QUANTUM UPPER TRIANGULAR MATRIX ALGEBRAS

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Abstract. Following the ideas in [12] and some inspiration from [9], we construct a bialgebra $T_q(n)$ and a pointed Hopf algebra $UT_q(n)$ which quantize the coordinate rings of the algebra of upper triangular matrices and of the group of invertible upper triangular matrices of size $n \geq 2$, respectively, where q is a nonzero parameter. The resulting structure on $UT_q(n)$ is neither commutative nor cocommutative, so we obtain a quantum group. The motivation comes from the idea of quantizing the incidence algebra of a finite poset, as the latter can be embedded as a subalgebra of the algebra of upper triangular matrices. After defining the bialgebra $T_q(n)$ and the Hopf algebra $UT_q(n)$, we study and compare their Lie algebras of derivations, their automorphism groups and their low degree Hochschild cohomology, in case $n = 2$.

Contents

Introduction

Incidence algebras and coalgebras, associated to a locally finite poset, are a fundamental object in combinatorics and number theory, as well as in topology, discrete geometry and representation theory (see e.g. [19, Chapter 3], [13]). For example, in [4] and [5] the authors show that the Hochschild cohomology of a simplicial complex associated to a finite poset is isomorphic, as a Gerstenhaber algebra, to the Hochschild cohomology of the corresponding incidence algebra (see also [7]). More recently, incidence algebras have become a fundamental tool in topological data analysis (see [3] for details). Both the algebra and the coalgebra structures of the incidence algebra have been instrumental in combinatorics and number theory, where the poset algebra plays a role analogous to that of the group algebra of a finite group.

Any finite poset has an extension into a totally ordered one, so incidence algebras of finite posets are subalgebras of the algebra of upper triangular matrices. As a first step towards the introduction of a *quantum incidence algebra*, in this paper we construct a noncommutative and noncocommutative bialgebra which is a quantum version of the coordinate ring of the algebra of upper triangular matrices. By formally setting an appropriate *quantum determinant* equal to one, we obtain a Hopf algebra $UT_q(n)$ which coacts on the uniparameter quantum affine space $A_n(q)$, where q is a fixed parameter from the base field. Despite being new, to our knowledge, the quantum group $UT_q(n)$ has a fairly straightforward description in terms of generators and relations, providing lots of interesting examples to study.

In this paper, after defining the bialgebra $T_q(n)$ and the Hopf algebra $UT_q(n)$, we explore a few of their properties, including the center and $*$-structures. Having *quantum symmetries* in mind, we study the low-dimensional Hochschild cohomology and the automorphism groups of $T_q(n)$ and of $UT_q(n)$ (both as algebras, bialgebras and Hopf algebras) in case $n = 2$.

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1. Preliminaries

Throughout the paper, K will denote a field of characteristic different from 2 and K^* its multiplicative group of units. All algebras, homomorphisms and tensor products will be considered over K, unless otherwise specified. Let A be an algebra. We will use the notation $Z(A)$ for the center of A, $\mathrm{Aut}(A)$ for its automorphism group, $\mathrm{Der}(A)$ for its Lie algebra of derivations, $\mathrm{IDer}(A)$ for the ideal of inner derivations and $\mathrm{HH}^k(A)$ for the k-th Hochschild cohomology group of A. So, in particular, $\mathrm{HH}^0(A) = Z(A)$ and $\mathrm{HH}^1(A) = \mathrm{Der}(A)/\,\mathrm{IDer}(A)$.

Given a set $X = \{x_i : i \in I\}$, the free unital associative algebra on X will be denoted by $K\langle X\rangle$ or $K\langle x_i : i \in I\rangle$. The identity map on the set X is written as id_X or id, if the set X is clear from the context. We denote by $\mathbb{Z}$ the set of integers and by $\mathbb{N}$ the set of nonnegative integers.

Recall (cf. [1]) that a (multiparameter) *quantum affine space* of dimension $n \geq 1$ is the quotient of $K\langle x_1, \dots, x_n\rangle$ by the relations

$$x_i x_j = q_{ij} x_j x_i,\ 1 \leq i, j \leq n, \tag{1}$$

where $Q := (q_{ij})_{i,j=1}^n \in (K^*)^{n^2}$ is a multiplicatively antisymmetric matrix; in other words, $q_{ji} = q_{ij}^{-1}$ and $q_{ii} = 1$, for all $1 \leq i, j \leq n$. We denote this algebra by $A_n(Q)$.

When $q_{ij} = q$ for all $i > j$, we write $A_n(q)$ instead of $A_n(Q)$. Thinking of $(x_i)_{i=1}^n$ in $(A_n(q))^n$ as a column vector, with the usual matricial order for the entries, we could represent the relations $x_i x_j = q x_j x_i$, for $i > j$, pictorially as $\begin{matrix} j \\ \uparrow q \\ i \end{matrix}$, which has the same meaning as $\begin{matrix} j \\ \downarrow q^{-1} \\ i \end{matrix}$.

Remark 1.1. Observe that $A_n(Q)$ is isomorphic to the iterated Ore extension $K[x_1][x_2;\sigma_2]\dots[x_n;\sigma_n]$, where σ_i is the automorphism of $K[x_1]\dots[x_{i-1};\sigma_{i-1}]$ given by $\sigma_i(x_j) = q_{ij}x_j$ for all $1 \leq j < i \leq n$. In particular, $A_n(Q)$ is a noetherian domain with K-basis formed by the (equivalence classes of) monomials $x^\nu := x_1^{\nu_1} \cdots x_n^{\nu_n}$, where $\nu \in \mathbb{N}^n$. Moreover, $A_n(Q)$ is a $\mathbb{Z}^n$-graded algebra whose homogeneous components are $A_n(Q)_\nu = Kx^\nu$, whenever $\nu \in \mathbb{N}^n$, and $A_n(Q)_\nu = \{0\}$, otherwise.

2. Quantum $T_q(n)$

Let $n \geq 2$ be an integer. We are going to propose a quantization of the algebra of upper triangular $n \times n$ matrices following the ideas in [12] (see also [8, Chapter IV]). Note that this is not a subalgebra of the algebra of quantum matrices (cf. [17, 2]), often denoted by $M_q(n)$ or $\mathcal{O}_q(M_n(K))$.

Let a_{ij}, $1 \leq i \leq j \leq n$, be variables that one can organize in the following upper triangular $n \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & a_{nn} \end{pmatrix}.$$

Fix the map ρ sending a_{ij} to $a_{n+1-j,n+1-i}$. In the language of matrices this is the reflection across the antidiagonal[1],

$$\rho(A) = \begin{pmatrix} a_{nn} & a_{n-1,n} & \dots & a_{1n} \\ 0 & a_{n-1,n-1} & \dots & a_{1,n-1} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & a_{11} \end{pmatrix}.$$

Fix $q \in K^*$ and consider the algebra $K\langle a_{ij} : 1 \leq i \leq j \leq n\rangle \otimes_K A_n(q)$. Following [12, 2, 8], we define

$$x_i' = \sum_{j=i}^n a_{ij} \otimes x_j \text{ and } x_i'' = \sum_{j=i}^n a_{n+1-j,n+1-i} \otimes x_j. \tag{2}$$

[1]It comes from the corresponding involution of the algebra of upper triangular $n \times n$ matrices.

Lemma 2.1. *One has* $x'_j x'_i = q x'_i x'_j$ *and* $x''_j x''_i = q x''_i x''_j$, $1 \le i < j \le n$, *if and only if*

$$a_{jk}a_{ik} = q a_{ik}a_{jk},\ \ i < j \le k, \tag{3}$$
$$a_{jk}a_{jl} = q a_{jl}a_{jk},\ \ j \le k < l, \tag{4}$$
$$a_{ik}a_{jl} = a_{jl}a_{ik},\ \ i < j \le l,\ \ i \le k < l, \tag{5}$$
$$a_{jk}a_{il} = q^2 a_{il}a_{jk},\ \ i < j \le k < l. \tag{6}$$

Proof. Let $i < j$. We have $x'_i = \sum_{k=i}^n a_{ik} \otimes x_k$ and $x'_j = \sum_{l=j}^n a_{jl} \otimes x_l$, so

$$\begin{aligned} x'_i x'_j &= \sum_{k=i}^n \sum_{l=j}^n a_{ik}a_{jl} \otimes x_k x_l \\ &= \sum_{k=j}^n a_{ik}a_{jk} \otimes x_k^2 + \sum_{i \le k < l,\ j \le l} a_{ik}a_{jl} \otimes x_k x_l + \sum_{j \le l < k} q a_{ik}a_{jl} \otimes x_l x_k \\ &= \sum_{k=j}^n a_{ik}a_{jk} \otimes x_k^2 + \sum_{i \le k < l,\ j \le l} a_{ik}a_{jl} \otimes x_k x_l + \sum_{j \le k < l} q a_{il}a_{jk} \otimes x_k x_l \\ &= \sum_{k=j}^n a_{ik}a_{jk} \otimes x_k^2 + \sum_{i \le k < j \le l} a_{ik}a_{jl} \otimes x_k x_l + \sum_{j \le k < l} (a_{ik}a_{jl} + q a_{il}a_{jk}) \otimes x_k x_l. \end{aligned}$$

Similarly,

$$x'_j x'_i = \sum_{k=j}^n a_{jk}a_{ik} \otimes x_k^2 + \sum_{i \le k < j \le l} q a_{jl}a_{ik} \otimes x_k x_l + \sum_{j \le k < l} (q a_{jl}a_{ik} + a_{jk}a_{il}) \otimes x_k x_l.$$

Thus, $x'_j x'_i = q x'_i x'_j$ if and only if

$$a_{jk}a_{ik} = q a_{ik}a_{jk},\ \ i < j \le k, \tag{7}$$
$$a_{jl}a_{ik} = a_{ik}a_{jl},\ \ i \le k < j \le l, \tag{8}$$
$$a_{ik}a_{jl} - a_{jl}a_{ik} = q^{-1}a_{jk}a_{il} - q a_{il}a_{jk},\ \ i < j \le k < l. \tag{9}$$

The equivalent conditions for $x''_j x''_i = q x''_i x''_j$ are obtained from (7)–(9) by applying ρ to each variable:

$$\begin{aligned} a_{k'j'}a_{k'i'} &= q a_{k'i'}a_{k'j'},\ \ k' \le j' < i', \\ a_{l'j'}a_{k'i'} &= a_{k'i'}a_{l'j'},\ \ l' \le j' < k' \le i', \\ a_{k'i'}a_{l'j'} - a_{l'j'}a_{k'i'} &= q^{-1}a_{k'j'}a_{l'i'} - q a_{l'i'}a_{k'j'},\ \ l' < k' \le j' < i', \end{aligned}$$

where $i' = n+1-i$, $j' = n+1-j$, $k' = n+1-k$ and $l' = n+1-l$. Renaming the indices (l', k', j', i') by (i, j, k, l), we obtain

$$a_{jk}a_{jl} = q a_{jl}a_{jk},\ \ j \le k < l, \tag{10}$$
$$a_{ik}a_{jl} = a_{jl}a_{ik},\ \ i \le k < j \le l, \tag{11}$$
$$a_{jl}a_{ik} - a_{ik}a_{jl} = q^{-1}a_{jk}a_{il} - q a_{il}a_{jk},\ \ i < j \le k < l. \tag{12}$$

In particular, we see that (8) is the same as (11), and, due to $\mathrm{char}(K) \ne 2$, (9) and (12) together are equivalent to

$$a_{ik}a_{jl} = a_{jl}a_{ik},\ a_{jk}a_{il} = q^2 a_{il}a_{jk},\ i < j \le k < l.$$

□

Definition 2.2. For $n \ge 2$, define $T_q(n)$ to be the quotient of $K\langle a_{ij} : 1 \le i \le j \le n\rangle$ by the ideal generated by relations (3)–(6).

As above for the quantum affine space, we can represent relations (3)–(6) pictorially as follows:

$$\begin{array}{ccc} (i,k) & \longrightarrow & (i,l) \\ \uparrow & \nearrow\!\!\!\nearrow & \uparrow \\ (j,k) & \longrightarrow & (j,l) \end{array} \tag{13}$$

The convention used in (13) is that the single horizontal and vertical arrows have label q, whereas the diagonal double arrow has label q^2; the absence of an edge means that the corresponding generators commute. Keeping the analogy with matrices, it is assumed that $i < j$ and $k < l$.

Proposition 2.3. *The map ρ sending a_{ij} to $a_{n+1-j,n+1-i}$ defines an automorphism of order 2 of $T_q(n)$.*

Proof. Although it could be checked directly that ρ preserves the defining relations of $T_q(n)$, this follows immediately from the symmetry of the diagram (13) under reflection across the antidiagonal. Clearly, $\rho^2 = \mathrm{id}_{T_q(n)}$ so ρ is an automorphism of order 2. □

Remark 2.4. When n is even, there exists an involution on the algebra of upper triangular matrices which is not equivalent to the reflection across the antidiagonal. This involution is given by the map $\theta(a_{ij}) = -a_{n+1-j,n+1-i}$, if $i \le \frac{n}{2} < j$, and $\theta(a_{ij}) = a_{n+1-j,n+1-i}$, otherwise. Applying θ to every variable in equalities (7)–(9), one obtains the same conditions given by equalities (10)–(12).

Corollary 2.5. *The algebra $T_q(n)$ is isomorphic to $A_{\binom{n+1}{2}}(Q)$ for some $Q = Q(q)$, where $\{a_{ij} : 1 \le i \le j \le n\}$ is ordered lexicographically.*

Example 2.6. The algebra $T_q(2)$ is the quotient of $K\langle a_{11}, a_{12}, a_{22}\rangle$ by the ideal generated by

$$a_{22}a_{12} = qa_{12}a_{22}, \tag{14}$$

$$a_{11}a_{12} = qa_{12}a_{11}, \tag{15}$$

$$a_{11}a_{22} = a_{22}a_{11}. \tag{16}$$

Observe that there are no relations of the form (6). It follows that $T_q(2)$ is isomorphic to $A_3(Q)$, where $Q = \begin{pmatrix} 1 & q & 1 \\ q^{-1} & 1 & q^{-1} \\ 1 & q & 1 \end{pmatrix}$. The relations (14)–(16) can be represented by the diagram $\begin{array}{ccc} a_{11} & \longrightarrow & a_{12} \\ & & \uparrow \\ & & a_{22} \end{array}$.

Unlike the case of the quantum full matrix algebra [8, Proposition IV.3.3], the center of $T_q(n)$ is trivial whenever q is not a root of unity.

Proposition 2.7. *If q is not a root of unity, then $Z(T_q(n)) = K$.*

Proof. By [1, p. 1788], $Z(T_q(n))$ is the K-space generated by those monomials

$$f = a_{11}^{\nu_{11}} \cdots a_{1n}^{\nu_{1n}} a_{22}^{\nu_{22}} \cdots a_{2n}^{\nu_{2n}} \cdots a_{n-1,n-1}^{\nu_{n-1,n-1}} a_{n-1,n}^{\nu_{n-1,n}} a_{nn}^{\nu_{nn}},$$

with $\nu_{kl} \in \mathbb{N}$, which are central.

Now, $a_{11}f = q^{\sum_{i=2}^n \nu_{1i}} f a_{11}$, so $\sum_{i=2}^n \nu_{1i} = 0$ and $\nu_{1i} = 0$ for all $2 \le i \le n$. Similarly, using the above, we have $a_{12}f = q^{-\nu_{11}-\nu_{22}} f a_{12}$, so $\nu_{11} = 0 = \nu_{22}$. This established the result in case $n = 2$. Suppose that $n > 2$. We have shown that f is in the subalgebra of $T_q(n)$ generated by $\{a_{ij} : 2 \le i \le j \le n\}$, and is thus a central monomial there. Since this subalgebra is isomorphic to $T_q(n-1)$, it follows by induction that $f = 1$, thus proving the claim. □

Let R be a K-algebra. An n-tuple $(r_1, \ldots, r_n) \in R^n$ is called an *R-point of $A_n(q)$* if $r_j r_i = q r_i r_j$ for all $i < j$. An $\binom{n+1}{2}$-tuple $(a_{ij})_{1 \le i \le j \le n} \in R^{\binom{n+1}{2}}$ is called an *R-point of $T_q(n)$* if the elements a_{ij} satisfy the relations (3)–(6) in R.

Remark 2.8. The R-points of $A_n(q)$ and $T_q(n)$ are in a one-to-one correspondence with the algebra morphisms $A_n(q) \to R$ and $T_q(n) \to R$, respectively.

The next fact is proved the same way as Lemma 2.1.

Remark 2.9. An $\binom{n+1}{2}$-tuple $A = (a_{ij})_{1\le i\le j\le n}$ in R is an R-point of $T_q(n)$ if and only if $AX := (\sum_{j=i}^n a_{ij} \otimes x_j)_{i=1}^n$ and $\rho(A)X := (\sum_{j=i}^n a_{n+1-j,n+1-i} \otimes x_j)_{i=1}^n$ are $(R \otimes A_n(q))$-points of $A_n(q)$, where $x_1, \dots, x_n$ are the canonical generators of $A_n(q)$.

The "only if" part of Remark 2.9 can be generalized as follows.

Remark 2.10. If $A = (a_{ij})_{1\le i\le j\le n}$ is an R-point of $T_q(n)$ and $X = (x_i)_{i=1}^n$ is an R'-point of $A_n(q)$, then $AX := (\sum_{j=i}^n a_{ij} \otimes x_j)_{i=1}^n$ and $\rho(A)X := (\sum_{j=i}^n a_{n+1-j,n+1-i} \otimes x_j)_{i=1}^n$ are $(R \otimes R')$-points of $A_n(q)$.

Lemma 2.11. *Let $A = (a_{ij})_{1\le i\le j\le n}$ and $B = (b_{ij})_{1\le i\le j\le n}$ be R-points of $T_q(n)$ such that $a_{ij}b_{kl} = b_{kl}a_{ij}$ for all $1 \le i \le j \le n$ and $1 \le k \le l \le n$. Then $AB := (\sum_{k=i}^j a_{ik}b_{kj})_{1\le i\le j\le n}$ is an R-point of $T_q(n)$.*

Proof. By Remark 2.9 the n-tuple $X' = (x'_i)_{i=1}^n := (\sum_{j=i}^n b_{ij} \otimes x_j)_{i=1}^n$ is an R'-point of $A_n(q)$, where $R' = R\otimes A_n(q)$. Applying Remark 2.10, we conclude that $X'' = (x''_i)_{i=1}^n := (\sum_{j=i}^n a_{ij}\otimes x'_j)_{i=1}^n = (\sum_{j=i}^n \sum_{k=i}^j a_{ik}\otimes (b_{kj}\otimes x_j))_{i=1}^n$ is an R''-point of $A_n(q)$, where $R'' = R\otimes R' = R\otimes(R\otimes A_n(q))$. Let $\varphi : R'' \to R'$ be the K-linear map given by $\varphi(a \otimes (b \otimes P)) = ab \otimes P$ for all $a, b \in R$ and $P \in A_n(q)$. Observe that for $t_1 = a_1 \otimes (b_1 \otimes P_1)$ and $t_2 = a_2 \otimes (b_2 \otimes P_2)$ with $a_2b_1 = b_1a_2$ one has

$$\begin{aligned}\varphi(t_1t_2) &= \varphi(a_1a_2 \otimes (b_1b_2 \otimes P_1P_2)) = a_1a_2b_1b_2 \otimes P_1P_2 = a_1b_1a_2b_2 \otimes P_1P_2\\ &= (a_1b_1 \otimes P_1)(a_2b_2 \otimes P_2) = \varphi(t_1)\varphi(t_2).\end{aligned}$$

It follows that $\varphi(x''_i x''_j) = \varphi(x''_i)\varphi(x''_j)$ for all $1 \le i, j \le n$. Hence, $(\varphi(x''_i))_{i=1}^n$ is an R'-point of $A_n(q)$. Since $\varphi(x''_i) = \sum_{j=i}^n \sum_{k=i}^j a_{ik}b_{kj} \otimes x_j = \sum_{j=i}^n (AB)_{ij} \otimes x_j$, we see that $(\varphi(x''_i))_{i=1}^n = (AB)X$, and thus $(AB)X$ is an R'-point of $A_n(q)$. Similarly one proves that $\rho(AB)X = (\rho(B)\rho(A))X$ is an R'-point of $A_n(q)$. By Remark 2.9 the tuple AB is an R-point of $T_q(n)$. □

We are finally ready to prove the main result of this section.

Theorem 2.12. *There are algebra morphisms $\Delta : T_q(n) \to T_q(n) \otimes T_q(n)$ and $\varepsilon : T_q(n) \to K$, given by $\Delta(a_{ij}) = \sum_{k=i}^j a_{ik} \otimes a_{kj}$ and $\varepsilon(a_{ij}) = \delta_{ij}$, $1 \le i \le j \le n$. These define a bialgebra structure on $T_q(n)$.*

Proof. We first show that Δ is a well-defined algebra morphism, i.e. $(\Delta(a_{ij}))_{1\le i\le j\le n}$ is a $(T_q(n) \otimes T_q(n))$-point of $T_q(n)$. Let $A = (a_{ij} \otimes 1)_{1\le i\le j\le n}$ and $B = (1 \otimes a_{ij})_{1\le i\le j\le n}$. It is clear that A and B are $(T_q(n) \otimes T_q(n))$-points of $T_q(n)$. Since $(a_{ij} \otimes 1)(1 \otimes a_{kl}) = a_{ij} \otimes a_{kl} = (1 \otimes a_{kl})(a_{ij} \otimes 1)$, then Lemma 2.11 guarantees that $AB = (\sum_{k=i}^j (a_{ik} \otimes 1)(1 \otimes a_{kj}))_{1\le i\le j\le n} = (\sum_{k=i}^j a_{ik} \otimes a_{kj})_{1\le i\le j\le n} = (\Delta(a_{ij}))_{1\le i\le j\le n}$ is a $(T_q(n) \otimes T_q(n))$-point of $T_q(n)$, as needed.

Similarly, ε is a well-defined algebra morphism because $(\delta_{ij})_{1\le i\le j\le n}$ is a K-point of $T_q(n)$ (due to $\delta_{ij}\delta_{kl} = 0$ for $i < j$ or $k < l$).

The coassociativity and counit axioms are verified the same way as for incidence coalgebras (see [18, Definition 3.2.6]). □

3. The Hopf algebra $UT_q(n)$

Now, we are going to introduce the analog of the well-known Hopf algebra $GL_q(n)$ (see [17], [8, IV.6, IV.10] and references therein).

Lemma 3.1. *Let $X \subseteq [1, n]$ and $1 \le i < j \le n$. Then the following equality holds in $T_q(n)$:*

$$a_{ij}\left(\prod_{x\in X} a_{xx}\right) = q^m \left(\prod_{x\in X} a_{xx}\right) a_{ij}, \tag{17}$$

where $m = -2|\{x \in X : i < x < j\}| - |X \cap \{i, j\}|$.

Proof. We have $a_{ij}a_{ii} = q^{-1}a_{ii}a_{ij}$, $a_{ij}a_{jj} = q^{-1}a_{jj}a_{ij}$, $a_{ij}a_{xx} = q^{-2}a_{xx}a_{ij}$ for all $i < x < j$ and $a_{ij}a_{xx} = a_{xx}a_{ij}$ for all $x \notin [i, j]$, whence (17). □

Set $\det_q(n) := \prod_{i=1}^n a_{ii} \in T_q(n)$.

Corollary 3.2. *Let $1 \le i \le j \le n$. Then*

$$\det{}_q(n) \cdot a_{ij} = q^{2(j-i)}a_{ij} \cdot \det{}_q(n). \tag{18}$$

Observe from Corollary 3.2 that if we were to define an analog of $GL_q(n)$ in the same way as was done in [8, IV.6] by taking the quotient of the polynomial algebra $T_q(n)[t]$ by the ideal $(t\det_q(n)-1)$ generated by $t\det_q(n)-1$ then, in case q is not a root of unity, we would get $a_{ij}=0$ in $T_q(n)[t]/(t\det_q(n)-1)$ for all $i<j$ because, unlike the case of $M_q(n)$, the element $\det_q(n)$ is not central in $T_q(n)$. Thus, to avoid working with a commutative and cocommutative Hopf algebra, it seems to be more reasonable to replace $T_q(n)[t]$ with the skew polynomial algebra $T_q(n)[t;\sigma]$, where σ is an automorphism of the algebra $T_q(n)$ that takes (18) into account.

Lemma 3.3. *Let $(A,\mu,\eta,\Delta,\varepsilon)$ be a bialgebra and $\sigma\in\mathrm{Aut}(A,\mu,\eta)$. Then $\Delta(t)=t\otimes t$ and $\varepsilon(t)=1$ extend the bialgebra structure to the skew polynomial algebra $A[t;\sigma]$ if and only if σ is a bialgebra automorphism of $(A,\mu,\eta,\Delta,\varepsilon)$.*

Proof. It is well-known (see, for example, [16, Exercise 2.1.19]) that the tensor product of two coalgebras $(C,\Delta_C,\varepsilon_C)$ and $(D,\Delta_D,\varepsilon_D)$ is a coalgebra under $\Delta_{C\otimes D}=(\mathrm{id}_C\otimes\tau_{C,D}\otimes\mathrm{id}_D)\circ(\Delta_C\otimes\Delta_D)$, where $\tau_{C,D}: C\otimes D\to D\otimes C$ is the flip map, and $\varepsilon_{C\otimes D}=\varepsilon_C\otimes\varepsilon_D$. Hence, there is a natural coalgebra structure on the vector space $A[t;\sigma]\cong A\otimes K[t]$ (vector space isomorphism!), such that

$$\Delta'(at^n)=\sum a_{(1)}t^n\otimes a_{(2)}t^n \text{ and } \varepsilon'(at^n)=\varepsilon(a) \tag{19}$$

(here we use Sweedler's notation for $\Delta(a)=\sum a_{(1)}\otimes a_{(2)}$ and identify $at^n\in A[t;\sigma]$ with $a\otimes t^n\in A\otimes K[t]$). Thus, $A[t;\sigma]$ is a bialgebra if and only if Δ' and ε' are algebra morphisms $A[t;\sigma]\to A[t;\sigma]\otimes A[t;\sigma]$ and $A[t;\sigma]\to K$, respectively. Observe that (19) already means that $\Delta'(at)=\Delta'(a)\Delta'(t)$ and $\varepsilon'(at)=\varepsilon'(a)\varepsilon'(t)$ for all $a\in A$. Therefore, Δ' is an algebra morphism if and only if

$$\Delta'(ta)=\Delta'(t)\Delta'(a) \tag{20}$$

for all $a\in A$. Now,

$$\Delta'(t)\Delta'(a)=\sum ta_{(1)}\otimes ta_{(2)}=\sum\sigma(a_{(1)})t\otimes\sigma(a_{(2)})t=((\sigma\otimes\sigma)\circ\Delta)(a)(t\otimes t),$$
$$\Delta'(ta)=\Delta'(\sigma(a)t)=\sum\sigma(a)_{(1)}t\otimes\sigma(a)_{(2)}t=(\Delta\circ\sigma)(a)(t\otimes t).$$

Thus, (20) is equivalent to $(\sigma\otimes\sigma)\circ\Delta=\Delta\circ\sigma$. Similarly, ε' is an algebra morphism if and only if $\varepsilon'(ta)=\varepsilon'(t)\varepsilon'(a)$ for all $a\in A$ if and only if $\varepsilon\circ\sigma=\varepsilon$. □

Lemma 3.4. *There is a bialgebra automorphism $\sigma\in\mathrm{Aut}(T_q(n))$ such that*

$$\sigma(a_{ij})=q^{2(i-j)}a_{ij} \tag{21}$$

for all $1\le i\le j\le n$.

Proof. It is obvious that $(q^{2(i-j)}a_{ij})_{1\le i\le j\le n}$ is a $T_q(n)$-point of $T_q(n)$, so there is a unique $\sigma\in\mathrm{End}(T_q(n))$ determined by (21). It is clearly bijective with $\sigma^{-1}(a_{ij})=q^{2(j-i)}a_{ij}$. We have

$$\begin{aligned}((\sigma\otimes\sigma)\circ\Delta)(a_{ij})&=\sum_{i\le k\le j}\sigma(a_{ik})\otimes\sigma(a_{kj})=\sum_{i\le k\le j}q^{2(i-k)}a_{ik}\otimes q^{2(k-j)}a_{kj}\\&=q^{2(i-j)}\sum_{i\le k\le j}a_{ik}\otimes a_{kj}=q^{2(i-j)}\Delta(a_{ij})=\Delta(\sigma(a_{ij})),\\(\varepsilon\circ\sigma)(a_{ij})&=\varepsilon\left(q^{2(i-j)}a_{ij}\right)=q^{2(i-j)}\delta_{ij}=\delta_{ij}=\varepsilon(a_{ij}),\end{aligned}$$

so σ is a bialgebra morphism. □

Corollary 3.5. *Let $\sigma\in\mathrm{Aut}(T_q(n))$ be given by (21). Then the bialgebra structure on $T_q(n)$ extends to a bialgebra structure on $T_q(n)[t;\sigma]$ by means of $\Delta(t)=t\otimes t$ and $\varepsilon(t)=1$.*

Remark 3.6. For all $b\in T_q(n)$, by Corollary 3.2 we have

$$\det_q(n)\cdot b=\sigma^{-1}(b)\cdot\det_q(n).$$

Moreover, $t\det_q(n)=\det_q(n)t$ is central in $T_q(n)[t;\sigma]$.

Let $(t\det_q(n)-1)$ be the ideal of the algebra $T_q(n)[t;\sigma]$ generated by $t\det_q(n)-1$.

Lemma 3.7. *The ideal $(t\det_q(n)-1)$ is also a coideal of the coalgebra $T_q(n)[t;\sigma]$.*

Proof. We have

$$\Delta(t\mathrm{det}_q(n)-1) = (t\otimes t)\prod_{i=1}^{n}(a_{ii}\otimes a_{ii}) - 1\otimes 1 = t\mathrm{det}_q(n)\otimes t\mathrm{det}_q(n) - 1\otimes 1$$
$$= t\mathrm{det}_q(n)\otimes (t\mathrm{det}_q(n)-1) + (t\mathrm{det}_q(n)-1)\otimes 1,$$

and $\varepsilon(t\mathrm{det}_q(n)-1) = 1-1 = 0$. □

Define $UT_q(n)$ as the quotient algebra $T_q(n)[t;\sigma]/(t\mathrm{det}_q(n)-1)$.

Corollary 3.8. *The algebra $UT_q(n)$ is a bialgebra under the comultiplication and counit induced by those from $T_q(n)[t;\sigma]$.*

The set $\Sigma = \{a_{11}^{\nu_{11}}a_{22}^{\nu_{22}}\cdots a_{nn}^{\nu_{nn}} : \nu_{ii}\in\mathbb{N},\ \text{for all } 1\le i\le n\}$ is a multiplicative set of normal [6, p. 214] elements in $T_q(n)$, hence an Ore set [6, p. 82]. Thus, since $T_q(n)$ is a domain, we can consider the corresponding localization $T_q(n)\Sigma^{-1}$. Since $\mathrm{det}_q(n) = \prod_i a_{ii}$, it follows that $T_q(n)\Sigma^{-1} = T_q(n)[\mathrm{det}_q(n)^{-1}]$, the localization of $T_q(n)$ at the powers of $\mathrm{det}_q(n)$. Since the elements a_{ii} are group-like, it follows that the bialgebra structure on $T_q(n)$ extends uniquely to $T_q(n)[\mathrm{det}_q(n)^{-1}]$ in such a way that $\Delta(a_{ii}^{-1}) = a_{ii}^{-1}\otimes a_{ii}^{-1}$ and $\varepsilon(a_{ii}^{-1}) = 1$, for all $1\le i\le n$.

Proposition 3.9. *There is a canonical bialgebra isomorphism $T_q(n)[\mathrm{det}_q(n)^{-1}]\to UT_q(n)$ sending a_{ij} to the class of a_{ij} in $UT_q(n)$, for all $1\le i\le j\le n$.*

Proof. Although this is straightforward, we will provide the details for the convenience of the reader less familiar with noncommutative localization.

Let $\Psi : T_q(n)\to UT_q(n)$ be the following composition

$$T_q(n)\xrightarrow{\ \iota\ } T_q(n)[t;\sigma]\xrightarrow{\ \pi\ }\frac{T_q(n)[t;\sigma]}{(t\mathrm{det}_q(n)-1)} = UT_q(n),$$

where ι is the inclusion and π is the canonical epimorphism. So $\Psi(a_{ij}) = a_{ij}+I$, where $I = (t\mathrm{det}_q(n)-1)$. We have that $\Psi(\mathrm{det}_q(n)) = \mathrm{det}_q(n)+I$ and $(t+I)(\mathrm{det}_q(n)+I) = t\mathrm{det}_q(n)+I = 1+I = (\mathrm{det}_q(n)+I)(t+I)$. Therefore, by the universal property of localization, Ψ extends uniquely to a bialgebra homomorphism $T_q(n)[\mathrm{det}_q(n)^{-1}]\to UT_q(n)$, still denoted by Ψ.

For the inverse map, consider the natural inclusion $\Phi : T_q(n)\to T_q(n)[\mathrm{det}_q(n)^{-1}]$. By Remark 3.6, for any $b\in T_q(n)$, we have

$$\mathrm{det}_q(n)^{-1}\Phi(b) = \mathrm{det}_q(n)^{-1}b = \sigma(b)\mathrm{det}_q(n)^{-1} = \Phi(\sigma(b))\mathrm{det}_q(n)^{-1}.$$

Thus, by the universal property of Ore extensions, Φ extends to $\Phi : T_q(n)[t;\sigma]\to T_q(n)[\mathrm{det}_q(n)^{-1}]$ so that $\Phi(t) = \mathrm{det}_q(n)^{-1}$. Moreover, $\Phi(t\mathrm{det}_q(n)-1) = \mathrm{det}_q(n)^{-1}\mathrm{det}_q(n)-1 = 0$, so Φ factors through a map $UT_q(n)\to T_q(n)[\mathrm{det}_q(n)^{-1}]$, which is easily seen to be inverse to Ψ, yielding the result. □

Remark 3.10. By Proposition 3.9, we can and will think of $UT_q(n)$ as the localization of $T_q(n)$ at powers of $a_{11},\dots,a_{nn}$ and identify $T_q(n)$ with a subalgebra of $UT_q(n)$.

Define the following elements in $T_q(n)$:

$$b_{ii} = \prod_{k\ne i} a_{kk},\ 1\le i\le n, \tag{22}$$

$$b_{ij} = \sum_{s=1}^{j-i}\ \sum_{i=i_0<\cdots<i_s=j}(-1)^s q^{2(j-i)-s}a_{i_0i_1}\dots a_{i_{s-1}i_s}\prod_{k\notin\{i_0,\dots,i_s\}}a_{kk},\ 1\le i<j\le n. \tag{23}$$

Remark 3.11. It can be easily checked that the elements $(b_{ij})_{1\le i\le j\le n}$ satisfy the following recurrence relation for $i<j$:

$$b_{ij} = -\sum_{k=i+1}^{j} q^{2(k-i)-1}a_{ik}b_{kj}a_{ii}^{-1} = -\sum_{k=i+1}^{j} q^{2(k-i)-1}a_{ik}a_{ii}^{-1}b_{kj}, \tag{24}$$

where the terms $b_{kj}a_{ii}^{-1}$ and $a_{ii}^{-1}b_{kj}$ in (24) belong to $T_q(n)$ and are equal since, for $k > i$, b_{kj} is a right and left multiple of a_{ii}, $T_q(n)$ is a domain and a_{ii} commutes with b_{kj} (see also Remark 3.10).

Recall the automorphism ρ from Proposition 2.3 and the automorphism σ from (21).

Lemma 3.12. *We have, for all $1 \le i \le j \le n$, $\rho(b_{ij}) = b_{n+1-j,n+1-i}$ and $\sigma(b_{ij}) = q^{2(i-j)}b_{ij}$.*

Proof. The result holds trivially in the case $i = j$. So, suppose that $i < j$. Notice first that the elements $a_{i_0 i_1}, \dots, a_{i_{s-1} i_s}$ appearing in (23) pairwise commute, and the same holds for the elements of the form a_{kk}. Given $1 \le i \le n$, set $\bar{i} = n + 1 - i$. Then we have

$$\begin{aligned}\rho(b_{ij}) &= \sum_{s=1}^{j-i} \sum_{i=i_0<\cdots<i_s=j} (-1)^s q^{2(j-i)-s} a_{\bar{i_s}\,\overline{i_{s-1}}} \dots a_{\bar{i_1}\,\bar{i_0}} \prod_{k\notin\{i_0,\dots,i_s\}} a_{\bar{k}\,\bar{k}} \\ &= \sum_{s=1}^{\bar{i}-\bar{j}} \sum_{\bar{j}=\bar{i_s}<\cdots<\bar{i_0}=\bar{i}} (-1)^s q^{2(\bar{i}-\bar{j})-s} a_{\bar{i_s}\,\overline{i_{s-1}}} \dots a_{\bar{i_1}\,\bar{i_0}} \prod_{\bar{k}\notin\{\bar{i_s},\dots,\bar{i_0}\}} a_{\bar{k}\,\bar{k}} \\ &= b_{\bar{j}\,\bar{i}}.\end{aligned}$$

The result for σ is straightforward. □

The elements $(b_{ij})_{1\le i\le j\le n}$ are the main ingredient for constructing an antipode map for $UT_q(n)$. This will be already perceived in the next result.

Lemma 3.13. *For all $1 \le i \le j \le n$, we have the following relation in $T_q(n)$:*

$$\sum_{i\le k\le j} b_{ik}a_{kj} = \det_q(n)\delta_{ij} = \sum_{i\le k\le j} q^{2(k-j)}a_{ik}b_{kj}. \tag{25}$$

Proof. We proceed by induction on $j - i \ge 0$. The case $i = j$ is obvious. If $i < j$ then, using (24), we get

$$\begin{aligned}\sum_{i\le k\le j} b_{ik}a_{kj} &= b_{ii}a_{ij} - \sum_{k=i+1}^{j}\sum_{l=i+1}^{k} q^{2(l-i)-1}a_{il}a_{ii}^{-1}b_{lk}a_{kj} \\ &= b_{ii}a_{ij} - \sum_{l=i+1}^{j} q^{2(l-i)-1}a_{il}a_{ii}^{-1}\sum_{k=l}^{j} b_{lk}a_{kj} \\ &= b_{ii}a_{ij} - \sum_{l=i+1}^{j} q^{2(l-i)-1}a_{il}a_{ii}^{-1}\det_q(n)\delta_{lj} \\ &= b_{ii}a_{ij} - q^{2(j-i)-1}a_{ij}a_{ii}^{-1}\det_q(n) \\ &= b_{ii}a_{ij} - q^{2(j-i)-1}a_{ij}b_{ii}.\end{aligned}$$

To finish the calculation, notice, using Remark 3.6, that

$$\begin{aligned}a_{ii}(b_{ii}a_{ij} - q^{2(j-i)-1}a_{ij}b_{ii}) &= \det_q(n)a_{ij} - q^{2(j-i)-1}a_{ii}a_{ij}b_{ii} \\ &= q^{2(j-i)}a_{ij}\det_q(n) - q^{2(j-i)}a_{ij}a_{ii}b_{ii} = 0.\end{aligned}$$

Since $T_q(n)$ is a domain, we conclude that $b_{ii}a_{ij} - q^{2(j-i)-1}a_{ij}b_{ii}=0$.

The other equality is proved similarly using the recurrence relation

$$b_{ij} = -\sum_{l=i}^{j-1} q^{2(j-l)-1}a_{lj}a_{jj}^{-1}b_{il}, \tag{26}$$

for $i < j$, which can be obtained from (24) applied to $b_{n+1-j,n+1-i}$ and then using the automorphism ρ. □

Our main goal is to show that $UT_q(n)$ is a Hopf algebra. To accomplish this, we must show that $\{tb_{ij}\}_{1\le i\le j\le n}$ is a $UT_q(n)$-point of $T_{q^{-1}}(n)$, which will follow from a series of technical lemmas which we start presenting.

Lemma 3.14. *Let $1 \le k \le n$ and $1 \le i < j \le n$. Then*

$$a_{kk}b_{ij} = q^m b_{ij}a_{kk}, \tag{27}$$

where

$$m = 2|\{x \in \{k\} : i < x < j\}| + |\{k\} \cap \{i,j\}| = \begin{cases} 0, & k \notin [i,j], \\ 1, & k \in \{i,j\}, \\ 2, & i < k < j. \end{cases} \tag{28}$$

Proof. In view of (23), it suffices to prove (27) with b_{ij} replaced with $p = a_{i_0i_1} \dots a_{i_{s-1}i_s}$, where $i = i_0 < \dots < i_s = j$.

Case 1: $k \notin [i,j]$. Then $a_{kk}a_{i_ui_{u+1}} = a_{i_ui_{u+1}}a_{kk}$ for all $0 \le u < s$, whence $a_{kk}p = pa_{kk}$.

Case 2: $k = i$. Then $a_{kk}a_{i_0i_1} = qa_{i_0i_1}a_{kk}$ and $a_{kk}a_{i_ui_{u+1}} = a_{i_ui_{u+1}}a_{kk}$ for all $u \neq 0$. Hence, $a_{kk}p = qpa_{kk}$.

Case 3: $k = j$. Follows by applying Case 2 to $a_{n+1-j,n+1-j}b_{n+1-j,n+1-i}$ and using ρ.

Case 4: $i < k < j$. There are two subcases.

Case 4.1: There is $0 < u < s$ such that $k = i_u$. Then $a_{kk}a_{i_{u-1}i_u} = qa_{i_{u-1}i_u}a_{kk}$, $a_{kk}a_{i_ui_{u+1}} = qa_{i_ui_{u+1}}a_{kk}$ and $a_{kk}a_{i_vi_{v+1}} = a_{i_vi_{v+1}}a_{kk}$ for all $v \notin \{u-1, u\}$. Hence, $a_{kk}p = q^2pa_{kk}$.

Case 4.2: There is $0 \le u < s$ such that $i_u < k < i_{u+1}$. Then $a_{kk}a_{i_ui_{u+1}} = q^2a_{i_ui_{u+1}}a_{kk}$ and $a_{kk}a_{i_vi_{v+1}} = a_{i_vi_{v+1}}a_{kk}$ for all $v \neq u$. Hence, $a_{kk}p = q^2pa_{kk}$. □

Lemma 3.15. *Let $1 \le k \le n$ and $1 \le i < j \le n$. Then*

$$b_{kk}\sigma(b_{ij}) = q^{-m}b_{ij}b_{kk}, \tag{29}$$

where $m = 2|\{x \in \{k\} : i < x < j\}| + |\{k\} \cap \{i,j\}|$, as in (28).

Proof. We have, using Lemma 3.14 and Remark 3.6,

$$\begin{aligned} a_{kk}(b_{kk}\sigma(b_{ij}) - q^{-m}b_{ij}b_{kk}) &= a_{kk}b_{kk}\sigma(b_{ij}) - q^{-m}a_{kk}b_{ij}b_{kk} \\ &= \det{}_q(n)\sigma(b_{ij}) - b_{ij}a_{kk}b_{kk} \\ &= \det{}_q(n)\sigma(b_{ij}) - b_{ij}\det{}_q(n) = 0. \end{aligned}$$

Since $T_q(n)$ is a domain, we get (29). □

Lemma 3.16. *Let $1 \le k < l \le n$ and $1 \le i < j \le n$. Then*

$$a_{kl}b_{ij} = q^m b_{ij}\sigma(a_{kl}), \tag{30}$$

where

$$m = \begin{cases} 0 & l < i \ \text{ or } \ k > j, \\ 1, & l = i \ \text{ or } \ k = j, \\ 2, & \text{otherwise.} \end{cases}$$

Proof. Let $p_1 = a_{i_0i_1} \dots a_{i_{s-1}i_s}$, where $i = i_0 < \dots < i_s = j$, and $p_2 = \prod_{x \in X} a_{xx}$, where $X = [1,n] \setminus \{i_0, \dots, i_s\}$.

Case 1: $l < i$. Then $a_{kl}a_{i_ui_{u+1}} = a_{i_ui_{u+1}}a_{kl}$ for all $0 \le u \le s-1$, whence $a_{kl}p_1 = p_1a_{kl}$. Since $[k,l] \subseteq X$, we have $a_{kl}p_2 = q^{2(k-l)}p_2a_{kl} = p_2\sigma(a_{kl})$ by Lemma 3.1.

Case 2: $l = i$. Then $a_{kl}p_1 = p_1a_{kl}$ as in Case 1. Since $\{x : k \le x < l\} \subseteq X$, but $l = i_0$, we have $a_{kl}p_2 = q^{-2(l-k)+1}p_2a_{kl} = qp_2\sigma(a_{kl})$ by Lemma 3.1.

Case 3: $i < l < j$.

Case 3.1: $k < i < l < j$.

Case 3.1.1: $i_u < l < i_{u+1}$ for some $0 \le u < s$. Then $a_{kl}a_{i_vi_{v+1}} = q^{-2}a_{i_vi_{v+1}}a_{kl}$ for all $0 \le v < u$ and $a_{kl}a_{i_vi_{v+1}} = a_{i_vi_{v+1}}a_{kl}$ for all $u \le v < s$, whence $a_{kl}p_1 = q^{-2u}p_1a_{kl}$. Since $|\{x \in X : k < x < l\}| = |\{k < x < l : x \notin \{i_0, \dots, i_u\}\}| = l-k-u-2$ and $\{k,l\} \subseteq X$, then $a_{kl}p_2 = q^{-2(l-k-u-1)}p_2a_{kl}$ by Lemma 3.1. Thus, $a_{kl}b_{ij} = q^2b_{ij}\sigma(a_{kl})$.

Case 3.1.2: $l = i_u$ for some $0 \le u < s$. Then $a_{kl}a_{i_vi_{v+1}} = q^{-2}a_{i_vi_{v+1}}a_{kl}$ for all $0 \le v < u-1$, $a_{kl}a_{i_{u-1}i_u} = q^{-1}a_{i_{u-1}i_u}a_{kl}$ and $a_{kl}a_{i_vi_{v+1}} = a_{i_vi_{v+1}}a_{kl}$ for all $u \le v < s$, whence $a_{kl}p_1 = q^{-2u+1}p_1a_{kl}$. Since $|\{x \in X : k < x < l\}| = |\{k < x < l : x \notin \{i_0, \dots, i_{u-1}\}\}| = l-k-u-1$ and $\{k,l\} \setminus \{i_0, \dots, i_s\} = \{k\}$, then $a_{kl}p_2 = q^{-2(l-k-u-1)-1}p_2a_{kl}$ by Lemma 3.1. Thus, $a_{kl}b_{ij} = q^2b_{ij}\sigma(a_{kl})$.

Case 3.2: $k = i < l < j$.

Case 3.2.1: $i_0 < l < i_1$. Then $a_{kl}a_{i_0i_1} = qa_{i_0i_1}a_{kl}$ and $a_{kl}a_{i_ui_{u+1}} = a_{i_ui_{u+1}}a_{kl}$ for all $0 < u < s$, whence $a_{kl}p_1 = qp_1a_{kl}$. Since $x \notin \{i_0, \dots, i_s\}$ for all $k < x < l$ and $\{k, l\} \setminus \{i_0, \dots, i_s\} = \{l\}$, then $a_{kl}p_2 = q^{-2(l-k-1)-1}p_2a_{kl}$ by Lemma 3.1. Thus, $a_{kl}b_{ij} = q^2b_{ij}\sigma(a_{kl})$.

Case 3.2.2: $i_u < l < i_{u+1}$ for some $0 < u < s$. Then $a_{kl}a_{i_0i_1} = q^{-1}a_{i_0i_1}a_{kl}$, $a_{kl}a_{i_vi_{v+1}} = q^{-2}a_{i_vi_{v+1}}a_{kl}$ for all $0 < v < u$ and $a_{kl}a_{i_vi_{v+1}} = a_{i_vi_{v+1}}a_{kl}$ for all $u \le v < s$, whence $a_{kl}p_1 = q^{-2u+1}p_1a_{kl}$. Since $|\{x \in X : k < x < l\}| = |\{k < x < l : x \notin \{i_1, \dots, i_u\}\}| = l - k - u - 1$ and $\{k, l\} \setminus \{i_0, \dots, i_s\} = \{l\}$, then $a_{kl}p_2 = q^{-2(l-k-u-1)-1}p_2a_{kl}$ by Lemma 3.1. Thus, $a_{kl}b_{ij} = q^2b_{ij}\sigma(a_{kl})$.

Case 3.2.3: $l = i_u$ for some $1 < u < s$. Then $a_{kl}a_{i_0i_1} = q^{-1}a_{i_0i_1}a_{kl}$, $a_{kl}a_{i_vi_{v+1}} = q^{-2}a_{i_vi_{v+1}}a_{kl}$ for all $0 < v < u-1$, $a_{kl}a_{i_{u-1}i_u} = q^{-1}a_{i_{u-1}i_u}a_{kl}$ and $a_{kl}a_{i_vi_{v+1}} = a_{i_vi_{v+1}}a_{kl}$ for all $u \le v < s$, whence $a_{kl}p_1 = q^{-2(u-1)}p_1a_{kl}$. Since $|\{x \in X : k < x < l\}| = |\{k < x < l : x \notin \{i_1, \dots, i_{u-1}\}\}| = l - k - u$ and $\{k, l\} \subseteq \{i_0, \dots, i_s\}$, then $a_{kl}p_2 = q^{-2(l-k-u)}p_2a_{kl}$ by Lemma 3.1. Thus, $a_{kl}b_{ij} = q^2b_{ij}\sigma(a_{kl})$.

Case 3.2.4: $l = i_1$. Then $a_{kl}a_{i_ui_{u+1}} = a_{i_ui_{u+1}}a_{kl}$ for all $0 \le u < s$, whence $a_{kl}p_1 = p_1a_{kl}$. Since $x \notin \{i_0, \dots, i_s\}$ for all $k < x < l$ and $\{k, l\} \subseteq \{i_0, \dots, i_s\}$, then $a_{kl}p_2 = q^{-2(l-k-1)}p_2a_{kl}$ by Lemma 3.1. Thus, $a_{kl}b_{ij} = q^2b_{ij}\sigma(a_{kl})$.

Case 3.3: $i < k < l < j$.

Case 3.3.1: $i_u < k < i_{u+1} \le i_v < l < i_{v+1}$ for some $0 \le u < v < s$. Then $a_{kl}a_{i_hi_{h+1}} = q^{-2}a_{i_hi_{h+1}}a_{kl}$, whenever $u < h < v$, and $a_{kl}a_{i_hi_{h+1}} = a_{i_hi_{h+1}}a_{kl}$, otherwise, whence $a_{kl}p_1 = q^{-2(v-u-1)}p_1a_{kl}$. Since $|\{x \in X : k < x < l\}| = |\{k < x < l : x \notin \{i_{u+1}, \dots, i_v\}\}| = l - k - v + u - 1$ and $\{k, l\} \subseteq X$, then $a_{kl}p_2 = q^{-2(l-k-v+u)}p_2a_{kl}$ by Lemma 3.1. Thus, $a_{kl}b_{ij} = q^2b_{ij}\sigma(a_{kl})$.

Case 3.3.2: $i_u = k < i_{u+1} \le i_v < l < i_{v+1}$ for some $0 < u < v < s$. Then $a_{kl}a_{i_ui_{u+1}} = q^{-1}a_{i_ui_{u+1}}a_{kl}$, $a_{kl}a_{i_hi_{h+1}} = q^{-2}a_{i_hi_{h+1}}a_{kl}$, whenever $u < h < v$, and $a_{kl}a_{i_hi_{h+1}} = a_{i_hi_{h+1}}a_{kl}$, otherwise, whence $a_{kl}p_1 = q^{-2(v-u)+1}p_1a_{kl}$. Since $|\{x \in X : k < x < l\}| = |\{k < x < l : x \notin \{i_{u+1}, \dots, i_v\}\}| = l - k - v + u - 1$ and $\{k, l\} \setminus \{i_0, \dots, i_s\} = \{l\}$, then $a_{kl}p_2 = q^{-2(l-k-v+u-1)-1}p_2a_{kl}$ by Lemma 3.1. Thus, $a_{kl}b_{ij} = q^2b_{ij}\sigma(a_{kl})$.

Case 3.3.3: $i_u < k < i_{u+1} \le i_v < l = i_{v+1}$ for some $0 \le u < v < s-1$. This case is obtained from Case 3.3.2 by applying the automorphism ρ.

Case 3.3.4: $i_u = k < i_{u+1} \le i_v < l = i_{v+1}$ for some $0 < u < v < s-1$. Then $a_{kl}a_{i_ui_{u+1}} = q^{-1}a_{i_ui_{u+1}}a_{kl}$, $a_{kl}a_{i_vi_{v+1}} = q^{-1}a_{i_vi_{v+1}}a_{kl}$, $a_{kl}a_{i_hi_{h+1}} = q^{-2}a_{i_hi_{h+1}}a_{kl}$ for $u < h < v$ and $a_{kl}a_{i_hi_{h+1}} = a_{i_hi_{h+1}}a_{kl}$, for $h \notin [u, v]$, whence $a_{kl}p_1 = q^{-2(v-u)}p_1a_{kl}$. Since $|\{x \in X : k < x < l\}| = |\{k < x < l : x \notin \{i_{u+1}, \dots, i_v\}\}| = l - k - v + u - 1$ and $\{k, l\} \subseteq \{i_0, \dots, i_s\}$, then $a_{kl}p_2 = q^{-2(l-k-v+u-1)}p_2a_{kl}$ by Lemma 3.1. Thus, $a_{kl}b_{ij} = q^2b_{ij}\sigma(a_{kl})$.

Case 3.3.5: $i_u < k < l < i_{u+1}$ for some $0 \le u < s$. Then $a_{kl}a_{i_ui_{u+1}} = q^2a_{i_ui_{u+1}}a_{kl}$ and $a_{kl}a_{i_hi_{h+1}} = a_{i_hi_{h+1}}a_{kl}$ for $h \ne u$, whence $a_{kl}p_1 = q^2p_1a_{kl}$. Since $[k, l] \subseteq X$, then $a_{kl}p_2 = q^{-2(l-k)}p_2a_{kl}$ by Lemma 3.1. Thus, $a_{kl}b_{ij} = q^2b_{ij}\sigma(a_{kl})$.

Case 3.3.6: $i_u = k < l < i_{u+1}$ for some $0 < u < s$. Then $a_{kl}a_{i_ui_{u+1}} = qa_{i_ui_{u+1}}a_{kl}$ and $a_{kl}a_{i_hi_{h+1}} = a_{i_hi_{h+1}}a_{kl}$ for $h \ne u$, whence $a_{kl}p_1 = qp_1a_{kl}$. Since $x \notin \{i_0, \dots, i_s\}$ for all $k < x < l$ and $\{k, l\} \setminus \{i_0, \dots, i_s\} = \{l\}$, then $a_{kl}p_2 = q^{-2(l-k-1)-1}p_2a_{kl}$ by Lemma 3.1. Thus, $a_{kl}b_{ij} = q^2b_{ij}\sigma(a_{kl})$.

Case 3.3.7: $i_u < k < l = i_{u+1}$ for some $0 \le u < s-1$. This case is obtained from Case 3.3.6 by applying the automorphism ρ.

Case 3.3.8: $i_u = k < l = i_{u+1}$ for some $0 < u < s-1$. Then $a_{kl}a_{i_hi_{h+1}} = a_{i_hi_{h+1}}a_{kl}$ for all $0 \le h < s$, whence $a_{kl}p_1 = p_1a_{kl}$. Since $x \notin \{i_0, \dots, i_s\}$ for all $k < x < l$ and $\{k, l\} \subseteq \{i_0, \dots, i_s\}$, then $a_{kl}p_2 = q^{-2(l-k-1)}p_2a_{kl}$ by Lemma 3.1. Thus, $a_{kl}b_{ij} = q^2b_{ij}\sigma(a_{kl})$.

Case 4: $l = j$.

Case 4.1: $k < i < l = j$. Suppose $s > 1$ in p_1. We have $a_{kl}a_{i_ui_{u+1}} = q^{-2}a_{i_ui_{u+1}}a_{kl}$ for $0 \le u < s-1$, by (6), and $a_{kl}a_{i_{s-1}i_s} = q^{-1}a_{i_{s-1}i_s}a_{kl}$, by (3). Thus, $a_{kl}p_1 = q^{-2s+1}p_1a_{kl}$. Note that this last equality is also true for $s = 1$.

Observe that $|\{x \in X : k < x < l\}| = (l-k-1)-s$. Thus, by Lemma 3.1, $a_{kl}p_2 = q^{-2[(l-k-1)-s]-1}p_2a_{kl} = q^{2(k-l)+2s+1}p_2a_{kl} = q^{2s+1}p_2\sigma(a_{kl})$. It follows that $a_{kl}b_{ij} = q^2b_{ij}\sigma(a_{kl})$.

Case 4.2: $i = k < l = j$. Suppose $s > 1$ in p_1. We have $a_{kl}a_{i_0i_1} = q^{-1}a_{i_0i_1}a_{kl}$, by (4), and $a_{kl}a_{i_{s-1}i_s} = q^{-1}a_{i_{s-1}i_s}a_{kl}$, by (3). Moreover, if $s > 2$, then, by (6), $a_{kl}a_{i_ui_{u+1}} = q^{-2}a_{i_ui_{u+1}}a_{kl}$ for $1 \le u < s-1$. Thus, $a_{kl}p_1 = q^{-2(s-1)}p_1a_{kl}$. Note that this last equality is also true for $s = 1$.

Observe that $|\{x \in X : i < x < j\}| = (j-i-1)-(s-1) = j-i-s = l-k-s$. Thus, by Lemma 3.1, $a_{kl}p_2 = q^{-2(l-k-s)}p_2a_{kl} = q^{2s}p_2\sigma(a_{kl})$. It follows that $a_{kl}b_{ij} = q^2b_{ij}\sigma(a_{kl})$.

Case 4.3: $i < k < l = j$. This follows from Case 3.2 by applying ρ.

Case 5: $j < l$.

Case 5.1: $k < i < j < l$. We have $a_{kl}a_{i_u i_{u+1}} = q^{-2}a_{i_u i_{u+1}}a_{kl}$ for $0 \leq u \leq s-1$, by (6), then $a_{kl}p_1 = q^{-2s}p_1 a_{kl}$. Since $|\{x \in X : k < x < l\}| = (l-k-1)-(s+1) = l-k-s-2$, then $a_{kl}p_2 = q^{-2(l-k-s-2)-2}p_2 a_{kl} = q^{2s+2}p_2\sigma(a_{kl})$, by Lemma 3.1. Therefore, $a_{kl}b_{ij} = q^2 b_{ij}\sigma(a_{kl})$.

Case 5.2: $i = k < j < l$. This follows from Case 4.1 by applying ρ.

Case 5.3: $i < k < j < l$. This follows from Case 3.1 by applying ρ.

Case 5.4: $i < j = k < l$. This follows from Case 2 by applying ρ.

Case 5.5: $i < j < k < l$. This follows from Case 1 by applying ρ. □

Lemma 3.17. *Let $1 \leq k < l \leq n$ and $1 \leq i < j \leq n$. Then*

$$b_{kl}\sigma(b_{ij}) = q^{-m}b_{ij}\sigma(b_{kl}), \tag{31}$$

where

$$m = \begin{cases} 1, & k = i < l < j \text{ or } i < k < l = j, \\ 2, & i < k < l < j, \\ -1, & i = k < j < l \text{ or } k < i < l = j, \\ -2, & k < i < j < l, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. This proof will be omitted. It consists in applying Lemmas 3.14 and 3.16 in each of the possible cases depending on the relative positions between $i < j$ and $k < l$. □

Theorem 3.18. *The bialgebra $UT_q(n)$ is a Hopf algebra with the antipode S given by*

$$S(a_{ij}) = tb_{ij} = \sigma(b_{ij})t, \text{ for all } 1 \leq i \leq j \leq n, \quad S(t) = \det_q(n) \quad \text{and} \quad S(1) = 1, \tag{32}$$

where b_{ij} is given by (22) *and* (23).

Proof. Let us first show that $\{S(a_{ij})\}_{1\leq i\leq j\leq n}$ is a $UT_q(n)$-point of $T_{q^{-1}}(n)$. Suppose that $1 \leq k \leq l \leq n$ and $1 \leq i < j \leq n$. Then,

$$S(a_{kl})S(a_{ij}) = tb_{kl}tb_{ij} = tb_{kl}\sigma(b_{ij})t = q^{-m}tb_{ij}\sigma(b_{kl})t = q^{-m}tb_{ij}tb_{kl} = q^{-m}S(a_{ij})S(a_{kl}),$$

where m is given in Lemmas 3.15 and 3.17. The claim now follows from relations (3)–(6).

By Remark 2.8 and the fact that $T_{q^{-1}}(n)$ is the opposite algebra of $T_q(n)$, there is a unique unital anti-homomorphism $S : T_q(n) \to UT_q(n)$ satisfying $S(a_{ij}) = tb_{ij}$. Moreover, since $\det_q(n)t = t\det_q(n)$ and $\det_q(n)\sigma(b_{ij}) = b_{ij}\det_q(n)$, then $S(a_{ij})\det_q(n) = \det_q(n)S(\sigma(a_{ij}))$ for all $1 \leq i \leq j \leq n$, so S extends uniquely to an anti-homomorphism $S : T_q(n)[t;\sigma] \to UT_q(n)$ satisfying (32).

Since $ta_{ii} = a_{ii}t$ for all $1 \leq i \leq t$, we have

$$S(t\det_q(n) - 1) = \left(\prod_{i=1}^{n}\left(t\prod_{j\neq i}a_{jj}\right)\right)\det_q(n) - 1 = t^n\det_q(n)^{n-1}\cdot\det_q(n) - 1 = (t\det_q(n))^n - 1 = 0.$$

Thus, S induces an algebra anti-homomorphism $S : UT_q(n) \to UT_q(n)$ satisfying (32).

Finally, using (25), we have, for all $1 \leq i \leq j \leq n$,

$$\sum_{i\leq k\leq j} S(a_{ik})a_{kj} = \sum_{i\leq k\leq j} tb_{ik}a_{kj} = t\det_q(n)\delta_{ij} = \delta_{ij} = \varepsilon(a_{ij}),$$

$$\sum_{i\leq k\leq j} a_{ik}S(a_{kj}) = \sum_{i\leq k\leq j} a_{ik}tb_{kj} = \sum_{i\leq k\leq j} q^{2(k-j)}a_{ik}b_{kj}t = \det_q(n)t\delta_{ij} = \delta_{ij} = \varepsilon(a_{ij}).$$

□

Remark 3.19. It can be shown, just as in the proof of [9, Proposition 3.1.1], that $UT_q(n)$ is a pointed Hopf algebra, i.e., all of its simple left and right comodules have dimension one.

Now that we have a Hopf structure on $UT_q(n)$, our next goal is to show that the antipode S has order two. Note that, in a Hopf algebra, the antipode is the inverse of the identity map relative to the convolution product. So, it would suffice to show that S^2 and S are mutual inverses with respect to the convolution product.

Lemma 3.20. *Let $\{x_{ij} : 1 \le i \le j \le n\} \subseteq UT_q(n)$ be such that, for all $1 \le i \le j \le n$,*

$$\sum_{i \le k \le j} x_{ik} S(a_{kj}) = \varepsilon(a_{ij}). \tag{33}$$

Then $x_{ij} = a_{ij}$ for all $1 \le i \le j \le n$.

Proof. The proof will be by induction on $j-i$. For the base case multiply $1 = \varepsilon(a_{ii}) = x_{ii}S(a_{ii})$ on the right by a_{ii}. Since S is an antipode, we have

$$a_{ii} = \varepsilon(a_{ii})a_{ii} = (x_{ii}S(a_{ii}))a_{ii} = x_{ii}(S(a_{ii})a_{ii}) = x_{ii}\varepsilon(a_{ii}) = x_{ii}.$$

Assume that $j - i > 0$ and $x_{st} = a_{st}$ for all $s \le t$ with $t - s < j - i$. Observe that $\sum_{i \le k < j} a_{ik}S(a_{kj}) = \varepsilon(a_{ij}) - a_{ij}S(a_{jj}) = -a_{ij}S(a_{jj})$, because S is an antipode. Then, in view of (33) and by the induction hypothesis,

$$0 = \varepsilon(a_{ij}) = \sum_{i \le k \le j} x_{ik}S(a_{kj}) = \sum_{i \le k < j} a_{ik}S(a_{kj}) + x_{ij}S(a_{jj}) = (x_{ij} - a_{ij})S(a_{jj}).$$

So, multiplying both sides by a_{jj} on the right and using $S(a_{jj})a_{jj} = \varepsilon(a_{jj}) = 1$, we get $x_{ij} - a_{ij} = 0$, which proves the induction step. □

Proposition 3.21. *The antipode S in $UT_q(n)$ satisfies $S^2 = \mathrm{id}$.*

Proof. Since S^2 is an algebra homomorphism, it suffices to show that $S^2(a_{ij}) = a_{ij}$, for all $1 \le i \le j \le n$, which will follow from using Lemma 3.20 with $x_{ij} = S^2(a_{ij})$.

If $i = j$ then $\sum_{i \le k \le i} S^2(a_{ik})S(a_{ki}) = S^2(a_{ii})S(a_{ii}) = S(a_{ii}S(a_{ii})) = S(\varepsilon(a_{ii})) = 1 = \varepsilon(a_{ii})$. By Lemmas 3.1, 3.14 and 3.16, it follows that $\sigma^{-1}(a_{kj})b_{ik} = qb_{ik}a_{kj}$, for all $i \le k \le j$, except if $i = k = j$. Thus, for $i < j$,

$$\begin{aligned}
\sum_{i \le k \le j} S^2(a_{ik})S(a_{kj}) &= S\left(\sum_{i \le k \le j} a_{kj}S(a_{ik})\right) = S\left(\sum_{i \le k \le j} a_{kj}tb_{ik}\right) \\
&= S\left(\sum_{i \le k \le j} t\sigma^{-1}(a_{kj})b_{ik}\right) = S\left(\sum_{i \le k \le j} qtb_{ik}a_{kj}\right) \\
&= qS\left(\sum_{i \le k \le j} S(a_{ik})a_{kj}\right) = qS(\varepsilon(a_{ij})) = 0 = \varepsilon(a_{ij}).
\end{aligned}$$

□

Proposition 3.22. *The automorphisms $\rho, \sigma \in \mathrm{Aut}(T_q(n))$ from Proposition 2.3 and Lemma 3.4, respectively, lift uniquely to algebra automorphisms of $UT_q(n)$. These lifts satisfy the following:*

(i) σ is a Hopf algebra automorphism of $UT_q(n)$;
(ii) ρ is a coalgebra antiautomorphism of $UT_q(n)$ that commutes with S.

Proof. Recall that, by Proposition 3.9 and Remark 3.10, $UT_q(n)$ is the localization of $T_q(n)$ at the powers of $\det_q(n)$. Since $\sigma(\det_q(n)) = \det_q(n) = \rho(\det_q(n))$ (where the last equality follows since the elements a_{ii} mutually commute), we conclude that σ and ρ extend to algebra automorphisms of the localization $T_q(n)[\det_q(n)^{-1}]$, with

$$\sigma(t) = \sigma(\det_q(n)^{-1}) = \sigma(\det_q(n))^{-1} = \det_q(n)^{-1} = t \tag{34}$$

and $\rho(t) = t$. Since σ is a bialgebra automorphism of $T_q(n)$ by Lemma 3.4, it is easily seen by (34) that its extension is a bialgebra (and hence, a Hopf algebra) automorphism of $UT_q(n)$.

Given $1 \leq i \leq n$, denote $\bar{i} = n+1-i$. Let τ be the flip map on $UT_q(n) \otimes UT_q(n)$. Then, for all $1 \leq i \leq j \leq n$,

$$(\rho \otimes \rho)(\Delta(a_{ij})) = \sum_{i \leq k \leq j} \rho(a_{ik}) \otimes \rho(a_{kj}) = \sum_{i \leq k \leq j} a_{\bar{k}\,\bar{i}} \otimes a_{\bar{j}\,\bar{k}} = \sum_{\bar{j} \leq \bar{k} \leq \bar{i}} a_{\bar{k}\,\bar{i}} \otimes a_{\bar{j}\,\bar{k}} = (\tau \circ \Delta)(a_{\bar{j}\,\bar{i}}) = (\tau \circ \Delta)(\rho(a_{ij})),$$

$$\varepsilon(\rho(a_{ij})) = \varepsilon(a_{\bar{j}\,\bar{i}}) = \delta_{\bar{j}\,\bar{i}} = \delta_{ij} = \varepsilon(a_{ij}).$$

Moreover, using Lemma 3.12, we have

$$\rho(S(a_{ij})) = \rho(tb_{ij}) = tb_{\bar{j}\,\bar{i}} = S(a_{\bar{j}\,\bar{i}}) = S(\rho(a_{ij})).$$

□

Remark 3.23. Let $X \subseteq \{1, \ldots, n\}$ and consider the unital associative subalgebra of $UT_q(n)$ generated by $\{a_{ii}^{\pm 1} : i \in X\}$. This is just the commutative Laurent polynomial algebra in $k := |X|$ variables. As $\Delta(a_{ii}^{\pm 1}) = a_{ii}^{\pm 1} \otimes a_{ii}^{\pm 1}$ and $S(a_{ii}^{\pm 1}) = a_{ii}^{\mp 1}$, this is a Hopf subalgebra. In fact, it is isomorphic to the Hopf group algebra of the free abelian group $\mathbb{Z}^k$.

It is straightforward to verify that $(\det_q(n) - 1)$ is a Hopf ideal of $UT_q(n)$, so we get a quotient Hopf algebra $UT_q(n)/(\det_q(n) - 1)$. Combining previous observations and Proposition 3.9, it follows that, in case q is not a root of unity, the Hopf algebra $UT_q(n)/(\det_q(n) - 1)$ is canonically isomorphic to the Hopf subalgebra described above for $X = \{1, \ldots, n-1\}$, isomorphic to the Hopf group algebra of $\mathbb{Z}^{n-1}$.

4. Hopf $*$-algebras

For an involution $*$ on a K-algebra A we denote $*(a)$ by a^* for each $a \in A$. Recall that a map $* : A \to A$ is an *involution* if $(a+b)^* = a^* + b^*$, $(ab)^* = b^* a^*$ and $(a^*)^* = a$ for all $a, b \in A$.

Consider the field K of characteristic different from 2 as a K-algebra. Suppose there is an involution $^- : K \to K$ such that $^- \neq \mathrm{id}_K$ and let

$$K_0 = \{\alpha \in K : \overline{\alpha} = \alpha\}.$$

Since $^-$ is an automorphism of order 2, by [15, Lemma 2.5], K_0 is a proper subfield of K such that $[K : K_0] = 2$ and there is a linear basis $\{1, i\}$ of K over K_0 such that $i^2 \in K_0$ and $\bar{i} = -i$.

For any K-vector space V, a map $\phi : V \to V$ is *antilinear* if $\phi(u+v) = \phi(u) + \phi(v)$ and $\phi(\alpha u) = \overline{\alpha}\phi(u)$ for all $u, v \in V$ and $\alpha \in K$. Clearly, every antilinear map is K_0-linear. An antilinear map between two K-algebras $\phi : A \to B$ will be called an *antilinear morphism* if $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in A$ and $\phi(1) = 1$.

Definition 4.1. Let $(H, \mu, \eta, \Delta, \varepsilon, S)$ be a Hopf K-algebra. We say that H is a *Hopf $*$-algebra* if there exists an antilinear involution $*$ on H satisfying the following conditions:

(i) $*$ is a morphism of K_0-coalgebras;
(ii) $(* \circ S)^2 = \mathrm{id}_H$.

Remark 4.2. If H is a Hopf K-algebra with an antilinear involution $*$, then $*$ is an antimorphism of K_0-algebras.

Lemma 4.3. *A Hopf algebra H has a Hopf $*$-algebra structure if and only if there exists an antilinear bijection γ on H such that*

(i) γ is a morphism of K_0-algebras and an antimorphism of K_0-coalgebras;
(ii) $\gamma^2 = (S \circ \gamma)^2 = \mathrm{id}_H$.

Proof. The proof is similar to that of [8, Lemma IV.8.2]. □

Let V be a K-vector space. Then the *conjugate* of V is the K-space $\overline{V}$ which coincides with $(V, +)$ as an abelian group, but is equipped with the following multiplication by scalars: $(\alpha, v) \mapsto \overline{\alpha} v$ for all $\alpha \in K$ and $v \in V$. Observe that antilinear maps $V \to W$ are exactly the same as linear maps $V \to \overline{W}$. If $(A, \cdot)$ is a K-algebra, then $(\overline{A}, \cdot)$ is also a K-algebra. As a consequence, we have the following.

Remark 4.4. For any K-algebra R there is a one-to-one correspondence between antilinear morphisms $T_q(n) \to R$ and $\overline{R}$-points of $T_q(n)$.

Proposition 4.5. *Assume that $q \in K_0$. Then there is an antilinear bijection $\gamma : UT_q(n) \to UT_q(n)$ which is a morphism of K_0-algebras and an antimorphism of K_0-coalgebras, such that*

$$\gamma(a_{ij}) = a_{n+1-j,n+1-i}. \tag{35}$$

Moreover, $\gamma^2 = \mathrm{id}$.

Proof. As in Proposition 2.3, since $q \in K_0$, it follows that $(\gamma(a_{ij}))_{1\le i\le j\le n}$ is a $\overline{UT_q(n)}$-point of $T_q(n)$, so there exists a unique antilinear morphism $\gamma : T_q(n) \to UT_q(n)$ mapping a_{ij} to $a_{\bar{j}\,\bar{i}}$, where $\bar{i} = n+1-i$. The same argument used in the proof of Proposition 3.22 (ii) shows that γ extends to an antilinear automorphism of the K-algebra $UT_q(n)$, which we still denote by γ, such that $\gamma(t) = t$. The proof that it is also an antimorphism of K_0-coalgebras is the same as the one for ρ in Proposition 3.22 (ii). Since $\gamma^2(a_{ij}) = a_{ij}$ and γ^2 is K-linear, then $\gamma^2 = \mathrm{id}$. □

Lemma 4.6. *Assume that $q \in K_0$. Then $\gamma \circ S = S \circ \gamma$.*

Proof. Since both $\gamma \circ S$ and $S \circ \gamma$ are antilinear, it suffices to show that $(\gamma \circ S)(a_{ij}) = (S \circ \gamma)(a_{ij})$, for all $1 \le i \le j \le n$. This follows exactly as for ρ in Proposition 3.22 (ii) (based on the computation in Lemma 3.12 which holds for γ because $q \in K_0$), showing that ρ and S commute. □

Theorem 4.7. *If $q \in K_0$, then there exists a Hopf $*$-algebra structure on $UT_q(n)$ given by $a^*_{ij} = (\gamma \circ S)(a_{ij})$.*

Proof. This follows from Lemma 4.3 and Proposition 4.5, where $(\gamma \circ S)^2 = \gamma^2 \circ S^2 = \mathrm{id}$ by Lemma 4.6 and Propositions 3.21 and 4.5. □

5. Derivations of $T_q(2)$ and $UT_q(2)$

In this section we assume that q is not a root of unity. Since $T_q(2)$ is a quantum affine space (see Example 2.6), we can specify the results of [1, Theorem 1.2] to this algebra.

For $(s,t) \in \{(1,1),(1,2),(2,2)\}$ and $\nu = (\nu_{11},\nu_{12},\nu_{22}) \in \mathbb{N}^3$ denote by $D_{st,\nu}$ the map $\{a_{11},a_{12},a_{22}\} \to K\langle a_{11},a_{12},a_{22}\rangle$ sending a_{st} to $a^\nu := a_{11}^{\nu_{11}} a_{12}^{\nu_{12}} a_{22}^{\nu_{22}}$ and a_{ij} to 0 for $(i,j) \ne (s,t)$. By the Leibniz rule, $D_{st,\nu}$ uniquely extends to a derivation of $K\langle a_{11},a_{12},a_{22}\rangle$. Moreover, by (14)–(16) (see also [1, p. 1789]), $D_{st,\nu}$ defines a derivation of $T_q(2)$ if and only if the following equalities hold in $T_q(2)$:

$$D_{st,\nu}(a_{22})a_{12} + a_{22}D_{st,\nu}(a_{12}) = q(D_{st,\nu}(a_{12})a_{22} + a_{12}D_{st,\nu}(a_{22})), \tag{36}$$
$$D_{st,\nu}(a_{11})a_{12} + a_{11}D_{st,\nu}(a_{12}) = q(D_{st,\nu}(a_{12})a_{11} + a_{12}D_{st,\nu}(a_{11})), \tag{37}$$
$$D_{st,\nu}(a_{11})a_{22} + a_{11}D_{st,\nu}(a_{22}) = D_{st,\nu}(a_{22})a_{11} + a_{22}D_{st,\nu}(a_{11}). \tag{38}$$

The following lemma characterizes this situation.

Lemma 5.1. *We have*

(i) $D_{11,\nu}$ defines a derivation of $T_q(2)$ if and only if $\nu \in \{(0,0,1),(1,0,0)\}$;
(ii) $D_{12,\nu}$ defines a derivation of $T_q(2)$ if and only if $\nu_{12} = 1$;
(iii) $D_{22,\nu}$ defines a derivation of $T_q(2)$ if and only if $\nu \in \{(0,0,1),(1,0,0)\}$.

Proof. *(i)* Let $(s,t) = (1,1)$. Observe that (37) is equivalent to $D_{11,\nu}(a_{11})a_{12} = qa_{12}D_{11,\nu}(a_{11})$, i.e., $a^\nu \cdot a_{12} = qa_{12} \cdot a^\nu \Leftrightarrow (q^{\nu_{22}} - q^{1-\nu_{11}})a_{11}^{\nu_{11}} a_{12}^{\nu_{12}+1} a_{22}^{\nu_{22}} = 0$. Since q is not a root of unity, the latter is equivalent to $\nu_{11} + \nu_{22} = 1 \Leftrightarrow \{\nu_{11},\nu_{22}\} = \{0,1\}$. Furthermore, (38) reduces to $D_{11,\nu}(a_{11})a_{22} = a_{22}D_{11,\nu}(a_{11}) \Leftrightarrow a^\nu \cdot a_{22} = a_{22} \cdot a^\nu \Leftrightarrow (1 - q^{\nu_{12}})a_{11}^{\nu_{11}} a_{12}^{\nu_{12}} a_{22}^{\nu_{22}+1} = 0 \Leftrightarrow \nu_{12} = 0$. Finally, (36) holds trivially for any ν.

(ii) Let $(s,t) = (1,2)$. We have (37) $\Leftrightarrow a_{11}D_{12,\nu}(a_{12}) = qD_{12,\nu}(a_{12})a_{11} \Leftrightarrow a_{11} \cdot a^\nu = qa^\nu \cdot a_{11} \Leftrightarrow (1 - q^{1-\nu_{12}})a_{11}^{\nu_{11}+1} a_{12}^{\nu_{12}} a_{22}^{\nu_{22}} = 0$, (36) $\Leftrightarrow a_{22}D_{12,\nu}(a_{12}) = qD_{12,\nu}(a_{12})a_{22} \Leftrightarrow a_{22} \cdot a^\nu = qa^\nu \cdot a_{22} \Leftrightarrow (q^{\nu_{12}} - q)a_{11}^{\nu_{11}} a_{12}^{\nu_{12}} a_{22}^{\nu_{22}+1} = 0$ and (38) is trivially satisfied, giving the unique condition $\nu_{12} = 1$.

(iii) Let $(s,t) = (2,2)$. We have (38) $\Leftrightarrow a_{11}D_{22,\nu}(a_{22}) = D_{22,\nu}(a_{22})a_{11} \Leftrightarrow a_{11} \cdot a^\nu = a^\nu \cdot a_{11} \Leftrightarrow (1 - q^{-\nu_{12}})a_{11}^{\nu_{11}+1} a_{12}^{\nu_{12}} a_{22}^{\nu_{22}} = 0$, (36) $\Leftrightarrow D_{22,\nu}(a_{22})a_{12} = qa_{12}D_{22,\nu}(a_{22}) \Leftrightarrow a^\nu \cdot a_{12} = qa_{12} \cdot a^\nu \Leftrightarrow (q^{\nu_{22}} - q^{1-\nu_{11}})a_{11}^{\nu_{11}} a_{12}^{\nu_{12}+1} a_{22}^{\nu_{22}} = 0$ and (37) is trivially satisfied, giving $\nu_{12} = 0$ and $\nu_{11} + \nu_{22} = 1$. □

Following [1, 1.2], set

$$D_{11} := D_{11,(1,0,0)}, \quad D_{12} := D_{12,(0,1,0)}, \quad D_{22} := D_{22,(0,0,1)}.$$

Furthermore, if

$$\Lambda_{st} = \{\nu \in \mathbb{N}^3 : \nu_{st} = 0 \text{ and } D_{st,\nu} \in \mathrm{Der}(T_q(2))\},$$

then Lemma 5.1 gives

$$\Lambda_{11} = \{(0,0,1)\}, \quad \Lambda_{12} = \emptyset, \quad \Lambda_{22} = \{(1,0,0)\},$$

so that

$$E := \mathrm{Span}_K\{D_{st,\nu} : 1 \le s \le t \le 2 \text{ and } \nu \in \Lambda_{st}\} = KD_{11,(0,0,1)} \oplus KD_{22,(1,0,0)}. \tag{39}$$

By [1, Theorem 1.2] and Proposition 2.7 we have the following characterization of $\mathrm{Der}(T_q(2))$.

Corollary 5.2.

$$\mathrm{Der}(T_q(2)) = \mathrm{IDer}(T_q(2)) \oplus KD_{11} \oplus KD_{12} \oplus KD_{22} \oplus KD_{11,(0,0,1)} \oplus KD_{22,(1,0,0)}. \tag{40}$$

In particular, $\dim(\mathrm{HH}^1(T_q(2))) = 5$.

Next, we tackle the derivation Lie algebra of the Hopf algebra $UT_q(2)$. We will see that $\mathrm{HH}^1(UT_q(2))$ is infinite dimensional, although free of rank 3 over $\mathrm{HH}^0(UT_q(2)) = Z(UT_q(2))$.

As before, we identify $UT_q(2)$ with the localization of $T_q(2)$ at the powers of a_{11} and a_{22}. We will also consider the subalgebra $\mathcal{P}_q$ generated by $a_{11}^{\pm1}$ and a_{12}, and the localization $\mathcal{T}_q$ of $\mathcal{P}_q$ at the powers of a_{12}. Notice that both $\mathcal{T}_q$ and $\mathcal{P}_q$ are localizations of the quantum plane $A_2(q^{-1})$ generated by a_{11} and a_{12}. For simplicity, we use the following more intuitive notation: $\mathcal{P}_q = K_q[a_{11}^{\pm1}, a_{12}]$ and $\mathcal{T}_q = K_q[a_{11}^{\pm1}, a_{12}^{\pm1}]$.

Lemma 5.3. *We have* $Z(\mathcal{T}_q) = Z(\mathcal{P}_q) = K$ *and* $Z(UT_q(2)) = K[z^{\pm1}]$*, where* $z = a_{11}a_{22}^{-1}$*. Moreover,* $UT_q(2) = \mathcal{P}_q[z^{\pm1}]$*, a (commutative) Laurent polynomial extension of* $\mathcal{P}_q$*.*

Proof. The commutation relations for $T_q(2)$ and $UT_q(2)$ are given by the matrix Q in Example 2.6. The corresponding matrix of exponents is $\mathcal{Q} = \left(\begin{smallmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{smallmatrix}\right)$ and $Z(UT_q(2))$ is the linear span of the monomials $a_{11}^{\nu_{11}}a_{12}^{\nu_{12}}a_{22}^{\nu_{22}}$ with $(\nu_{11},\nu_{12},\nu_{22}) \in \mathbb{Z}\times\mathbb{N}\times\mathbb{Z}$ in the null space of $\mathcal{Q}$. Since this null space in $\mathbb{Z}^3$ is $\{k(1,0,-1) : k \in \mathbb{Z}\}$, it follows that $Z(UT_q(2))$ is the Laurent polynomial ring $K[z^{\pm1}]$. Similarly, since the matrix of exponents of the commutation relations for $\mathcal{T}_q$ and $\mathcal{P}_q$ is $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, which has trivial null space, it follows that $Z(\mathcal{T}_q) = Z(\mathcal{P}_q) = K$.

Clearly, $\mathcal{P}_q[z^{\pm1}] \subseteq UT_q(2)$. Note that $\mathcal{P}_q[z^{\pm1}]$ is generated as a vector space by $\{a_{11}^i a_{12}^j a_{11}^k a_{22}^{-k} : i,k \in \mathbb{Z}, j \in \mathbb{N}\}$. Up to nonzero scalars, this set is just $\{a_{11}^i a_{12}^j a_{22}^k : i,k \in \mathbb{Z}, j \in \mathbb{N}\}$, which is linearly independent, hence a basis of $\mathcal{P}_q[z^{\pm1}]$. But the latter is also a basis of $UT_q(2)$, showing the equality $UT_q(2) = \mathcal{P}_q[z^{\pm1}]$ and the algebraic independence of $z^{\pm1}$ over $\mathcal{P}_q$. Note in particular that $a_{22}^{\pm1} = a_{11}^{\pm1}z^{\mp1}$. □

Note that any derivation D of $T_q(2)$ extends uniquely to a derivation of $UT_q(2)$, still denoted D, by localization, using $D(x^{-1}) = -x^{-1}D(x)x^{-1}$, for $x = a_{11}^k, a_{22}^k$ ($k \in \mathbb{N}$) and the Leibniz rule. In particular, the (extended) derivations D_{11}, D_{12} and D_{22} play an important role in the description of $\mathrm{Der}(UT_q(2))$.

Theorem 5.4. *We have*

$$\mathrm{Der}(UT_q(2)) = \mathrm{IDer}(UT_q(2)) \oplus K[z^{\pm1}]D_{11} \oplus K[z^{\pm1}]D_{12} \oplus K[z^{\pm1}]D_{22}. \tag{41}$$

In particular, $\mathrm{HH}^1(UT_q(2))$ *is a free module of rank* 3 *over* $Z(UT_q(2)) = K[z^{\pm1}]$*.*

Proof. We begin by computing and decomposing the derivation Lie algebra of $\mathcal{P}_q = K_q[a_{11}^{\pm1}, a_{12}]$. Since $Z(\mathcal{T}_q) = K$, it follows from [14, Corollary 2.6] that

$$\mathrm{Der}(\mathcal{P}_q) = \mathrm{IDer}(\mathcal{P}_q) \oplus Kd_{11} \oplus Kd_{12},$$

where $d_{11}(a_{11}) = a_{11}$, $d_{11}(a_{12}) = 0$, $d_{12}(a_{11}) = 0$ and $d_{12}(a_{12}) = a_{12}$.

Now, using the equality $UT_q(2) = \mathcal{P}_q[z^{\pm1}]$ from Lemma 5.3 and [11, Theorem 2.1] (where we need to use an analogous version for central Laurent extensions, instead of polynomial extensions, but which has the same proof), we can conclude that

$$\mathrm{Der}(\mathcal{P}_q[z^{\pm1}]) = \mathrm{IDer}(\mathcal{P}_q[z^{\pm1}]) \oplus K[z^{\pm1}]\overline{d_{11}} \oplus K[z^{\pm1}]\overline{d_{12}} \oplus K[z^{\pm1}]\partial_z,$$

where $\overline{d_{11}}$ and $\overline{d_{12}}$ are d_{11} and d_{12} extended to $\mathcal{P}_q[z^{\pm1}]$ by setting $\overline{d_{11}}(z) = 0 = \overline{d_{12}}(z)$, $\partial_z(\mathcal{P}_q) = 0$ and $\partial_z(z) = 1$.

Using the fact that $a_{22} = a_{11}z^{-1}$, we can compute the values of these three derivations at a_{22}:

$$\overline{d_{11}}(a_{22}) = \overline{d_{11}}(a_{11}z^{-1}) = \overline{d_{11}}(a_{11})z^{-1} + a_{11}\overline{d_{11}}(z^{-1}) = a_{11}z^{-1} = a_{22},$$
$$\overline{d_{12}}(a_{22}) = \overline{d_{12}}(a_{11}z^{-1}) = \overline{d_{12}}(a_{11})z^{-1} + a_{11}\overline{d_{12}}(z^{-1}) = 0,$$
$$\partial_z(a_{22}) = \partial_z(a_{11}z^{-1}) = \partial_z(a_{11})z^{-1} + a_{11}\partial_z(z^{-1}) = -a_{11}z^{-2}.$$

The next table summarizes all the relevant information:

	$\overline{d_{11}}$	$\overline{d_{12}}$	∂_z
a_{11}	a_{11}	0	0
a_{12}	0	a_{12}	0
a_{22}	a_{22}	0	$-z^{-2}a_{11}$

It follows that $\overline{d_{11}} = D_{11} + D_{22}$, $\overline{d_{12}} = D_{12}$ and $\partial_z = -z^{-1}D_{22}$. Conversely, $D_{11} = \overline{d_{11}} + z\partial_z$, $D_{12} = \overline{d_{12}}$ and $D_{22} = -z\partial_z$, so

$$K[z^{\pm 1}]\overline{d_{11}} \oplus K[z^{\pm 1}]\overline{d_{12}} \oplus K[z^{\pm 1}]\partial_z = K[z^{\pm 1}]D_{11} \oplus K[z^{\pm 1}]D_{12} \oplus K[z^{\pm 1}]D_{22}.$$

□

6. Automorphisms of $T_q(2)$ and $UT_q(2)$

In this final section, we will give a complete description of the groups of algebra automorphisms of $T_q(2)$ and of $UT_q(2)$ in the case that q is not a root of unity.

We start with the following remark, which is a consequence of Remark 2.8.

Remark 6.1. Any $\varphi \in \mathrm{End}(T_q(2))$ is uniquely determined by $\varphi(a_{ij}) = p_{ij}$ satisfying

$$p_{22}p_{12} = qp_{12}p_{22}, \tag{42}$$
$$p_{11}p_{12} = qp_{12}p_{11}, \tag{43}$$
$$p_{11}p_{22} = p_{22}p_{11}. \tag{44}$$

Moreover, in case $\varphi \in \mathrm{End}(UT_q(2))$, we additionally need to require that p_{11} and p_{22} be invertible in $UT_q(2)$, thus (nonzero scalar multiples of) monomials in a_{11} and a_{22}.

In particular, note that for any choice of $\alpha_{11}, \alpha_{12}, \alpha_{22} \in K^*$, there are automorphisms of $T_q(2)$ and of $UT_q(2)$ such that $a_{ij} \mapsto \alpha_{ij}a_{ij}$, for all $1 \le i \le j \le 2$. We call these the *diagonal* automorphisms. The set of all such automorphisms forms a group isomorphic to $(K^*)^3$. We also recall the automorphism ρ defined in Proposition 2.3, which fixes a_{12} and interchanges a_{11} and a_{22}. Since, in general, ρ does not commute with the diagonal automorphisms, it follows that $\mathrm{Aut}(T_q(2))$ and $\mathrm{Aut}(UT_q(2))$ are nonabelian.

Lemma 6.2. *Suppose that $q \neq 1$. Let A be either $T_q(2)$ or $UT_q(2)$ and $\varphi \in \mathrm{Aut}(A)$. Then there is a unit $u \in A$ such that $\varphi(a_{12}) = ua_{12}$.*

Proof. Consider $I = (a_{12})$ the ideal of A generated by a_{12}. We have $T_q(2)/I \cong K[a_{11}, a_{22}]$ (respectively, $UT_q(2)/I \cong K[a_{11}^{\pm 1}, a_{22}^{\pm 1}]$), a commutative polynomial ring (respectively, Laurent polynomial ring) in two variables. In particular, A/I is a commutative domain of Gelfand–Kirillov dimension 2.

The automorphism φ induces an isomorphism between A/I and $A/\varphi(I)$, hence $A/\varphi(I)$ is also a commutative domain of Gelfand–Kirillov dimension 2. In particular, in $A/\varphi(I)$ we have

$$\overline{0} = [\overline{a_{12}}, \overline{a_{11}}] = \overline{[a_{12}, a_{11}]} = (1-q)\overline{a_{12}a_{11}} = (1-q)\overline{a_{12}}\ \overline{a_{11}}.$$

As $q \neq 1$ and $A/\varphi(I)$ is a domain, we conclude that either $a_{12} \in \varphi(I)$ or $a_{11} \in \varphi(I)$. A similar computation shows that either $a_{12} \in \varphi(I)$ or $a_{22} \in \varphi(I)$. As a_{11} and a_{22} are units in $UT_q(2)$, it immediately follows that $a_{12} \in \varphi(I)$ in case $A = UT_q(2)$. So assume that $A = T_q(2)$. If $a_{12} \notin \varphi(I)$, we must have $a_{11}, a_{22} \in \varphi(I)$. Hence $T_q(2)/\varphi(I)$ is isomorphic to a quotient of $T_q(2)/(a_{11}, a_{22}) \cong K[a_{12}]$. By [10, Lemma 3.1] this would imply that the Gelfand–Kirillov dimension of $T_q(2)/\varphi(I)$ is at most 1, which is a contradiction. Thus, $a_{12} \in \varphi(I)$.

We now have $I = (a_{12}) \subseteq \varphi(I)$. Since φ was arbitrary, the same holds for φ^{-1}, whence $I \subseteq \varphi^{-1}(I)$. Thus $(a_{12}) = I = \varphi(I) = (\varphi(a_{12}))$. As a_{12} is normal in A (i.e., $a_{12}A = Aa_{12}$), we are done. □

We are now ready to describe $\mathrm{Aut}(T_q(2))$. The following notion will be helpful. An automorphism $\varphi \in \mathrm{Aut}(T_q(2))$ is called *linear* if

$$\varphi(\mathrm{Span}_K\{a_{11}, a_{12}, a_{22}\}) = \mathrm{Span}_K\{a_{11}, a_{12}, a_{22}\}.$$

Theorem 6.3. *Suppose that q is not a root of unity. Then any automorphism of $T_q(2)$ is linear. Moreover,*

$$\mathrm{Aut}(T_q(2)) \cong GL_1(K) \times GL_2(K) = K^* \times GL_2(K).$$

Proof. Let $\lambda \in GL_1(K) = K^*$ and $A = (\mu_{ij}) \in GL_2(K)$. The pair $(\lambda, A) \in GL_1(K) \times GL_2(K)$ corresponds to the assignment $p_{12} = \lambda a_{12}$, $p_{11} = \mu_{11}a_{11} + \mu_{21}a_{22}$ and $p_{22} = \mu_{12}a_{11} + \mu_{22}a_{22}$. It is immediate to check that the elements p_{ij} satisfy (42)–(44), so they define $\varphi_{\lambda,A} \in \mathrm{End}(T_q(2))$ with $\varphi_{\lambda,A}(a_{ij}) = p_{ij}$. Clearly, $\varphi_{\lambda,A}$ is invertible, with inverse $\varphi_{\lambda^{-1},A^{-1}}$. This proves that $GL_1(K) \times GL_2(K)$ embeds naturally in the group of linear automorphisms of $T_q(2)$.

Conversely, let $\varphi \in \mathrm{Aut}(T_q(2))$. Note that both ρ and the diagonal automorphisms defined earlier are linear, so we can work modulo these automorphisms.

Since the group of units of $T_q(2)$ is reduced to scalars, we conclude from Lemma 6.2 that $\varphi(a_{12}) = \lambda a_{12}$, for some $\lambda \in K^*$. Composing with an appropriate diagonal automorphism, we can suppose that $\lambda = 1$, so that φ fixes a_{12}.

Write $\varphi(a_{11}) = \sum_{k,l\geq 0} g_{kl}(a_{12})a_{11}^k a_{22}^l$, for some polynomials $g_{kl}(y) \in K[y]$. Using (43) we obtain

$$\begin{aligned}\sum_{k,l\geq 0} q^{k+l} a_{12} g_{kl}(a_{12}) a_{11}^k a_{22}^l &= \left(\sum_{k,l\geq 0} g_{kl}(a_{12}) a_{11}^k a_{22}^l\right) a_{12} = q a_{12} \left(\sum_{k,l\geq 0} g_{kl}(a_{12}) a_{11}^k a_{22}^l\right) \\ &= q \sum_{k,l\geq 0} a_{12} g_{kl}(a_{12}) a_{11}^k a_{22}^l.\end{aligned}$$

As q is not a root of unity, we conclude that $g_{kl}(y) = 0$, whenever $k + l \neq 1$. Thus, we can write $\varphi(a_{11}) = g_{10}(a_{12})a_{11} + g_{01}(a_{12})a_{22}$. Similarly, $\varphi(a_{22}) = h_{10}(a_{12})a_{11} + h_{01}(a_{12})a_{22}$, for some $h_{10}(y), h_{01}(y) \in K[y]$.

Next we use (44). Note that $a_{ii}p(a_{12}) = p(qa_{12})a_{ii}$ for $i = 1, 2$ and all $p(y) \in K[y]$. Thus, the (left) coefficient (in $K[a_{12}]$) of a_{11}^2 in the product $\varphi(a_{11})\varphi(a_{22})$ is $g_{10}(a_{12})h_{10}(qa_{12})$. Similarly, the coefficient of a_{11}^2 in the product $\varphi(a_{22})\varphi(a_{11})$ is $h_{10}(a_{12})g_{10}(qa_{12})$. Equating these terms we obtain

$$g_{10}(a_{12})h_{10}(qa_{12}) = h_{10}(a_{12})g_{10}(qa_{12}). \tag{45}$$

Claim: Let $g, h \in K[y]$ satisfy

$$g(y)h(qy) = h(y)g(qy). \tag{46}$$

Then one of g or h is a scalar multiple of the other.

Proof of the Claim: Without loss of generality, we can assume that $g, h \neq 0$ and that they are monic. Moreover, if $d \in K[y]$ is a common divisor of g and h, then we can replace g and h by g/d and h/d in (46), so we can assume that g and h are coprime. Then, as $g(y)$ divides $h(y)g(qy)$ and is coprime to $h(y)$, $g(y)$ must divide $g(qy)$. For degree considerations, $g(y) = \mu g(qy)$, for some $\mu \in K^*$. But the fact that q is not a root of unity implies that g is a monomial in y. The same argument applies to h. By coprimeness, one of these polynomials is 1 and the other y^m, for some $m \in \mathbb{N}$. Then using (46) we conclude that $q^m = 1$, so also $^{-1} = 0$. This completes the proof of the claim.

Applying the Claim to g_{10} and h_{10}, by (45) we conclude that g_{10} and h_{10} are proportional. Similarly, comparing (left) coefficients of a_{22}^2 in (44), we conclude that g_{01} and h_{01} are also proportional. Up to composing with ρ, we can assume that $g_{10} \neq 0$. So $h_{10} = \lambda g_{10}$, for some $\lambda \in K$. Also, g_{01} and h_{01} cannot both be 0 (otherwise $\varphi(a_{22}) = \lambda\varphi(a_{11})$), so assume $h_{01} \neq 0$ (the case $g_{01} \neq 0$ is similar), whence $g_{01} = \mu h_{01}$, for some $\mu \in K$.

Since φ^{-1} fixes a_{12} and acts on a_{ii}, $i = 1, 2$, similarly to φ, it follows from $\varphi(\varphi^{-1}(a_{ii})) = a_{ii}$, $i = 1, 2$, that the matrix $A = \left(\begin{smallmatrix} g_{10} & h_{10} \\ g_{01} & h_{01}\end{smallmatrix}\right)$ is invertible over $K[a_{12}]$. So, $\det(A) \in K^*$. By the previous paragraph, $\det(A)$ is a multiple of $g_{10}h_{01}$, which implies that $g_{10}, h_{01} \in K^*$ and thus $A \in GL_2(K)$ and $\varphi = \varphi_{\lambda,A}$, as needed. □

Remark 6.4. Observe that $\varphi \in \mathrm{Aut}(T_q(2))$ is a bialgebra automorphism if and only if $\varphi(a_{12}) = \lambda a_{12}$ and $\varphi(a_{ii}) = a_{ii}$, $i = 1, 2$. Thus, the group of bialgebra automorphisms of $T_q(2)$ is isomorphic to K^*.

Next, we tackle the automorphism group of $UT_q(2)$. Although the group of units of $UT_q(2)$ is larger, there is a bit more rigidity in $\mathrm{Aut}(UT_q(2))$ in the sense that, modulo the subgroup $\langle\rho\rangle \cong \mathbb{Z}/2\mathbb{Z}$ generated by ρ, the automorphisms of $UT_q(2)$ are in a certain sense diagonal (see below).

Theorem 6.5. *The following is a subgroup of* $\mathrm{Aut}(UT_q(2))$*:*

$$\mathcal{G} = \{\varphi \in \mathrm{End}(UT_q(2)) : \varphi(a_{12}) = \lambda_{12} a_{11}^k a_{22}^l a_{12},\ \varphi(a_{ii}) = \lambda_{ii} z^j a_{ii},\ i \in \{1,2\}, \lambda_{ij} \in K^*, j,k,l \in \mathbb{Z}\},$$

where $z = a_{11}a_{22}^{-1}$*. Moreover, if q is not a root of unity, then*

$$\mathrm{Aut}(UT_q(2)) = \mathcal{G} \rtimes \langle\rho\rangle.$$

Proof. Using (42)–(44), it is routine to check that $\mathcal{G} \subseteq \mathrm{End}(UT_q(2))$ and that $\mathcal{G}$ is a submonoid of $\mathrm{End}(UT_q(2))$. Moreover, it is not hard to check that in fact $\mathcal{G} \subseteq \mathrm{Aut}(UT_q(2))$ is a subgroup. This can also be done explicitly as follows. Identify $\mathcal{G}$ as a set with $(K^*)^3 \times \mathbb{Z}^3$ (note that these are not isomorphic as groups, because $\mathcal{G}$ is nonabelian), where $\varphi \in \mathcal{G}$ with $\varphi(a_{12}) = \lambda_{12}a_{11}^k a_{22}^l a_{12}$ and $\varphi(a_{ii}) = \lambda_{ii}z^j a_{ii}$, for $i = 1,2$, is identified with $(\lambda_{12}, \lambda_{11}, \lambda_{22}, j, k, l) \in (K^*)^3 \times \mathbb{Z}^3$. Then, by transport of structure, the operation induced on $(K^*)^3 \times \mathbb{Z}^3$ via composition in $\mathcal{G}$ is given by

$$\begin{aligned}&(\lambda_{12}, \lambda_{11}, \lambda_{22}, j_1, k_1, l_1) * (\mu_{12}, \mu_{11}, \mu_{22}, j_2, k_2, l_2)\\ &= \left(\lambda_{12}\mu_{12}\lambda_{11}^{k_2}\lambda_{22}^{l_2}, \lambda_{11}\mu_{11}\left(\frac{\lambda_{11}}{\lambda_{22}}\right)^{j_2}, \lambda_{22}\mu_{22}\left(\frac{\lambda_{11}}{\lambda_{22}}\right)^{j_2}, j_1 + j_2, k_1 + k_2 + j_1(k_2 + l_2), l_1 + l_2 - j_1(k_2 + l_2)\right).\end{aligned} \tag{47}$$

Thus the identity morphism corresponds to $(1,1,1,0,0,0)$ and $(\lambda_{12}, \lambda_{11}, \lambda_{22}, j, k, l)^{-1}$ is

$$\left(\lambda_{12}^{-1}\lambda_{11}^{k-j(k+l)}\lambda_{22}^{l+j(k+l)}, \lambda_{11}^{-1}\left(\frac{\lambda_{11}}{\lambda_{22}}\right)^{j}, \lambda_{22}^{-1}\left(\frac{\lambda_{11}}{\lambda_{22}}\right)^{j}, -j, j(k+l) - k, -j(k+l) - l\right).$$

Moreover, if $\varphi \in \mathcal{G}$ is represented by $(\lambda_{12}, \lambda_{11}, \lambda_{22}, j, k, l)$, then $\rho\circ\varphi\circ\rho$ is represented by $(\lambda_{12}, \lambda_{22}, \lambda_{11}, -j, l, k)$, so ρ normalizes $\mathcal{G}$ and clearly $\mathcal{G} \cap \langle\rho\rangle = \mathrm{id}_{UT_q(2)}$.

Assume now that q is not a root of unity. It remains to show that $\mathrm{Aut}(UT_q(2))$ is generated by $\mathcal{G}$ and ρ. Note that $\mathcal{G}$ contains the diagonal automorphisms of $UT_q(2)$, so we can work modulo ρ and the diagonal automorphisms. Let $\varphi \in \mathrm{Aut}(UT_q(2))$. Since the group of units of $UT_q(2)$ is $\{\lambda a_{11}^k a_{22}^l : \lambda \in K^*, k, l \in \mathbb{Z}\}$, we conclude from Lemma 6.2 that, modulo a diagonal automorphism, $\varphi(a_{12}) = a_{11}^k a_{22}^l a_{12}$, for some $k, l \in \mathbb{Z}$.

Since a_{11} is a unit, there exist $i, j \in \mathbb{Z}$ and $\lambda \in K^*$ such that $\varphi(a_{11}) = \lambda a_{11}^i a_{22}^j$; as before, we can assume that $\lambda = 1$. Using the relation (43) and the fact that q is not a root of unity, we conclude, as in the proof of Theorem 6.3, that $i+j = 1$, so $\varphi(a_{11}) = a_{11}^i a_{22}^{1-i} = a_{11}z^{i-1}$. Similarly, we have $\varphi(a_{22}) = a_{11}^j a_{22}^{1-j} = a_{22}z^j$, for some $j \in \mathbb{Z}$. The automorphism φ must restrict to an automorphism of $Z(UT_q(2))$ and $Z(UT_q(2)) = K[z^{\pm 1}]$, by Lemma 5.3. We have

$$\varphi(z) = \varphi(a_{11}a_{22}^{-1}) = a_{11}z^{i-1}a_{22}^{-1}z^{-j} = z^{i-j},$$

whence $i-j = \pm 1$. Interchanging φ with $\varphi\circ\rho$, if necessary, we can assume that $i-j = 1$. Thus $\varphi(a_{11}) = a_{11}z^j$ and $\varphi(a_{22}) = a_{22}z^j$, so $\varphi \in \mathcal{G}$. □

Remark 6.6. Let $\mathcal{G}$ be the subgroup of $\mathrm{Aut}(UT_q(2))$ from Theorem 6.5. Then each $(\lambda_{12}, \lambda_{11}, \lambda_{22}, j, k, l) \in \mathcal{G}$ can be uniquely written as

$$\begin{aligned}(\lambda_{12}, \lambda_{11}, \lambda_{22}, j, k, l) &= (1,1,1,j,k,l) * (\lambda_{12}, \lambda_{11}, \lambda_{22}, 0,0,0)\\ &= (1,1,1,0,k,l) * (1,1,1,j,0,0) * (\lambda_{12}, \lambda_{11}, \lambda_{22}, 0,0,0)\\ &= (1,1,1,j,0,0) * (1,1,1,0,k-j(k+l), l+j(k+l)) * (\lambda_{12}, \lambda_{11}, \lambda_{22}, 0,0,0).\end{aligned} \tag{48}$$

More precisely, $\mathcal{G}$ is the Zappa–Szép product $(G_1 \rtimes G_2)G_3$ of its subgroups, where

$$\begin{aligned}G_1 &= \{(\lambda_{12}, \lambda_{11}, \lambda_{22}, j, k, l) \in \mathcal{G} : \lambda_{12} = \lambda_{11} = \lambda_{22} = 1 \text{ and } j = 0\} \cong \mathbb{Z} \times \mathbb{Z},\\ G_2 &= \{(\lambda_{12}, \lambda_{11}, \lambda_{22}, j, k, l) \in \mathcal{G} : \lambda_{12} = \lambda_{11} = \lambda_{22} = 1 \text{ and } k = l = 0\} \cong \mathbb{Z},\\ G_3 &= \{(\lambda_{12}, \lambda_{11}, \lambda_{22}, j, k, l) \in \mathcal{G} : j = k = l = 0\} \cong (K^*)^3.\end{aligned}$$

Indeed, all the decompositions of (48) are straightforward by (47), and their uniqueness is obvious by (48) itself. It is directly verified that G_3 is a subgroup of $\mathcal{G}$ isomorphic to $(K^*)^3$ and that $\mathcal{G} = HG_3$, where

$$H = \{(\lambda_{12}, \lambda_{11}, \lambda_{22}, j, k, l) \in \mathcal{G} : \lambda_{12} = \lambda_{11} = \lambda_{22} = 1\}$$

is a subgroup of $\mathcal{G}$ with $G_3 \cap H = \{(1, 1, 1, 0, 0, 0)\}$. Now, by the second and third equalities of (48) we have $H = G_1G_2 = G_2G_1$ and $G_1 \trianglelefteq H$. Since, obviously, $G_1 \cap G_2 = \{(1, 1, 1, 0, 0, 0)\}$, we conclude that $H = G_1 \rtimes G_2$, whence $\mathcal{G} = (G_1 \rtimes G_2)G_3$. Finally, the isomorphisms $G_1 \cong \mathbb{Z} \times \mathbb{Z}$ and $G_2 \cong \mathbb{Z}$ are evident by (47).

Corollary 6.7. *Assume that q is not a root of unity and let $z = a_{11}a_{22}^{-1}$. Then the group of Hopf algebra automorphisms of $UT_q(2)$ is*

$$\mathcal{H} = \{\varphi \in \mathrm{Aut}(UT_q(2)) : \varphi(a_{12}) = \lambda z^j a_{12},\ \varphi(a_{ii}) = z^j a_{ii},\ i \in \{1, 2\}, \lambda \in K^*, j \in \mathbb{Z}\} \cong K^* \times \mathbb{Z}.$$

Proof. Let $\varphi \in \mathrm{Aut}(UT_q(2))$. By Theorem 6.5 there are two cases.

Case 1: $\varphi \in \mathcal{G}$. Then $\varphi(a_{12}) = \lambda_{12}a_{11}^k a_{22}^l a_{12}$, $\varphi(a_{ii}) = \lambda_{ii}z^j a_{ii}$, where $i \in \{1, 2\}$, $\lambda_{ij} \in K^*$ and $j, k, l \in \mathbb{Z}$. For all $i \in \{1, 2\}$ we have

$$\begin{aligned}(\varphi \otimes \varphi)(\Delta(a_{ii})) &= \varphi(a_{ii}) \otimes \varphi(a_{ii}) = \lambda_{ii}z^j a_{ii} \otimes \lambda_{ii}z^j a_{ii} = \lambda_{ii}^2 z^j a_{ii} \otimes z^j a_{ii},\\ \Delta(\varphi(a_{ii})) &= \Delta(\lambda_{ii}z^j a_{ii}) = \lambda_{ii}z^j a_{ii} \otimes z^j a_{ii},\end{aligned}$$

so that $(\varphi \otimes \varphi)(\Delta(a_{ii})) = \Delta(\varphi(a_{ii})) \Leftrightarrow \lambda_{ii} = 1$. Hence, assume $\lambda_{11} = \lambda_{22} = 1$. Then

$$\begin{aligned}(\varphi \otimes \varphi)(\Delta(a_{12})) &= \varphi(a_{11}) \otimes \varphi(a_{12}) + \varphi(a_{12}) \otimes \varphi(a_{22}) = z^j a_{11} \otimes \lambda_{12}a_{11}^k a_{22}^l a_{12} + \lambda_{12}a_{11}^k a_{22}^l a_{12} \otimes z^j a_{22},\\ \Delta(\varphi(a_{12})) &= \Delta(\lambda_{12}a_{11}^k a_{22}^l a_{12}) = \lambda_{12}(a_{11}^k a_{22}^l \otimes a_{11}^k a_{22}^l)(a_{11} \otimes a_{12} + a_{12} \otimes a_{22}), \end{aligned} \tag{49}$$

so that $(\varphi \otimes \varphi)(\Delta(a_{12})) = \Delta(\varphi(a_{12})) \Leftrightarrow z^j = a_{11}^k a_{22}^l$, in which case $\varphi(a_{12}) = \lambda_{12}z^j a_{12}$. Thus, φ respects Δ $\Leftrightarrow \varphi \in \mathcal{H}$. Since $\varepsilon \circ \varphi = \varepsilon$ for any $\varphi \in \mathcal{H}$, we are done.

Case 2: $\varphi \in \mathcal{G}\rho$. Then $\varphi(a_{12}) = \lambda_{12}a_{11}^k a_{22}^l a_{12}$, $\varphi(a_{ii}) = \lambda_{ii}z^j a_{\bar{i}\bar{i}}$, where $i \in \{1, 2\}$, $\bar{i} = 3 - i$, $\lambda_{ij} \in K^*$ and $j, k, l \in \mathbb{Z}$. As in Case 1 one has $(\varphi \otimes \varphi)(\Delta(a_{ii})) = \Delta(\varphi(a_{ii})) \Leftrightarrow \lambda_{ii} = 1$ for all $i \in \{1, 2\}$. Assuming $\lambda_{11} = \lambda_{22} = 1$, we have

$$(\varphi \otimes \varphi)(\Delta(a_{12})) = \varphi(a_{11}) \otimes \varphi(a_{12}) + \varphi(a_{12}) \otimes \varphi(a_{22}) = z^j a_{22} \otimes \lambda_{12}a_{11}^k a_{22}^l a_{12} + \lambda_{12}a_{11}^k a_{22}^l a_{12} \otimes z^j a_{11},$$

while $\Delta(\varphi(a_{12}))$ is given by (49) as in Case 1. Hence $(\varphi \otimes \varphi)(\Delta(a_{12})) = \Delta(\varphi(a_{12})) \Leftrightarrow z^j a_{22} = a_{11}^{k+1}a_{22}^l$ and $z^j a_{11} = a_{11}^k a_{22}^{l+1}$, which is impossible. Thus, φ is not a bialgebra morphism.

The isomorphism $\mathcal{H} \cong K^* \times \mathbb{Z}$ is obvious. □

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