A GENERALIZATION OF A SMIRNOV'S INEQUALITY

RUI A. C. FERREIRA
CMUP, FACULDADE DE CIÊNCIAS, UNIVERSIDADE DO PORTO, RUA DO CAMPO ALEGRE 687,
4169–007 PORTO, PORTUGAL.
EMAIL: RUI.AC.FERREIRA@FC.UP.PT

Abstract. We extend Smirnov's inequality for the trigonometric function $\cos(\sqrt{x})$ using a Mittag-Leffler function.

In [1, 4], the following inequality was proved: for every positive integer n,

$$\left|\frac{d^n}{dx^n}\cos\sqrt{x}\right| \le \frac{n!}{(2n)!}, \qquad x>0.$$

This estimate is sharp, since equality is attained at the origin.

The purpose of this note is to prove an analogue of this inequality by using the Mittag-Leffler function. Recall that

$$E_\alpha(z)=\sum_{m=0}^{\infty}\frac{z^m}{\Gamma(\alpha m+1)}, \qquad \alpha>0,\ z\in\mathbb{C},$$

and

$$E_{\alpha,\beta}(z)=\sum_{m=0}^{\infty}\frac{z^m}{\Gamma(\alpha m+\beta)}, \qquad \alpha,\beta>0,\ z\in\mathbb{C},$$

from which it is easily seen that $E_2(-x)=\cos(\sqrt{x})$.

The proof follows the elementary spirit of the integral-representation method used in [2]. We first derive an iterated integral formula for the derivatives of $x\mapsto E_\alpha(-x)$. This formula is obtained from a simple identity connecting $E_{\alpha,\alpha}$ and E_α, together with the Beta integral. We then use a cosine-transform representation of $E_\alpha(-x^\alpha)$ to deduce

$$|E_\alpha(-x)|\le 1, \qquad x\ge 0.$$

Combining these two ingredients yields the desired derivative estimate; see Theorem 3 below.

We now proceed to our main results.

Theorem 1. *Let $1<\alpha\le 2$. Then, for every $n\in\mathbb{N}$ and $x\ge 0$, we have*

$$\frac{d^n}{dx^n}E_\alpha(-x)=\frac{(-1)^n}{\big(\alpha\Gamma(\alpha-1)\big)^n}\int_{[0,1]^n}\left(\prod_{j=1}^{n-1}u_j^{\alpha(n-j)}\right)\left(\prod_{j=1}^{n}(1-u_j)^{\alpha-2}\right)E_\alpha\left(-x\prod_{j=1}^{n}u_j^{\alpha}\right)du_1\cdots du_n.$$

Proof. We start with the identity

$$E_{\alpha,\alpha}(-x)=\frac{1}{\Gamma(\alpha-1)}\int_0^1(1-u)^{\alpha-2}E_\alpha(-u^\alpha x)\,du,$$

which follows immediately from an obvious change of variables in [3, Equality (3.28)].

2010 *Mathematics Subject Classification.* 33E12, 26D05.

Key words and phrases. Mittag-Leffler's function, Smirnov's inequality, Integral transform.

Since

$$\frac{d}{dz}E_\alpha(z) = \frac{1}{\alpha}E_{\alpha,\alpha}(z),$$

we obtain

$$\frac{d}{dx}E_\alpha(-x) = -\frac{1}{\alpha\Gamma(\alpha-1)}\int_0^1 (1-u)^{\alpha-2}E_\alpha(-u^\alpha x)\,du.$$

Differentiating once more gives

$$\frac{d^2}{dx^2}E_\alpha(-x) = \frac{1}{\alpha^2(\Gamma(\alpha-1))^2}\int_{[0,1]^2} u_1^\alpha(1-u_1)^{\alpha-2}(1-u_2)^{\alpha-2}E_\alpha(-u_1^\alpha u_2^\alpha x)\,du_1du_2,$$

and proceeding inductively yields the aforementioned representation. $\square$

We need the following representation of the Mittag-Leffler function, which should be of independent interest, and that actually we have not found in the literature.

Theorem 2. *Let $1<\alpha<2$. Then*

$$E_\alpha(-x^\alpha) = \frac{2\sin(\alpha\pi/2)}{\pi}\int_0^\infty \cos(tx)\frac{t^{\alpha-1}}{t^{2\alpha}+2t^\alpha\cos(\alpha\pi/2)+1}\,dt, \qquad x\geq 0. \tag{1}$$

Proof. Let

$$\beta = \frac{\alpha}{2}\in(0,1).$$

Then, by the Laplace transform for the Mittag-Leffler function (cf. [3, Equality (3.13)]), we have for $s>0$,

$$\frac{s^{\beta-1}}{s^\beta+1} = \int_0^\infty e^{-st}E_\beta(-t^\beta)\,dt.$$

Using the standard integral representation of $E_\beta(-t^\beta)$ (cf. [3, Equality (4.55)])), we obtain

$$\frac{s^{\beta-1}}{s^\beta+1} = \int_0^\infty e^{-st}\frac{\sin(\beta\pi)}{\pi}\int_0^\infty e^{-tr}\frac{r^{\beta-1}}{r^{2\beta}+2r^\beta\cos(\beta\pi)+1}\,dr\,dt.$$

By Fubini's theorem,

$$\frac{s^{\beta-1}}{s^\beta+1} = \int_0^\infty \frac{\sin(\beta\pi)}{\pi}\frac{r^{\beta-1}}{r^{2\beta}+2r^\beta\cos(\beta\pi)+1}\int_0^\infty e^{-t(s+r)}\,dt\,dr.$$

Therefore,

$$\frac{s^{\beta-1}}{s^\beta+1} = \int_0^\infty \frac{\sin(\beta\pi)}{\pi}\frac{r^{\beta-1}}{r^{2\beta}+2r^\beta\cos(\beta\pi)+1}\frac{1}{s+r}\,dr.$$

Now,

$$\frac{s^{\alpha-1}}{s^\alpha+1} = s\frac{(s^2)^{\beta-1}}{(s^2)^\beta+1}.$$

Consequently,

$$\frac{s^{\alpha-1}}{s^\alpha+1} = \int_0^\infty \frac{\sin(\beta\pi)}{\pi}\frac{r^{\beta-1}}{r^{2\beta}+2r^\beta\cos(\beta\pi)+1}\frac{s}{s^2+r}\,dr.$$

With the change of variables $r=u^2$, this becomes

$$\frac{s^{\alpha-1}}{s^\alpha+1} = \int_0^\infty \frac{2\sin(\alpha\pi/2)}{\pi}\frac{u^{\alpha-1}}{u^{2\alpha}+2u^\alpha\cos(\alpha\pi/2)+1}\frac{s}{s^2+u^2}\,du.$$

Since

$$\frac{s}{s^2+u^2} = \int_0^\infty e^{-st}\cos(ut)\,dt,$$

we get

$$\frac{s^{\alpha-1}}{s^\alpha+1} = \int_0^\infty \frac{2\sin(\alpha\pi/2)}{\pi} \frac{u^{\alpha-1}}{u^{2\alpha}+2u^\alpha\cos(\alpha\pi/2)+1} \int_0^\infty e^{-st}\cos(ut)\,dt\,du.$$

Thus,

$$\frac{s^{\alpha-1}}{s^\alpha+1} = \int_0^\infty e^{-st} \int_0^\infty \cos(ut) \frac{2\sin(\alpha\pi/2)}{\pi} \frac{u^{\alpha-1}}{u^{2\alpha}+2u^\alpha\cos(\alpha\pi/2)+1}\,du\,dt,$$

and the result follows from the uniqueness of the Laplace transform. □

Before we prove the main result we need the following

Corollary 1. *Let* $1<\alpha<2$. *Then*

$$|E_\alpha(-x)| \le 1, \qquad x \ge 0.$$

Proof. The result follows immediately from (1) by noticing that

$$|E_\alpha(-x^\alpha)| \le \frac{2\sin(\alpha\pi/2)}{\pi} \int_0^\infty \frac{t^{\alpha-1}}{t^{2\alpha}+2t^\alpha\cos(\alpha\pi/2)+1}\,dt = 1.$$

□

We are ready for our main result.

Theorem 3. *Suppose that* $1<\alpha<2$. *For every positive integer* n, *one has*

$$\left|\frac{d^n}{dx^n}E_\alpha(-x)\right| \le \frac{(n-1)!}{\alpha\Gamma(\alpha n)}, \qquad x \ge 0.$$

Proof. By Theorem 1 and Corollary 1 we have

$$\left|\frac{d^n}{dx^n}E_\alpha(-x)\right| \le \frac{1}{\alpha^n\Gamma(\alpha-1)^n} \int_{[0,1]^n} \left(\prod_{j=1}^{n-1} u_j^{\alpha(n-j)}\right)\left(\prod_{j=1}^{n}(1-u_j)^{\alpha-2}\right)du.$$

The integral factorizes as

$$\int_0^1 (1-u)^{\alpha-2}\,du = \frac{1}{\alpha-1},$$

and

$$\int_0^1 u^{\alpha k}(1-u)^{\alpha-2}\,du = \frac{\Gamma(\alpha k+1)\Gamma(\alpha-1)}{\Gamma(\alpha(k+1))}, \qquad k=1,\dots,n-1.$$

Therefore,

$$\left|\frac{d^n}{dx^n}E_\alpha(-x)\right| \le \frac{1}{\alpha^n\Gamma(\alpha)} \prod_{k=1}^{n-1} \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha(k+1))}.$$

Finally,

$$\Gamma(\alpha k+1) = \alpha k\Gamma(\alpha k),$$

so that

$$\prod_{k=1}^{n-1} \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha(k+1))} = \frac{\alpha^{n-1}(n-1)!\Gamma(\alpha)}{\Gamma(\alpha n)},$$

which yields

$$\left|\frac{d^n}{dx^n}E_\alpha(-x)\right| \le \frac{(n-1)!}{\alpha\Gamma(\alpha n)}.$$

This completes the proof. □

Corollary 2 (Smirnov's inequality). *We have*

$$\left|\frac{d^n}{dx^n}\cos(\sqrt{x})\right| \le \frac{n!}{(2n)!}, \qquad x \ge 0,\ n \in \mathbb{N}.$$

Proof. In Theorem 3, let $\alpha \to 2^-$ to obtain

$$\left|\frac{d^n}{dx^n}\cos(\sqrt{x})\right| \le \frac{(n-1)!}{2\Gamma(2n)} = \frac{n(n-1)!}{2n\Gamma(2n)} = \frac{n!}{(2n)!}.$$

The proof is done. □

Remark 1. *In [4] a Gronwall inequality is proved, namely,*

$$\left|\frac{d^n}{dx^n}\frac{\sin x}{x}\right| \le \frac{1}{n+1}, \qquad x \in \mathbb{R},\ n \in \mathbb{N}. \tag{2}$$

Though

$$E_{2,2}(-x^2) = \frac{\sin x}{x},$$

we have, with

$$f_\alpha(x) = E_{\alpha,\alpha}(-x^\alpha), \qquad \alpha \in (1,2),$$

that

$$f_\alpha''(x) = \sum_{k=1}^{\infty} \frac{(-1)^k \alpha k(\alpha k-1)x^{k\alpha-2}}{\Gamma(k\alpha+\alpha)}.$$

From this we conclude that

$$\lim_{x\to 0^+} f_\alpha''(x) = -\infty.$$

Therefore, a direct analogue of Smirnov's estimate for the family $E_{\alpha,\alpha}(-x^\alpha)$ cannot hold, since the second derivative is unbounded near the origin for every $1 < \alpha < 2$.

Acknowledgments

Rui A. C. Ferreira was supported by the "Fundação para a Ciência e a Tecnologia (FCT)" through the program "Stimulus of Scientific Employment, Individual Support—6th Edition Call" with reference 2023.05967.CEECIND/CP2833/CT0001.

DOI: https://doi.org/10.54499/2023.05967.CEECIND/CP2833/CT0001.

Declarations

There is no conflict or competing interests.

References

[1] J. Gélinas and A. G. Smirnov, Integral representation for derivatives of $\cos(\sqrt{x})$, *Amer. Math. Monthly* **126** (2019), no. 6, 561.
[2] C. Liu, New proof of a Gronwall-type trigonometric inequality, *Amer. Math. Monthly* **127** (2020), no. 2, 188.
[3] P. Van Mieghem, *The Mittag-Leffler and Gamma Function*, Oxford Mathematical Monographs, 2026.
[4] A. G. Smirnov, A Gronwall-type trigonometric inequality, *Amer. Math. Monthly* **125** (2018), no. 5, 453–456.