Going hyperbolic

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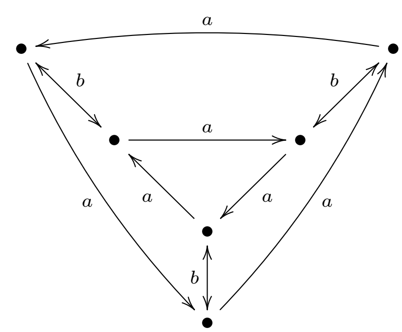
1 Introduction

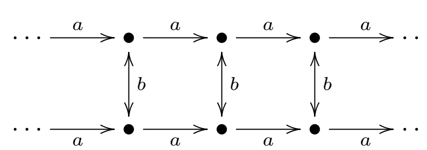
The theories of finite and infinite groups have developed different approaches over the years. On the one hand, finite groups can be viewed as subgroups of finite symmetric groups and relate naturally to combinatorics. On the other hand, infinite groups tend to be viewed as quotients of free groups (through generators and relators) and have strong connections to geometry.

Indeed, the seminal work of the geometer Max Dehn [9] is very much at the source of what is nowadays known as geometric group theory, when he performed an algorithmic study of the so called *surface groups* (fundamental groups of surfaces). His ideas were later generalized by Martin Greendlinger with his small cancellation theory [10].

Cayley graphs turned out to be an important tool in this context, which we now proceed to define. Given a group G generated by a set A, the *Cayley graph* $\mathrm{Cay}_A(G)$ has vertex set G and labelled directed edges of the form $g \xrightarrow{a} ga$ for all $g \in G$ and $a \in A^{\pm 1}$ (and the A-labelled edges determine the A^{-1}-labelled ones). For instance, for the symmetric group S_3, generated by the permutations $a = (123)$ and $b = (12)$, we get the Cayley graph



For the direct product $\mathbb{Z} \times \mathbb{Z}_2$, generated by $a = (1, 0)$ and $b = (0, 1)$, we get the Cayley graph

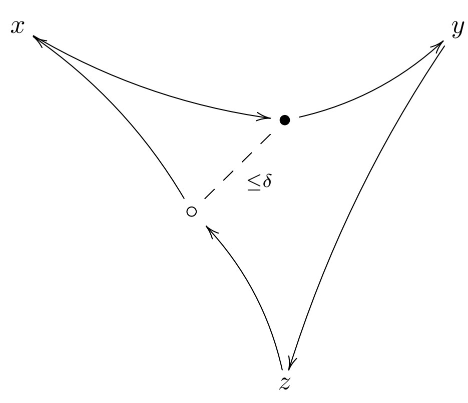


Note that there is no need to identify the vertices since Cayley graphs have a transitive automorphism group.

In the eighties, Mikhail Gromov developed a brilliant new idea [11]: if we consider $\mathrm{Cay}_A(G)$ as a (geodesic) metric space for some group G finitely generated by A, its geometric properties may imply good algorithmic properties, and hyperbolic geometry means excellent news! This thought is somewhat disturbing... we haven't established yet what is the geometry of *our* universe: is it hyperbolic (negative curvature), spherical (positive curvature) or euclidean (flat)? It is absolutely irrelevant for our daily life, but we are not used to think that one alternative may be better than the other... surprise, surprise: that is precisely what happens in the realm of finitely generated groups. What does this mean exactly?

Defining a metric d_A on $G = \langle A \rangle$ is easy: since $\mathrm{Cay}_A(G)$ is connected, we define $d_A(g,h)$ as the length of the shortest path from g to h, and any such path is called a *geodesic*. If we actually imagine the edges of the Cayley graph as real lines (as we often do in our mind when we think about graphs), then every geodesic of length n becomes isometric to the interval $[0,n] \subset \mathbb{R}$, and $\mathrm{Cay}_A(G)$ becomes what is known as a *geodesic metric space*.

A popular way of defining hyperbolic geometry in this context is through *geodesic triangles* (a collection of 3 geodesics $\{[x,y],[y,z],[z,x]\}$ connecting 3 vertices x,y,z). Given $\delta \geq 0$, we say that this triangle is δ-*thin* if every point in one of the geodesics it at distance $\leq \delta$ from some point in one of the other two geodesics.



The geodesic metric space is *hyperbolic* if there exists some $\delta \geq 0$ such that every geodesic triangle is δ-thin.

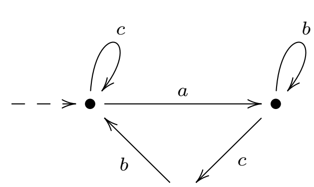
Algebraic structures have not earned a reputation of robustness, to say the least... a slight deformation on an abelian group and oops, it is not abelian anymore. On the contrary, hyperbolic geometry is certainly robust: within geodesic metric spaces, hyperbolicity is preserved through quasi-isometry. An *isometry* $\varphi : (X,d) \to (X',d')$ is a bijection preserving distance, i.e. $d'(\varphi(x),\varphi(y)) = d(x,y)$ for all $x,y \in X$. In a *quasi-isometry*, all these notions are relaxed within constant bounds, for instance through inequalities of the form $d'(\varphi(x),\varphi(y)) \leq Kd(x,y) + L$ and so on. A suggestive image is that quasi-isometric spaces look the same if we watch them from very far away...

Now if A and B are two alternative finite generating sets for a group G, then $\mathrm{Cay}_A(G)$ and $\mathrm{Cay}_B(G)$ are quasi-isometric, thus we can define G to be a *hyperbolic group* if $\mathrm{Cay}_A(G)$ is hyperbolic for some (every) finite generating set A. Using closure under quasi-isometry, one can also prove that if H is a finite index subgroup of G, then G is hyperbolic if and only if H is hyperbolic.

But how abundant in *nature* are hyperbolic groups anyway? Finite groups are trivially hyperbolic, and so are free groups of finite rank (since their Cayley graph with respect to a basis is a tree). This implies that finitely generated *virtually free* groups (characterized by having a free subgroup of finite index and finite rank) are also hyperbolic. And so are fundamental groups of compact riemannian manifolds with negative (not necessarily constant) sectional curvature. On the other hand, no group containing $\mathbb{Z} \times \mathbb{Z}$ (the fundamental group of the torus) as a subgroup is hyperbolic. This might lead us to suspect that hyperbolicity is not such a common property... is it?

A group G is *finitely presented* if it can be defined by finitely many generators and relators, say $\langle A \mid R \rangle$. This means that G is isomorphic to the quotient of the free group F_A on the finite set A by the normal subgroup generated by the finite subset R of F_A. Most group theory is about finitely presented groups, really. It happens that Alexander Ol'shanskii proved in 1992 [12] (but Gromov announced this fact previously) that the probability of a finitely presented group being hyperbolic tends to 1 (under reasonable assumptions)! So we do not know whether our universe is hyperbolic, but the universe of finitely presented groups certainly is...

A surprising feature of hyperbolic groups are the amazing algorithmic properties they satisfy, making computations easy with respect to other groups. Let $\mathrm{Geo}_A(G)$ consist of all words labelling geodesics in $\mathrm{Cay}_A(G)$ starting at the identity 1. Then $\mathrm{Geo}_A(G)$ turns out to be a *rational language*, that is, it is the language of a *finite automaton*. Given a finite automaton $\mathcal{A}$, the language $L(\mathcal{A})$ is the set of all words which can be read in the automaton. For instance, the language of the automaton



is $\{c, ab^*cb\}^*ab^*c$, where X^* denotes the submonoid generated by X. The fact of $\mathrm{Geo}_A(G)$ being rational does not ensure in itself the algorithmic properties we have been boasting about, but the fact is that there exist also finite automata which encode somehow the action by right and left multiplication of each letter on $\mathrm{Geo}_A(G)$. It follows that hyperbolic groups are actually *biautomatic* groups, which have solvable word problem and solvable conjugacy problem (indeed they are the only biautomatic groups where $\mathrm{Geo}_A(G)$ has these properties). Thus there exist efficient algorithms for deciding whether two arbitrary words on $A^{\pm 1}$ represent the same element or conjugate elements of G. And hyperbolic groups are finitely presentable. We should note that all these problems were actually solved my Max Dehn for surface groups!

Another unexpected theorem arises by considering isoperimetric functions. Consider a presentation $\langle A \mid R \rangle$ of a group G and $w \in F_A$. If $w = 1$ in G, then w is the product of conjugates of elements of $R^{\pm 1}$. Let $\lambda(w)$ denote the minimum number of such factors in such a product. An *isoperimetric function* for $\langle A \mid R \rangle$ is a function $f : \mathbb{N} \to \mathbb{R}^+$ such that $\lambda(w) \leq f(|w|)$ for every $w \in F_A$ representing the identity in G (where $|w|$ denotes the length of w in reduced form). Gromov proved that a finitely presented group is hyperbolic if and only if it admits a linear isoperimetric function (and if and only if it admits a subquadratic isoperimetric function). Therefore we have a theorem that establishes an equivalence between a geometric property and computational complexity.

We already know that hyperbolic groups are deeply tied to geometry, what about topology? Given $G = \langle A \rangle$, the topology of the metric space (G, d_A) is discrete and would not seem very promising. But the concept of boundary introduced by Gromov (for any hyperbolic geodesic metric space, actually) changed the whole game. Assume that $G = \langle A \rangle$ is hyperbolic. A *geodesic ray* is an infinite path in $\mathrm{Cay}_A(G)$ starting at the identity such that every finite subpath is a geodesic. Two geodesic rays are equivalent if the Hausdorff distance between their sets of vertices is finite (i.e., every vertex of one of them is at bounded distance from some vertex in the other ray). The *Gromov boundary* ∂G is the set of all equivalence classes of geodesic rays. Gromov defined a topology $\mathcal{T}$ on $\overline{G} = G \cup \partial G$ with several important properties:

- both $\overline{G}$ and ∂G are compact for $\mathcal{T}$ and its restriction;
- $\mathcal{T}$ is metrizable for a family of metrics d on G called *visual metrics*;
- the completion of any visual metric on G induces the topology $\mathcal{T}$ on $\overline{G}$;
- ∂G is invariant under quasi-isometry, so we do not need to specify the finite generating set of G.

The boundary is particularly simple to describe in the case of a free group since it consists of all (right) infinite reduced words $a_1 a_2 a_3 \ldots$ (and is a Cantor set). If we consider the *hyperbolic plane* $\mathbb{H}^2$ via the Poincaré disk model,

its boundary is the outlining circumference. We should note that $\mathbb{H}^2$ plays an important role in the theory of hyperbolic groups due to the following alternative theorem due to Bonk and Kleiner [3]: every hyperbolic group is either virtually free or there exists a quasi-isometric embedding of $\mathbb{H}^2$ into its Cayley graph.

2 Hyperbolicity in Porto

I started working with free groups around 20 years ago and later on I moved on to virtually free groups. My approach was essentially automata-theoretic and involved also *transducers* (automata with output). For instance, in [14] I used these ideas to study the dynamics of continuous extensions of endomorphisms to the boundary of a virtually free group. In fact, it is the boundary which becomes interesting from the dynamical viewpoint (recall that the topology of the group itself is discrete). An endomorphism φ of a finitely generated virtually free group G admits a continuous extension Φ to ∂G if and only if it is uniformly continuous for some (any) visual metric. I proved that this happens if and only if φ is *virtually injective* (that is, it has finite kernel). Now the fixed points of Φ are divided into two disjoint subclasses: singular and regular. The *sin-*

gular fixed points arise as the topological closure of the fixed points of φ, which constitute a finitely generated subgroup $\text{Fix}(\varphi)$ of G (I provided an automata-theoretic proof in [14]). The remaining fixed points are called *regular*, and I proved that the set $\text{Reg}(\Phi)$ of regular fixed points is in some sense finitely generated: if we consider the natural action of $\text{Fix}(\varphi)$ on $\text{Reg}(\Phi)$, it turns out that it has only finitely many orbits. In the automorphism case, it was shown that the regular fixed points are either exponentially stable attractors or exponentially stable repellers.

During my stay at Salvador da Bahia, I worked with Vítor Araújo (UFBA, formerly UP) in problems related to hyperbolic geometry and hyperbolic groups.

One of the motivations for our work was the possibility of defining new pseudometrics on the automorphism group of an arbitrary hyperbolic group. This led us to consider Hölder conditions in [1]. A mapping $\varphi : (X, d) \to (X', d')$ between metric spaces satisfies a *Hölder condition* of exponent $r > 0$ if there exists a constant $K > 0$ such that

$$d'(\varphi(x), \varphi(y)) \leq K(d(x, y))^r$$

holds for all $x, y \in X$. Such a condition clearly implies uniform continuity.

As a preliminary result, we showed that all visual metrics on a hyperbolic group are Hölder equivalent. Our main theorem establishes several equivalent conditions for a nontrivial endomorphism φ of a hyperbolic group $G = \langle A \rangle$ and a visual metric d on G:

- φ satisfies a Hölder condition with respect to d;
- φ admits a continuous extension to ∂G satisfying a Hölder condition with respect to the natural extension of d;
- φ is a quasi-isometric embedding of (G, d_A) into itself (where d_A denotes de geodesic metric);
- φ is virtually injective and $\varphi(G)$ is a quasiconvex subgroup of G.

We should note that quasiconvex subgroups play a major role in the theory of hyperbolic groups. We say that $H \leq G$ is *quasiconvex* if every point in a geodesic of $\text{Cay}_A(G)$ with endpoints in H lies at bounded geodesic distance from some vertex of H. It is known that every quasiconvex subgroup of a hyperbolic groups is hyperbolic (but the converse implication does not hold).

In [1], we also proved that if the hyperbolic group G is either virtually free or torsion-free co-hopfian, then φ is uniformly continuous if and only if it satisfies a Hölder condition if and only if it is virtually injective.

Hyperbolic groups are defined upon the concept of hyperbolic geodesic metric space, which admits several equivalent definitions. In this text, we only considered so far thin geodesic triangles, but other important characterizations involve the concept of mesh or the Gromov product. Another goal of ours was to develop a theory for an appropriate subclass of hyperbolic geodesic metric spaces which would play a similar role for the subclass of finitely generated virtually free groups. In [2], we introduced four equivalent geometric conditions on a geodesic metric space such that a finitely generated group $G = \langle A \rangle$ is virtually free if and only $\mathrm{Cay}_A(G)$ satisfies any one of these geometric conditions. Since these conditions are preserved through quasi-isometry, this does not depend on the finite generating set considered.

In the condition which is the analogue of thinness of geodesic triangles, we replaced triangles by arbitrary polygons. Consider a *geodesic polygon* $\{[x_0, x_1], [x_1, x_2], \ldots [x_{n-1}, x_n], [x_n, x_0]\}$. Given $\delta \geq 0$, we say that this polygon is δ-thin if every point in one of the geodesics it at distance $\leq \delta$ from some point in one of the other geodesics of the polygon. The geodesic metric space is *polygon hyperbolic* if there exists some $\delta \geq 0$ such that every geodesic polygon is δ-thin.

A few years later, my former PhD student André Carvalho developed also an interest on hyperbolic groups. A major achievement consisted on generalizing the *bounded reduction property* (BRP) of free groups to hyperbolic groups [5]. In the free group case, this is a quite useful property: if $\varphi : F_A \to F_A$ is an injective homomorphism, then for every reduced product $uv \in F_A$, there is a bounded number of letters cancelled in the reduction of $\varphi(u)\varphi(v)$. Carvalho presented some equivalent geometric characterizations of the BRP and showed that an endomorphism of a hyperbolic group satisfies the BRP if and only if it preserves a *coarse median* (a certain type of ternary operation defined by Brian Bowditch for metric spaces [4]). He also proved that uniformly continuous endomorphisms of a hyperbolic group (for a visual metric) always satisfy the BRP. As a consequence, he generalized a theorem, proved by Frédéric Paulin for automorphisms [13], showing that every uniformly continuous endomorphism of a hyperbolic group has a finitely generated subgroup of fixed points. And he showed that every uniformly continuous endomorphism of a hyperbolic group satisfies a Hölder condition, solving an open problem proposed on [1].

It should be noted that Carvalho extended these ideas beyond the realm of hyperbolic groups, proving theorems on the BRP for automatic groups [8], and the dynamics of continuous extensions to the boundary for endomorphisms of certain classes of graph groups such as $F_n \times \mathbb{Z}^m$ and $F_n \times F_m$ [6, 7]. In both these cases, the Gromov completion is appropriately replaced by the *Roller completion*.

Hyperbolic groups have been generalized over the years in different directions: automatic groups, semihyperbolic groups, (weakly) relatively hyperbolic groups hyperbolic semigroups... however, the exquisite harmony we can find in the theory of hyperbolic groups, where ideas from completely different areas converge to produce unexpected results, has yet to be replicated in these more general settings. If the great Japanese writer Yukio Mishima were a mathematician, he would probably be fond of hyperbolic groups!

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