SONINE PAIRS, DISCRETE CONVOLUTIONS AND DISCRETE MITTAG-LEFFLER FUNCTIONS

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Abstract. We investigate discrete Sonine pairs associated with Volterra-type convolution operators. We prove that a kernel admits a unique Sonine partner if and only if its initial value is nonzero, and derive explicit representations of the corresponding partner through recursive convolution expansions. In the second part of the paper, we study the discrete Mittag-Leffler function arising as the solution of a linear Caputo-type fractional difference equation. We establish its monotonicity with respect to the fractional order for the full range of the parameter for which such monotonicity holds. We then address an open problem concerning the sign pattern of the solution in the case of negative parameters below the stability threshold. In particular, we prove that, when the parameter is less than minus one, the solution either oscillates for all indices or is eventually negative, depending on a sharp threshold which in turn depends on the fractional order. Finally, we determine the asymptotic behaviour of the discrete Mittag-Leffler function in all parameter regimes.

1. Introduction

The study of fractional difference equations is a matter of the XXI century. The books [4, 10] provide a good account of the results developed so far within the theory. Nevertheless, the theory continues to evolve, both in the classical fractional-difference setting [2, 8, 9, 12, 13]) and in broader frameworks involving nonlocal and Volterra-type discrete operators. Recently the authors initiated the theory of discrete convolution equations using the notion of Sonine kernels [5].

Sonine pairs play an important role in fractional calculus and in the theory of Volterra-type convolution equations. In the continuous setting, a pair of kernels is called a Sonine pair when their convolution is equal to the constant function one. Such identities are closely related to the inversion of fractional integral operators and to the construction of fractional derivatives.

The study of discrete Sonine pairs and fractional difference equations (or more generally, Volterra difference equations [3, Chapter 6]) was initiated and fairly developed in [5, 6].

For the sake of clarity, we now introduce the concepts nedeed in the sequel. We work on the lattice

$$\mathbb{N}_a = \{a, a+1, a+2, \ldots\}, \quad a \in \mathbb{R}, \quad \mathbb{N}_1 = \mathbb{N}.$$

Given two functions $f, g : \mathbb{N}_a \to \mathbb{R}$, we consider the discrete convolution

$$(f * g)_a(t) = \sum_{\tau=a}^{t-1} f(t - \tau - 1 + a)g(\tau), \qquad t \in \mathbb{N}_a.$$

Throughout this text we assume that empty sums are equal to zero, i.e., $\sum_{k=a}^{a-1} f(k) = 0$, for every $a \in \mathbb{R}$ and any function f. Therefore, $(f * g)_a(a) = 0$ for all functions f and g. It is well known that the convolution is commutative and associative.

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We say that f and g form a discrete Sonine pair if

$$(f*g)_a(t)=1, \qquad t\in\mathbb{N}_{a+1}.$$

This definition is natural from the point of view of discrete Volterra equations: the problem of finding a Sonine partner of a given kernel is precisely one of the matters solved in this work. Indeed, we prove that a function $f:\mathbb{N}_a\to\mathbb{R}$ admits a unique Sonine partner if and only if

$$f(a)\neq 0.$$

The proof is elementary and relies on the recursive structure of the convolution. We then obtain an explicit representation of the Sonine partner in terms of iterated discrete convolutions. Although this formula is fully explicit, it may be difficult to evaluate in concrete examples. For this reason, we also emphasize the use of generating functions, which provide a simple and effective way of constructing Sonine pairs.

In the second part of the paper we consider the fractional difference equation

$${}^{C}\nabla_0^{\alpha}u(n)=-\lambda u(n), \quad \lambda\neq -1,\ n\in\mathbb{N}_1,\ 0<\alpha<1,$$

with initial condition $u(0)=1$, whose solution $u_{\alpha,\lambda}(n)$ is called *discrete Mittag-Leffler function* (here, ${}^{C}\nabla_0^{\alpha}$ denotes the nabla Caputo fractional difference, see [10]). In [6] the author showed that, if $\lambda>0$ and $0<\alpha<1$, then $u_{\alpha,\lambda}(n)$ is completely monotonic, i.e., $(-1)^m\Delta^m u_{\alpha,\lambda}(n)\geq 0$ for all $m,n\in\mathbb{N}_0$, where Δ^m is the mth order forward difference operator (cf. [4]). Motivated by the recent results in [7] concerning the classical Mittag-Leffler function, we investigate here the monotonicity of the discrete Mittag-Leffler function as a function of the fractional parameter α. We characterize such monotonicity for the full range of λ and n.

Another relevant part of this work consists in characterizing the geometry of $u_{\alpha,\lambda}$ when $\lambda<-1$. This issue was left as an open problem in [6] and is herein solved by using complex analysis methods. Finally, as a consequence of the results proved to solve the mentioned open problem, we revisit and characterize the asymptotic behaviour of $u_{\alpha,\lambda}(n)$ as $n\to\infty$.

The paper is organized as follows. In Section 2.1 we delve into discussing Sonine pairs, specifically proving an existence and uniqueness theorem, and derive explicit examples using both convolution expansions and generating functions. In Section 2.2 we study the monotonicity of the discrete Mittag-Leffler function with respect to the parameter α. Some bounds for the solution of the aformentioned initial value problem are made explicit. In Section 2.3 we solve the aforementioned open problem and in Section 2.4 we present the asymptotic behaviour of $u_{\alpha,\lambda}$ for every (possible) choice of the parameters α and λ.

2. Main results

2.1. **Sonine pairs.** Bearing in mind the definition of Sonine pairs, a natural question arises: given a function $f:\mathbb{N}_a\to\mathbb{R}$, does it have a Sonine pair g? If so, is it unique? It turns out that the answer is positive to both questions. That is the content of the next

Theorem 1. *Fix $f:\mathbb{N}_a\to\mathbb{R}$. Then, there exists a unique $g:\mathbb{N}_a\to\mathbb{R}$ such that $(f*g)_a(t)=1$ for all $t\in\mathbb{N}_{a+1}$, if and only if $f(a)\neq 0$.*

Proof. Suppose firstly that $(f*g)_a(t)=1$ for all $t\in\mathbb{N}_{a+1}$ holds. Then, in particular, $(f*g)_a(a+1)=f(a)g(a)=1$, hence $f(a)\neq 0$.

Now, assume that $f(a)\neq 0$. Then, for $t\in\mathbb{N}_{a+1}$, we get

$$\begin{aligned}
&(f*g)_a(t)=1\\
&\Leftrightarrow \sum_{\tau=a}^{t-1} f(t-\tau-1+a)g(\tau)=1\\
&\Leftrightarrow f(a)g(t-1)=1-\sum_{\tau=a}^{t-2} f(t-\tau-1+a)g(\tau)\\
&\Leftrightarrow g(t-1)=\frac{1}{f(a)}-\frac{1}{f(a)}\sum_{\tau=a}^{t-2} f(t-\tau-1+a)g(\tau), \qquad (1)
\end{aligned}$$

from which we immediately conclude that g is uniquely determined. The proof is done. □

We can actually determine explicitly the Sonine partner of a function f. In order to do it, we introduce the following notation: For a function q defined on $\mathbb{N}_a$, we put $q_a^{(1)}(t)=q(t)$ and $q_a^{(n)}(t)=(q*q_a^{(n-1)})_a(t),\ n\in\mathbb{N}_2$.

Corollary 1. *Fix $f:\mathbb{N}_a\to\mathbb{R}$. Let $q(t)=f(t+1)$, $t\in\mathbb{N}_a$, and suppose $f(a)\neq 0$. Then the solution g of $(f*g)_a(t)=1,\ t\in\mathbb{N}_{a+1}$, is given by*

$$g(t)=\frac{1}{f(a)}\left(1+\sum_{k=a}^{t-1}\left(-\frac{1}{f(a)}\right)^{k-a+1}\sum_{s=k}^{t-1} q_a^{(k-a+1)}(s)\right),\quad t\in\mathbb{N}_a. \qquad (2)$$

Proof. By (1) we have that g satisfies the equation

$$g(t)=\frac{1}{f(a)}-\frac{1}{f(a)}\sum_{\tau=a}^{t-1} f(t-\tau+a)g(\tau),\quad t\in\mathbb{N}_a,$$

which is equivalent to

$$g(t)=\frac{1}{f(a)}-\frac{1}{f(a)}\sum_{\tau=a+1}^{t} q(t-\tau+a)g(\tau-1),\quad t\in\mathbb{N}_a.$$

Now the result follows by (3.4)–(3.5) in [5]. □

Even with simple functions f, the right hand side of equality (2) might be cumbersome to compute. For example, for $f(t)=1/(t+1)$, $t\in\mathbb{N}_0$, we know from [9, Example 9] that, with $g(t)=\frac{1}{t!}\int_0^1\frac{\Gamma(x+t)}{\Gamma(x)}dx$, we have $(f*g)_0(t)=1$ for all $t\in\mathbb{N}_1$. By uniqueness of solutions to the linear Volterra difference equation, we immediately deduce from Corollary 1 that

$$\frac{1}{t!}\int_0^1\frac{\Gamma(x+t)}{\Gamma(x)}dx=1+\sum_{k=0}^{t-1}(-1)^{k+1}\sum_{s=k}^{t-1} q_0^{(k+1)}(s),\quad t\in\mathbb{N}_0, \qquad (3)$$

where $q(t)=1/(t+2)$. In passing, we emphasize the novelty of equality (3), at least to the best of our knowledge.

We would like to obtain $g(t)=\frac{1}{t!}\int_0^1\frac{\Gamma(x+t)}{\Gamma(x)}dx$ in a different manner from what was done in [9, Example 9]. To do that, we use generating functions. So, denote by $F_u(z)=\sum_{k=0}^{\infty}u(k)z^k$ the generating function of $u:\mathbb{N}_0\to\mathbb{R}$. It is readily seen by the convolution theorem that $(f*g)_0(t)=1$ is equivalent to $F_g(z)=1/((1-z)F_f(z)),\ |z|<1$. Since $F_f(z)=-\frac{\log(1-z)}{z}$, then

$$g(t)=\frac{1}{t!}\left(-\frac{z}{(1-z)\log(1-z)}\right)^{(t)}(0),\quad t\in\mathbb{N}_0.$$

Now we just have to observe that

$$-\frac{z}{(1-z)\log(1-z)}=\int_0^1(1-z)^{-x}dx,$$

to obtain the desired formula.

Remark 1. *We may write the left hand side of* (3) *using Stirling numbers, obtaining*

$$\frac{1}{t!}\int_0^1\frac{\Gamma(x+t)}{\Gamma(x)}dx=\frac{1}{t!}\int_0^1\sum_{k=0}^{t}\begin{bmatrix}t\\k\end{bmatrix}x^k dx=\frac{1}{t!}\sum_{k=0}^{t}\frac{\begin{bmatrix}t\\k\end{bmatrix}}{k+1}.$$

2.2. **Monotonicity in the parameter of the discrete Mittag-Leffler function.** In [5], an explicit representation for the solution of

$$\begin{aligned}{}^{C}\nabla_0^{\alpha}u(n)&=-\lambda u(n),\quad n\in\mathbb{N},\ \lambda\neq-1,\ 0<\alpha\leq 1,\\ u(0)&=1,\end{aligned}$$

or, equivalently, of the Volterra difference equation

$$u(n)=1-\lambda\sum_{k=1}^{n}\frac{(n-k+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}u(k),\quad n\in\mathbb{N}_0,\tag{4}$$

was given. It reads as

$$u_{\alpha,\lambda}(n)=\frac{1}{1+\lambda}\left(1+\sum_{k=1}^{n-1}\left(\frac{-\lambda}{1+\lambda}\right)^k\sum_{r=0}^{k-1}(-1)^r\binom{k}{r}\frac{n^{\overline{(k-r)\alpha}}-k^{\overline{(k-r)\alpha}}}{\Gamma((k-r)\alpha+1)}\right),\ n\in\mathbb{N}.\tag{5}$$

The solution to (4) is known by the discrete Mittag-Leffler function. In [6] the author shows that, if $\lambda>0$ then the discrete Mittag-Leffler function is completely monotonic with respect to the variable n.

More recently, in [7] the authors study monotonicity properties with respect to the parameter α of the Mittag-Leffler function E_α. We wonder about the monotonic behaviour of the discrete Mittag-Leffler function with respect to the parameter α, when λ and n are fixed.

If $\lambda<-1$ there is no monotonicity in general: See Figure 1, for when $\lambda=-2$ and $n=3$.

Observe that, if $n=1$ then $u_{\alpha,\lambda}=\frac{1}{1+\lambda}$ and, therefore, there is nothing to study in this case.

Now we state and prove our main result in this section.

Theorem 2. *Let* $n\in\mathbb{N}_2$.

(1) If $-1<\lambda<0$, *then* $\alpha\to u_{\alpha,\lambda}(n)$ *is strictly increasing on* $\alpha\in(0,1]$;

(2) If $\lambda>0$, *then* $\alpha\to u_{\alpha,\lambda}(n)$ *is strictly decreasing on* $\alpha\in(0,1]$.

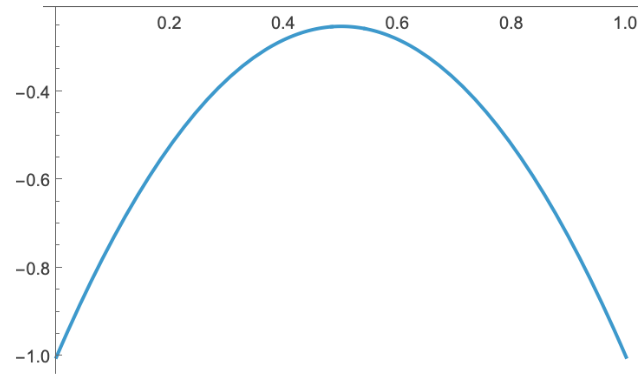

FIGURE 1. $u_{\alpha,-2}(3)$ for $\alpha \in (0,1)$.

Proof. Let $-1 < \lambda < 0$. By [10, Lemma 3.141] the solution of (4) may be represented by the series:

$$u_{\alpha,\lambda}(n) = \sum_{k=0}^{\infty} (-\lambda)^k \frac{\Gamma(k\alpha + n)}{\Gamma(n)\Gamma(k\alpha+1)}. \tag{6}$$

Now, differentiate with respect to α to obtain[1]

$$\frac{d}{d\alpha} u_{\alpha,\lambda}(n) = \frac{1}{\Gamma(n)} \sum_{k=1}^{\infty} k(-\lambda)^k \frac{\Gamma(k\alpha+n)}{\Gamma(k\alpha+1)} (\psi(k\alpha+n) - \psi(k\alpha+1)) > 0, \tag{7}$$

in view that the digamma function $\psi(x)$ is strictly increasing for $x > 0$. Therefore, (1) is proved.

Suppose now that $\lambda > 0$. By [6, Formula (11)], we have

$$F_{u_{\alpha,\lambda}}(z) = \frac{(1-z)^{-1} + \lambda(1-z)^{-\alpha}}{1 + \lambda(1-z)^{-\alpha}}, \quad |z| < 1,$$

or

$$F_{u_{\alpha,\lambda}}(z) = \frac{(1-z)^{\alpha-1} + \lambda}{(1-z)^{\alpha} + \lambda}. \tag{8}$$

Let us differentiate (8) with respect to α. We get, after some simplifications

$$\frac{d}{d\alpha} F_{u_{\alpha,\lambda}}(z) = \lambda z \log(1-z) \frac{(1-z)^{\alpha-1}}{(\lambda + (1-z)^{\alpha})^2}.$$

Define now

$$\sigma_{\alpha,\lambda}(s) = \frac{s^{\alpha-1}}{(\lambda + s^{\alpha})^2}, \quad s > 0.$$

The function $s \to s^{\alpha-1}$ is completely monotonic (in this context, a function $f : \mathbb{R}^+ \to \mathbb{R}$ is completely monotonic if $(-1)^n f^{(n)}(s) \geq 0$ for all $s > 0$ and $n \in \mathbb{N}_0$) being, therefore, $s \to s^{\alpha}$ a nonnegative function with a completely monotonic derivative. Since

$$s \to \frac{1}{(\lambda+s)^2},$$

[1]The following series is absolutely convergent. Indeed, since

$$\frac{\Gamma(k\alpha+n)}{\Gamma(k\alpha+1)} = \Pi_{j=1}^{n-1}(k\alpha+j) \leq c_n k^{n-1} (c_n > 0), \ \psi(k\alpha+n) - \psi(k\alpha+1) = \sum_{j=1}^{n-1} \frac{1}{k\alpha+j} \leq H_{n-1},$$

we get

$$\sum_{k=1}^{\infty} k(-\lambda)^k \frac{\Gamma(k\alpha+n)}{\Gamma(k\alpha+1)} (\psi(k\alpha+n) - \psi(k\alpha+1)) \leq c_n H_{n-1} \sum_{k=1}^{\infty} k^n |\lambda|^k,$$

from which the assertion follows.

is completely monotonic, then by Theorems 1 and 2 in [14] we conclude that $\sigma_{\alpha,\lambda}$ is completely monotonic. Since

$$\frac{(1-z)^{\alpha-1}}{(\lambda+(1-z)^{\alpha})^2}=\sum_{k=0}^{\infty}\frac{(-1)^k\sigma^{(k)}_{\alpha,\lambda}(1)}{k!}z^k,$$

and

$$z\log(1-z)=-\sum_{k=1}^{\infty}\frac{z^{k+1}}{k},$$

we get

$$\frac{d}{d\alpha}F_{u_{\alpha,\lambda}}(z)=-\lambda\sum_{j=1}^{\infty}\frac{z^{j+1}}{j}\sum_{k=0}^{\infty}\frac{(-1)^k\sigma^{(k)}_{\alpha,\lambda}(1)}{k!}z^k,$$

from which we infer that (taking the coefficient of z^n)

$$\frac{d}{d\alpha}u_{\alpha,\lambda}(n)=-\lambda\left(\sum_{j=1}^{n-2}\frac{\frac{(-1)^{n-j-1}\sigma^{(n-j-1)}_{\alpha,\lambda}(1)}{(n-j-1)!}}{j}+\frac{\sigma_{\alpha,\lambda}(1)}{n-1}\right)<0,$$

and the proof is concluded. □

Remark 2. *Observe that $\alpha=0$ is allowed in* (5), *and we obtain $u_{0,\lambda}(n)=1/(1+\lambda)$. Since $u_{1,\lambda}(n)=1/(1+\lambda)^n$ we get, by Theorem 2, the following bounds:*

(1) If $-1<\lambda<0$,

$$\frac{1}{1+\lambda}<u_{\alpha,\lambda}(n)<\frac{1}{(1+\lambda)^n},\quad n\in\mathbb{N}_2,$$

(2) If $\lambda>0$,

$$\frac{1}{(1+\lambda)^n}<u_{\alpha,\lambda}(n)<\frac{1}{1+\lambda},\quad n\in\mathbb{N}_2.$$

2.3. **Answer to an Open Problem.** In [6] a problem was left open at the end of the manuscript. Essentially, it was asked about the geometry of the solution to (4) when the parameter $\lambda<-1$. For $\lambda>-1$ the solution is known to be, e.g., positive.

Recall that, when $\alpha=1$ we get,

$$u_{1,\lambda}(n)=\frac{1}{(1+\lambda)^n},\quad n\in\mathbb{N}_0.$$

It is readily seen that, for $\lambda<-1$, the sequence oscillates in the sense that $u_{1,\lambda}(n)u_{1,\lambda}(n+1)<0$ for all $n\in\mathbb{N}_0$. For the fractional problem, i.e., for when $0<\alpha<1$ what is the behaviour of $u_{\alpha,\lambda}(n)$? In the rest of this section we characterize exactly that pattern, thus answering positively to what was a numerical evidence provided in [6].

We begin by deducting a novel representation for $u_{\alpha,\lambda}(n)$.

Lemma 1. *Let $0<\alpha<1$ and $\lambda<-1$. Put $\mu=-\lambda>1$ and define $r=\mu^{1/\alpha}-1>0$.*

Then

$$u_{\alpha,\mu}(n)=(-1)^n c_{\alpha,\mu}(n),\quad n\in\mathbb{N}_0,$$

where

$$c_{\alpha,\mu}(n)=\frac{1}{\alpha r^n}+(-1)^{n+1}I_{\alpha,\mu}(n),$$

and

$$I_{\alpha,\mu}(n) = \frac{1}{\pi}\mu\sin(\pi\alpha)\int_0^\infty \frac{t^{\alpha-1}}{\mu^2 - 2\mu\cos(\pi\alpha)t^\alpha + t^{2\alpha}}(t+1)^{-n}dt.$$

Moreover,

$$0 < I_{\alpha,\mu}(n) \le \frac{1-\alpha}{\alpha} \qquad \textit{for every } n \in \mathbb{N}_0.$$

Proof. We start with the generating function (8), introducing the notation:

$$W_{\alpha,\mu}(z) = F_{u_{\alpha,\mu}}(-z) = \frac{\mu - (1+z)^{\alpha-1}}{\mu - (1+z)^\alpha}.$$

Observe that $W_{\alpha,\mu}(z) = \sum_{k=0}^\infty (-1)^k u_{\alpha,\mu}(k) z^k$ so, if we define $c_{\alpha,\mu}$ to be the coefficient of z^n in $W_{\alpha,\mu}$, then

$$c_{\alpha,\mu}(n) = (-1)^n u_{\alpha,\mu}(n).$$

If we define by $G_{\alpha,\mu,n}(z) = W_{\alpha,\mu}(z)/z^{n+1}$ we get, by Cauchy's coefficient formula,

$$c_{\alpha,\mu}(n) = \frac{1}{2\pi i}\int_{|z|=\rho} G_{\alpha,\mu,n}(z)dz, \quad 0 < \rho < r,$$

and then,

$$c_{\alpha,\mu}(n) = \operatorname{Res}_{z=0} G_{\alpha,\mu,n}(z). \tag{9}$$

Let now $R > \max\{1, r\}$ and $\varepsilon > 0$. Define the slit annular-type region

$$D_{R,\varepsilon} = \{z \in \mathbb{C} : |z| < R,\ |z+1| > \varepsilon\} \setminus \{x \in \mathbb{R} : -R < x < -1-\varepsilon\}.$$

The contour $\Gamma_{R,\varepsilon}$ is the positively oriented boundary of $D_{R,\varepsilon}$ and is depicted in Figure 2. It consists of four pieces: the large circular arc C_R, the upper side of the cut $L^+_{R,\varepsilon}$, the small clockwise circular arc around the branch point C_ε and the lower side of the cut $L^-_{R,\varepsilon}$.

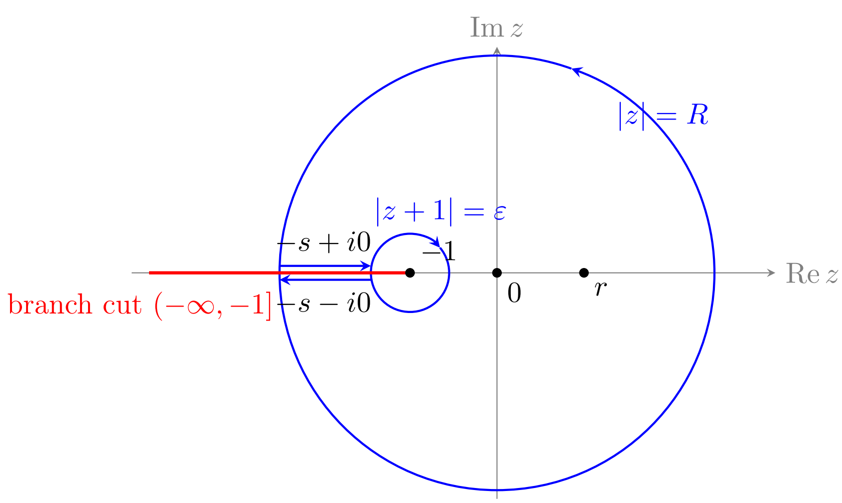

FIGURE 2. Keyhole contour around the branch cut $(-\infty, -1]$.

We have,

$$\int_{\Gamma_{R,\varepsilon}} G_{\alpha,\mu,n}(z)\,dz = \int_{C_R} G_{\alpha,\mu,n}(z)\,dz + \int_{L^+_{R,\varepsilon}} G_{\alpha,\mu,n}(z)\,dz + \int_{C_\varepsilon} G_{\alpha,\mu,n}(z)\,dz + \int_{L^-_{R,\varepsilon}} G_{\alpha,\mu,n}(z)\,dz. \tag{10}$$

The two cut integrals combine to

$$\int_{L^+_{R,\varepsilon}} G_{\alpha,\mu,n}(z)\,dz + \int_{L^-_{R,\varepsilon}} G_{\alpha,\mu,n}(z)\,dz = \int_{1+\varepsilon}^{R} \frac{W_{\alpha,\mu}(-s+i0) - W_{\alpha,\mu}(-s-i0)}{(-s)^{n+1}}\,ds.$$

Since $(-s)^{n+1} = (-1)^{n+1}s^{n+1}$ and $W_{\alpha,\mu}(-s-i0) = \overline{W_{\alpha,\mu}(-s+i0)}$, we end up with

$$2i(-1)^{n+1}\int_{1+\varepsilon}^{R} \operatorname{Im} W_{\alpha,\mu}(-s+i0)s^{-n-1}\,ds. \tag{11}$$

We have,

$$\lim_{\varepsilon\to 0^+, R\to\infty}\int_{1+\varepsilon}^{R} \operatorname{Im} W_{\alpha,\mu}(-s+i0)s^{-n-1}ds < \infty.$$

To prove the convergence of the above integrals, define

$$J_n = \int_1^\infty \operatorname{Im} W_{\alpha,\mu}(-s+i0)s^{-n-1}ds.$$

A direct computation yields

$$\operatorname{Im} W_{\alpha,\mu}(-s+i0) = \frac{\mu\sin(\pi\alpha)t^{\alpha-1}(t+1)}{\mu^2 - 2\mu\cos(\pi\alpha)t^\alpha + t^{2\alpha}},\quad t = s-1,$$

hence,

$$J_n = \mu\sin(\pi\alpha)\int_0^\infty \frac{t^{\alpha-1}}{\mu^2 - 2\mu\cos(\pi\alpha)t^\alpha + t^{2\alpha}}(t+1)^{-n}dt.$$

Since $\mu^2 - 2\mu\cos(\pi\alpha)t^\alpha + t^{2\alpha} = (\mu - \cos(\pi\alpha)t^\alpha)^2 + (\sin(\pi\alpha)t^\alpha)^2 > 0$ for $t > 0$, then $J_n > 0$. Moreover,

$$J_n \le J_0 = \frac{\pi(1-\alpha)}{\alpha},$$

i.e. J_n are well defined for all $n \in \mathbb{N}_0$.

Moving on to the remaining two integrals in (10), we obtain upon using Theorem 4.3.1 and Theorem 4.3.2 in [1],

$$\lim_{\varepsilon\to 0^+}\int_{C_\varepsilon} G_{\alpha,\mu,n}(z)\,dz = 0, \tag{12}$$

and

$$\lim_{R\to\infty}\int_{C_R} G_{\alpha,\mu,n}(z)\,dz = 0. \tag{13}$$

Thus,

$$\lim_{\varepsilon\to 0^+, R\to\infty}\int_{\Gamma_{R,\varepsilon}} G_{\alpha,\mu,n}(z)\,dz = 2i(-1)^{n+1}\int_1^\infty \operatorname{Im} W_{\alpha,\mu}(-s+i0)s^{-n-1}\,ds. \tag{14}$$

By the residue theorem, we obtain

$$2i(-1)^{n+1}J_n = 2\pi i\left(\operatorname{Res}_{z=0} G_{\alpha,\mu,n}(z) + \operatorname{Res}_{z=r} G_{\alpha,\mu,n}(z)\right),$$

as r is a pole of $W_{\alpha,\mu}$. It is easilly seen that $z = r$ is a simple pole and $\operatorname{Res}_{z=r} G_{\alpha,\mu,n}(z) = -\frac{1}{\alpha r^n}$.

The proof is done. $\square$

Remark 3. *In view of* (5) *and the previous Lemma, we get the interesting equality:*

$$\frac{1}{1-\mu}\left(1+\sum_{k=1}^{n-1}\left(\frac{\mu}{1-\mu}\right)^k\sum_{r=0}^{k-1}(-1)^r\binom{k}{r}\frac{n^{\overline{(k-r)\alpha}}-k^{\overline{(k-r)\alpha}}}{\Gamma((k-r)\alpha+1)}\right)$$
$$=\frac{(-1)^n}{\alpha(\mu^{1/\alpha}-1)^n}-\frac{1}{\pi}\mu\sin(\pi\alpha)\int_0^\infty\frac{t^{\alpha-1}}{\mu^2-2\mu\cos(\pi\alpha)t^\alpha+t^{2\alpha}}(t+1)^{-n}dt,\quad n\in\mathbb{N}.$$

We are now ready to state and prove the main result within this section.

Theorem 3. *Suppose that* $0<\alpha<1$.

(1) If $-2^\alpha\le\lambda<-1$, *then* $u_{\alpha,\lambda}(n)u_{\alpha,\lambda}(n+1)<0$ *for all* $n\in\mathbb{N}_0$.

(2) If $\lambda<-2^\alpha$, *then* $u_{\alpha,\lambda}(2n+1)<0$ *for all* $n\in\mathbb{N}_0$ *and* $u_{\alpha,\lambda}(n)<0$ *for all* $n\ge n_0$, *where* n_0 *is some positive integer.*

Proof. Let $-2^\alpha\le\lambda<-1$ Then, $1<\mu\le 2^\alpha$ and $0<r\le 1$ in Lemma 1. Therefore, $\frac{1}{\alpha r^n}\ge\frac{1}{\alpha}$.

If n is even, then

$$c_{\alpha,\mu}(n)=\frac{1}{\alpha r^n}-I_{\alpha,\mu}(n)\ge\frac{1}{\alpha}-\frac{1-\alpha}{\alpha}=1>0.$$

If n is odd, then

$$c_{\alpha,\mu}(n)=\frac{1}{\alpha r^n}+I_{\alpha,\mu}(n)>0.$$

Therefore,

$$u_{\alpha,\mu}(n)u_{\alpha,\mu}(n+1)=(-1)^n c_{\alpha,\mu}(n)(-1)^{n+1}c_{\alpha,\mu}(n+1)=(-1)^{2n+1}c_{\alpha,\mu}(n)c_{\alpha,\mu}(n+1)<0,\quad n\in\mathbb{N}_0,$$

and (1) is proved.

Assume now that $\lambda<-2^\alpha$, i.e., $\mu>2^\alpha$. Then, $r>1$.

If n is odd then, again, $c_{\alpha,\mu}(n)>0$, hence $u_{\alpha,\mu}(n)=-c_{\alpha,\mu}(n)<0$.

If n is even then

$$c_{\alpha,\mu}(n)=\frac{1}{\alpha r^n}-I_{\alpha,\mu}(n),$$

and we will show that $c_{\alpha,\mu}(n)<0$ for $n\ge n_0$. Since

$$|\mu^2-2\mu\cos(\pi\alpha)t^\alpha+t^{2\alpha}|\le\mu^2+2\mu,\quad 0<t<1,$$

then,

$$I_{\alpha,\mu}(n)\ge\frac{\sin(\pi\alpha)}{\pi(\mu+2)}\int_0^1 t^{\alpha-1}(t+1)^{-n}dt.$$

Moreover, for $t<1/n$ with $n\ge1$, we have

$$(t+1)^{-n}>\left(1+\frac{1}{n}\right)^{-n}>\frac{1}{e},$$

hence,

$$I_{\alpha,\mu}(n)\ge\frac{\sin(\pi\alpha)}{e\pi(\mu+2)}\int_0^{\frac{1}{n}}t^{\alpha-1}dt=\frac{\sin(\pi\alpha)}{e\pi\alpha(\mu+2)}n^{-\alpha}.$$

Since $r>1$ we immediately conclude that there exists $n_0\in\mathbb{N}$ such that

$$c_{\alpha,\mu}(n)=\frac{1}{\alpha r^n}-I_{\alpha,\mu}(n)\le\frac{1}{\alpha r^n}-\frac{\sin(\pi\alpha)}{e\pi\alpha(\mu+2)}n^{-\alpha}<0,\quad n\ge n_0.$$

The theorem is now proved. □

2.4. **Asymptotics.** In this final section we would like to study the asymptotic behaviour of $u_{\alpha,\lambda}$.

From the results obtained in [6] (cf., in particular, Page 8), we know that

$$\text{If } \lambda > 0, \text{ then } \lim_{n\to\infty} u_{\alpha,\lambda}(n) = 0. \tag{15}$$

Moreover,

$$\text{If } -1 < \lambda < 0, \text{ then } \lim_{n\to\infty} u_{\alpha,\lambda}(n) = +\infty. \tag{16}$$

Indeed, by [6, Lemma 1 and Corollary 2]

$$u_{\alpha,\lambda}(n) = \int_0^\infty e^{-t}\frac{t^{n-1}}{\Gamma(n)}E_\alpha(-\lambda t^\alpha)dt,$$

where E_α denotes the Mittag-Leffler function of parameter α. Since $-1 < \lambda < 0$, we have

$$E_\alpha(-\lambda t^\alpha) = \sum_{k=0}^\infty \frac{(-\lambda t^\alpha)^k}{\Gamma(k\alpha+1)} \geq \frac{-\lambda t^\alpha}{\Gamma(\alpha+1)}, \quad t > 0,$$

hence

$$\int_0^\infty e^{-t}\frac{t^{n-1}}{\Gamma(n)}E_\alpha(-\lambda t^\alpha)dt \geq \frac{-\lambda}{\Gamma(\alpha+1)}\int_0^\infty e^{-t}\frac{t^{n-1+\alpha}}{\Gamma(n)}dt = \frac{-\lambda}{\Gamma(\alpha+1)}\frac{\Gamma(n+\alpha)}{\Gamma(n)} \to +\infty, \text{ as } n \to +\infty.$$

The previous aqssertions (15) and (16) were proved about 10 years ago by other methods [11].

We end this work providing the asymptotic characterization of $u_{\alpha,\lambda}$ for when $\lambda < -1$.

Theorem 4. *Let $0 < \alpha < 1$ and $\lambda < -1$.*

(1) If $-2^\alpha < \lambda < -1$, then $\lim_{n\to\infty} u_{\alpha,\lambda}(2n) = +\infty$ and $\lim_{n\to\infty} u_{\alpha,\lambda}(2n+1) = -\infty$.
(2) If $\lambda = -2^\alpha$, then $\lim_{n\to\infty} u_{\alpha,\lambda}(2n) = 1/\alpha$ and $\lim_{n\to\infty} u_{\alpha,\lambda}(2n+1) = -1/\alpha$.
(3) If $\lambda < -2^\alpha$, then $\lim_{n\to\infty} u_{\alpha,\lambda}(n) = 0$.

Proof. Recall that

$$I_{\alpha,\mu}(n) = \frac{1}{\pi}\mu\sin(\pi\alpha)\int_0^\infty \frac{t^{\alpha-1}}{\mu^2 - 2\mu\cos(\pi\alpha)t^\alpha + t^{2\alpha}}(t+1)^{-n}\,dt,$$

where $0 < \alpha < 1$ and $\mu > 1$. Since

$$\mu^2 - 2\mu\cos(\pi\alpha)t^\alpha + t^{2\alpha} = (t^\alpha - \mu\cos(\pi\alpha))^2 + \mu^2\sin^2(\pi\alpha) \geq \mu^2\sin^2(\pi\alpha), \quad t \geq 0,$$

we obtain

$$0 \leq I_{\alpha,\mu}(n) \leq \frac{1}{\pi(\mu\sin(\pi\alpha))}\int_0^\infty t^{\alpha-1}(t+1)^{-n}\,dt = \frac{1}{\pi(\mu\sin(\pi\alpha))}\Gamma(\alpha)\frac{\Gamma(n-\alpha)}{\Gamma(n)}, \quad n \in \mathbb{N}.$$

But $\lim_{n\to\infty}\frac{\Gamma(n-\alpha)}{\Gamma(n)} = 0$, hence $\lim_{n\to\infty} I_{\alpha,\mu}(n) = 0$.

By Lemma 1,

$$\lim_{n\to\infty} u_{\alpha,\mu}(n) = \lim_{n\to\infty}\left[(-1)^n\frac{1}{\alpha r^n}\right],$$

where, again, $r = \mu^{1/\alpha} - 1$.

Now, if $-2^\alpha < \lambda < -1$, then $0 < r < 1$ and (1) follows immediately.

If $\lambda = -2^\alpha$, then $r = 1$ and (2) holds.

If $\lambda < -2^\alpha$, then $r > 1$ and (3) follows.

The proof is done. $\square$

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