi}$ induces a holomorphic reduction σ_L of the structure group of E to L, and that φ is fixed implies that φ is contained in $E(B)_{\sigma,\chi}^- \otimes L$. $\square$

A.10. **Automorphism groups of polystable pairs.** In this section we include a result which is required for the proof of Theorem 3.12. We also find it interesting by itself and think it might be of use in other context. Let (E, φ) be an L-twisted pair. Let $\mathrm{Aut}(E, \varphi)$ denote the holomorphic automorphisms of (E, φ), i.e., the holomorphic gauge transformations $g : E \to E$ such that $\phi \circ g^L = \phi$, where $\phi : E^L \to B$ is the antiequivariant map corresponding to φ and $g^L : E \times_X L \to E \times_X L$ is the transformation acting as g in the E factor and the identity in the L factor.

The group $\mathrm{Aut}(E, \varphi)$ carries a natural structure of Lie group with Lie algebra equal to $\mathrm{aut}(E, \phi)$.

Lemma A.18. *Let (E, φ) be an α-polystable pair. Then $\mathrm{Aut}(E, \varphi)$ is a reductive Lie group.*

Proof. If (E, φ) is α-polystable, then by Theorem A.17 there exists a reduction $h \in \Gamma(E(H^{\mathbb{C}}/H))$ of the structure group satisfying equation (A.74). By the arguments in the proof of Theorem A.17 this can be interpreted as the vanishing of the moment map of the action of $\mathscr{G}$ (the gauge group of E_h) on $\mathscr{A} \times \mathscr{S}$ at the point (A, φ), where A is the Chern connection of E and h. It follows (see for example Proposition 1.6 in [51]) that $\mathrm{Aut}(E, \phi)$ is the complexification of $\mathrm{Aut}(E, \phi) \cap \mathscr{G}$. Any $g \in \mathrm{Aut}(E, \phi) \cap \mathscr{G}$ preserves simultaneously the complex structure of E and the reduction h, hence it also preserves the Chern connection A. But the group of gauge transformations in $\mathscr{G}$ preserving a given connection can be identified with a closed subgroup of the automorphisms of the fiber of

E_h at any given point, and consequently is a compact Lie group. Hence $\mathrm{Aut}(E, \phi) \cap \mathscr{G}$ is a compact Lie group, so by the previous argument $\mathrm{Aut}(E, \phi)$ is reductive. $\qquad\square$

Appendix B. Twisted G-Higgs pairs and simplified stability

B.1. **Twisted G-Higgs pairs.** Let G be a real reductive Lie group, let $H \subset G$ be a maximal compact subgroup and let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be a Cartan decomposition, so that the Lie algebra structure on $\mathfrak{g}$ satisfies

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \qquad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \qquad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}.$$

The group H acts linearly on $\mathfrak{m}$ through the adjoint representation, and this action extends to a linear holomorphic action of $H^{\mathbb{C}}$ on $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m} \otimes \mathbb{C}$ (this is the isotropy representation). Furthermore, the Killing form on $\mathfrak{g}$ induces on $\mathfrak{m}^{\mathbb{C}}$ a Hermitian structure which is preserved by the action of H.

Let X be a compact Riemann surface and let L be a holomorphic line bundle on X. We define an L-twisted G-Higgs pair to be a pair (E, φ), where E is a holomorphic $H^{\mathbb{C}}$-principal bundle over X and φ is a holomorphic section of $E(\mathfrak{m}^{\mathbb{C}}) \otimes L$. Here $E(\mathfrak{m}^{\mathbb{C}})$ is the $\mathfrak{m}^{\mathbb{C}}$-bundle associated to E via the isotropy representation. Let $\mathfrak{z}$ be the centre of $\mathfrak{h}^{\mathbb{C}}$ and let $\alpha \in i\mathfrak{h} \cap \mathfrak{z}$. The notions of α-stability, semistability and polystability given in Section A.5 apply naturally to L-twisted G-Higgs pairs. A polystable L-twisted G-Higgs pair satisfies the following.

Proposition B.1. *Let (E, φ) be an L-twisted G-Higgs pair which is α-polystable but not α-stable. Then the Jordan–Hölder reduction of (E, φ) is an L-twisted G'-Higgs pair for some reductive subgroup $G' \subset G$.*

Proof. Recall from Section A.8 that in the Jordan–Hölder reduction $(E', \varphi', H', (\mathfrak{m}^{\mathbb{C}})')$ of $(E, \varphi, H, \mathfrak{m}^{\mathbb{C}})$ the subgroup $H' \subset H$ is defined as the centralizer of a torus $T \subset H$ and that $(\mathfrak{m}^{\mathbb{C}})'$ is the fixed point set of T acting on $\mathfrak{m}^{\mathbb{C}}$. So it suffices to prove that the Lie algebra structure on $\mathfrak{h} \oplus \mathfrak{m}$ induces a structure of Cartan pair on $(\mathfrak{h}', (\mathfrak{m}^{\mathbb{C}})' \cap \mathfrak{m})$. The action of T on $\mathfrak{h}$ and $\mathfrak{m}$ induces decompositions

$$\mathfrak{h} = \bigoplus_{\eta \in T^{\vee}} \mathfrak{h}_\eta \qquad \text{and} \qquad \mathfrak{m} = \bigoplus_{\eta \in T^{\vee}} \mathfrak{m}_\eta,$$

where $T^{\vee}$ denotes the group of characters of T (for which we use additive notation). Then one has, as usual,

$$[\mathfrak{h}_\eta, \mathfrak{h}_\mu] \subset \mathfrak{h}_{\eta+\mu}, \qquad [\mathfrak{h}_\eta, \mathfrak{m}_\mu] \subset \mathfrak{m}_{\eta+\mu}, \qquad [\mathfrak{m}_\eta, \mathfrak{m}_\mu] \subset \mathfrak{h}_{\eta+\mu}$$

for any pair of characters $\eta, \mu \in T^{\vee}$. Taking $\eta = \mu = 0$ and observing that $\mathfrak{h}' = \mathfrak{h}_0$ and $(\mathfrak{m}^{\mathbb{C}})' \cap \mathfrak{m} = \mathfrak{m}_0$, it follows that

$$[\mathfrak{h}', \mathfrak{h}'] \subset \mathfrak{h}', \qquad [\mathfrak{h}', (\mathfrak{m}^{\mathbb{C}})' \cap \mathfrak{m}] \subset (\mathfrak{m}^{\mathbb{C}})' \cap \mathfrak{m}, \qquad [(\mathfrak{m}^{\mathbb{C}})' \cap \mathfrak{m}, (\mathfrak{m}^{\mathbb{C}})' \cap \mathfrak{m}] \subset \mathfrak{h}',$$

so that $\mathfrak{h}', (\mathfrak{m}^{\mathbb{C}})' \cap \mathfrak{m})$ is certainly a Cartan pair.

We can make a more precise statement: defining G' as the centralizer of T inside G we have proved that the Jordan–Hölder reduction of (E, φ) is an L-twisted G'-Higgs bundle. $\qquad\square$

B.2. L-twisted $\mathrm{Sp}(2n, \mathbb{R})$-Higgs pairs. Let $G = \mathrm{Sp}(2n, \mathbb{R})$. The maximal compact subgroup of G is $H = \mathrm{U}(n)$ and hence $H^{\mathbb{C}} = \mathrm{GL}(n, \mathbb{C})$. Now, if $\mathbb{V} = \mathbb{C}^n$ is the fundamental representation of $\mathrm{GL}(n, \mathbb{C})$, then the isotropy representation space is:

$$\mathfrak{m}^{\mathbb{C}} = S^2 \mathbb{V} \oplus S^2 \mathbb{V}^*.$$

An L-twisted $\mathrm{Sp}(2n, \mathbb{R})$-Higgs pair is thus a pair consisting of a rank n holomorphic vector bundle V over X and a section

$$\varphi = (\beta, \gamma) \in H^0(L \otimes S^2 V \oplus L \otimes S^2 V^*).$$

Let α be a real number. Following Sections A.5 and A.6 (see also [10]), (V, φ) is said to be α-**semistable** if for any filtration by holomorphic subbundles

$$\mathscr{V} = (0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_k = V),$$

the following condition holds. For any sequence of real numbers $\lambda = (\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k)$ define the subbundle

$$N(\mathscr{V}, \lambda) = \sum_{\lambda_i + \lambda_j \leq 0} L \otimes V_i \otimes_S V_j \oplus \sum_{\lambda_i + \lambda_j \geq 0} L \otimes V_{i-1}^\perp \otimes_S V_{j-1}^\perp \subset L \otimes (S^2 V \oplus S^2 V^*),$$

where, if V', V'' are subbundles of V, $V' \otimes_S V''$ denotes the subbundle of $S^2 V$ induced by $V' \otimes V''$ under the projection $V \otimes V \to S^2 V$. (This is the same as the bundle $L \otimes E(B)_{\sigma, \chi}^-$ of Appendix A; we use the notation $N(\mathscr{V}, \lambda)$ for convenience.) Define also

$$d(\mathscr{V}, \lambda, \alpha) = \lambda_k(\deg V_k - \alpha n_k) + \sum_{j=1}^{k-1} (\lambda_j - \lambda_{j+1})(\deg V_j - \alpha n_j),$$

where $n_j = \mathrm{rk}\, V_j$ (this expression is equal to $\deg(E)(\sigma, \chi) - \langle \alpha, \chi \rangle$). Then, if $\varphi \in H^0(N(\mathscr{V}, \lambda))$, we must have

(B.76) $$d(\mathscr{V}, \lambda, \alpha) \geq 0.$$

The pair (V, φ) is α-**stable** if it is α-semistable and furthermore, for any choice of $\mathscr{V}$ and λ for which there is a $j < k$ such that $\lambda_j < \lambda_{j+1}$, whenever $\varphi \in H^0(N(\mathscr{V}, \lambda))$, we have

(B.77) $$d(\mathscr{V}, \lambda, \alpha) > 0.$$

It is well known that when $\varphi = 0$, the α-(semi)stability is equivalent to $\alpha = \mu(V)$ (where $\mu(V) = \deg V / \mathrm{rk}\, V$ is the slope of V) and V being (semi)stable. The next two theorems give a generalization of this fact for general φ, providing a much simpler (semi)stability condition for quadratic pairs. It is important to notice that in the statement of the theorems, the inclusions in the filtration of V are not necessarily strict, in contrast to the original definition. The proofs of these theorems will be given in Subsections B.4 and B.5.

Theorem B.2. *Let (V, φ) be an L-twisted $\mathrm{Sp}(2n, \mathbb{R})$-Higgs pair. The pair (V, φ) is α-semistable if and only if for any filtration of holomorphic subbundles $0 \subset V_1 \subset V_2 \subset V$ such that*

(B.78) $$\varphi = (\beta, \gamma) \in H^0(L \otimes ((S^2 V_2 + V_1 \otimes_S V) \oplus (S^2 V_1^\perp + V_2^\perp \otimes_S V^*)))$$

we have

(B.79) $$\deg V - \deg V_2 - \deg V_1 \geq \alpha(n - n_2 - n_1),$$

where $n = \mathrm{rk}\, V$.

Remark B.3. The statement of the Theorem also covers the case $\varphi = 0$, as we shall now explain. If $0 = V_1 = V_2$, then the condition (B.78) is equivalent to $\beta = 0$ and the inequality (B.79) reads $\deg V \geq \alpha n$. If $V_1 = V_2 = V$, then (B.78) is equivalent to $\gamma = 0$ and the inequality (B.79) says that $\deg V \leq \alpha n$. Consequently, if $\varphi = (\beta, \gamma) = 0$, then α-semistability implies $\alpha = \deg V / \operatorname{rk} V = \mu(V)$. In this case, taking $V_1 = 0$ and $V_2 \subset V$ any subbundle, the condition (B.79) is equivalent to $\mu(V_2) \leq \mu(V)$, so V is semistable. On the other hand one can check that if V is semistable and $\alpha = \mu(V)$, then the condition (B.79) is satisfied for any filtration $0 \subset V_1 \subset V_2 \subset V$.

Theorem B.4. *Let (V, φ) be an L-twisted $\operatorname{Sp}(2n, \mathbb{R})$-Higgs pair. The pair (V, φ) is α-stable if and only if the following condition is satisfied. For any filtration of holomorphic subbundles $0 \subset V_1 \subset V_2 \subset V$ such that*

$$\varphi \in H^0(L \otimes ((S^2 V_2 + V_1 \otimes_S V) \oplus (S^2 V_1^\perp + V_2^\perp \otimes_S V^*)))$$

the following holds: if at least one of the subbundles V_1 and V_2 is proper (that is, non-zero and different from V) then

$$\deg V - \deg V_2 - \deg V_1 > \alpha(n - n_2 - n_1),$$

(where $n = \operatorname{rk} V$), and in any other case

$$\deg V - \deg V_2 - \deg V_1 \geq \alpha(n - n_2 - n_1).$$

Remark B.5. Arguing as in Remark B.3 we deduce from the previous theorem that if $\varphi = 0$, then $(V, 0)$ is α-stable if and only if $\alpha = \deg V / \operatorname{rk} V$ and V is a stable vector bundle.

B.3. **Some results on convex sets.** Let W be an n dimensional vector space over $\mathbb{R}$. We denote the convex hull of any subset $S \subset W$ by $\operatorname{CH}(S) \subset W$. Let $h_1, h_2, \ldots, h_l$ be elements of the dual space W^*. We assume that $l \geq n$ and that the first n elements $h_1, \ldots, h_n$ are a basis of W^*. Define for any $h \in W^*$ the set

$$\{h \leq a\} = \{v \in W \mid h(v) \leq a\} \subset W,$$

and define $\{h = a\} \subset W$ similarly.

Consider the convex subset of W

$$C = \bigcap_i \{h_i \leq 0\}$$

(here and below if no range is specified for the index then it is supposed to be the whole set $\{1, \ldots, l\}$).

Remark B.6. The fact that $\{h_1, \ldots, h_l\}$ span W^* is equivalent to the condition that C does not contain any positive dimensional vector subspace of W. Indeed, if $h \in W^*$ and $Z \subset W$ is a subspace contained in $\{h \leq 0\}$, then Z is contained in $\{h = 0\}$. Consequently any vector subspace of W contained in C has to lie in $\bigcap_i \{h_i = 0\} = 0$.

Lemma B.7. $C = \operatorname{CH}(\partial C)$.

Proof. For any $\alpha \leq 0$ define $C_\alpha = C \cap \{h_1 + \cdots + h_n = \alpha\}$. Since for any $x \in C$ we have $h_i(x) \leq 0$ and furthermore $h_1, \ldots, h_n$ is a basis of W^*, we deduce that C_α is compact. Hence $C_\alpha = \operatorname{CH}(\partial C_\alpha)$. Now, take any $x \in C$ and set $\alpha = h_1(x) + \cdots + h_n(x)$. Then $x \in C_\alpha = \operatorname{CH}(\partial C_\alpha) \subset \operatorname{CH}(\partial C)$. This proves the inclusion $C \subset \operatorname{CH}(\partial C)$. The other inclusion follows from the fact that C is convex. $\square$

Now we have $\partial C = \bigcup_i C_i$, where $C_i = \{h_i = 0\} \cap C$. On the other hand, for any i the collection of elements $h_1, \ldots, h_l$ induce elements $h'_1, \ldots, h'_l$ on the dual of $\{h_i = 0\}$ which obviously span. Hence we may apply again the lemma to C_i and deduce that $C_i = \mathrm{CH}(\partial C_i)$. Proceeding recursively, we deduce that C is the convex hull of the union of the sets

$$C_I = \bigcap_{i \in I} \{h_i = 0\} \cap C$$

where I runs over the collection of subets of $\{1, \ldots, l\}$ satisfying

(B.80) $|I| = n - 1$ and the vectors $\{h_i \mid i \in I\}$ are linearly independent.

Each such subset C_I is a halfline.

Lemma B.8. *Fix a basis $e_1, \ldots, e_n$ of W, and denote by $e_1^*, \ldots, e_n^*$ the dual basis. Assume that any h_i can be written either as $e_a^* - e_b^*$ or $\pm(e_a^* + e_b^*)$ for some indices a, b depending on i. Then for any I satisfying (B.80) there are disjoint subsets $P, N \subset \{1, \ldots, n\}$ so that defining the element $c_I = \sum_{i \in P} e_i - \sum_{j \in N} e_j$ we have $C_I = \mathbb{R}_{\geq 0} c_I$.*

Proof. Pick some I satisfying (B.80), so that $C_I = \bigcap_{i \in I} \{h_i = 0\}$ is one dimensional, and let $c_I \in W$ be an element such that $C_I = \mathbb{R}_{\geq 0} c_I$. Write $c_I = \sum \lambda_j e_j$ and take some nonzero $\lambda \in \{\lambda_1, \ldots, \lambda_n\}$. Define $P_\lambda = \{j \mid \lambda_j = \lambda\}$ and $N_\lambda = \{j \mid \lambda_j = -\lambda\}$. We want to prove that for any $j \notin P_\lambda \cup N_\lambda$, $\lambda_j = 0$. Suppose the contrary. Then

$$c'_I = \sum_{j \in P_\lambda \cup N_\lambda} 2\lambda_j e_j + \sum_{j \notin P_\lambda \cup N_\lambda} \lambda_j e_j$$

does not belong to $\mathbb{R} c_I$. However, for any pair of indices a, b we clearly have

$$(e_a^* - e_b^*) c_I = 0 \implies (e_a^* - e_b^*) c'_I = 0 \qquad \text{and} \qquad (e_a^* + e_b^*) c_I = 0 \implies (e_a^* + e_b^*) c'_I = 0.$$

This implies by our assumption that $c'_I \in \bigcap_{i \in I} \{h_i = 0\} = C_I$, in contradiction with the fact that C_I is one dimensional. $\square$

B.4. **Proof of Theorem B.2.** As already mentioned, when $\varphi = 0$ the pair $(V, 0)$ is α-semistable if and only if $\alpha = \mu(V)$ and V is semistable. Thus, by Remark B.3, it suffices to consider the case $\varphi \neq 0$. Let $\mathscr{V}$ be any filtration of V, and define

$$\Lambda(\mathscr{V}, \varphi) = \{\lambda \in \mathbb{R}^k \mid \lambda_1 \leq \cdots \leq \lambda_k, \ \varphi \in N(\mathscr{V}, \lambda)\}.$$

The pair (V, φ) is α-semistable if for any $\lambda \in \Lambda(\mathscr{V}, \varphi)$ we have

$$d(\mathscr{V}, \lambda, \alpha) \geq 0.$$

But $d(\mathscr{V}, \lambda, \alpha)$ is clearly a linear function on λ, so to check stability it suffices to verify that $d(\mathscr{V}, \lambda, \alpha) \geq 0$ for any λ belonging to a set $\Lambda' \subset \mathbb{R}^k$ whose convex hull is $\Lambda(\mathscr{V}, \varphi)$. Define for any i, j the subbundles

$$D_{i,j} = V_i \otimes_S V_j + V_{i-1} \otimes_S V + V \otimes_S V_{j-1} \subset S^2 V$$

and

$$D_{i,j}^* = V_{i-1}^\perp \otimes_S V_{j-1}^\perp + V_i^\perp \otimes_S V^* + V^* \otimes_S V_j^\perp \subset S^2 V^*.$$

A tuple $\lambda_1 \leq \cdots \leq \lambda_k$ belongs to $\Lambda(\mathscr{V}, \varphi)$ if and only if these two conditions holds:

- for any i, j such that β is contained in $H^0(L \otimes D_{i,j})$ but is not contained in the sum $H^0(L \otimes D_{i-1,j}) + H^0(L \otimes D_{i,j-1})$, we have $\lambda_i + \lambda_j \leq 0$.
- for any i, j such that γ is contained in $H^0(L \otimes D_{i,j}^*)$ but is not contained in the sum $H^0(L \otimes D_{i+1,j}^*) + H^0(L \otimes D_{i,j+1}^*)$, we have $\lambda_i + \lambda_j \geq 0$.

Hence $\Lambda(\mathscr{V}, \varphi) \subset \mathbb{R}^k$ is the intersection of halfspaces of the form $\{\lambda_i - \lambda_{i+1} \leq 0\}$ and, $\{\lambda_a + \lambda_b \leq 0\}$ (for at least one pair (a, b), if $\beta \neq 0$) or $\{\lambda_c + \lambda_d \geq 0\}$ (for at least one pair (c, d), if $\gamma \neq 0$). Since the only nonzero vector subspace included in the set $\Lambda = \{\lambda_1 \leq \cdots \leq \lambda_k\}$ is the line generated by $(1, \ldots, 1)$ and the set $\Lambda(\mathscr{V}, \varphi)$ is contained and Λ and furthermore satisfies at least one equation of the form $\lambda_a + \lambda_b \geq 0$ or $\lambda_c + \lambda_d \leq 0$, it follows that $\Lambda(\mathscr{V}, \varphi)$ does not contain any nonzero vector subspace.

So by the arguments in the previous subsection $\Lambda(\mathscr{V}, \varphi)$ is the convex hull of a collection of half lines of the form $\mathbb{R}_{\geq 0} \lambda_I$, and by Lemma B.8 we can assume that the coordinates of λ_I are 0 and ± 1. But if $\lambda_I \in \Lambda(\mathscr{V}, \varphi)$ we necessarily must have $c_I = (-1, \ldots, -1, 0, \ldots, 0, 1, \ldots, 1)$, say a copies of -1, b of 0 and $k - (a + b)$ of 1. Consider first the case when $0 < a < a + b < k$. Define now the filtration

$$\mathscr{V}' = (0 \subsetneq V_a \subsetneq V_{a+b} \subsetneq V).$$

One can easily check that

$$d(\mathscr{V}, \lambda_I, \alpha) = d(\mathscr{V}', (-1, 0, 1), \alpha) = \deg V - \deg V_a - \deg V_{a+b} - \alpha(n - n_a - n_{a+b}),$$

and that $N(\mathscr{V}, \lambda) = L \otimes ((S^2 V_{a+b} + V_a \otimes_S V) \oplus (S^2 V_a^\perp + V_{a+b}^\perp \otimes_S V^*))$.

Next we need to consider the cases where one or more of the inequalities in the condition $0 < a < a + b < k$ becomes an equality, in which case some of the inclusions in $0 \subsetneq V_a \subsetneq V_{a+b} \subsetneq V$ will not be strict. Since in the semistability condition one has to consider strict inclusions, *a priori* we should consider separately each case (so for example, if $0 < a < a + b = k$, we consider the filtration $0 \subsetneq V_a \subsetneq V$ with weights $\lambda = (-1, 0)$, and so on). In the following table we list the possible degenerations (apart from the case $a = a + b = k = 0$, which is impossible since $k \geq 1$) and the corresponding form of the conditions $\varphi \in H^0(N(\mathscr{V}, \lambda))$ and $d(\mathscr{V}, \lambda, \alpha) \geq 0$.

Degeneration	$\varphi \in H^0(N(\mathscr{V}, \lambda))$	$d(\mathscr{V}, \lambda, c) \geq 0$
$0 = a < a + b = k$	always satisfied	always satisfied
$0 = a = a + b < k$	$\beta = 0$	$\deg V \geq \alpha n$
$0 < a = a + b = k$	$\gamma = 0$	$\deg V \leq \alpha n$
$0 < a < a + b = k$	$\gamma \in H^0(L \otimes S^2 V_a^\perp)$	$\deg V_a \leq \alpha n_a$
$0 < a = a + b < k$	$\varphi \in H^0(L \otimes (V_a \otimes V \oplus V_a^\perp \otimes V^*))$	$\deg V - 2 \deg V_a \geq \alpha(n - 2n_a)$
$0 < a < a + b < k$	$\beta \in H^0(L \otimes S^2 V_{a+b})$	$\deg V - \deg V_{a+b} \geq \alpha(n - n_{a+b})$

TABLE B.1. Semistability conditions for degenerate filtrations

Inspecting each of these cases in turn we see that they correspond to instances of the α-semistability condition stated in the Theorem with some inclusions not being strict. More precisely, in each case the subbundle $N(\mathscr{V}, \lambda)$ turns out to coincide with $L \otimes ((S^2 V_{a+b} + V_a \otimes_S V) \oplus (S^2 V_a^\perp + V_{a+b}^\perp \otimes_S V^*))$, and the degree $d(\mathscr{V}, \lambda, \alpha)$ is equal to $\deg V - \deg V_a - \deg V_{a+b} - \alpha(n - n_a - n_{a+b})$.

B.5. **Proof of Theorem B.4.** The proof is exactly like that of Theorem B.2, except that we have to distinguish the cases in which stability implies strict inequality. We assume that $\varphi \neq 0$. Following the notation of Subsection B.5, these are the cases in which λ contains at least two different values. If $\lambda_I = (-1, \ldots, -1, 0, \ldots, 0, 1, \ldots, 1)$ contains a copies of -1, b copies of 0 and $k - (a + b)$ copies of 1, admitting that some of the numbers a, b or $k - (a + b)$ is equal to 0, the condition that λ_I contains at least two different numbers is

equivalent to asking that at least one of the bundles V_a and V_{a+b} is a proper subbundle of V (this happens in the last three rows of Table B.1). Using the fact that $N(\mathcal{V}, c)$ is the positive span of vectors of the form λ_I (because $\varphi \neq 0$), the theorem follows.

B.6. Polystable $\mathrm{Sp}(2n, \mathbb{R})$**-Higgs pairs.** Let α be a real number. According to Sections A.5 and A.6, a twisted $\mathrm{Sp}(2n, \mathbb{R})$-Higgs pair (V, φ) with $\varphi = (\beta, \gamma) \in H^0(L \otimes S^2 V \oplus L \otimes S^2 V^*)$ is said to be α**-polystable** if it is semistable and for any filtration by holomorphic strict subbundles

$$\mathcal{V} = (0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_k = V),$$

and sequence of strictly increasing real numbers $\lambda = (\lambda_1 < \cdots < \lambda_k)$ such that $\varphi \in H^0(N(\mathcal{V}, \lambda))$ and $d(\mathcal{V}, \lambda, \alpha) = 0$ there is a splitting of vector bundles

$$V \simeq V_1 \oplus V_2/V_1 \oplus \cdots \oplus V_k/V_{k-1}$$

with respect to which

$$\beta \in H^0\Big(\bigoplus_{\lambda_i + \lambda_j = 0} L \otimes V_i/V_{i-1} \otimes_S V_j/V_{j-1} \Big)$$

and

$$\gamma \in H^0\Big(\bigoplus_{\lambda_i + \lambda_j = 0} L \otimes (V_i/V_{i-1})^* \otimes_S (V_j/V_{j-1})^* \Big).$$

This implies that if (V, φ) is α-polystable but not α-stable, then it can be decomposed as the sum of a polystable vector bundle, L-twisted $\mathrm{U}(p, q)$-Higgs pairs (arising from pairs $0 \neq \lambda_i = -\lambda_j$ with $i \neq j$), and lower rank twisted symplectic Higgs pairs (arising in case there is some $\lambda_i = 0$). Furthermore, by the results in Section A.8 each of these pieces is α-polystable, so the procedure can be repeated until one reaches a decomposition all of whose pieces are α-stable. Again by the results in Section A.8, such decomposition is unique up to isomorphism, and is in fact the Jordan-Hölder reduction of (V, φ).

B.7. L**-twisted** $\mathrm{Sp}(2n, \mathbb{C})$**-Higgs pairs.** Consider now the case $G = \mathrm{Sp}(2n, \mathbb{C})$. A maximal compact subgroup of G is $H = \mathrm{Sp}(2n)$ and hence $H^{\mathbb{C}}$ coincides with $\mathrm{Sp}(2n, \mathbb{C})$. Now, if $\mathbb{W} = \mathbb{C}^{2n}$ is the fundamental representation of $\mathrm{Sp}(2n, \mathbb{C})$ and ω denotes the standard symplectic form on $\mathbb{W}$, the isotropy representation space is

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{sp}(\mathbb{W}) = \mathfrak{sp}(\mathbb{W}, \omega) := \{\xi \in \mathrm{End}(\mathbb{W}) \mid \omega(\xi \cdot, \cdot) + \omega(\cdot, \xi \cdot) = 0\} \subset \mathrm{End}\,\mathbb{W},$$

so it coincides with the adjoint representation of $\mathrm{Sp}(2n, \mathbb{C})$ on its Lie algebra. An L-twisted $\mathrm{Sp}(2n, \mathbb{C})$-Higgs pair is thus a pair consisting of a rank $2n$ holomorphic symplectic vector bundle (W, Ω) over X (so Ω is a holomorphic section of $\Lambda^2 W^*$ whose restriction to each fiber of W is nondegenerate) and a section

$$\Phi \in H^0(L \otimes \mathfrak{sp}(W)),$$

where $\mathfrak{sp}(W)$ is the vector bundle whose fiber over x is given by $\mathfrak{sp}(W_x, \Omega_x)$.

Define for any filtration by holomorphic subbundles

$$\mathcal{W} = (0 = W_0 \subsetneq W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_k = W)$$

satisfying $W_{k-i} = W_i^{\perp_\Omega}$ for any i (here $\perp_\Omega$ denotes the perpendicular with respect to Ω) the set

$$\Lambda(\mathcal{W}) = \{(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{R}^k \mid \lambda_i \leq \lambda_{i+1} \text{ and } \lambda_{k-i+1} + \lambda_i = 0 \text{ for any } i \}.$$

For any $\lambda \in \Lambda(\mathcal{W})$ define the following subbundle of $L \otimes \operatorname{End} W$:

$$N(\mathcal{W}, \lambda) = L \otimes \mathfrak{sp}(W) \cap \sum_{\lambda_i \geq \lambda_j} L \otimes \operatorname{End}(W_i, W_j).$$

Define also

$$d(\mathcal{W}, \lambda) = \sum_{j=1}^{k-1} (\lambda_j - \lambda_{j+1}) \deg W_j$$

(note that since W carries a symplectic structure we have $W \simeq W^*$ and hence $\deg W = \deg W_k = 0$).

Following again Sections A.5 and Section A.6, the pair $((W, \Omega), \Phi)$ is said to be

- **semistable** if for any filtration $\mathcal{W}$ as above and any $\lambda \in \Lambda(\mathcal{W})$ such that $\Phi \in H^0(N(\mathcal{W}, \lambda))$, the following inequality holds: $d(\mathcal{W}, \lambda) \geq 0$.
- **stable** if it is semistable and furthermore, for any choice of filtration $\mathcal{W}$ and $\lambda \in \Lambda(\mathcal{W})$ which is not identically zero (so for which there is a $j < k$ such that $\lambda_j < \lambda_{j+1}$), and such that $\Phi \in H^0(N(\mathcal{W}, \lambda))$, we have $d(\mathcal{W}, \lambda) > 0$.
- **polystable** if it is semistable and for any filtration $\mathcal{W}$ as above and $\lambda \in \Lambda(\mathcal{W})$ satisfying $\lambda_i < \lambda_{i+1}$ for each i, $\psi \in H^0(N(\mathcal{W}, \lambda))$ and $d(\mathcal{W}, \lambda) = 0$, there is an isomorphism

$$W \simeq W_1 \oplus W_2/W_1 \oplus \cdots \oplus W_k/W_{k-1}$$

such that the pairing via Ω any element of the summand W_i/W_{i-1} with an element of the summand W_j/W_{j-1} is zero unless $i + j = k + 1$; furthermore, via the isomorphism above,

$$\Phi \in H^0(\bigoplus_i L \otimes \operatorname{Hom}(W_i/W_{i-1}, W_i/W_{i-1})).$$

We now prove an analog of Theorems B.2 and B.4, which implies that the definition of (semi)stability which we have given coincides with the usual one in the literature. Recall that if (W, Ω) is a symplectic vector bundle, a subbundle $W' \subset W$ is said to be isotropic if the restriction of Ω to W' is identically zero.

Theorem B.9. *An L-twisted $\operatorname{Sp}(2n, \mathbb{C})$-Higgs pair $((W, \Omega), \Phi)$ is semistable if and only if for any isotropic subbundle $W' \subset W$ such that $\Phi(W') \subset L \otimes W'$ we have $\deg W' \leq 0$. Furthermore, $((W, \Omega), \Phi)$ is stable if for any nonzero and strict isotropic subbundle $0 \neq W' \subset W$ such that $\Phi(W') \subset L \otimes W'$ we have $\deg W' < 0$. Finally, $((W, \Omega), \Phi)$ is polystable if it is semistable and for any nonzero and strict isotropic subbundle $W' \subset W$ such that $\Phi(W') \subset L \otimes W'$ and $\deg W' = 0$ there is another isotropic subbundle $W'' \subset W$ such that $\Phi(W'') \subset L \otimes W''$ and $W = W' \oplus W''$.*

Proof. The proof follows the same ideas as the proofs of Theorems B.2 and B.4, so we just give a sketch. Take an L-twisted $\operatorname{Sp}(2n, \mathbb{C})$-Higgs pair $((W, \Omega), \Phi)$, and assume that for any isotropic subbundle $W' \subset W$ such that $\Phi(W') \subset L \otimes W'$ we have $\deg W' \leq 0$. We want to prove that $((W, \Omega), \Phi)$ is semistable. Choose any filtration $\mathcal{W} = (0 \subsetneq W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_k = W)$ satisfying $W_{k-i} = W_i^{\perp_\Omega}$ for any i. We have to understand the geometry of the convex set

$$\Lambda(\mathcal{W}, \Phi) = \{\lambda \in \Lambda(\mathcal{W}) \mid \Phi \in N(\mathcal{W}, \lambda)\} \subset \mathbb{R}^k.$$

Define for that $\mathcal{J} = \{j \mid \Phi(W_j) \subset L \otimes W_j\} = \{j_1, \ldots, j_r\}$. One checks easily that if $\lambda = (\lambda_1, \ldots, \lambda_k) \in \Lambda(\mathcal{W})$ then

(B.81) $\lambda \in \Lambda(\mathcal{W}, \Phi) \Longleftrightarrow \lambda_a = \lambda_b$ for any $j_i \leq a \leq b \leq j_{i+1}$.

We claim that the set of indices $\mathcal{J}$ is symmetric:

(B.82) $j \in \mathcal{J} \Longleftrightarrow k - j \in \mathcal{J}$.

To check this it suffices to prove that $\Phi(W_j) \subset L \otimes W_j$ implies that $\Phi(W_j^{\perp\Omega}) \subset L \otimes W_j^{\perp\Omega}$. Suppose that this is not true, so that for some j we have $\Phi W_j \subset L \otimes W_j$ and there exists some $w \in W_j^{\perp\Omega}$ such that $\Phi w \notin L \otimes W_j^{\perp\Omega}$. Then there exists $v \in W_j$ such that $\Omega(v, \Phi w) \neq 0$. However, since $\Phi \in H^0(L \otimes \mathfrak{sp}(W))$, we must have $\Omega(v, \Phi w) = -\Omega(\Phi v, w)$, and the latter vanishes because by assumption Φv belongs to W_j. So we have reached a contradiction.

Let $\mathcal{J}' = \{j \in \mathcal{J} \mid 2j \leq k\}$ and define for any $j \in \mathcal{J}'$ the vector

$$L_j = -\sum_{c \leq j} e_c + \sum_{d \geq k-j+1} e_d,$$

where $e_1, \ldots, e_k$ is the canonical basis of $\mathbb{R}^k$. It follows from (B.81) and (B.82) that $\Lambda(\mathcal{W}, \Phi)$ is the positive span of the vectors $\{L_j \mid j \in \mathcal{J}'\}$. Consequently, we have

$$d(\mathcal{W}, \lambda) \geq 0 \text{ for any } \lambda \in \Lambda(\mathcal{W}, \Phi) \iff d(\mathcal{W}, L_j) \geq 0 \text{ for any } j.$$

One computes $d(\mathcal{W}, L_j) = -\deg W_{k-j} - \deg W_j$. On the other hand, since we have an exact sequence $0 \to W_{k-j} \to W^* \to W_j^* \to 0$ (the injective arrow is given by the pairing with Ω) we have $0 = \deg W^* = \deg W_{k-j} + \deg W_j^*$, so $\deg W_{k-j} = \deg W_j$ and consequently $d(\mathcal{W}, L_j) \geq 0$ is equivalent to $\deg W_j \leq 0$, which holds by assumption. Hence $((W, \Omega), \Phi)$ is semistable.

The converse statement, namely, that if $((W, \Omega), \Phi)$ is semistable then for any isotropic subbundle $W' \subset W$ such that $\Phi(W') \subset L \otimes W'$ we have $\deg W' \leq 0$ is immediate by applying the stability condition of the filtration $0 \subset W' \subset W'^{\perp\Omega} \subset W$.

Finally, the proof of the second statement on stability is very similar to case of semistability, so we omit it. The statement on polystability is also straightforward. $\square$

B.8. L-twisted $\mathrm{SL}(n, \mathbb{C})$-Higgs pairs.

If $G = \mathrm{SL}(n, \mathbb{C})$ then the maximal compact subgroup of G is $H = \mathrm{SU}(n)$ and hence $H^{\mathbb{C}}$ coincides with $\mathrm{SL}(n, \mathbb{C})$. Now, if $\mathbb{W} = \mathbb{C}^n$ is the fundamental representation of $\mathrm{SL}(n, \mathbb{C})$, the isotropy representation space is given by the traceless endomorphisms of $\mathbb{W}$

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{sl}(\mathbb{W}) = \{\xi \in \mathrm{End}(\mathbb{W}) \mid \mathrm{Tr}\, \xi = 0\} \subset \mathrm{End}\, \mathbb{W},$$

so it coincides again with the adjoint representation of $\mathrm{SL}(n, \mathbb{C})$ on its Lie algebra. An L-twisted $\mathrm{SL}(n, \mathbb{C})$-Higgs pair is thus a pair consisting of a rank n holomorphic vector bundle W over X endowed with a trivialization $\det W \simeq \mathcal{O}$ and a holomorphic section

$$\Phi \in H^0(L \otimes \mathrm{End}_0 W),$$

where $\mathrm{End}_0 W$ denotes the bundle of traceless endomorphisms of W.

Define for any filtration by holomorphic subbundles

$$\mathcal{W} = (0 = W_0 \subsetneq W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_k = W)$$

the convex set

$$\Lambda(\mathcal{W}) = \{(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{R}^k \mid \lambda_i \leq \lambda_{i+1} \text{ for any } i \text{ and } \sum_i \operatorname{rk} W_i(\lambda_i - \lambda_{i+1}) = 0\}.$$

For any $\lambda \in \Lambda(\mathcal{W})$ define the following subbundle of $L \otimes \operatorname{End} W$:

$$N(\mathcal{W}, \lambda) = L \otimes \operatorname{End}_0 W \cap \sum_{\lambda_i \geq \lambda_j} L \otimes \operatorname{End}(W_i, W_j).$$

Define also

$$d(\mathcal{W}, \lambda) = \sum_{j=1}^{k-1} (\lambda_j - \lambda_{j+1}) \deg W_j$$

(since $\det W$ is trivial we have $\deg W = \deg W_k = 0$).

Following again Sections A.5 and A.6, (W, Φ) is said to be:

- **semistable** if for any filtration $\mathcal{W}$ and $\lambda \in \Lambda(\mathcal{W})$ such that $\Phi \in H^0(N(\mathcal{W}, \lambda))$, we have $d(\mathcal{W}, \lambda) \geq 0$.
- **stable** if it is semistable and furthermore, for any choice of filtration $\mathcal{W}$ and $\lambda \in \Lambda(\mathcal{W})$ which is not identically zero (so for which there is a $j < k$ such that $\lambda_j < \lambda_{j+1}$), and such that $\Phi \in H^0(N(\mathcal{W}, \lambda))$, we have $d(\mathcal{W}, \lambda) > 0$.
- **polystable** if it is semistable and for any filtration $\mathcal{W}$ as above and $\lambda \in \Lambda(\mathcal{W})$ satisfying $\lambda_i < \lambda_{i+1}$ for each i, $\psi \in H^0(N(\mathcal{W}, \lambda))$ and $d(\mathcal{W}, \lambda) = 0$, there is an isomorphism

$$W \simeq W_1 \oplus W_2/W_1 \oplus \cdots \oplus W_k/W_{k-1}$$

with respect to which

$$\Phi \in H^0(\bigoplus_i L \otimes \operatorname{Hom}(W_i/W_{i-1}, W_i/W_{i-1})).$$

Again we have a result as Theorem B.9 implying that the present notions of (semi)stability coincide with the usual ones.

Theorem B.10. *An L-twisted $\operatorname{SL}(n, \mathbb{C})$-Higgs pair (W, Φ) is semistable if and only if for any subbundle $W' \subset W$ such that $\Phi(W') \subset L \otimes W'$ we have $\deg W' \leq 0$. Furthermore, (W, Φ) is stable if for any nonzero and strict subbundle $W' \subset W$ such that $\Phi(W') \subset L \otimes W'$ we have $\deg W' < 0$. Finally, (W, Φ) is polystable if it is semistable and for each subbundle $W' \subset W$ such that $\Phi(W') \subset L \otimes W'$ and $\deg W' = 0$ there is another subbundle $W'' \subset W$ satisfying $\Phi(W'') \subset L \otimes W''$ and $W = W' \oplus W''$.*

The proof of Theorem B.10 is very similar to that of Theorem B.9, so we omit it.

B.9. L-twisted $\operatorname{GL}(n, \mathbb{R})$-Higgs pairs. We study now L-twisted G-Higgs pairs for $G = \operatorname{GL}(n, \mathbb{R})$. When $L = K^2$, these will be related to maximal degree $\operatorname{Sp}(2n, \mathbb{R})$-Higgs bundles.

A maximal compact subgroup of $\operatorname{GL}(n, \mathbb{R})$ is $H = \operatorname{O}(n)$ and hence $H^{\mathbb{C}} = \operatorname{O}(n, \mathbb{C})$. Now, if $\mathbb{W}$ is the standard n-dimensional complex vector space representation of $\operatorname{O}(n, \mathbb{C})$, then the isotropy representation space is:

$$\mathfrak{m}^{\mathbb{C}} = S^2 \mathbb{W}.$$

An L-twisted $\mathrm{GL}(n,\mathbb{R})$-Higgs pair over X is thus a pair $((W,Q),\psi)$ consisting of a holomorphic $\mathrm{O}(n,\mathbb{C})$-bundle, i.e. a rank n holomorphic vector bundle W over X equipped with a non-degenerate quadratic form Q, and a section

$$\psi \in H^0(L \otimes S^2 W).$$

Note that when $\psi = 0$ a twisted $\mathrm{GL}(n,\mathbb{R})$-Higgs pair is simply an orthogonal bundle.

Since the centre of $\mathfrak{o}(n)$ is trivial, $\alpha = 0$ is the only possible value for which stability of an L-twisted $\mathrm{GL}(n,\mathbb{R})$-Higgs pair is defined. The stability condition is formulated as follows.

For any filtration of vector bundles

$$\mathcal{W} = (0 = W_0 \subsetneq W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_k = W)$$

of satisfying $W_j = W_{k-j}^{\perp_Q}$ (here $W_{k-j}^{\perp_Q}$ denotes the orthogonal complement of W_{k-j} with respect to Q) define

$$\Lambda(\mathcal{W}) = \{(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{R}^k \mid \lambda_i \leq \lambda_{i+1} \text{ and } \lambda_i + \lambda_{k-i+1} = 0 \text{ for any } i \}.$$

Define for any $\lambda \in \Lambda(\mathcal{W})$ the following bundle.

$$N(\mathcal{W}, \lambda) = \sum_{\lambda_i + \lambda_j \leq 0} L \otimes W_i \otimes_S W_j.$$

Also we define

$$d(\mathcal{W}, \lambda) = \sum_{j=1}^{k-1} (\lambda_j - \lambda_{j+1}) \deg W_j$$

(note that the quadratic form Q induces an isomorphism $W \simeq W^*$ so $\deg W = \deg W_k = 0$).

According to Sections A.5 and A.6, an L-twisted $\mathrm{GL}(n,\mathbb{R})$-Higgs pair (W, Q, ψ) is said to be

- **semistable** if for all filtrations $\mathcal{W}$ as above and all $\lambda \in \Lambda(\mathcal{W})$ such that $\psi \in H^0(N(\mathcal{W}, \lambda))$, we have $d(\mathcal{W}, \lambda) \geq 0$,
- **stable** if it is semistable and for any choice of filtration $\mathcal{W}$ and nonzero $\lambda \in \Lambda(\mathcal{W})$ such that $\psi \in H^0(N(\mathcal{W}, \lambda))$, we have $d(\mathcal{W}, \lambda) > 0$,
- **polystable** if it is semistable and for any filtration $\mathcal{W}$ as above and $\lambda \in \Lambda(\mathcal{W})$ satisfying $\lambda_i < \lambda_{i+1}$ for each i, $\psi \in H^0(N(\mathcal{W}, \lambda))$ and $d(\mathcal{W}, \lambda) = 0$, there is an isomorphism

$$W \simeq W_1 \oplus W_2/W_1 \oplus \cdots \oplus W_k/W_{k-1}$$

such that pairing via Q any element of the summand W_i/W_{i-1} with an element of the summand W_j/W_{j-1} is zero unless $i + j = k + 1$; furthermore, via this isomorphism,

$$\psi \in H^0(\bigoplus_{\lambda_i + \lambda_j = 0} L \otimes (W_i/W_{i-1}) \otimes_S (W_j/W_{j-1})).$$

There is a simplification of the stability condition for orthogonal pairs analogous to Theorem B.2 and Theorem B.4.

Theorem B.11. *The L-twisted $\mathrm{GL}(n,\mathbb{R})$-Higgs pair $((W,Q),\psi)$ is semistable if and only if for any isotropic subbundle $W' \subset W$ such that $\psi \in H^0(S^2W'^{\perp Q} \oplus W' \otimes_S W \otimes L)$ the inequality $\deg W' \leq 0$ holds. Furthermore, $((W,Q),\psi)$ is stable if it is semistable and for any isotropic strict subbundle $0 \neq W' \subset W$ such that $\psi \in H^0(S^2W'^{\perp Q} \oplus W' \otimes_S W \otimes L)$ we have $\deg W' < 0$ holds. Finally, $((W,Q),\psi)$ is polystable if it is semistable and for any isotropic strict subbundle $0 \neq W' \subset W$ such that $\psi \in H^0(S^2W'^{\perp Q} \oplus W' \otimes_S W \otimes L)$ and $\deg W' = 0$ there is another isotropic subbundle $W'' \subset W$ such that $\psi \in H^0(S^2W''^{\perp Q} \oplus W'' \otimes_S W \otimes L)$ and $W = W' \oplus W''$.*

Proof. The proof is analogous to the proofs of Theorems B.2 and B.4. Take an L-twisted $\mathrm{GL}(n,\mathbb{R})$-Higgs pair $((W,Q),\psi)$, and assume that for any isotropic subbundle $W' \subset W$ such that $\psi \in H^0(S^2W'^{\perp Q} \oplus W' \otimes_S W \otimes L)$ the inequality $\deg W' \leq 0$ holds. We also assume that ψ is nonzero, for otherwise the result follows from the usual characterization of (semi)stability for $\mathrm{SO}(n,\mathbb{C})$-principal bundles due to Ramanathan (see [41]). We want to prove that $((W,Q),\psi)$ is semistable. Choose any filtration $\mathcal{W} = (0 \subsetneq W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_k = W)$ satisfying $W_{k-i} = W_i^{\perp\Omega}$ for any i. Consider the convex set

$$\Lambda(\mathcal{W},\psi) = \{\lambda \in \Lambda(\mathcal{W}) \mid \psi \in N(\mathcal{W},\lambda)\} \subset \mathbb{R}^k.$$

Define for any i,j the subbundle

$$D_{i,j} = W_i \otimes_S W_j + W_{i-1} \otimes_S W + W \otimes_S W_{j-1} \subset S^2W.$$

A tuple $\lambda = (\lambda_1, \ldots, \lambda_k) \in \Lambda(\mathcal{W})$ belongs to $\Lambda(\mathcal{W},\psi)$ if and only if:

> for any i,j such that ψ is contained in $H^0(L \otimes D_{i,j})$ but is not contained in the sum $H^0(L \otimes D_{i-1,j}) + H^0(L \otimes D_{i,j-1})$, we have $\lambda_i + \lambda_j \leq 0$.

Hence $\Lambda(\mathcal{W},\psi)$ is the intersection of $\Lambda(\mathcal{W})$ with the set of points in $\mathbb{R}^k$ satisfying a collection of inequalities of the form $\lambda_a + \lambda_b \leq 0$ and $\lambda_c + \lambda_d \geq 0$ (the latter follow from the restrictions $\lambda_i + \lambda_{k-i+1} = 0$). Since $\Lambda(\mathcal{W})$ does not contain any line, a fortiori $\Lambda(\mathcal{W},\psi)$ neither does, so (using Lemma B.8) $\Lambda(\mathcal{W},\psi)$ is the convex hull of a set of half lines $\{\mathbb{R}_{\geq 0}L_i \mid i \in \mathcal{I}\}$, where $L_i = (-1, \ldots, -1, 0, \ldots, 0, 1, \ldots, 1)$ contains i copies of -1 and i copies of 1. Consequently, we have

$$d(\mathcal{W},\lambda) \geq 0 \text{ for any } \lambda \in \Lambda(\mathcal{W},\psi) \iff d(\mathcal{W},L_i) \geq 0 \text{ for any } i \in \mathcal{I} .$$

It follows from the definition that $N(\mathcal{W},L_i) = W_i \otimes_S W + S^2W_{k-i}$ and since $W_{k-i} = W_i^{\perp Q}$ the condition $L_i \in \Lambda(\mathcal{W},\psi)$ can be translated into the condition

$$\psi \in H^0(S^2W_i^{\perp Q} \oplus W_i \otimes_S W \otimes L).$$

One computes $d(\mathcal{W},L_i) = -\deg W_{k-i} - \deg W_i$. On the other hand, since we have an exact sequence $0 \to W_{k-i} \to W^* \to W_i* \to 0$ (the injective arrow is given by the pairing with the quadratic form Q) we have $0 = \deg W^* = \deg W_{k-i} + \deg W_i^*$, so $\deg W_{k-i} = \deg W_i$ and consequently $d(\mathcal{W},L_i) \geq 0$ is equivalent to $\deg W_i \leq 0$, which holds by assumption. Hence $((W,Q),\psi)$ is semistable.

The converse statement, namely, that if $((W,Q),\psi)$ is semistable then for any isotropic subbundle $W' \subset W$ such that $\Phi(W') \subset L \otimes W'$ we have $\deg W' \leq 0$ is immediate by applying the stability condition of the filtration $0 \subset W' \subset W'^{\perp Q} \subset W$.

Finally, the proof of the second statement on stability is very similar to the case of semistability, so we omit it. The statement on polystability is also straightforward. $\square$

Remark B.12. The condition $\psi \in H^0(S^2 W_1^{\perp_Q} \oplus W_1 \otimes_S W \otimes L)$ is equivalent to $\tilde{\psi}(W_1) \subseteq W_1 \otimes L$, where $\tilde{\psi} = \psi \circ Q \colon W \to W \otimes L$. The reasoning is analogous to the proof of Corollary 4.2.

<div align="center">References</div>

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