One-dimensional particle clouds with elastic collisions

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Abstract

We study an interacting particle system of a finite number of labelled particles on the integer lattice, in which particles have intrinsic masses and left/right jump rates. If a particle is the minimal-label particle at its site when it tries to jump left, the jump is executed. If not, 'momentum' is transferred to increase the rate of jumping left of the minimal-label particle. Similarly for jumps to the right. The collision rule is 'elastic' in the sense that the net rate of flow of mass is independent of the present configuration, in contrast to the exclusion process, for example. We show that the particle masses and jump rates determine explicitly, via a concave majorant of a simple 'potential' function associated to the masses and jump rates, a unique partition of the system into maximal stable subsystems. The internal configuration of each stable subsystem remains tight, while the location of each stable subsystem obeys a strong law of large numbers with an explicit speed. We indicate connections to adjacent models, including diffusions with rank-based coefficients.

Key words: Interacting particle system, elastic collisions, lattice Atlas model, asymptotic speeds, partial stability, invariant measures, exclusion process.

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1 Definitions and main results

We consider dynamics of an interacting system of N ordered particles performing continuous-time nearest-neighbour random walks on the integer lattice $\mathbb{Z}$ with *elastic collisions*. Each particle $i \in \{1, 2, \ldots, N\}$ (labelled left to right) is endowed with intrinsic jump rates $a_i, b_i \in \mathbb{R}_+ := [0, \infty)$ and an intrinsic *mass* $m_i \in (0, \infty)$.

The dynamics of the system are as follows. If the site occupied by particle i is occupied by no other particle, then particle i jumps to the left with rate a_i and to the right with rate b_i, independently of the other particles; hence particles perform independent continuous-time random walks as long as they avoid each other. When two or more particles occupy the same site, only the particle with the smallest label in the stack can jump left, and only the particle with the largest label in the stack can jump right, and the rates are modified by what we call the elastic collision rule. Specifically, if particles $k, \ldots, k+\ell$ (and no other) are at a given site then particle k jumps to the left with rate $m_k^{-1} \sum_{j=k}^{k+\ell} m_j a_j$, while particle $k+\ell$ jumps to the right with rate $m_{k+\ell}^{-1} \sum_{j=k}^{k+\ell} m_j b_j$.

Denote by $X_i(t) \in \mathbb{Z}$ the position of particle i at time $t \in \mathbb{R}_+$, with initial configuration $X_1(0) \leq X_2(0) \leq \cdots \leq X_N(0)$; observe that the collision rule preserves the weak order $X_1(t) \leq$

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$$b_1 \qquad b_2 + b_3 \qquad\qquad b_4 + b_5 + b_6$$

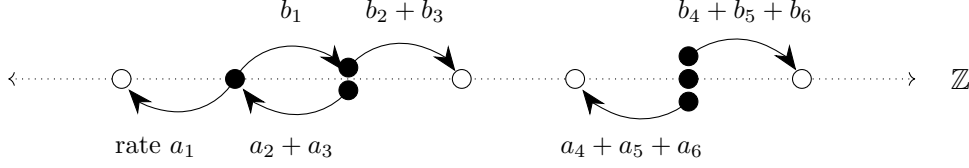$$\text{rate } a_1 \qquad a_2 + a_3 \qquad\qquad a_4 + a_5 + a_6$$

FIGURE 1: Schematic for the $N = 6$ elastic particle system with identical masses $m_i \equiv m \in (0, \infty)$ for all i. Pictured is the configuration $X_1(t) = 0$, $X_2(t) = X_3(t) = 1$, $X_4(t) = X_5(t) = X_6(t) = 4$ and the transition rates from this configuration are indicated on the arrows. In the case of multiple particles occupying the same site, we imagine that particles are stacked in increasing order of index, and it is the bottom and top particles that are allowed to move: the transition to the left would move the particle from the base of the stack, while the transition to the right would move the particle from the top of the stack. To contrast the elastic dynamics with exclusion dynamics, we refer to Figure 4 below; note that in the identical-mass setting, the total activity rate of the elastic system is $\sum_i (a_i + b_i)$ independently of the current configuration.

$X_2(t) \leq \cdots \leq X_N(t)$ for all $t \in \mathbb{R}_+$. We give a slightly more formal definition shortly, and refer to Figure 1 for a schematic.

We use the word '*elastic*' to describe the distinguishing property of this interacting-particle system that *the total momentum is independent of the present configuration*, where momentum is the (net) rate at which mass moves to the right (see Lemma 1.7 below and surrounding discussion). This model is equivalent to one proposed in §6.2 of [36] as a lattice particle system possessing the elastic collision property of the continuum *Atlas model*, in contrast to the classical simple *exclusion process* which does not have the elastic property, since the exclusion dynamics suppresses movement of particles in some configurations (see §2 for a description of the model of [36] and its equivalence to the one here, and for background on the exclusion process and the Atlas model).

To give a physical motivation to the elastic collision rule, note that, first, if we interpret transitions rates as speeds, when it does not share occupancy with any other particle, particle i has 'momentum to the left' $m_i a_i$ and 'momentum to the right' $m_i b_i$. On the other hand, if particle i is part of a stack of multiple particles, the momentum generated by the stack to the left and right is obtained by summing the individual momenta of the particles in the stack. For example, if particles $k, k+1, \ldots, k+\ell$ are the occupants of a particular site, then the elastic collision rule imparts to particle k a 'momentum to the left' of $m_k \cdot m_k^{-1} \sum_{j=k}^{k+\ell} m_j a_j = \sum_{j=k}^{k+\ell} m_j a_j$, while particles $k+1, \ldots, k+\ell$ have 'momentum to the left' of zero. Similarly to the right. In this sense, 'momentum' is conserved.

We introduce some more notation and describe our main results. For $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$, define $[n] := \{i \in \mathbb{Z} : 1 \leq i \leq n\}$. The configuration space of the system of $N \in \mathbb{N}$ particles is $\mathbb{X}_N \subset \mathbb{Z}^N$, given by

$$\mathbb{X}_N = \left\{ (x_i)_{i \in [N]} \in \mathbb{Z}^N : x_1 \leq \cdots \leq x_N \right\}. \tag{1.1}$$

We write $X(t) \in \mathbb{X}_N$ for the state at time $t \in \mathbb{R}_+$, with coordinates $X(t) = (X_i(t))_{i \in [N]}$. We start the system from a deterministic (but arbitrary) initial state $X(0) \in \mathbb{X}_N$. The process $X := (X(t))_{t \in \mathbb{R}_+}$ is a continuous-time Markov chain on the countable state space $\mathbb{X}_N$, in which $X_i(t)$ jumps to $X_i(t)-1$ at rate $A_i(X(t))$ and $X_i(t)$ jumps to $X_i(t) + 1$ at rate $B_i(X(t))$ where, for $x = (x_i)_{i \in [N]} \in \mathbb{X}_N$,

$$A_i(x) := \mathbb{1}_{\{x_{i-1} < x_i\}} \sum_{j=i}^{N} \frac{m_j a_j}{m_i} \mathbb{1}_{\{x_j = x_i\}}, \quad B_i(x) := \mathbb{1}_{\{x_{i+1} > x_i\}} \sum_{j=1}^{i} \frac{m_j b_j}{m_i} \mathbb{1}_{\{x_j = x_i\}}, \tag{1.2}$$

with conventions $x_0 = -\infty$ and $x_{N+1} = +\infty$. Denote the intrinsic velocities of the particles

$$u := (u_i)_{i \in [N]}, \text{ where } u_i := b_i - a_i, \text{ for } i \in [N]. \tag{1.3}$$

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We then define $U_0 := 0$, $M_0 := 0$, and, for $k \in [N]$,

$$U_k := -\sum_{i=1}^{k} m_i u_i, \text{ and } M_k := \sum_{i=1}^{k} m_i. \tag{1.4}$$

Also, for $\ell \in \{0, 1, \ldots, k\}$ define

$$U_{\ell,k} := U_k - U_\ell = -\sum_{i=\ell+1}^{k} m_i u_i, \text{ and } M_{\ell,k} := M_k - M_\ell = \sum_{i=\ell+1}^{k} m_i. \tag{1.5}$$

Remark 1.1. Because of the elastic interaction mechanism, many of the a_is and b_is can be set to zero and the model still be non-trivial. For example, one can take $b_1 > 0$, $a_N > 0$, and all other $a_i, b_i = 0$, and the elastic process still, with positive probability, moves all particles any finite distance to the left or right, while, in contrast, the exclusion process with the same parameters (see §2.4) would reach an absorbing state in finitely many steps.

Remark 1.2. The reason for the negative sign in the definition of U_k in (1.4) is to conform with the interpretation of U_k/M_k as a sort of *potential* function associated with the system. Roughly speaking, the evolution of the system will favour moving particles in directions of negative gradient of the potential; hence the choice of sign. We elaborate on this following Definition 1.8 below.

For non-empty $\mathfrak{C} \subseteq [N]$ denote the span of particles labelled by $\mathfrak{C}$ as

$$\Delta_{\mathfrak{C}}(t) := \sup_{i \in \mathfrak{C}} X_i(t) - \inf_{i \in \mathfrak{C}} X_i(t) = X_{\max \mathfrak{C}}(t) - X_{\min \mathfrak{C}}(t), \text{ for } t \in \mathbb{R}_+. \tag{1.6}$$

Our first main result is a criterion for *stability* of the system, meaning that the system evolves as a single *cloud* of particles, within which the inter-particle distances remain exponentially tight, with a single characteristic *speed*. The formal statement is the following.

Theorem 1.3 (Stability criterion)**.** *Let $N \in \mathbb{N}$, $m_i \in (0, \infty)$ for all $i \in [N]$, and $a_i, b_i \in \mathbb{R}_+$ for all $i \in [N]$. With U_k, M_k defined at (1.4), suppose that*

$$\frac{U_k}{M_k} < \frac{U_N}{M_N}, \text{ for all } k \in [N-1]. \tag{1.7}$$

Then the system is a single stable cloud in the following sense.

(a) **Limiting speed.** *For every $i \in [N]$, there holds the strong law of large numbers*

$$\lim_{t \to \infty} t^{-1} X_i(t) = -\frac{U_N}{M_N}, \text{ a.s.} \tag{1.8}$$

(b) **Relative stability.** *There exist constants $C \in \mathbb{R}_+$ and $\delta > 0$ (depending on the a_i, b_i, and m_i) such that*

$$\sup_{t \in \mathbb{R}_+} \mathbb{P}\left[\Delta_{[N]}(t) \geq s\right] \leq C\left[1 + e^{C\Delta_{[N]}(0)}\right] e^{-\delta s}, \text{ for all } s \in \mathbb{R}_+,$$

and, moreover,

$$\limsup_{t \to \infty} \frac{\Delta_{[N]}(t)}{\log t} < \infty, \text{ a.s.}$$

Furthermore, condition (1.7) is necessary for (b) to hold.

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Remark 1.4. That condition (1.7) is sufficient for relative stability was conjectured in [36] (for a modified but equivalent version of the process, as explained in §2.3), and is a lattice analogue of the stability result for Atlas-type models (e.g. Theorem 8 of [39]). That condition (1.7) is necessary for relative stability (b) is a consequence of the more general partial stability result, Theorem 1.10, that we present below. That result also shows that (a) *can* hold if (1.7) is violated, in a system with multiple stable clouds all with the same characteristic speed, a simple example being the homogeneous-speed case in which $u_i \equiv u$ for all $i \in [N]$ (cf. Example 1.11 below).

For a configuration $x = (x_1, \ldots, x_N) \in \mathbb{X}_N$, define its vector of inter-particle distances

$$D(x) := (D_i(x))_{i \in [N-1]} \in \mathbb{Z}_+^{N-1}, \text{ where } D_i(x) := x_{i+1} - x_i \text{ for } i \in [N-1]. \tag{1.9}$$

Then associated to process X is the process $\eta(t) := (\eta_i(t))_{i \in [N-1]}$, on $\mathbb{Z}_+^{N-1}$, where

$$\eta(t) := D(X(t)), \text{i.e., } \eta_i(t) = X_{i+1}(t) - X_i(t) \text{ for } i \in [N-1]. \tag{1.10}$$

Clearly $(X_1(t), \eta(t))$ defines a Markov process on $\mathbb{Z} \times \mathbb{Z}_+^{N-1}$ that contains the same information as the original process X. Moreover, it is not hard to see that $\eta := (\eta(t))_{t \in \mathbb{R}_+}$ is itself a Markov process, describing the configuration relative to the left-most particle. The relative stability part of Theorem 1.3 has the following interpretation in terms of the process η.

Corollary 1.5. *The Markov chain η on $\mathbb{Z}_+^{N-1}$ is positive recurrent if and only if (1.7) holds. If positive recurrent, then η is geometrically ergodic, meaning that the stationary distribution has a finite exponential moment.*

The following example covers systems with few particles; a contrast with the exclusion process described in §2.4 is given in Example 2.1 below.

Example 1.6 (Small systems). The case $N = 1$ is trivial; then (1.7) holds vacuously, $U_1 = m_1(a_1 - b_1)$ and (1.8) reduces to the ordinary strong law for a single random walker, $\lim_{t \to \infty} t^{-1} X_1(t) = u_1$, a.s. When $N = 2$, the stability criterion (1.7) reduces to the simple condition $u_1 > u_2$ (regardless of the masses), meaning that if the two particles did not interact, the leftmost would overtake the rightmost. For $N = 3$, the masses of the particles enter, and the stability criterion (1.7) is

$$(m_2 + m_3)u_1 > m_2 u_2 + m_3 u_3 \text{ and } m_1 u_1 + m_2 u_2 > (m_1 + m_2)u_3;$$

note that a consequence of the above two inequalities is that $u_1 > u_3$, and a sufficient condition for stability is $u_1 > u_2 > u_3$. △

Before moving on to the general case in which (1.7) is not satisfied, we give some intuition behind the speed $-U_N/M_N$ appearing in (1.8), and the origin of condition (1.7). Let

$$G := (G(t))_{t \in \mathbb{R}_+}, \text{ where } G(t) := \frac{1}{M_N} \sum_{i \in [N]} m_i X_i(t), \text{ for } t \in \mathbb{R}_+, \tag{1.11}$$

the *centre of mass* process associated with the particle system. In general G is not itself a Markov process, since the transition law of $G(t)$ depends on the whole configuration $X(t)$. Nevertheless, we show below (see Theorem 3.1) that there is a well-defined local *speed* of $G(t)$ which is always equal to $-U_N/M_N$. Moreover, in the equal-mass case where $m_i \equiv m$ independently of i, it is immediate from (1.2) to verify the following stronger fact, which says that G *is* Markov. This is a discrete analogue of a similar observation for diffusion systems (e.g. Lemma 7 of [39], and §2.6 below).

Lemma 1.7. *If $m_i \equiv m \in (0, \infty)$ for all $i \in [N]$, then G performs a continuous-time, spatially homogeneous random walk on $N^{-1}\mathbb{Z}$ which jumps $-m/N$ at rate $\sum_{i=1}^{N} a_i$ and $+m/N$ at rate $\sum_{i=1}^{N} b_i$.*

In particular, Lemma 1.7 together with the strong law for Poisson processes shows that $\lim_{t \to \infty} t^{-1} G(t) = -U_N/M_N$ in this case; we show below the same is true more generally (see Theorem 3.1). This fact is true regardless of the stability or otherwise of the particle system, but it does explain why, in Theorem 1.3, when the system is stable, the limiting speed of the cloud as given by (1.8) has to be $-U_N/M_N$.

To state our general stability result, Theorem 1.10 below, we need some additional definitions.

Definition 1.8 (Concave majorant, boundary, slopes). (i) Given $M_0, \ldots, M_N$ and $U_0, \ldots, U_M$, a continuous, concave function $u : [0, M_N] \to \mathbb{R}$ is a *concave majorant* if $U_k \leq u(M_k)$ for every $k \in \{0, 1, \ldots, N\}$. The (unique) *least concave majorant* $\overline{u} : [0, M_N] \to \mathbb{R}$ is a concave majorant such that $\overline{u}(x) \leq u(x)$ for all $x \in [0, M_N]$ and every concave majorant u. See Figure 2 for a picture.

(ii) It is not hard to see that $\overline{u}$ is piecewise linear. Denote

$$V := \big\{ k \in \{0, 1, \ldots, N\} : U_k = \overline{u}(M_k) \big\}, \tag{1.12}$$

the set of *boundary* indices for $\overline{u}$; note that it is always the case that $\{0, N\} \subseteq V$. In other words, $V = \{k_0, k_1, \ldots, k_\nu\}$ ($\nu \in [N]$) is the unique set with $0 = k_0 < \cdots < k_\nu = N$,

$$\frac{U_{k_{j-1}, k_j}}{M_{k_{j-1}, k_j}} \geq \frac{U_{k_j, k_{j+1}}}{M_{k_j, k_{j+1}}}, \text{ for all } j \in [\nu - 1], \text{ and} \tag{1.13}$$

$$\frac{U_{k_j, m}}{M_{k_j, m}} < \frac{U_{k_j, k_{j+1}}}{M_{k_j, k_{j+1}}} \text{ for all } 0 \leq j < \nu \text{ and all } k_j < m < k_{j+1}, \tag{1.14}$$

where the convention is $U_0/M_0 := \infty$.

(iii) Given boundary indices $V = \{k_0, k_1, \ldots, k_\nu\}$, denote the sequence of *slopes* of successive boundary segments by

$$v_j := \frac{U_{k_{j-1}, k_j}}{M_{k_{j-1}, k_j}}, \text{ for } j \in [\nu]; \tag{1.15}$$

then (1.13) says that $v_j \geq v_{j+1}$ for all $j \in [\nu - 1]$.

Remark 1.9. If in (1.13), we demand *strict* inequality, then we obtain the set of *vertices* of the least concave majorant, and the line segments between successive vertices are its *faces* which have strictly ordered slopes. The set V from Definition 1.8 can also include non-vertex boundary points, in cases where the path (M_k, V_k) touches its concave majorant at the interior of a face, corresponding to equality in (1.13).

The next result will demonstrate a *cloud decomposition*, which means an ordered partition $\mathfrak{C}_1 < \cdots < \mathfrak{C}_\nu$ of $[N]$, where the *clouds* $\mathfrak{C}_k$ are non-empty, pairwise disjoint, have union $[N]$, and where $\mathfrak{C} < \mathfrak{C}'$ for $\mathfrak{C}, \mathfrak{C}' \subseteq [N]$ means that $i < j$ for every $i \in \mathfrak{C}$ and $j \in \mathfrak{C}'$. Then for every $i \in [N]$ there is a unique $j \in [\nu]$ such that $i \in \mathfrak{C}_j$; we say that particle i belongs to cloud j. The next result shows that the particles in each cloud are typically close, in the sense that all inter-particle distances within the same cloud remain exponentially tight, but different clouds either diverge ballistically (if their speeds differ) or are very often well-separated (if they have the same speed, see Remark 1.12(ii)). For this last part, we will need also the following non-degeneracy condition:

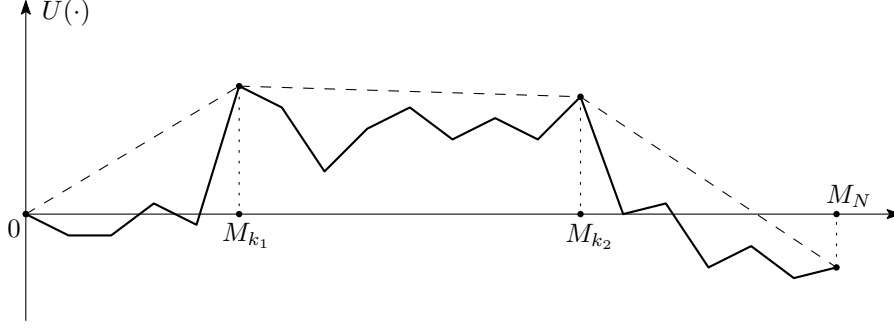$$\sum_{i \in \mathfrak{C}_j} (a_i + b_i) > 0 \text{ for every } j \in [\nu], \tag{1.16}$$

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FIGURE 2: Identifying the stable clouds: here, these are $\{1, \ldots, k_1\}$, $\{k_1 + 1, \ldots, k_2\}$, and $\{k_2 + 1, \ldots, N\}$ (with the speeds being equal to minus the corresponding slopes).

which says that every cloud has some intrinsic activity. For $j \in [\nu - 1]$, we write

$$L_j(t) := \min_{i \in \mathfrak{C}_{j+1}} X_i(t) - \max_{i' \in \mathfrak{C}_j} X_{i'}(t) = X_{\min \mathfrak{C}_{j+1}}(t) - X_{\max \mathfrak{C}_j}(t) \in \mathbb{Z}_+, \text{ for } t \in \mathbb{R}_+, \qquad (1.17)$$

the minimal distance between particles from clouds $\mathfrak{C}_j$ and $\mathfrak{C}_{j+1}$ at time t.

Theorem 1.10 (Cloud decomposition). *Let $N \in \mathbb{N}$, $m_i \in (0, \infty)$ for all $i \in [N]$, and $a_i, b_i \in \mathbb{R}_+$ for all $i \in [N]$. Suppose that (1.16) holds. Suppose that $V = \{k_0, k_1, \ldots, k_\nu\}$ is the set of boundary indices for the least concave majorant, as in Definition 1.8. Then for the cloud decomposition $\mathfrak{C}_1, \ldots, \mathfrak{C}_\nu$ given by $\mathfrak{C}_j := \{i \in [N] : k_{j-1} < i \leq k_j\}$, the following hold.*

(a) **Cloud speeds.** *For every $j \in [\nu]$ and every $i \in \mathfrak{C}_j$, with v_j the slope defined at (1.15),*

$$\lim_{t \to \infty} t^{-1} X_i(t) = -v_j, \ \ a.s.$$

(b) **Clouds are stable.** *Recall that $\Delta_{\mathfrak{C}}$ is defined at (1.6). There exist constants $C \in \mathbb{R}_+$ and $\delta > 0$ (depending on the $a_i, b_i,$ and m_i) such that for every $j \in [\nu]$,*

$$\sup_{t \in \mathbb{R}_+} \mathbb{P}\left[\Delta_{\mathfrak{C}_j}(t) \geq s\right] \leq C\left[1 + \mathrm{e}^{C\Delta_{\mathfrak{C}_j}(0)}\right]\mathrm{e}^{-\delta s}, \text{ for all } s \in \mathbb{R}_+,$$

and, moreover,

$$\limsup_{t \to \infty} \frac{\Delta_{\mathfrak{C}_j}(t)}{\log t} < \infty, \ \ a.s.$$

(c) **Cloud separations.** *Let $j \in [\nu - 1]$. If $v_j > v_{j+1}$, then $\lim_{t \to \infty} t^{-1} L_j(t) = |v_{j+1} - v_j| > 0$, a.s. If $v_j = v_{j+1}$, then there exists $\varepsilon \in (0, 1/2)$ such that*

$$\lim_{t \to \infty} \frac{1}{t^{1-\varepsilon}} \int_0^t \mathbb{1}\{L_j(s) \leq t^\varepsilon\}\mathrm{d}s = 0, \ \ a.s., \qquad (1.18)$$

while nevertheless

$$\liminf_{t \to \infty} L_j(t) = 0, \ \ a.s. \qquad (1.19)$$

Example 1.11 (Homogeneous speeds). Suppose that $u_i \equiv u \in \mathbb{R}$ for all $i \in [N]$. Then $U_k/M_k = -u$ for all $k \in [N]$, meaning that $V = \{1, 2, \ldots, N\}$ and the cloud decomposition consists of $\nu = N$

singleton "clouds" all with the same speed $-v_j = u$. The condition (1.16) holds (and hence so does the separation result in (c) above) unless $a_i = b_i = 0$ for all $i \in [N]$, in which case (1.18) will fail, since the particles do not move at all. This, and similar examples, shows that the hypothesis (1.16) cannot be removed in Theorem 1.10(c). $\triangle$

Remarks 1.12. (i) Recalling Definition 1.8, we have that $V = \{0, N\}$, i.e., $n = 1$ and the concave majorant $\overline{u}$ is strictly above (M_i, U_i) for every $i \in [N-1]$, if and only if (1.7) holds. Hence Theorem 1.3 follows from Theorem 1.10, although we prove them in the other order.

(ii) Statement (1.18) says that each gap between clouds with equal speeds is all but a vanishing proportion of the time growing at least as a small power of t, while (1.19) says that it is, on the other hand, recurrent. A natural (but difficult) question concerns recurrence or transience of $L_A(t) := (L_j(t))_{j \in A}$ for $A \subseteq [\nu - 1]$ with $v_j = v_{j+1}$ for every $j \in A$. For $N = 3$ particles in $\nu = 3$ singleton clouds, criteria for recurrence and transience can be deduced from results for random walks on $\mathbb{Z}_+^2$ [7] as described in Example 2.18 of [36] for the related model with exclusion interaction, but we leave the general case as an open problem.

(iii) Theorem 1.10 identifies stable subsystems of a Markov chain that may not be stable as a whole; interest in such *partial stability* results [2, 22, 36] has been stimulated by queueing theory especially, and there is a queueing interpretation of the present model, as we explain in §2.5 below.

(iv) In the construction of Definition 1.8, note that if we multiply all masses m_i by the same positive constant, then the $U_{\ell,k}$, $M_{\ell,k}$ are multiplied by that same constant, hence their ratios remain the same, and so the boundary indices $\{k_0, \ldots, k_\nu\}$ remain unchanged, as do the v_j at (1.15). Similarly, if we multiply all rates a_i, b_i by the same positive constant, then the $U_{\ell,k}$ are multiplied but not the $M_{\ell,k}$, so the boundary indices $\{k_0, \ldots, k_\nu\}$ remain unchanged once more, but the v_i are all multiplied by the common rate factor. Moreover, in the special case where masses $m_i \equiv m \in (0, \infty)$ are constant, the boundary indices also remain the same if one *adds* the same quantity to all the a_i's (and/or all the b_i's), corresponding to an affine transformation $U_k \mapsto U_k + \alpha k$ which preserves the concave majorant (cf. Figure 2).

(v) Suppose that parameters $\omega := (a_i, b_i)_{i \in [N]}$ are determined before the dynamics begins by sampling a *random environment*. That is, take $(a_1, b_1), \ldots, (a_N, b_N)$ are independent draws from a law $\mathbf{P}$, and then, given the realization of the environment ω, define the Markov chain X through rates (1.2), under a law that we now call $\mathbb{P}_\omega$ to indicate the dependence on ω. Theorem 1.10 is then a *quenched* result (for given ω) but also of interest is behaviour for typical ω. For simplicity, suppose that $m_i \equiv 1 \in (0, \infty)$ (constant masses). Then the cloud decomposition is determined by the concave majorant of the N-step random walk (under $\mathbf{P}$) $U_0, U_1, \ldots, U_N$ with increments $-u_1, \ldots, -u_N$, by (1.4). A classical result on concave majorants of random walks (see [1, 46] and references therein) says that, assuming u_i has a density under $\mathbf{P}$, the expected number of slopes of the concave majorant is asymptotically equivalent to $\log N$, and hence the (random) number of clouds ν in the decomposition of Theorem 1.10 has $\mathbf{E}\nu \sim \log N$ as $N \to \infty$ as well.

We indicate two quite broad directions for potential future work, in addition to the recurrence/transience question raised in Remark 1.12(ii).

- We have dealt here with finite systems of particles with elastic collisions, but it is a natural open problem to also consider semi-infinite systems, with particles enumerated by the natural numbers. Here one expects some progress can be made by comparison with large finite systems, but also new and rich phenomena; see e.g. [38], and references therein, for the setting of exclusion interaction (as described in §2.4 in the finite case).

- In contrast to the case of exclusion interaction, where explicit product-geometric invariant distributions are known in the stable case (see [36]), the results above in the elastic case give no explicit form for invariant measures. There are reasons to expect that, in the elastic case, simple explicit formulas are not available generically, but only in some special cases of the parameters, that as yet remain to be classified. We make some observations in this direction in §5, but leave fuller study of invariant measures as an open problem.

The outline of the rest of the paper is as follows. In §2 we describe equivalent models to the model described above (including in §2.3 the original formulation of the particle model from [36] in which no site can be occupied by more than one particle). In §2.4 we contrast the elastic model with the simple exclusion process, which has a non-elastic collision mechanism and is one of the most studied models of interacting particle systems, while in §2.6 we draw parallels with a well-studied continuum model of mutually-reflecting diffusions. In §2.5 we formulate a queueing model equivalent to the elastic particle model. In the exclusion-process context, a similar translation gives a Jackson network, for which partial stability results due to Goodman and Massey [22] were a key component to developing the corresponding cloud decomposition [36]; as far as we know, no such results are available for the class of queueing models we end up with here, and so we need a different approach. The proofs of our main results are divided between proof of, loosely speaking, stability (Theorem 1.3) in §3 and instability and hence the cloud decomposition (Theorem 1.10) in §4. The arguments for stability, including the proof of Theorem 1.3 given in §3.1, use some Lyapunov function ideas presented in §3.2. The most involved part of the proof of instability concerns the case where there are several different clouds of the same speed, and here we make use of some apparently novel tools from martingale defocusing that we present in a self-contained appendix (§A). The proof of Theorem 1.10 is then given in §4.3. In §5 we present some partial results on invariant distributions and pose some open questions in that context.

2 Discussion of equivalent and adjacent models

2.1 Random walks with rank-dependent rates

Here is an alternative construction of the model formulated in §1. Consider a system of N labelled particles with identical masses $m_i \equiv m \in (0, \infty)$ for all $i \in [N]$. Let $Y(t) := (Y_1(t), \ldots, Y_N(t)) \in \mathbb{Z}^N$, where $Y_i(t) \in \mathbb{Z}$ denotes the position of the particle labelled i at time $t \in \mathbb{R}_+$, started from $Y(0) \in \mathbb{Z}^N$ (not necessarily in label order). Let $\sigma_t(i)$ denote the permutation on $[N]$ giving the time-t *rank* of the particle labelled i among all the particles, using lexicographic order in case of ties (rank 1 being leftmost, rank N rightmost):

$$\sigma_t(i) := \sum_{j \in [N]} \mathbb{1}\{Y_j(t) < Y_i(t)\} + \sum_{j \in [i]} \mathbb{1}\{Y_j(t) = Y_i(t)\}.$$

Then define dynamics by declaring that at time t, particle i jumps -1 at rate $A_i^{\sigma_t}(Y(t))$ and jumps $+1$ at rate $B_i^{\sigma_t}(Y(t))$, independently of other particles, given the ranks, where A^σ and B^σ, for a permutation σ with inverse σ^{-1}, are defined similarly to (1.2): for $y \in \mathbb{Z}^{\mathbb{N}}$,

$$A_i^\sigma(y) := \mathbb{1}_{\{y_{\sigma^{-1}(\sigma(i)-1)} < y_i\}} \sum_{j=\sigma(i)}^N a_j \mathbb{1}_{\{y_{\sigma^{-1}(j)} = y_i\}}, \quad B_i^\sigma(y) := \mathbb{1}_{\{y_{\sigma^{-1}(\sigma(i)+1)} > y_i\}} \sum_{j=1}^{\sigma(i)} b_j \mathbb{1}_{\{y_{\sigma^{-1}(j)} = y_i\}},$$

with conventions $\sigma^{-1}(0) = 0$, $\sigma^{-1}(N + 1) = N + 1$, $y_0 = -\infty$ and $y_{N+1} = +\infty$. Considering the *ordered* particles $X_i(t) := Y_{\sigma_t^{-1}(i)}(t)$, we recover the identical-mass elastic particle system $X(t)$

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defined in §1. In words, the elastic collision mechanism can be obtained by starting with a system of particles that perform independent random walks at rate parameters determined by their ranks, and tracking the ordered configuration of particles. This is a discrete analogue of a construction of Atlas-type diffusion models (see e.g. [9, 28, 32, 39] and §2.6 below).

2.2 Discrete time

Associated to the continuous-time Markov process X defined in §1 is the discrete-time jump chain, obtained by sampling the continuous-time process at the times at which it changes configuration. Since the jump rates of the continuous-time process are uniformly bounded above, and uniformly bounded below if we exclude the trivial case where $a_i = b_i \equiv 0$ for all $i \in [N]$, statements about stability, partial stability, exponential bounds, etc., from Theorems 1.3 and 1.10 apply equally well to the discrete-time version of the process.

The relationship between the *speeds* is a little more involved. In the particular case when all masses are equal, i.e., $m_i \equiv m \in (0, \infty)$ for all $i \in [N]$, a consequence of (1.2) is that the total activity rate of the continuous-time process is constant, regardless of the configuration:

$$\lim_{h \to 0} \frac{\mathbb{P}[X(t+h) \neq X(t) \mid X(t) = x]}{h} = \sum_{i \in [N]} (a_i + b_i), \text{ for every } x \in \mathbb{X}. \tag{2.1}$$

Consequently, ergodic properties of the discrete-time and continuous-time chains are equivalent up to a constant multiplicative factor in terms of the right-hand side of (2.1): this means that speeds are all related by the same multiplicative factor, and stationary distributions coincide exactly. In the case of general masses m_i, the time-change between discrete and continuous time is less explicit, as total activity rate depends on the ergodic behaviour internally to each stable cloud, which is not explicitly quantified since stationary distributions are not known explicitly (see §5 below).

2.3 Expanded elastic system excluding multiple occupancy

Take the elastic process described in §1 in the case when all masses are equal, i.e., $m_i \equiv m \in (0, \infty)$ for all $i \in [N]$. Define $\widetilde{X}_i(t) := X_i(t) + i - 1$ for every $i \in [N]$ and all $t \in \mathbb{R}_+$. The Markov process $\widetilde{X}(t) := (\widetilde{X}_i(t))_{i \in [N]}$ can be described as a particle system on $\mathbb{Z}$, with no more than one particle per site, in which the particle at position $\widetilde{X}_i$ attempts to jump left at rate a_i and right at rate b_i. If no particle is occupying the site of an attempted jump, the jump is executed. If a particle is occupying the target site of an attempted jump, instead *that* blocking particle immediately attempts to jump to *its* neighbouring site in the same direction; and so on. In this way 'momentum' is transferred to the outermost particles of contiguous blocks, as represented in Figure 3. For the case of non-identical masses m_i, a similar interpretation is possible.

We call $\widetilde{X}(t)$ the *expanded* elastic particle system, and, when contrast is needed, we refer to the model of §1 as the *contracted* elastic particle system. Up to the bijection between the two sets of configurations by the transformation $(x_1, \ldots, x_N) \mapsto (x_1, x_2 + 1, \ldots, x_N + N - 1)$, the two models are equivalent. Thus Theorem 1.3 and Theorem 1.10 apply, *mutatis mutandis*, to the expanded elastic model as well. The expanded elastic model was first proposed, as far as the authors are aware, in §6.2 of [36], where the stability criterion in Theorem 1.3 was conjectured.

2.4 Simple exclusion process with particle-wise heterogeneity

The exclusion process, originating with [35, 44], is one of the most intensively studied interacting particle models, but it differs crucially from the elastic model of §1 because collisions suppress

Rate labels above arrows: rate a_1 b_1 $a_2 + a_3$ $b_2 + b_3$ $a_4 + a_5 + a_6$ $b_4 + b_5 + b_6$

$X_1(t)$ $X_2(t)$ $X_3(t)$ $X_4(t)$ $X_5(t)$ $X_6(t)$
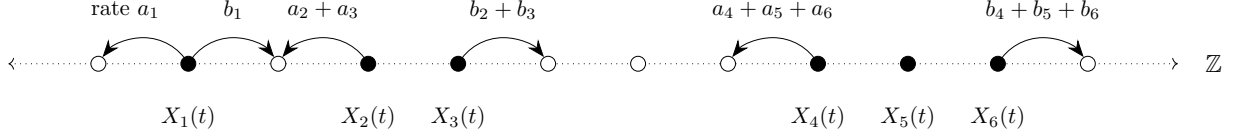
FIGURE 3: *Expanded elastic dynamics.* Take the configuration of $N = 6$ equal-mass particles represented in Figure 1, and apply the transformation $\widetilde{X}_i(t) := X_i(t) + i - 1$ for every $i \in [N]$ to obtain a configuration in which the 'stacks' of multiple occupancy sites are expanded into contiguous blocks of particles. This gives an equivalent version of the elastic particle system, in which no site can be occupied by more than one particle, and in which 'momentum' of attempted jumps is transferred to the outermost particles of contiguous blocks.

activity of the system, as we will explain. The version of the exclusion process that serves as a comparator for our model (in the case of identical masses $m_i \equiv m$) consists of N particles, with configurations in $\mathbb{X}_N$, and particle i has jump rates a_i, b_i to the left, right, when it is on its own, exactly like the model of §1. The difference is that when multiple particles occupy the same site, the minimal index particle jumps left at only its intrinsic rate, rather than the sum of all rates of the particle stack. Similarly for jumps to the right; thus the exclusion collision rule replaces the rates A_i, B_i from (1.2) by simply $A_i^{\mathrm{exc}}(x) := \mathbb{1}_{\{x_{i-1} < x_i\}} a_i$ and $B_i^{\mathrm{exc}}(x) := \mathbb{1}_{\{x_{i+1} > x_i\}} b_i$. See Figure 4 for an illustration.

Rate labels: b_1 b_3 b_6 rate a_1 a_2 a_4
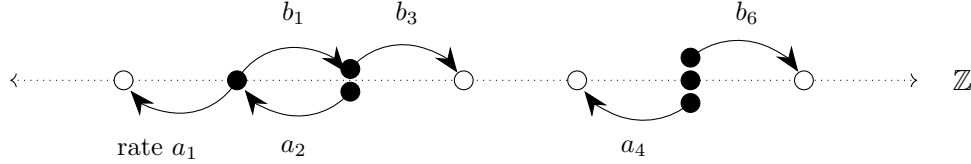
FIGURE 4: *Contracted exclusion dynamics.* Schematic showing a configuration of $N = 6$ equal-mass particles and transition rates indicated on the arrows. In the case of multiple particles occupying the same site, we imagine that they are stacked in increasing order of index, and it is the bottom and top particles that are allowed to move: the transition to the left would move the particle from the base of the stack, while the transition to the right would move the particle from the top of the stack. To translate a configuration to the more standard exclusion configuration, shift particle i to the right by $i - 1$ sites; this change of variables shows equivalence of this 'contracted' exclusion process to the classical exclusion process in which sites are occupied by at most one particle.

It should be noted that the exclusion process is usually described in terms of configurations expanded using the transformation $(x_1, \ldots, x_N) \mapsto (x_1, x_2 + 1, \ldots, x_N + N - 1)$, but, similarly to §2.3, there is an equivalence exactly as between the contracted and expanded elastic models (cf. Figures 1 and 3).

Example 2.1 (Small systems, continued)**.** To emphasize that there is no simple monotonicity relation between the exclusion and elastic interactions, we give examples (with $m_i \equiv m \in (0, \infty)$ constant) using the same rate parameters where (i) the model with exclusion interaction is stable but with elastic interaction is not, and (ii) vice versa.

For case (i), consider $N = 3$ particles with intrinsic rates $a_1 = a_2 = a_3 = b_1 = 1$, $b_2 = 2$, and $b_3 = 1/3$. Example 2.10 of [36] shows that under exclusion dynamics, this system is stable, i.e., $\{1, 2, 3\}$ is a single cloud. On the other hand Theorem 1.10 above shows that, under elastic dynamics, the stable clouds are $\{1\}$ and $\{2, 3\}$ with corresponding speeds $-v_1 = 0$ and $-v_2 = 1/6$; in the elastic case, particle 2 can push particle 3 faster to the right.

For case (ii), consider $N = 3$ particles with intrinsic rates $a_1 = b_1 = b_3 = 1$, $a_2 = 1/2$, $b_2 = 3/2$,

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and $a_3 = 3$. Now Example 2.10 of [36] shows that under exclusion dynamics, the stable clouds are $\{1\}$ and $\{2, 3\}$, while Example 1.6 (or Theorem 1.3) shows that, under elastic dynamics, the whole system is stable. $\triangle$

There is a bijection between the N-particle exclusion process and an *open Jackson network* of $N-1$ queues (see §3 of [36] and references therein). In the work of Malyshev and the present authors [36], this bijection was used to apply results of Goodman and Massey [22] on partial stability for Jackson queueing networks to obtain the cloud decomposition of the exclusion system. The partial stability results (Theorems 2.1 and 2.3 of [36]) are analogous to the present Theorems 1.3 and 1.10, but the algebra of cloud speeds in the exclusion/Jackson case is rather more complicated, since the elastic property is absent. On the other hand, the exclusion/Jackson setting turns out to be simpler from the point of view of exhibiting *reversibility* that allows invariant distributions to be computed explicitly as product-geometric distributions. We discuss invariant distributions and the generic absence of reversibility in the elastic model in §5 below. Rather than describe in detail the queueing model that corresponds to the exclusion systems (see [36, §3] for that), in §2.5 we instead describe a similar correspondence for the elastic model. We refer to [38], and references therein, for exclusion interaction among semi-infinite systems of particles.

2.5 Queueing networks with resource redeployment, random walks in orthants

Once more, take the elastic process described in §1 in the case when all masses are equal, i.e., $m_i \equiv m \in (0, \infty)$ for all $i \in [N]$. The Markov process $\eta = (\eta(t))_{t \in \mathbb{R}_+}$ of inter-particle distances defined by (1.10) has an interpretation as the process of queue-lengths in a network of $N-1$ queues in series, with a specific form of pooling of server resource.

In this interpretation, $\eta_i(t) \in \mathbb{Z}_+$ counts the number of customers in queue $i \in [N-1]$ at time $t \in \mathbb{R}_+$. Customers enter and leave the network only at the two extremal queues, 1 and $N-1$. Arrivals are according to a Poisson arrival stream with rate a_1 at queue 1, and an arrival stream of rate b_N at queue $N-1$. If queue $i \in [N-1]$ is non-empty, then the ith server serves the customer at the head of queue i at rate $b_i + a_{i+1}$. When a customer is served, it is routed to queue $i-1$ with probability $\frac{b_i}{b_i + a_{i+1}}$ or to queue $i+1$ with probability $\frac{a_{i+1}}{b_i + a_{i+1}}$; being routed to queue 0 or N means that the customer leaves the system.

The above part of the dynamics is identical to the Jackson network that corresponds to the exclusion process: the difference for the elastic model will be that when a server's queue is empty, the server will *redeploy* resource to serve the nearest occupied queues in each direction. Precisely, if queue $i \in [N-1]$ is empty, then server i lends service rate b_i to the nearest occupied queue to the right, if there is one, or to augment the arrival stream at queue $N-1$ if there are no occupied queues to the right, and lends service rate a_{i+1} to the nearest occupied queue to the left, or to the arrival stream at queue 1 if there are no occupied queues to the left. The interpretation of the constant activity rate in (2.1) in this setting is that the total service effort in the queueing network is independent of the current configuration, unlike in the Jackson network. (Here 'service' includes the server recruiting extra arrivals to the extremal queues.)

The process η can also be viewed as a continuous-time, reflected random walk on the orthant $\mathbb{Z}_+^{N-1}$, with nearest-neighbour jumps. For $N \in \{2, 3\}$, there are exhaustive criteria for classifying stability of such walks (see e.g. [17, 18][1]) in terms of first and second moments of increments. For $N \in \{4, 5\}$ the generic classification requires precise knowledge of quantities which are hard to compute, namely stationary distributions for lower-dimensional projections [17, 29]. For

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[1]These works are written for discrete-time random walks, but they immediately apply to this case also, since the rates normalize to probabilities with the same normalizing constant from (2.1).

$N \geq 6$, the generic case is intractable [19]. The family of random walks here, with the explicit stability criterion in Corollary 1.5, gives an example where there is a complete stability description for any N; the model of [36] provides another class of examples.

2.6 Atlas model of rank-dependent interacting diffusions

Continuum (diffusion) models, in which N particles perform mutually-reflecting Brownian motions with drifts and diffusion coefficients determined by their ranks, have been extensively studied in recent years, and include the *Atlas model* and its relatives: see e.g. [8, 9, 28, 32, 39, 42, 43] and references therein.

In one of the most general settings for finite systems, the system of N diffusing particles is described by a system of stochastic differential equations (SDEs) with reflections. If $x_1(t) \leq x_2(t) \leq \cdots \leq x_N(t)$ denote the ordered positions on $\mathbb{R}$ of the N particles at time $t \in \mathbb{R}_+$, and parameters $q_j^-, q_j^+ \in (0,1)$ satisfy $q_{j+1}^+ + q_j^- = 1$ for all $j \in [N-1]$, consider the dynamics satisfying

$$\mathrm{d}x_i(t) = \mu_i \mathrm{d}t + \sigma_i \mathrm{d}W_i(t) + q_i^+ \mathrm{d}L^i(t) - q_i^- \mathrm{d}L^{i+1}(t), \text{ for every } i \in [N], \tag{2.2}$$

where $W_1, \ldots, W_N$ are independent standard Brownian motions, and $L^i(t)$ is the local time at 0 of $x_i - x_{i-1}$ up to time t, with the convention $L^1 \equiv L^{N+1} \equiv 0$. The local-time terms in (2.2) maintain the order of the x_i, while each particle x_i has its own intrinsic drift (μ_i) and diffusion (σ_i) coefficients. The coefficients q_j^-, q_j^+ multiplying the local time terms control the relative impact of collisions on each of the two particles involved, and play a comparable role to the *masses* in the model of §1, as we now explain; see also the discussion in §3 of [32].

We can identify 'masses' in the system (2.2) by defining, for $k \in [N]$,

$$m_k := \prod_{i \in [k]} \left(\frac{1 - q_i^+}{q_i^+} \right).$$

Then some algebra shows that, for $k \in [N]$.

$$\frac{m_{k-1}}{m_k + m_{k-1}} = q_k^+, \text{ and } \frac{m_{k+1}}{m_k + m_{k+1}} = 1 - q_{k+1}^+ = q_k^-,$$

where we set $m_0 = m_{N+1} \equiv 0$. In other words, we can re-write (2.2) as

$$\mathrm{d}x_i(t) = \mu_i \mathrm{d}t + \sigma_i \mathrm{d}W_i(t) + \frac{m_{i-1}}{m_i + m_{i-1}} \mathrm{d}L^i(t) - \frac{m_{i+1}}{m_i + m_{i+1}} \mathrm{d}L^{i+1}(t), \text{ for every } i \in [N]. \tag{2.3}$$

The continuum system (2.2) is *elastic* in the sense that the total flow of mass is constant. Formally, define $M := \sum_{k \in [N]} m_k$ and the centre of mass $g(t) := M^{-1} \sum_{i \in [N]} m_i x_i(t)$. By summing (2.3) and using the conditions on the q_j^-, q_j^+, we deduce that g has dynamics

$$\mathrm{d}g(t) = \overline{\mu} \mathrm{d}t + \overline{\sigma} \mathrm{d}\overline{W}(t),$$

where $\overline{W}$ is standard Brownian motion, and the drift and diffusion coefficients satisfy

$$\overline{\mu} = \frac{1}{M} \sum_{i \in [N]} m_i \mu_i, \text{ and } \overline{\sigma}^2 = \frac{N}{M^2} \sum_{i \in [N]} m_i^2 \sigma_i^2.$$

(Compare Theorem 3.1 below for the elastic particle system.)

The elastic continuum model just described serves as a scaling limit of a class of particle system models, under certain near-critical parameter scaling, that includes both the elastic model of §1 and the 'inelastic' simple exclusion process of §2.4: see [32, §3] and [36, §6.3] for more detail. The equivalent scaling limit when the discrete-space model is viewed as a queueing network (as in §2.5) is a *heavy traffic* limit in which every queue is asymptotically critically loaded, and the scaling limit is a reflected diffusion on an orthant; see e.g. [20, 41].

As far as the authors are aware, long-time stability of the continuum system defined by (2.2) above has been studied only as far as classifying whether or not the whole system is stable (e.g. [28, 32, 39, 42]), rather than identifying a full cloud decomposition as we do in Theorem 1.10; in the continuum setting there are additional complexities as one must consider the analogue of the 'corner trapping' phenomenon for reflected diffusion in an orthant [47].

2.7 Further literature on elastic collisions

Physical systems of particles exhibit (approximately) elastic collisions, in which kinetic energy and momentum are both (approximately) conserved, if there are no external forces and internal energy states of particles can be (approximately) ignored. A number of probabilistic models for such physical systems have been studied in the literature, including Kac's uniformly mixing model for kinetic theory of gases [31] (which neglects space), and the work of Jepsen, Harris and Spitzer [25, 30, 45] on elastic models of colliding particles on the line with deterministic or Brownian free-space trajectories. To give a flavour of the sorts of phenomena observed for the spatial models, suppose that at time 0 a system of particles with equal masses is given according to a stationary Poisson process on $\mathbb{R}$, with one much more massive tagged particle at the origin; independently of the positions, the atoms are given i.i.d. velocities, and undergo elastic collisions when they meet. If one observes the tagged particle in an infinite system started from a given density profile, possibly in the presence of a hard barrier, there are results that establish convergence to a Brownian, Ornstein–Uhlenbeck, or other Gaussian process, and also results when the tagged particle is much less massive than the rest: see e.g. [10, 11, 14, 16, 21, 34]. An informative overview of this and other literature is given in the introduction to [39].

3 Proofs: Stability

3.1 Global speed and local stability

In this section we state two key ingredients to our proofs. Theorem 3.1 states that the drift of the centre of mass of the system is independent of the current configuration, which is a distinguishing feature of the elastic collision mechanism. Proposition 3.2 gives a criterion for identifying (not necessarily maximal) stable subsystems of particles. These two results will combine to give Theorem 1.3 (the short deduction appears after their statement), and are also crucial in the proof of Theorem 1.10. The proofs of both Theorem 3.1 and Proposition 3.2 are given in §3.2 below.

Recall the definition of the centre of mass process G from (1.11). The next result shows that G travels with speed $-U_N/M_N$.

Theorem 3.1 (Centre of mass)**.** *For the process G as defined at* (1.11)*, it holds that*

$$\mathbb{E}[G(t + h) - G(t) \mid X(t) = x] = -hU_N/M_N, \text{ for all } t, h \in \mathbb{R}_+ \text{ and all } x \in \mathbb{X}_N. \qquad (3.1)$$

Moreover, $\lim_{t \to \infty} t^{-1} G(t) = -U_N/M_N$*, a.s.*

The next important ingredient is a *local stability* result, presented in Proposition 3.2 below. This result will yield the 'stability' part of Theorem 1.3 directly. Moreover, the result says that if a contiguous subset of particles in the system would, considered as an isolated system, satisfy the stability criterion from Theorem 1.3, then that subset is stable also as a subsystem of the full system. The intuition is that the presence of additional particles to the left or right can only stabilize the subsystem further, by preventing the extreme particles from diverging, and/or by contributing stabilizing inwards momentum.

For $\ell, r \in [N]$ with $\ell \leq r$, we write $[\ell; r] := \{x \in [N] : \ell \leq x \leq r\}$, a *discrete interval*. Note that $[1; r] = [r]$ in our previous notation. Recall the definition of Δ_C from (1.6).

Proposition 3.2 (Local stability). *Suppose that $\ell, r \in [N]$, $\ell < r$, and it holds that*

$$\frac{U_{\ell,k}}{M_{\ell,k}} < \frac{U_{\ell,r}}{M_{\ell,r}} \ \text{for all } k \in [\ell; r-1]. \tag{3.2}$$

Then there exist constants $C \in \mathbb{R}_+$ and $\delta > 0$ (depending on the a_i, b_i, and m_i) such that

$$\sup_{t \in \mathbb{R}_+} \mathbb{P}\left[\Delta_{[\ell+1;r]}(t) \geq s\right] \leq C\left[1 + e^{C\Delta_{[\ell+1;r]}(0)}\right] e^{-\delta s}, \ \text{for all } s \in \mathbb{R}_+, \tag{3.3}$$

and, moreover,

$$\limsup_{t \to \infty} \frac{\Delta_{[\ell+1;r]}(t)}{\log t} < \infty, \ \text{a.s.} \tag{3.4}$$

Remark 3.3. Note that condition (3.2) involving the functions U_k, M_k over values $k \in [\ell; r]$ translates to stability of particles with labels $[\ell + 1; k]$ at (3.3). This is similar to the stability criterion in Theorem 1.10, reflecting that there is no particle with label 0.

Proof of Theorem 1.3. The statement of Proposition 3.2 with $\ell = 0$ and $r = N$ shows that the condition (1.7) is sufficient for stability, in the sense expressed in part (b) of the theorem. Moreover, since $\max_{i \in [N]} |X_i(t) - G(t)| \leq \Delta_{[N]}(t)$ for all $t \in \mathbb{R}_+$, it follows also that $\limsup_{t \to \infty} t^{-1}|X_i(t) - G(t)| = 0$, a.s., for every $i \in [N]$. Together with Theorem 3.1, this yields part (a) of the theorem.

The final statement of the theorem, that (1.7) is necessary for part (b), is most easily deduced from Theorem 1.10(c) (the proof of that result comes only in §4, but nowhere uses this final part of Theorem 1.3). Indeed, if (1.7) fails then there is at least one $j \in [\nu - 1]$ for which $v_j \geq v_{j+1}$. If $v_j > v_{j+1}$, then Theorem 1.10(c) shows that $\liminf_{t \to \infty} t^{-1}\Delta_{[N]}(t) \geq v_j - v_{j+1} > 0$, contradicting part (b). If $v_j = v_{j+1}$, then Theorem 1.10(c) shows that, for some $\varepsilon > 0$, $\limsup_{t \to \infty} t^{-\varepsilon}\Delta_{[N]}(t) \geq 1$, a.s., say, which also contradicts part (b). $\square$

Before moving on to the proofs of Theorem 3.1 and Proposition 3.2, we introduce some notation that will allow us to work with increments of functionals of particle system configurations. Define

$$\tau := \inf\{t \in \mathbb{R}_+ : X(t) \neq X(0)\}, \tag{3.5}$$

the first jump time of the Markov chain. Write $\mathbb{P}_x, \mathbb{E}_x$ for probabilities and expectations in the case where the initial configuration is $X(0) = x \in \mathbb{X}_N$. The key step in the proof of Proposition 3.2 is the following Lyapunov function estimate. Recall the definitions of the inter-particle distance function $D : \mathbb{X}_N \to \mathbb{Z}_+^{N-1}$ from (1.9) and $\eta(t) = D(X(t))$ defined at (1.10). Observe that τ defined by (3.5) is also the first jump time of the Markov chain η.

Lemma 3.4. *Suppose that $\ell, r \in [N]$, $\ell < r$, and that (3.2) holds. Then there exist a function $F : \mathbb{Z}_+^{N-1} \to \mathbb{R}_+$ and constants $B, R \in \mathbb{R}_+$ and $\varepsilon > 0$ such that*

$$\mathbb{E}_x F(\eta(t)) < \infty, \text{ for all } x \in \mathbb{X}_N \text{ and all } t \in \mathbb{R}_+; \tag{3.6}$$

$$\mathbb{P}_x[|F(\eta(\tau)) - F(\eta(0))| \le B] = 1, \text{ for all } x \in \mathbb{X}_N; \tag{3.7}$$

$$\mathbb{E}_x[F(\eta(\tau)) - F(\eta(0))] \le -\varepsilon, \text{ for all } x \in \mathbb{X}_N \text{ with } \|D(x)\| > R. \tag{3.8}$$

The function F will be a (weighted) norm describing the displacement of the particles with labels in $[\ell+1; r]$ relative to their centre of mass. In §3.2 we give the formal definition of F, which requires introducing notation for a suitable weighted norm and inner product, and then prove Lemma 3.4 and deduce Proposition 3.2. The Markov-chain computations required to establish Lemma 3.4 also allow us to prove Theorem 3.1, so that proof is also in §3.2.

3.2 Lyapunov function computations

For $x, y \in \mathbb{R}^N$ and $\ell, r \in [N]$ with $\ell \le r$, consider the weighted positive semi-definite form

$$\langle x, y \rangle_{\ell, r} := \sum_{i \in [\ell; r]} m_i x_i y_i$$

and the associated seminorm $\|x\|_{\ell, r} := \sqrt{\langle x, x \rangle_{\ell, r}}$. In the special case where $\ell = 1$ and $r = N$, we write $\langle x, y \rangle_N := \langle x, y \rangle_{1,N}$ and $\|x\|_N := \|x\|_{1,N}$.

Let $\mathbf{1}$ be the vector in $\mathbb{R}^N$ with all components equal to 1. Observe that $\langle \mathbf{1}, \mathbf{1} \rangle_{\ell+1, r} = M_r - M_\ell = M_{\ell, r}$ in the notation at (1.5). In particular, $\langle \mathbf{1}, \mathbf{1} \rangle_N = \langle \mathbf{1}, \mathbf{1} \rangle_{1,N} = M_N$, and the centre of mass of a configuration $x \in \mathbb{X}_N$ is

$$g(x) := M_N^{-1} \langle x, \mathbf{1} \rangle_N = \frac{1}{M_N} \sum_{i \in [N]} m_i x_i. \tag{3.9}$$

For a configuration $x \in \mathbb{X}_N$, let R_x be the total jump rate at x, i.e.,

$$R_x := \lim_{h \to 0} h^{-1} \mathbb{P}_x(X(h) \ne x) = \sum_{i \in [N]} \big(A_i(x) + B_i(x) \big), \tag{3.10}$$

where A_i, B_i are defined at (1.2). It is not hard to check (see formula (3.22) below) that

$$\inf_{x \in \mathbb{X}_N} R_x > 0 \text{ provided } \sum_{i \in [N]} (a_i + b_i) > 0. \tag{3.11}$$

We will tacitly assume that $\sum_{i \in [N]} (a_i + b_i) > 0$ for the rest of this section; the case $\sum_{i \in [N]} (a_i + b_i) = 0$ being trivial (but formally covered by the results of §1). When a jump occurs from x, the Markov chain transitions to some $x \pm e_k$, where $e_k \in \mathbb{X}_N$ is the unit vector with 1 in coordinate k. The associated transition probabilities are, for every $k \in [N]$,

$$\mathbb{P}_x[X(\tau) = x - e_k] = \frac{A_k(x)}{R_x}, \text{ and } \mathbb{P}_x[X(\tau) = x + e_k] = \frac{B_k(x)}{R_x}, \tag{3.12}$$

where A_k, B_k are defined at (1.2) and R_x is given by (3.10). In what follows we write

$$\mathbf{e} := X(\tau) - X(0) \tag{3.13}$$

for the first increment, so that $\mathbb{P}_x(\mathbf{e} = e)$ describe the transition probabilities for the (discrete-time) jump chain as given by (3.12). In particular, for $H : \mathbb{X}_N \to \mathbb{R}_+$,

$$\mathbb{E}_x[H(X(\tau))] = \mathbb{E}_x \, H(x + \mathbf{e}), \text{ for all } x \in \mathbb{X}_N. \tag{3.14}$$

Given $x \in \mathbb{X}_N$ and $k \in [N]$, we define $k_-, k_+ \in [N]$ by

$$k_- := k_-(x) := \min\{i \in [N] : x_i = x_k\}, \ k_+ := k_+(x) := \max\{i \in [N] : x_i = x_k\}, \tag{3.15}$$

the minimal and maximal labels of particles sharing the same site as k in configuration x. We also define an 'off-lattice' extension of $\mathbb{X}_N$ defined at (1.1) by

$$\mathbb{X}'_N := \{(x_i)_{i \in [N]} \in \mathbb{R}^N : x_1 \leq x_2 \cdots \leq x_N\}. \tag{3.16}$$

Write $a := (a_i)_{i \in [N]}$ and $b := (b_i)_{i \in [N]}$ for the jump rates, and recall from (1.3) that $u = b - a$. The following lemma enables us to work with expected functional increments; for example, the hypotheses on x, y in the statement are satisfied whenever $y_k = f(x_k)$ for all $k \in [N]$ and some non-decreasing $f : \mathbb{Z} \to \mathbb{R}$.

Lemma 3.5. *Let $x \in \mathbb{X}_N$ and assume that $y \in \mathbb{X}'_N$ is such that $y_k = y_{k+1}$ for all $k \in [N]$ such that $x_k = x_{k+1}$. Then for $\mathbf{e}$ as defined at (3.13), it holds that*

$$\mathbb{E}_x\langle y, \mathbf{e}\rangle_N = R_x^{-1}\langle y, u\rangle_N. \tag{3.17}$$

More generally, for $\ell, r \in [N]$, with $\ell \leq r$ and $\ell_\pm, r_\pm$ as defined in (3.15),

$$R_x \mathbb{E}_x\langle y, \mathbf{e}\rangle_{\ell,r} = \langle y, u\rangle_{\ell,r} + \mathbb{1}_{\{\ell > \ell_-\}}\langle y, b\rangle_{\ell_-,\ell-1} + \mathbb{1}_{\{\ell > \ell_-\}}\langle y, a\rangle_{\ell_-,\ell_+} - \mathbb{1}_{\{r < r_+\}}\langle y, a\rangle_{r+1,r_+}$$
$$- \mathbb{1}_{\{r < r_+\}}\langle y, b\rangle_{r_-,r_+}. \tag{3.18}$$

In particular, if $\min_{k \in [N]} y_k \geq 0$, we have the inequalities

$$\mathbb{E}_x\langle y, \mathbf{e}\rangle_{1,r} \leq R_x^{-1}\langle y, u\rangle_{1,r}, \tag{3.19}$$

$$\mathbb{E}_x\langle y, \mathbf{e}\rangle_{\ell,N} \geq R_x^{-1}\langle y, u\rangle_{\ell,N}. \tag{3.20}$$

To prepare for the proof, we introduce some convenient terminology for describing configurations. Take $s \in [N]$ and $h_1, \ldots, h_s \in \mathbb{N}$ such that $\sum_{j \in [s]} h_j = N$. For $k \in [s]$, write $H_k := \sum_{j=1}^{k} h_j$, so that $H_0 = 0$ and $H_s = N$. We say that $x = (x_1, \ldots, x_N) \in \mathbb{X}_N$ is a configuration with s *stacks* of *heights* $h := (h_1, \ldots, h_s)$, and write $x \in \mathbb{X}_N^{s,h}$, if

$$x_1 < x_{H_1+1} < \cdots < x_{H_{s-1}+1}, \text{ and, for all } j \in [s] \text{ and } i \in [h_j], \ x_{H_{j-1}+i} = x_{H_{j-1}+1}. \tag{3.21}$$

Figure 1 shows a configuration $x \in \mathbb{X}_6^{3,h}$ of 6 particles in $s = 3$ stacks of heights $h = (1, 2, 3)$.

Proof of Lemma 3.5. Suppose that $x \in \mathbb{X}_N^{s,h}$ for $s \in [N]$ and $h = (h_1, \ldots, h_s)$, as at (3.21). First observe, by (3.10) and (1.2), we may write

$$R_x = \sum_{j=1}^{s} \sum_{i=1}^{h_j} \left(a_{H_{j-1}+i} \frac{m_{H_{j-1}+i}}{m_{H_{j-1}+1}} + b_{H_{j-1}+i} \frac{m_{H_{j-1}+i}}{m_{H_j}} \right), \tag{3.22}$$

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and that, by (3.12), the increment $\mathbf{e}$ is equal to $-e_{H_{j-1}+1}$ with probability $R_x^{-1} \sum_{i=1}^{h_j} a_{H_{j-1}+i} \frac{m_{H_{j-1}+i}}{m_{H_{j-1}+1}}$ and to e_{H_j} with probability $R_x^{-1} \sum_{i=1}^{h_j} b_{H_{j-1}+i} \frac{m_{H_{j-1}+i}}{m_{H_j}}$. Therefore,

$$R_x \mathbb{E}_x \langle y, \mathbf{e} \rangle_{\ell, r} = \langle y, u \rangle_{\ell+1, r_- - 1} + \langle y, b - a \mathbb{1}_{\{\ell = \ell_-\}} \rangle_{\ell_-, \ell_+} + \langle y, b \mathbb{1}_{\{r = r_+\}} - a \rangle_{r_-, r_+}.$$

This yields (3.18). Taking $\ell = 1$ (which certainly has $\ell = \ell_-$) in (3.18) we get

$$R_x \mathbb{E}_x \langle y, \mathbf{e} \rangle_{\ell, r} = \langle y, u \rangle_{\ell, r} - \mathbb{1}_{\{r < r_+\}} \langle y, b \rangle_{r_-, r_+} - \mathbb{1}_{\{r < r_+\}} \langle y, a \rangle_{r+1, r_+} \leq \langle y, u \rangle_{\ell_-, r_+},$$

provided all $y_k \geq 0$, as claimed in (3.19). Similarly, taking $r = r_+ = N$ in (3.18) yields (3.20). Taking both $\ell = 1$ and $r = N$ in (3.18), we verify (3.17). $\qquad\square$

Define $\tau_0 := 0$ and $\tau_1 := \tau$ as at (3.5), and then, iteratively, define

$$\tau_{n+1} := \inf\{t \geq \tau_n : X(t) \neq X(\tau_n)\}, \text{ for every } n \in \mathbb{Z}_+, \tag{3.23}$$

so that $0 = \tau_0 < \tau_1 < \tau_2 < \cdots$ are the (a.s. finite) jump times of X. Let $\mathcal{F}_t := \sigma(X(s) : 0 \leq s \leq t)$, the σ-algebra generated by X up to time $t \in \mathbb{R}_+$. By (3.10), we have that, given $\mathcal{F}_{\tau_n}$, the holding time $\tau_{n+1} - \tau_n$ is exponentially distributed with parameter $R_{X(\tau_n)}$, which by (3.10)–(3.11) satisfies $\delta < R_x < \delta^{-1}$ for every $x \in \mathbb{X}_N$ and some constant $\delta \in (0, 1)$ depending only on the a_i, b_i, and m_i. Observe that since $\eta = D(X)$ jumps if and only if X jumps, the τ_n given by (3.23) are also the jump times of η, a fact that we will use in the proof of Proposition 3.2 below. First we can complete the proof of Theorem 3.1.

Proof of Theorem 3.1. Comparison of notation (1.11) for G and (3.9) for g reveals that $G(t) = g(X(t))$. Then by (3.14) and an application of Lemma 3.5 with $y = \mathbf{1}$ shows, using the formula (3.9),

$$\begin{aligned}
\lim_{h \to 0} h^{-1} \mathbb{E}[G(t+h) - G(t) \mid X(t) = x] &= R_x \, \mathbb{E}_x[g(X(\tau)) - g(x)] \\
&= R_x \big(\mathbb{E}_x \, g(x + \mathbf{e}) - g(x) \big) \\
&= R_x \, \mathbb{E}_x \langle \mathbf{e}, \mathbf{1} \rangle_N \\
&= -\frac{U_N}{M_N}, \text{ for all } x \in \mathbb{X}_N. \tag{3.24}
\end{aligned}$$

Since the final expression in (3.24) is independent of x, we verify (3.1). Moreover, considering the associated jump chain, for every $k \in \mathbb{Z}_+$,

$$\mathbb{E}[g(X(\tau_{k+1})) - g(X(\tau_k)) \mid \mathcal{F}_{\tau_k}] = -\frac{U_N}{M_N R_{X(\tau_k)}}, \text{ a.s.} \tag{3.25}$$

As argued below (3.23), the holding times $\tau_{k+1} - \tau_k$ are exponentially distributed with uniformly bounded rates. A straightforward Azuma–Hoeffding bound and application of the Borel–Cantelli lemma then shows that

$$\lim_{k \to \infty} \frac{1}{k} [\tau_k - A_k] = 0, \text{ a.s.}, \tag{3.26}$$

where $A_0 := 0$, and $A_k := \sum_{i=0}^{k-1} 1/R_{X(\tau_i)}$ for $k \in \mathbb{N}$. Since, a.s., $\delta < k^{-1} A_k < \delta^{-1}$ for all $k \in \mathbb{N}$, it follows from (3.26) that $\tau_k/A_k \to 1$, a.s., as $k \to \infty$. Furthermore, there exists $B \in \mathbb{R}_+$ such that $|g(X(\tau_{k+1})) - g(X(\tau_k))| \leq B$ for all $k \in \mathbb{Z}_+$, a.s. The standard Doob construction, with compensator given via (3.25), then shows that $M_k := g(X(\tau_k)) + (U_N/M_N)A_k$ defines a martingale

with uniformly bounded increments, so another application of the Azuma–Hoeffding inequality yields

$$\lim_{k \to \infty} \frac{1}{k} \left[g(X(\tau_k)) + (U_N/M_N)A_k \right] = 0, \text{ a.s.} \tag{3.27}$$

Combining (3.26) and (3.27), we conclude that $\lim_{k \to \infty} g(X(\tau_k))/\tau_k = -U_N/M_N$, a.s. A standard interpolation argument now shows that $\lim_{t \to \infty} t^{-1}G(t) = -U_N/M_N$, a.s. $\qquad\square$

Next we turn to the proof of Lemma 3.4.

Proof of Lemma 3.4. For $x \in \mathbb{X}_N$, write $x_{[\ell;r]} = (x_{[\ell;r]}^{(1)}, \ldots, x_{[\ell;r]}^{(N)}) \in \mathbb{X}_N$, where

$$x_{[\ell;r]}^{(k)} = x_k \mathbb{1}_{\{k \in [\ell;r]\}},$$

i.e., projection onto the subspace spanned by coordinates $[\ell;k]$. Define $\psi_{\ell,r} : \mathbb{X}_N \to \mathbb{X}'_N$ by

$$\psi_{\ell,r}(x) := x_{[\ell+1;r]} - M_{\ell,r}^{-1} \langle x, \mathbf{1} \rangle_{\ell+1,r} \mathbf{1}_{[\ell+1;r]}, \text{ for } x \in \mathbb{X}_N, \tag{3.28}$$

i.e., projection of x onto coordinates $[\ell+1;r]$, translated by its centre of mass. Note that $\psi_{\ell,r}$ is invariant under translation of x, since if $y_i = x_i + \alpha$ for all $i \in [N]$ and some $\alpha \in \mathbb{R}$, $\langle y, \mathbf{1} \rangle_{\ell,r} = \langle x, \mathbf{1} \rangle_{\ell,r} + \alpha M_{\ell,r}$.

Since $\psi_{\ell,r}(x)$ depends only on the $[\ell+1;r]$ coordinates of x, we could view $\psi_{\ell,r}$ as acting from $\mathbb{X}_{r-\ell}$ to $\mathbb{X}'_{r-\ell}$, but we prefer to keep the full vectors (including all zeros) since coordinate labels correspond to specific masses. The Lyapunov function F that appears in (3.8) will be derived from

$$\Psi(x) := \|\psi_{\ell,r}(x)\|_{\ell+1,r}, \text{ for all } x \in \mathbb{X}_N, \tag{3.29}$$

where $\psi_{\ell,r}$ is given by (3.28) (we suppress dependence on ℓ, r in the notation Ψ). Since Ψ is invariant under translation of x, $\Psi(x)$ can be written as a function of $D(x)$ as given at (1.9); indeed, define $D^{-1} : \mathbb{Z}_+^{N-1} \to \mathbb{X}_N$ by $D^{-1}(z_1, \ldots, z_{N-1}) := (0, z_1, z_1 + z_2, \ldots, z_1 + \cdots + Z_{N-1})$, and then

$$F(z) := \Psi(D^{-1}(z)) \text{ for all } z \in \mathbb{Z}_+^{N-1} \tag{3.30}$$

will be the F that appears in (3.8). We also note from (3.29) and (3.28) that

$$\Psi(x)^2 = \|\psi_{\ell,r}(x)\|_{\ell+1,r}^2 = M_{\ell,r} \sum_{i \in [\ell+1;r]} \frac{m_i}{M_{\ell,r}} \left[x_i - \sum_{j \in [\ell+1;r]} \frac{m_j}{M_{\ell,r}} x_j \right]^2, \tag{3.31}$$

which exhibits Ψ^2 as the variance of the mass distribution if particles $\ell+1, \ldots, r$ are placed at locations $x_{\ell+1}, \ldots, x_r$. It follows from (3.31) that

$$\Psi(x)^2 \leq M_{\ell,r} \left| \max_{i \in [\ell+1;r]} x_i - \min_{i \in [\ell+1;r]} x_i \right|^2 = M_{\ell,r}(x_r - x_{\ell+1})^2. \tag{3.32}$$

In the other direction, since $a^2 + b^2 \geq (a+b)^2/2$, we get from (3.31) that

$$\Psi(x)^2 \geq m_{\ell+1}(\bar{x} - x_{\ell+1})^2 + m_r(x_r - \bar{x})^2 \geq \frac{\min(m_{\ell+1}, m_r)}{2}(x_r - x_{\ell+1})^2, \tag{3.33}$$

where we wrote $\bar{x} := M_{\ell,r}^{-1} \sum_{j \in [\ell+1;r]} m_j x_j$. One can interpret bounds (3.32)–(3.33) as quantifying the fact that $x_r - x_{\ell+1} = \max_{i,j \in [\ell+1;r]} |x_i - x_j|$ is equivalent to the norm $\|\psi_{\ell,r}(x)\|_{\ell+1,r}$.

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We need to study the increments of Ψ applied to Markov process X. Since $\langle \mathbf{1}, \mathbf{1} \rangle_{\ell+1,r} = M_{\ell,r}$, $\psi_{\ell,r}$ given by (3.28) satisfies the orthogonality relation

$$\langle \psi_{\ell,r}(x), \mathbf{1}_{[\ell+1;r]} \rangle_{\ell+1,r} = 0, \text{ for all } x \in \mathbb{X}_N. \tag{3.34}$$

Let $e = \pm e_j$ be (plus or minus) one of the coordinate unit vectors. We have $\psi_{\ell,r}(x + e) = \psi_{\ell,r}(x)$ whenever $j \notin [\ell+1;r]$. So fix $j \in [\ell+1;r]$. Then we compute

$$\begin{aligned}
\|\psi_{\ell,r}(x + e)\|^2_{\ell+1,r} &= \left\| x + e - M^{-1}_{\ell,r} \langle x + e, \mathbf{1} \rangle_{\ell+1,r} \mathbf{1}_{[\ell+1;r]} \right\|^2_{\ell+1,r} \\
&= \left\| \psi_{\ell,r}(x) + e - M^{-1}_{\ell,r} \langle e, \mathbf{1} \rangle_{\ell+1,r} \mathbf{1}_{[\ell+1;r]} \right\|^2_{\ell+1,r} \\
&= \|\psi_{\ell,r}(x)\|^2_{\ell+1,r} + 2 \langle \psi_{\ell,r}(x), e \rangle_{\ell+1,r} + m_j - M^{-1}_{\ell,r} m_j^2,
\end{aligned}$$

using the orthogonality (3.34) of $\psi_{\ell,r}(x)$ and $\mathbf{1}_{[\ell+1;r]}$. With $\mathbf{e}$ being the random increment vector at $x \in \mathbb{X}_N$, it follows from the above that

$$\Psi(x + \mathbf{e})^2 - \Psi(x)^2 = 2 \langle \psi_{\ell,r}(x), \mathbf{e} \rangle_{\ell+1,r} + T_{\ell,r}(x, \mathbf{e}),$$

where $|T_{\ell,r}(x, e)|$ is uniformly bounded for all $x \in \mathbb{X}_N$ and all unit vectors e. Moreover, $|\Psi(x + e) - \Psi(x)|$ is also uniformly bounded for all $x \in \mathbb{X}_N$ and all unit vectors e. It follows that there is a constant $B < \infty$ such that, a.s.,

$$\Psi(x + \mathbf{e}) - \Psi(x) = \frac{\Psi(x + \mathbf{e})^2 - \Psi(x)^2}{\Psi(x + \mathbf{e}) + \Psi(x)} \leq \frac{\langle \psi_{\ell,r}(x), \mathbf{e} \rangle_{\ell+1,r}}{\Psi(x)} + \frac{B}{\Psi(x)},$$

for all $x \in \mathbb{X}_N$. In particular, for all $x \in \mathbb{X}_N$,

$$\mathbb{E}_x \left[\Psi(x + \mathbf{e}) - \Psi(x) \right] \leq \frac{1}{\Psi(x)} \left[\mathbb{E}_x \langle \psi_{\ell,r}(x), \mathbf{e} \rangle_{\ell+1,r} + B \right]. \tag{3.35}$$

The remaining part of the proof is to obtain a (sufficiently negative) upper bound for the right-hand expectation in (3.35). To this end, define

$$V_k := U_{\ell,k} - M_{\ell,k} \frac{U_{\ell,r}}{M_{\ell,r}}, \text{ for } k \in [\ell;r]. \tag{3.36}$$

Then $V_\ell = V_r = 0$, and the condition (3.2) implies that $V_k < 0$ for all $k \in [\ell+1;r-1]$ (and $U_{\ell,r} > 0$). Hence, under the hypothesis (3.2),

$$\varepsilon := - \max_{k \in [\ell+1;r-1]} V_k \tag{3.37}$$

satisfies $\varepsilon > 0$. Moreover, since $U_k - U_{k-1} = m_k u_k$, a straightforward calculation implies that

$$V_{k-1} - V_k = m_k \left[u_k + \frac{U_{\ell,r}}{M_{\ell,r}} \right], \text{ for all } k \in [\ell+1;r]. \tag{3.38}$$

We will next apply Lemma 3.5 with $y = \psi_{\ell,r}(x)$. Note first that for $k \in [\ell_-, \ell_+]$, we have $x_k = x_\ell = \min_{i \in [\ell+1;r]} x_i$, from which it follows that $y_k \leq 0$ for all $k \in [\ell_-;\ell_+]$. Similarly, $y_k \geq 0$ for all $k \in [r_-, r_+]$. Hence we deduce from (3.18) that $R_x \mathbb{E}_x \langle \psi_{\ell,r}(x), \mathbf{e} \rangle_{\ell+1,r} \leq \langle y, u \rangle_{\ell+1,r}$. Since $\sum_{k \in [\ell+1;r]} y_k m_k = 0$, by (3.34), it follows that

$$R_x \mathbb{E}_x \langle \psi_{\ell,r}(x), \mathbf{e} \rangle_{\ell+1,r} \leq \sum_{k \in [\ell+1;r]} y_k m_k \left[u_k + \frac{U_{\ell,r}}{M_{\ell,r}} \right] = \sum_{k \in [\ell+1;r]} y_k \left[V_{k-1} - V_k \right],$$

by (3.38). Then, by partial summation and the fact that $V_\ell = V_r = 0$,

$$R_x \mathbb{E}_x \langle \psi_{\ell,r}(x), \mathbf{e} \rangle_{\ell+1,r} \leq \sum_{k \in [\ell+1;r-1]} V_k(y_{k+1} - y_k) \leq -\varepsilon \sum_{k \in [\ell+1;r-1]} (y_{k+1} - y_k),$$

using (3.37) and the fact that $y_{k+1} - y_k = x_{k+1} - x_k \geq 0$ for $k \in [\ell+1; r-1]$. Hence

$$R_x \mathbb{E}_x \langle \psi_{\ell,r}(x), \mathbf{e} \rangle_{\ell+1,r} \leq -\varepsilon(y_r - y_{\ell+1}) = -\varepsilon(x_r - x_{\ell+1}).$$

Using the inequality (3.32) we conclude that there exists $\varepsilon > 0$ such that

$$\mathbb{E}_x \langle \psi_{\ell,r}(x), \mathbf{e} \rangle_{[\ell+1;r]} \leq -\varepsilon \Psi(x), \text{ for all } x \in \mathbb{X}_N. \tag{3.39}$$

Combining (3.35) and (3.39), we see that, for some $R \in \mathbb{R}_+$,

$$\mathbb{E}_x(\Psi(X(\tau)) - \Psi(X(0))) \leq -\varepsilon, \text{ for all } x \in \mathbb{X}_N \text{ with } \|D(x)\| > R.$$

Since Ψ is translation invariant, from here and (3.30) we deduce (3.8). $\qquad\square$

To prove Proposition 3.2, we will show that the Lyapunov function F from Lemma 3.4 satisfies a version of a geometric drift condition (see e.g. Definition 14.1.5 of [15]). The theory of *geometric ergodicity* for Markov chains (c.f. Theorem 15.1.5 of [15]) does not apply directly for us, as through F we are working with a *projection* of a Markov process, considering the relative stability of some (typically, not all) coordinates. Nevertheless, we need less than the full power of geometric ergodicity; the Lyapunov function technique relies on martingale rather than Markov structure, and we can obtain what we need from robust results for adapted processes from Hajek [24]. Since those results are formulated in discrete time, it is convenient to work with the jump chain.

Proof of Proposition 3.2. Consider the discrete-time jump chain associated with the Markov chain η on $\mathbb{Z}_+^{N-1}$, i.e., the process $\eta(\tau_k)$ where τ_k is defined at (3.23). As explained below (3.23), the holding times $\tau_{k+1} - \tau_k$ are exponentially distributed with uniformly bounded rates, and so discrete-time results will be easily transferred to the continuous-time process.

Define $V : \mathbb{Z}_+^{N-1} \to (0, \infty)$ by $V(z) := \exp(\delta F(z))$, where F is defined by (3.30). Then

$$V(\eta(\tau)) = V(\eta(0)) \cdot \exp(\delta(F(\eta(\tau)) - F(\eta(0)))).$$

Given $B \in \mathbb{R}_+$ and $\varepsilon > 0$, we can find $\delta \in (0, \infty)$ small enough such that

$$e^{\delta y} \leq 1 + \delta\left(y + \frac{\varepsilon}{2}\right), \text{ for all } y \in [-B, B].$$

Hence from (3.7) and (3.8), we have, for all $x \in \mathbb{X}_N$,

$$\mathbb{E}_x V(\eta(\tau)) \leq V(D(x))\left[1 + \delta\left(\mathbb{E}_x[F(\eta(\tau)) - F(D(x))] + \frac{\varepsilon}{2}\right)\right]$$

$$\leq V(D(x))\left(1 - \frac{\delta\varepsilon}{2}\right), \text{ for all } \|D(x)\| > B.$$

Now Theorem 2.3 of [24] shows that there exist constants $C \in \mathbb{R}_+$ and $\delta > 0$ such that

$$\mathbb{P}_x[F(\eta(\tau_k)) \geq r] \leq \left[C + e^{\delta(F(D(x))-k)}\right] e^{-\delta r}, \text{ for all } r \in \mathbb{R}_+ \text{ and all } k \in \mathbb{Z}_+. \tag{3.40}$$

Consequently, for $r \in \mathbb{R}_+$,

$$\sup_{t \in \mathbb{R}_+} \mathbb{P}_x[F(\eta(t)) \geq r] \leq \sup_{t \in \mathbb{R}_+} \mathbb{P}_x \left[\bigcup_{k \in \mathbb{Z}_+} \{t \in [\tau_k, \tau_{k+1})\} \cap \{F(\eta(\tau_k)) \geq r\} \right]$$

$$\leq \sum_{k \in \mathbb{Z}_+} \mathbb{P}_x[F(\eta(\tau_k)) \geq r]$$

$$\leq C \left[1 + e^{\delta F(D(x))} \right] e^{-\delta r}, \text{ for all } r \in \mathbb{R}_+,$$

by (3.40). A consequence of the inequalities (3.32)–(3.33), and the fact that $\Delta_{[\ell+1;r]}(t) = X_r(t) - X_{\ell+1}(t)$ from (1.6), and $F(\eta(t)) = \Psi(X(t))$ by (3.30), is that there exists $a > 0$ such that, a.s., $aF(\eta(t)) \leq \Delta_{[\ell+1;r]}(t) \leq a^{-1}F(\eta(t))$ for all $t \in \mathbb{R}_+$, and hence we obtain (3.3).

Finally, we deduce (3.4). Let N_t denote the number of times that Markov chain η jumps during time interval $[t, t+1]$. Since the jump rate is uniformly bounded, $\mathbb{P}[N_t \geq z] \leq e^{-\delta z}$ for some $\delta > 0$. Combined with (3.3) and the fact that $|\Delta(\tau) - \Delta(0)| \leq B$, a.s., for some $B \in \mathbb{R}_+$, this shows that, for a given $\Delta(0)$, we can find $A \in \mathbb{R}_+$ large enough so that

$$\mathbb{P} \left[\sup_{s \in [t, t+1]} \Delta(s) \geq 2A \log t \right] \leq \mathbb{P}[N_t \geq (A/B) \log t] + \mathbb{P}[\Delta(t) \geq A \log t] = O(t^{-2}),$$

say. The Borel–Cantelli lemma then shows that a.s., $\sup_{t \leq s \leq t+1} \Delta(s) \leq 2A \log t$ for all but finitely many $t \in \mathbb{Z}_+$, which implies (3.4). □

4 Proofs: Instability and cloud decomposition

4.1 Overview

The proof of the cloud decomposition, Theorem 1.10, is divided into several parts. In the putative cloud decomposition $\mathfrak{C}_1, \ldots, \mathfrak{C}_\nu$, each cloud $\mathfrak{C}_j$ is locally stable (by Proposition 3.2) and would, were it to be an isolated system with none of the other clouds present, possess the intrinsic speed $-v_j$, with the order $v_1 \geq v_2 \geq \cdots \geq v_\nu$ as at (1.15). The proof of Theorem 1.10 requires us to establish (i) that these intrinsic cloud speeds are indeed replicated in the full system, and (ii) that clouds are typically well-separated. The structure of the argument to do this is as follows.

- We first establish the result for the system in which $v_1 = v_2 = \cdots = v_\nu$. This is the most substantial part of the proof, and requires several steps in itself. In §4.2 we show that the centres of mass of the leftmost and rightmost clouds separate at least diffusively (Proposition 4.1), using a submartingale defocusing result from §A.

- Then in §4.3 we use a sort of induction to show *all* adjacent clouds typically separate, in a quantified sense. Roughly speaking, if there are $\nu \geq 3$ clouds, then since the two extreme clouds spend most of their time far apart, and clouds are tight, there must be another pair of clouds that also spends much time far apart, and so all but a vanishing fraction of the time the system evolves as two independent subsystems, both strictly smaller than the original. Iterating a formal version of this argument gives the result that all clouds typically separate in a quantified sense (Proposition 4.6).

- Having established separation of adjacent clouds, it is not hard to show that no cloud's speed can be significantly perturbed from its intrinsic speed, and hence (Lemma 4.7) all clouds do indeed travel at the same speed.

- Having settled the case of systems where all clouds have the same speed, the general case follows relatively easily by decomposing the system into subsystems consisting of same-speed clouds. Subsystems then have strictly ordered intrinsic speeds which they would maintain as isolated systems (by Lemma 4.7) and a coupling argument finishes the proof of the cloud decomposition, particle speeds, and diffusive separation.

- The recurrence of adjacent same-speed clouds in (1.19) is established by a separate supermartingale argument via Lemma 4.4.

4.2 Same-speed clouds: Centre-of-mass dynamics

We start the scheme outlined in §4.1. For $\nu \geq 2$ and $n \in \mathbb{Z}_+$, define

$$L(n) := \min_{i \in \mathfrak{C}_\nu} X_i(\tau_n) - \max_{i' \in \mathfrak{C}_1} X_{i'}(\tau_n) = X_{\min \mathfrak{C}_\nu}(\tau_n) - X_{\max \mathfrak{C}_1}(\tau_n), \tag{4.1}$$

the distance between the leftmost particle in the rightmost cloud and the rightmost particle in the leftmost cloud at time τ_n defined at (3.23). The first result of this section states that, in the case where $v_1 = \cdots = v_\nu$, $L(n)$ grows at least diffusively, meaning that the two extreme clouds are well-separated for all but a vanishing proportion of time (the case of two particles of the same intrinsic speeds shows that this result can be essentially sharp).

Proposition 4.1. *Suppose that (1.16) holds, that $\nu \geq 2$, and that $v_1 = \cdots = v_\nu$. Then for every $\gamma \in (0, 1/2)$, there exists $\varepsilon_0 := \varepsilon_0(\gamma) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$ and all n sufficiently large,*

$$\mathbb{P}\left[\sum_{i=0}^n \mathbb{1}\{L(i) \leq n^\gamma\} > n^{\frac{1}{2} + \gamma + 5\varepsilon}\right] \leq e^{-n^\varepsilon}.$$

The basic element in the proof of Proposition 4.1 is to show that the distance between the extreme clouds (precisely, their centres of mass) is a submartingale. This result, Lemma 4.2 below, does not need to assume $v_1 = v_\nu$. The intuition for this result is that the two extreme clouds have their own intrinsic speeds, v_1 (for the leftmost) and v_ν (for the rightmost), and, as isolated systems, these intrinsic speeds would be exactly the speeds of the centres of mass of the clouds (cf. Theorem 3.1). In the full system, the presence of other particles in between these two clouds should make it harder for them to move towards each other.

To state the result, let $\mathfrak{C} = [\ell; r]$ for $1 \leq \ell < r \leq N$. Then for $x \in \mathbb{X}_N$, define

$$g_{\mathfrak{C}}(x) := \langle \mathbf{1}, x \rangle_{\ell, r} / \langle \mathbf{1}, \mathbf{1} \rangle_{\ell, r}, \text{ and, for } t \in \mathbb{R}_+, \ G_j(t) := g_{\mathfrak{C}_j}(X(t)). \tag{4.2}$$

Also set

$$\Gamma(n) := G_\nu(\tau_n) - G_1(\tau_n) = g_{[k_{\nu-1}+1; k_\nu]}(X(\tau_n)) - g_{[1; k_1]}(X(\tau_n)). \tag{4.3}$$

Lemma 4.2. *Suppose that $\nu \geq 2$, and recall from (1.15) that $v_1 \geq v_\nu$. Then, for every $n \in \mathbb{Z}_+$,*

$$\mathbb{E}[\Gamma(n+1) - \Gamma(n) \mid \mathcal{F}_{\tau_n}] \geq (v_1 - v_\nu)/R_{X(\tau_n)}, \ a.s. \tag{4.4}$$

Suppose, additionally, that (1.16) holds. Then there exists $\delta > 0$ such that

$$\mathbb{E}[(\Gamma(n+1) - \Gamma(n))^2 \mid \mathcal{F}_{\tau_n}] \geq \delta, \ a.s. \tag{4.5}$$

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Remark 4.3 (Non-monotonicity). It does not appear straightforward to use the intuition for Lemma 4.2 expressed above directly, via a stochastic monotonicity argument, for example. The exception is in the case where all $m_i \equiv m \in (0, \infty)$ are equal, in which case the natural coupling shows that the presence of an additional particle to the right (say) of a system cannot increase the speed of the centre of mass of the system. For systems with different masses, the non-constant total activity rate (i.e. violation of (2.1)) means the natural coupling does not work.

Proof of Lemma 4.2. Recall from Definition 1.8 and (1.12) that that $\mathfrak{C}_1 = \{k_0 + 1, \ldots, k_1\}$ and $\mathfrak{C}_\nu = \{k_{\nu-1} + 1, \ldots, k_\nu\}$, with $k_0 = 0$ and $k_\nu = N$, are the leftmost and rightmost clouds, respectively. By the definition of $G_{\ell,r}$ from (4.2) and Lemma 3.5, with $\mathbf{e}$ as defined at (3.13),

$$\mathbb{E}[G_\nu(\tau_{n+1}) - G_\nu(\tau_n) \mid \mathcal{F}_{\tau_n}] = M_{k_{\nu-1},N}^{-1} \mathbb{E}_{X(\tau_n)} \langle \mathbf{1}, \mathbf{e} \rangle_{k_{\nu-1}+1, N} \geq -\frac{U_{k_{\nu-1},N}}{M_{k_{\nu-1},N} R_{X(\tau_n)}}, \quad \text{a.s.}$$

Similarly,

$$\mathbb{E}[G_1(\tau_{n+1}) - G_1(\tau_n) \mid \mathcal{F}_{\tau_n}] = M_{0,k_1}^{-1} \mathbb{E}_{X(\tau_n)} \langle \mathbf{1}, e \rangle_{1,k_1} \leq -\frac{U_{0,k_1}}{M_{0,k_1} R_{X(\tau_n)}}, \quad \text{a.s.}$$

It follows from (4.3) that (recall that $k_0 = 0$ and $k_\nu = N$)

$$\mathbb{E}[\Gamma(n+1) - \Gamma(n) \mid \mathcal{F}_{\tau_n}] \geq \frac{1}{R_{X(\tau_n)}} \left(\frac{U_{k_0,k_1}}{M_{k_0,k_1}} - \frac{U_{k_{\nu-1},k_\nu}}{M_{k_{\nu-1},k_\nu}} \right) = \frac{v_1 - v_\nu}{R_{X(\tau_n)}}, \quad \text{a.s.,}$$

by (1.15), verifying (4.4). Since $k_1 < k_{\nu-1} + 1$, at most one of $G_\nu(\tau_{n+1}) - G_\nu(\tau_n)$ and $G_1(\tau_{n+1}) - G_1(\tau_n)$ can be non-zero. Hence

$$\mathbb{E}[(\Gamma(n+1) - \Gamma(n))^2 \mid \mathcal{F}_{\tau_n}] = \mathbb{E}[(G_1(\tau_{n+1}) - G_1(\tau_n))^2 \mid \mathcal{F}_{\tau_n}] \\ + \mathbb{E}[(G_\nu(\tau_{n+1}) - G_\nu(X(\tau_n)))^2 \mid \mathcal{F}_{\tau_n}].$$

The quantity $G_1(\tau_{n+1}) - G_1(\tau_n)$ is non-zero if and only if one of the particles with labels in $\mathfrak{C}_1 = [1; k_1]$ jumps, and then, by (3.12),

$$\mathbb{E}[(G_1(\tau_{n+1}) - G_1(\tau_n))^2 \mid \mathcal{F}_{\tau_n}] = M_{0,k_1}^{-1} \mathbb{E}_{X(\tau_n)}[\langle \mathbf{1}, \mathbf{e} \rangle_{1,k_1}^2] \\ \geq R_{X(\tau_n)}^{-1} M_{0,k_1}^{-1} \sum_{i \in \mathfrak{C}_1} (a_i + b_i),$$

which, under hypothesis (1.16), is uniformly positive. This implies (4.5). □

Recall from (4.2) that $G_j(t)$ is the centre of mass of cloud $\mathfrak{C}_j$ at time $t \in \mathbb{R}_+$. Define

$$\Gamma_j(n) := G_{j+1}(\tau_n) - G_j(\tau_n), \quad \text{for } j \in [\nu - 1] \text{ and } n \in \mathbb{Z}_+, \tag{4.6}$$

the distance between the centres of mass of clouds j and $j + 1$ at time τ_n.

Lemma 4.4. *Suppose that $\nu \geq 2$, and that $j \in [\nu - 1]$ is such that $v_j = v_{j+1}$, with the notation from (1.15). Then, for every $n \in \mathbb{Z}_+$,*

$$\mathbb{E}[\Gamma_j(n+1) - \Gamma_j(n) \mid \mathcal{F}_{\tau_n}] \leq 0, \quad \text{on } \{\Gamma_j(n) > 0\}. \tag{4.7}$$

Proof. Suppose that $j \in [\nu - 1]$ is such that $v_j = v_{j+1}$. Then G_j, G_{j+1} are the centres of mass processes for clouds $\mathfrak{C}_j = \{k_{j-1} + 1, \ldots, k_j\}$ and $\mathfrak{C}_{j+1} = \{k_j + 1, k_{j+1}\}$, respectively. From (4.2),

$$\mathbb{E}[G_j(X(\tau_{n+1})) - G_j(X(\tau_n)) \mid \mathcal{F}_n] = M_{k_{j-1},k_j}^{-1} \, \mathbb{E}_{X(\tau_n)} \langle \mathbf{1}, \mathbf{e} \rangle_{k_{j-1}+1,k_j}$$

On the event $\{\Gamma_j(n) > 0\}$, recalling the definitions of $k_{\pm} = k_{\pm}(X(\tau_n))$ from (3.15), we have $(k_j + 1)_- = k_j + 1$ and $(k_j)_+ = k_j$. Hence, from (3.18) in Lemma 3.5, we have that, on $\{\Gamma_j(n) > 0\}$,

$$R_{X(\tau_n)} \, \mathbb{E}_{X(\tau_n)} \langle \mathbf{1}, \mathbf{e} \rangle_{k_j+1,k_{j+1}} \leq U_{k_j,k_{j+1}}, \text{ and } R_{X(\tau_n)} \, \mathbb{E}_{X(\tau_n)} \langle \mathbf{1}, \mathbf{e} \rangle_{k_{j-1}+1,k_j} \geq U_{k_{j-1},k_j},$$

by (1.5). It follows from (4.6) that, on $\{\Gamma_j(n) > 0\}$,

$$\mathbb{E}[\Gamma_j(n+1) - \Gamma_j(n) \mid \mathcal{F}_{\tau_n}] = R_{X(\tau_n)}^{-1} \left(M_{k_j,k_{j+1}}^{-1} \langle \mathbf{1}, \mathbf{e} \rangle_{k_j+1,k_{j+1}} - M_{k_{j-1},k_j}^{-1} \langle \mathbf{1}, \mathbf{e} \rangle_{k_{j-1}+1,k_j} \right)$$

$$\leq R_{X(\tau_n)}^{-1} \left(\frac{U_{k_j,k_{j+1}}}{M_{k_j,k_{j+1}}} - \frac{U_{k_{j-1},k_j}}{M_{k_{j-1},k_j}} \right) = 0,$$

by (1.15) and the assumption that $v_j = v_{j+1}$. This proves (4.7). $\qquad\square$

Recall the definition of $\Delta_{\mathfrak{C}}(t)$ from (1.6). The following straightforward consequence of Proposition 3.2 will be useful in showing that to control dynamics of clouds it suffices, on large scales, to control dynamics of their centres of mass.

Lemma 4.5. *Suppose that $\nu \in \mathbb{N}$ and cloud decomposition $\mathfrak{C}_1, \ldots, \mathfrak{C}_\nu$ has speeds $v_1 = \cdots = v_\nu \in \mathbb{R}$. Then, for every $\varepsilon > 0$, for all n sufficiently large,*

$$\mathbb{P}\left[\max_{j \in [\nu]} \max_{0 \leq i \leq n} \Delta_{\mathfrak{C}_j}(\tau_i) \geq n^{2\varepsilon} \right] \leq \mathrm{e}^{-n^\varepsilon}.$$

Proof. The proof of Proposition 3.2 up to (3.40) shows that, for every $\varepsilon > 0$,

$$\max_{j \in [\nu]} \sup_{i \in \mathbb{Z}_+} \mathbb{P}[\Delta_{\mathfrak{C}_j}(\tau_i) \geq n^{2\varepsilon}] \leq \mathrm{e}^{-n^\varepsilon},$$

for all n sufficiently large. The claimed result follows from a union bound. $\qquad\square$

Proof of Proposition 4.1. We apply Lemma A.1 with $X_n = \Gamma(n)$ as defined at (4.3). Then the bounded-increments hypothesis (A.2) is satisfied, and Lemma 4.2 shows that the submartingale hypothesis (A.3) and uniform lower bound on second moments (A.4) both hold (here we use hypothesis (1.16)). Then Lemma A.1 shows that, for any $\gamma \in (0, 1/2)$ and $\varepsilon \in (0, \frac{1-2\gamma}{4})$,

$$\mathbb{P}\left[\sum_{i=0}^{n} \mathbb{1}\{\Gamma(i) \leq n^\gamma\} > n^{\frac{1}{2}+\gamma+4\varepsilon} \right] \leq \mathrm{e}^{-n^\varepsilon}. \tag{4.8}$$

Since $L(n)$ defined at (4.1) satisfies $|L(n) - \Gamma(n)| \leq \Delta_{\mathfrak{C}_1}(\tau_n) + \Delta_{\mathfrak{C}_\nu}(\tau_n)$, we combine Lemma 4.5 with (4.8) to get

$$\mathbb{P}\left[\sum_{i=0}^{n} \mathbb{1}\{L(i) \leq n^\gamma - 2n^{2\varepsilon}\} > n^{\frac{1}{2}+\gamma+4\varepsilon} \right] \leq \mathbb{P}\left[\sum_{i=0}^{n} \mathbb{1}\{\Gamma(i) \leq n^\gamma\} > n^{\frac{1}{2}+\gamma+4\varepsilon} \right]$$

$$+ \mathbb{P}\left[\max_{0 \leq i \leq n} (\Delta_{\mathfrak{C}_1}(\tau_i) + \Delta_{\mathfrak{C}_\nu}(\tau_i)) \geq 2n^{2\varepsilon} \right] \leq 2\mathrm{e}^{-n^\varepsilon},$$

which yields the result, provided $\varepsilon < \gamma/2$. $\qquad\square$

4.3 Same-speed clouds: Quantified separation

As outlined in §4.1, the key result in the proof of Theorem 1.10 is the following quantitative separation result on systems of multiple clouds all of the same intrinsic speed, which shows that not only do the two extreme clouds stay well separated most of the time, but *all* adjacent pairs of clouds do. Recall from (1.17) that for $j \in [\nu - 1]$, $L_j(t) \in \mathbb{Z}_+$ denotes the separation of clouds $\mathfrak{C}_j$ and $\mathfrak{C}_{j+1}$ at time t.

Proposition 4.6. *Suppose that* (1.16) *holds, that* $\nu \geq 2$, *and that* $v_1 = \cdots = v_\nu$. *Then there exists* $\varepsilon > 0$ *such that, for all n sufficiently large,*

$$\mathbb{P}\left[\sum_{i=0}^{n} \mathbb{1}\left\{\min_{1 \leq j \leq \nu - 1} L_j(\tau_i) \leq n^\varepsilon\right\} > n^{1-\varepsilon}\right] \leq e^{-n^\varepsilon}.$$

Proof. Suppose $\nu \geq 2$. For $\gamma \in (0, 1/2)$, take $\varepsilon \in (0, \frac{1-2\gamma}{12})$ and, for $k \in \mathbb{N}$, define

$$\gamma_k := \gamma^k, \quad \alpha_k := 1 - 2^{1-k}\gamma^{k(k-1)/2}\left(\frac{1}{2} - \gamma - 6\varepsilon\right), \text{ and } \varepsilon_k := 2^{1-k}\varepsilon. \tag{4.9}$$

For $k \in \{1, 2, \ldots, \nu - 1\}$, say time $i \in \mathbb{Z}_+$ is *k-good* if $\#\{j \in [\nu - 1] : L_j(i) \geq 2^{-k}n^{\gamma_k}\} \geq k$ (the definition of k-good also depends on n, but we assume n is fixed and sufficiently large, as determined in the subsequent argument, and keep the n-dependence tacit). Let $\mathcal{G}_k(i)$ denote the event that i is k-good, and note that $\mathcal{G}_{k+1}(i) \subseteq \mathcal{G}_k(i) \in \mathcal{F}_i$ for $1 \leq k \leq \nu - 2$. Define, for $k \in [\nu - 1]$,

$$B_k(A) := \sum_{i \in A} \mathbb{1}_{\mathcal{G}_k^c(i)},$$

the number of $i \in A \subseteq [n]$ that are not k-good. If $\nu = 2$, note that $L_1(\tau_n) = L(n)$ as defined at (4.1), and we have from Proposition 4.1 that, for $\varepsilon > 0$ as given, for all n large enough,

$$\mathbb{P}[B_1([n]) \geq n^{\frac{1}{2}+\gamma_1+5\varepsilon}] \leq e^{-n^\varepsilon},$$

which completes the proof if $\nu = 2$. The general case is a finite induction, for which we suppose $\nu > 2$ and consider the inductive hypothesis

$$\mathbb{P}[B_k([n]) \geq n^{\alpha_k}] \leq e^{-n^{\varepsilon_k}}, \text{ where } k \in [\nu - 1], \tag{4.10}$$

which we have verified for $k = 1$ with $\alpha_1 = \frac{1}{2} + \gamma_1 + 6\varepsilon_1$ and $\varepsilon_1 = \varepsilon$ from above, as in (4.9).

The heart of the induction is to show that, with very high probability, most k-good times can be upgraded to $(k+1)$-good times, as follows. If a time $t \in [n]$ is k-good, then the next n_k times will have k gaps that remain large (where n_k is chosen appropriately of order n^{γ_k}), since particles only move to their nearest neighbours, meaning gaps cannot shrink too fast. Over that relatively long time, the $k + 1$ subsystems (separated by the gaps that made the original time k-good) evolve independently. As long as $k < \nu - 1$, at least one of those subsystems contains at least two clouds, and so we can apply Proposition 4.1 to conclude that the extreme clouds of that subsystem separate for most times in the interval $[t, t + n_k]$. Since clouds are tight by the cloud concentration result in Lemma 4.5, this means that there is at least one (additional) large gap between clouds at most of those times. This additional separation allows us to upgrade most k-good times to $(k + 1)$-good times, with high probability. Iterating this shows that, with high probability, most times are $(\nu - 1)$-good, which means all clouds are well separated. We give the details, starting by explaining

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how a particular k-good time leads to many $(k+1)$-good times, and then using a blocking argument prove (4.10) for each $1 \le k \le \nu - 1$.

First we show that, for every $k \in [\nu - 2]$, for all $t \in [n]$ and all n large enough,

$$\mathbb{P}\Big[B_{k+1}\big([t, t+n_k]\big) \ge n^{\alpha_1 \gamma_k} \,\Big|\, \mathcal{F}_t\Big] \le \mathrm{e}^{-n^{\varepsilon_1}}, \text{ on } \mathcal{G}_k(t), \tag{4.11}$$

where $n_k := \lfloor 2^{-k-1} n^{\gamma_k} \rfloor$. Fix any k-good time $t \in [n]$. By definition of the k-good property, that means there exists $J_t \subset [\nu]$ with size $\#J_t = k$ for which $L_j(t) \ge 2^{-k} n^{\gamma_k}$ for every $j \in J_t$. (For definiteness, in case of a choice, choose the J_t that has the smallest possible maximal element.) Since $|L_j(n+1) - L_j(n)| \le 1$ for all $n \in \mathbb{Z}_+$, it follows that

$$L_j(i) \ge 2^{-k-1} n^{\gamma_k} \text{ for all } i \in [t, t+n_k] \text{ and all } j \in J_t. \tag{4.12}$$

Suppose that t is k-good. List J_t in order as $J_t = \{j_1, j_2, \ldots, j_k\}$ and define a corresponding ordered partition of $[\nu]$, denoted $\mathfrak{B}_1, \ldots, \mathfrak{B}_{k+1}$, by

$$\mathfrak{B}_1 := \bigcup_{j=1}^{j_1} \mathfrak{C}_j, \quad \mathfrak{B}_2 := \bigcup_{j=j_1+1}^{j_2} \mathfrak{C}_j, \quad \ldots \quad, \mathfrak{B}_{k+1} := \bigcup_{j=j_k+1}^{\nu} \mathfrak{C}_j.$$

Each $\mathfrak{B}_i$ is a union of one or more clouds $\mathfrak{C}_j$. Since $k + 1 \le \nu - 1$, there is at least one $\mathfrak{B}_\ell = \cup_{j=j_{\ell-1}+1}^{j_\ell} \mathfrak{C}_j$ (with convention $j_0 = 0$ and $j_{k+1} = \nu$) that is the union of at least two clouds; choose and fix such $1 \le \ell \le k+1$ (the $\mathfrak{B}_i$ and ℓ depend on t but we omit that from the notation). Recall the property (4.12), which states that the collections of clouds $\mathfrak{B}_1, \ldots, \mathfrak{B}_{k+1}$ stay well-separated over time interval $[t, t+n_k]$. The same is true if we ran a system with the same configuration at time t and independent evolution of subsystems $\mathfrak{B}_1, \ldots, \mathfrak{B}_{k+1}$. Hence there is a natural coupling over time $[t, t+n_k]$ of the original system to a collection of independent subsystems $\mathfrak{B}_1, \ldots, \mathfrak{B}_{k+1}$ (which do not interact). In particular, over time $[t, t+n_k]$, the part of the full system corresponding to $\mathfrak{B}_\ell$, when it jumps, has the same law as an isolated system containing only the particles in $\mathfrak{B}_\ell$.

Consider an isolated system containing only the particles corresponding to $\mathfrak{B}_\ell$, started (at time 0) from the configuration inherited from the full system at time t. Denote by $\tau_0^\ell, \tau_1^\ell, \ldots$ the jump times of this isolated system, analogously to (3.23). We retain the original labels for the particles and clusters, and set
$$L^\ell(n) := \min_{i \in \mathfrak{C}_{j_\ell}} X_i(\tau_n^\ell) - \max_{i' \in \mathfrak{C}_{j_{\ell-1}}} X_{i'}(\tau_n^\ell),$$

the separation of the two extreme clouds of the isolated system, according to its internal jump clock. Proposition 4.1 applied to this isolated system yields, for $\varepsilon > 0$ small enough,

$$\mathbb{P}\left[\sum_{i=0}^{n} \mathbb{1}\Big\{L^\ell(i) \le n^\gamma\Big\} > n^{\frac{1}{2}+\gamma+5\varepsilon}\right] \le \mathrm{e}^{-n^\varepsilon}.$$

In addition, write

$$\Delta^\ell(n) := \sum_{j=j_{\ell-1}+2}^{j_\ell-1} \Delta_{\mathfrak{C}_j}(\tau_n^\ell),$$

for the total span of all clouds other than the two extreme clouds. An application of Lemma 4.5 shows that $\mathbb{P}[\max_{0 \le i \le n} \Delta^\ell(i) \ge n^{2\varepsilon}] \le \mathrm{e}^{-n^\varepsilon}$. In n steps of the full system, the subsystem corresponding to $\mathfrak{B}_\ell$ takes at most n steps. On the other hand, a binomial concentration bound shows that, with probability at least $1 - \mathrm{e}^{-n^\varepsilon}$, the subsystem corresponding to $\mathfrak{B}_\ell$ takes at least δn steps,

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where $\delta > 0$ is a positive constant depending only on the a_i, b_i, whose existence is guaranteed by (1.16). Hence, since $\gamma \in (0, 1/2)$ was arbitrary, by the coupling described above, we deduce for the full system that, for any $\gamma \in (0, 1/2)$, there exists $\varepsilon > 0$ so that, for all n large enough,

$$\mathbb{P}\left[\sum_{i=t}^{t+n} \mathbb{1}\left\{\max_{j_{\ell-1}+1 \leq j \leq j_\ell-1} L_j(i) \leq n^\gamma\right\} > n^{\alpha_1} \;\middle|\; \mathcal{F}_t\right] \leq \mathrm{e}^{-n^\varepsilon}, \text{ on } \mathcal{G}_k(t),$$

where $\alpha_1 = \frac{1}{2} + \gamma + 6\varepsilon$ as at (4.9). Combined with (4.12), this verifies (4.11).

Now we can complete the induction. Suppose that (4.10) is true for some k with $1 \leq k \leq \nu - 2$, $\gamma_k \in (0, 1/2)$ and $\alpha_k \in (1/2, 1)$ given by (4.9), and some $\varepsilon_k > 0$. Define $s_j := jn_k$, so that $0 = s_0 < s_1 < \cdots < s_{w_k} \leq n$, where $w_k := \lfloor n/n_k \rfloor$ has $w_k \sim 2^{k+1} n^{1-\gamma_k}$ as $n \to \infty$. Consider time intervals $i_p := [s_{p-1}, s_p - 1]$ for $p \in [w_k]$ and $i_{w_k+1} := [s_{w_k}, n]$. On the event $B_k([n]) < n^{\alpha_k}$, there can be no more than $n^{\alpha_k(1-\gamma_k)}$ intervals i_p ($p \in [w_k + 1]$) for which $B_k(i_p) > n^{\alpha_k\gamma_k}$. Write $\mathcal{P} := \{p \in [w_k + 1] : B_k(i_p) \leq n^{\alpha_k\gamma_k}\}$. Then, since each i_p has size $n_k < n^{\gamma_k}$,

$$B_{k+1}([n]) \leq n^{\alpha_k(1-\gamma_k)} n^{\gamma_k} + \sum_{p \in \mathcal{P}} B_{k+1}([i_p]), \text{ on } \{B_k([n]) < n^{\alpha_k}\}. \tag{4.13}$$

Consider any $p \in \mathcal{P}$. Then there is some k-good time t_p (for definiteness, choose the earliest) with $s_{p-1} \leq t_p < s_p$ and $t_p \leq s_{p-1} + n^{\alpha_k\gamma_k}$. Hence, by (4.11) and the fact that $\#\mathcal{P} = w_k + 1 \leq 2^\nu n^{1-\gamma_k}$,

$$\mathbb{P}\left[\sum_{p \in \mathcal{P}} B_{k+1}([i_p]) \geq 2^{1+\nu} n^{1-\gamma_k(1-\alpha_k)}\right] \leq 2^\nu n^{1-\gamma_k} \sup_{1 \leq p \leq w_k+1} \mathbb{P}[B_{k+1}([i_p]) \geq 2n^{\alpha_k\gamma_k}]$$

$$\leq 2^\nu n^{1-\gamma_k} \sup_{1 \leq p \leq w_k+1} \mathbb{P}[B_{k+1}([i_p]) \geq n^{\alpha_k\gamma_k} + n^{\alpha_1\gamma_k}] \leq \mathrm{e}^{-n^{\varepsilon_1}},$$

using also that $\alpha_k \geq \alpha_1$ by (4.9). It follows from the induction hypothesis (4.10) and the fact that $\gamma_k < 1/2$ and $\varepsilon_k \leq \varepsilon_1$ that

$$\mathbb{P}[B_{k+1}([n]) \geq 2^{2+\nu} n^{1-\gamma_k(1-\alpha_k)}]$$
$$\leq \mathbb{P}[B_{k+1}([n]) \geq 2^{1+\nu} n^{1-\gamma_k(1-\alpha_k)} + n^{\alpha_k+\gamma_k(1-\alpha_k)}, B_k(n) < n^{\alpha_k}] + \mathbb{P}[B_k([n]) \geq n^{\alpha_k}]$$
$$\leq \mathbb{P}\left[\sum_{p \in \mathcal{P}} B_{k+1}([i_p]) \geq 2n^{1-\gamma_k(1-\alpha_k)}\right] + \mathrm{e}^{-n^{\varepsilon_k}} \leq 2\mathrm{e}^{-n^{\varepsilon_k}},$$

using (4.13). Using the fact that $\alpha_{k+1} > 1 - \gamma_k(1 - \alpha_k)$ by (4.9) we verify $\mathbb{P}[B_{k+1}([n]) \geq n^{\alpha_{k+1}}] \leq \mathrm{e}^{-n^{\varepsilon_{k+1}}}$, for $\varepsilon_{k+1} = \varepsilon_k/2$ and all n sufficiently large, which completes the inductive step. $\square$

Lemma 4.7. *Suppose that (1.16) holds, that $\nu \geq 2$, and that $v_1 = \cdots = v_\nu = v \in \mathbb{R}$. Then, for every $i \in [N]$, $\lim_{t\to\infty} t^{-1} X_i(t) = -v$, a.s.*

Proof. If $\nu = 1$ then the result is contained in Theorem 1.3, so it suffices to suppose $\nu \geq 2$. We couple the original particle system to a system in which each cloud behaves completely independently of the others. The two systems can be coupled to have the identical initial configurations, and that whenever ($n \in \mathbb{Z}_+$) event $E_n := \{\min_{1 \leq j \leq \nu-1} L_j(\tau_n) > 0\}$ occurs, i.e., no particles from different clouds are together in the original system, they take precisely the same holding times and increments. Write $\widetilde{X} = (\widetilde{X}(t))_{t \in \mathbb{R}_+}$ for the system with independent clouds, maintaining the labelling of the original system, so that $(\widetilde{X}_i(t))_{i \in \mathfrak{C}_j}$ behave as independent elastic systems for each j, and weak order is maintained within each $\mathfrak{C}_j$, but particles from different 'clouds' can swap places due to the independence.

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Consider $G_j(t)$ the centre of mass of cloud $\mathfrak{C}_j$ in the original system and $\widetilde{G}_j(t)$ the centre of mass of cloud $\mathfrak{C}_j$ in the independent system. For fixed j, the subsystem $(\widetilde{X}_i(t))_{i \in \mathfrak{C}_j}$ (with suitable mapping of notations) satisfies the hypotheses of Theorem 3.1, and so, in particular,

$$\lim_{t \to \infty} t^{-1}\widetilde{G}_j(t) = -v_j, \text{ a.s.} \tag{4.14}$$

Fix $j \in [\nu]$ and define $\tau_{j,0} := 0$ and, for $n \in \mathbb{Z}_+$,

$$\tau_{j,n+1} := \inf\{t \geq \tau_{j,n} : G_j(t) \neq G_j(\tau_{j,n})\},$$

the jump times of G_j (a subsequence of $\tau_0, \tau_1, \ldots$ defined at (3.23)). Define $\widetilde{\tau}_{j,0}, \widetilde{\tau}_{j,1}, \ldots$ analogously for $\widetilde{G}_j$. Also define $N_{j,t} := \sup\{n \in \mathbb{Z}_+ : \tau_{j,n} \leq t\}$ and $\widetilde{N}_{j,t} := \sup\{n \in \mathbb{Z}_+ : \widetilde{\tau}_{j,n} \leq t\}$, the numbers of jumps, up to time t, of G_j and $\widetilde{G}_j$, respectively. The process $\widetilde{G}_j$ can jump several times between the jumps of G_j, but the coupling guarantees that if E_i occurs in the coupled system at time $\tau_{j,i}$, then $G_j(\tau_{j,i}) - G_j(\tau_{j,i+1}) = \widetilde{G}_j(\tau_{j,i+1}) - \widetilde{G}_j(\tau_{j,i})$. It follows that, for a constant $C < \infty$,

$$\begin{aligned}
\left|G_j(t) - \widetilde{G}_j(t)\right| &\leq \sum_{i=0}^{N_{j,t}-1} \left(G_j(\tau_{j,i+1}) - G_j(\tau_{j,i})\right)\mathbb{1}_{E_i^c} + \sum_{i=0}^{N_{j,t}-1} \left(\widetilde{G}_j(\tau_{j,i+1}) - \widetilde{G}_j(\tau_{j,i})\right)\mathbb{1}_{E_i^c} \\
&\leq C \sum_{i=0}^{\max(N_{j,t}, \widetilde{N}_{j,t})-1} \mathbb{1}_{E_i^c},
\end{aligned}$$

since $G_j(0) = \widetilde{G}_j(0)$ and both G_j and $\widetilde{G}_j$ have bounded increments. Since jump rates are uniformly bounded away from 0 and ∞, there is a constant $B < \infty$ such that, a.s., for all t large enough, both $N_{j,t} \leq Bt$ and $t^{-1}\widetilde{N}_{j,t} \leq Bt$. Hence $\lim_{t \to \infty} t^{-1}|G_j(t) - \widetilde{G}_j(t)| = 0$, a.s., by Proposition 4.6. Also by Lemma 4.5, for every $i \in \mathfrak{C}_j$, $n^{-1}|X_i(\tau_n) - G_j(n)| \to 0$, a.s. $\qquad \square$

Finally, we can complete the proof of the cloud decomposition, Theorem 1.10.

Proof of Theorem 1.10. The cloud stability result, part (b), is a consequence of Proposition 3.2. For the remaining parts, we use another coupling. Split the system into subsystems of constant-speed clouds, so for example, $\mathfrak{B}_1 = \mathfrak{C}_1 \cup \cdots \cup \mathfrak{C}_{i_1}$ is the leftmost subsystem, where $i_1 = \max\{j \in [\nu] : v_j = v_1\}$. This gives subsystems $\mathfrak{B}_1, \ldots, \mathfrak{B}_K$, say ($K \in [\nu]$). Consider a system that starts from the same initial configuration as the original system, but in which the subsystems $\mathfrak{B}_j$ evolve independently of each other. In the independent system, each $\mathfrak{B}_j$ evolves as a system with same-speed clouds, and so satisfies the strong law from Lemma 4.7 and the cloud separation result from Proposition 4.6. We couple the independent system and the original system such that whenever the original system is in a configuration in which no two subsystems have particles at the same site, we use the same holding times and jumps as the independent system. By the strict ordering of the speeds, in particular, the strong law shows that there is an a.s.-finite time τ for which, for all $t \geq \tau$, the two systems evolve with exactly the same increments. It follows that the maximum displacement between the particles in the original system and their partners in the independent system remains bounded by a finite random variable. This verifies the speeds for the original system as stated in part (a) of the theorem, and also means that the separation result from Proposition 4.6 remains valid for the original system, completing the proof of part (c) of the theorem. $\qquad \square$

5 Comments on invariant distributions

5.1 Irreversibility of dynamics

A continuous-time Markov chain on a countable state space $\mathbb{X}$, with transition rates $q_{i,j}$, $i, j \in \mathbb{X}$, is *reversible* with respect to an invariant measure $\pi = (\pi_i)_{i \in \mathbb{X}}$ if the detailed balance relation $\pi_i q_{i,j} = \pi_j q_{j,i}$ holds for all $i, j \in \mathbb{X}$. For any cycle of states $i_1, i_2, \dots, i_n, i_{n+1}$ with $i_1 = i_{n+1}$ and no other pair of states equal, taking a product of the detailed balance relations $\pi_{i_k} q_{i_k, i_{k+1}} = \pi_{i_{k+1}} q_{i_{k+1}, i}$ over $k \in \{1, \dots, n\}$ implies that

$$\prod_{k=1}^{n} q_{i_k, i_{k+1}} = \prod_{k=1}^{n} q_{i_{k+1}, i_k}, \tag{5.1}$$

i.e., the product of the transition rates along the forward cycle is the same as the product of the transition rates along the backward cycle. In fact, a criterion of Kolmogorov says that the Markov chain is reversible if and only if (5.1) holds for *every* cycle [33, p. 23].

It is well known that the simple exclusion process, as described in §2.4, is reversible, and this is key to the explicit computation of invariant measures. For the elastic model studied in the present paper, the generic situation, provided $N \geq 3$, is that reversibility fails: see Figure 5 for an example. Thus it seems that, generically, it may be difficult to obtain an explicit expression for the invariant measure in the stable case.

Remark 5.1 (Discrete vs. continuous time). Consider the elastic particle system with $m_i \equiv m \in (0, \infty)$ for all $i \in [N]$. As mentioned in §2.2, since the total activity rate (2.1) of the elastic model with constant masses is independent of the current configuration, the discrete-time jump chain associated with the continuous-time Markov chain is stable precisely when the continuous-time chain is stable, and the stationary measures coincide. The Kolmogorov cycle criterion has a direct discrete-time analogue, where in (5.1) one replaces the transition rates by transition *probabilities*. Since the total activity rate is constant, the same example in Figure 5 also shows that reversibility fails in the discrete-time setting, as it must. By contrast, the exclusion process does not have constant total activity rate, but nevertheless it turns out that the jump-chain of the exclusion process is *also* reversible (with a different invariant measure from the continuous-time version).

5.2 Exact computations

Suppose that the elastic particle system is stable, i.e., satisfies the criterion (1.7) from Theorem 1.3. Then the Markov process of inter-particle separations $\eta(t) := (\eta_1(t), \dots, \eta_{N-1}(t))$ has an invariant distribution $\pi^{(N-1)}(x)$, $x \in \mathbb{Z}_+^{N-1}$ (cf. Corollary 1.5). Reversibility often enables one to explicitly compute the corresponding invariant measure; for example, the exclusion process (§2.4) is reversible and stationary measures are product-geometric (see e.g. [36]). For non-reversible systems exact computation of stationary distributions is typically much harder. Generically, as in Figure 5, there is no reversibility for the elastic particle system for $N \geq 3$, and exact computation of stationary distributions remains an open problem (we comment on this in §5.3 below). However, the case $N = 2$ *is* reversible (since it reduces to a birth-death random walk on $\mathbb{Z}_+$). In this section we record this special case, and we finish with some open problems in §5.3, including to identify exceptional parameter values for the general $N \geq 3$ case where reversibility holds.

Case $N = 2$. For two particles (see Theorem 1.3 and Example 1.6) the system is stable if and only if $u_1 > u_2$, where u_i is as defined at (1.3) (note that this implies $b_1 + a_2 > 0$). The inter-particle
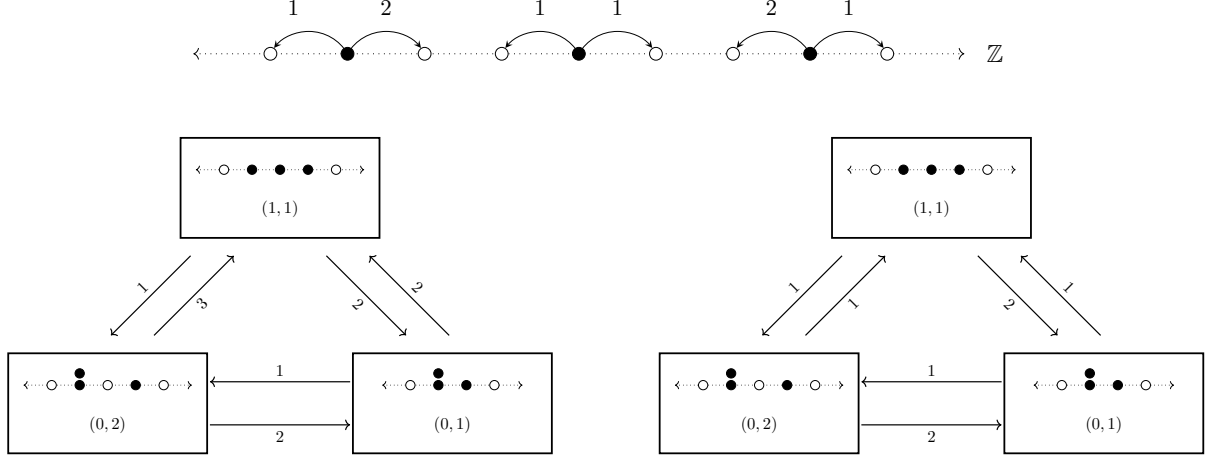
FIGURE 5: An example of an $N = 3$ particle system showing that the dynamics for the elastic process is not reversible. Take jump rates (a_i, b_i) for particles in free space given by $(1, 2), (1, 1), (2, 1)$ (*top picture*). Pictured are the transition rates for the cycle $(1, 1) \leftrightarrow (0, 2) \leftrightarrow (0, 1) \leftrightarrow (1, 1)$ in the elastic model (*bottom left*) as opposed to the exclusion process (*bottom right*). In the exclusion process the rates for sequence $(1, 1) \rightarrow (0, 2) \rightarrow (0, 1) \rightarrow (1, 1)$ and its reversal $(1, 1) \leftarrow (0, 2) \leftarrow (0, 1) \leftarrow (1, 1)$ have the same product: $1 \cdot 2 \cdot 1 = 2 \cdot 1 \cdot 1$. In the elastic process, the rates for sequence $(1, 1) \rightarrow (0, 2) \rightarrow (0, 1) \rightarrow (1, 1)$ have product $1 \cdot 2 \cdot 2 = 4$ while its reversal $(1, 1) \leftarrow (0, 2) \leftarrow (0, 1) \leftarrow (1, 1)$ has product $2 \cdot 1 \cdot 3 = 6$.

distance is a birth-death random walk on $\mathbb{Z}_+$ and, in the positive-recurrent case, the stationary distribution $\pi : \mathbb{Z}_+ \to [0, 1]$ can be computed explicitly by solving

$$(a_1 + a_2 + b_1 + b_2)\pi_0 = (b_1 + a_2)\pi_1,$$
$$(a_1 + a_2 + b_1 + b_2)\pi_1 = (b_1 + a_2)\pi_2 + (a_1 + a_2 + b_1 + b_2)\pi_0$$
$$(a_1 + a_2 + b_1 + b_2)\pi_k = (b_1 + a_2)\pi_{k+1} + (a_1 + b_2)\pi_{k-1}, \ k \geq 2.$$

The solution obtained (cf. e.g. [5, pp. 197–8]) is a *zero-modified geometric* distribution

$$\pi_0 = \frac{u_1 - u_2}{2(b_1 + a_2)}, \ \text{and} \ \pi_k = \frac{(u_1 - u_2)(a_1 + a_2 + b_1 + b_2)(a_1 + b_2)^{k-1}}{2(b_1 + a_2)^{k+1}}, \ k \geq 1.$$

An alternative argument uses speeds and 'ergodic' considerations. By stability, the long-run average speed of any particle must be the same as that of the centre of mass of the system, which, by Theorem 3.1, is always $-U_2/M_2 = (u_1 + u_2)/2$. The leftmost particle travels at its intrinsic speed u_1 unless $\eta_1(t) = 0$, in which case it cannot jump to the right but its speed of jumping left is increased. It is not hard to show then that

$$\frac{u_1 + u_2}{2} = u_1 - (b_1 + a_2) \lim_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{1}\{\eta_1(s) = 0\} \mathrm{d}s = u_1 - (b_1 + a_2)\pi_0,$$

recovering the formula for π_0; considering the rightmost particle gives the same conclusion.

Case $N \geq 3$. Without reversibility, it appears challenging to proceed in general. Even using the 'ergodic' property and speeds, as in the last example, to compute stationary probabilities of one or more coordinates being zero does not seem straightforward: considering the speeds gives $N + 1$ equations and there are 2^{N-1} unknowns (each coordinate can be zero or non-zero), but, as in the above example, the $N + 1$ equations are not linearly independent, so even for $N = 3$ the system is undetermined.

5.3 Open problems

It is known in the setting of Atlas models (see §2.6 and [39]) that the *skew symmetry* condition of Harrison & Williams [26, 27, 48], transferred from the theory of reflected diffusions in the orthant, yields a necessary and sufficient condition on parameters of the model in order for it to have a product-exponential invariant distribution, and possess a relative of the reversibility property known as *strong duality*. We suspect that there should be a related picture in our setting, given the strong parallels with the Atlas model, but it is not clear what condition is needed to play the role of skew symmetry: the Atlas model is a certain heavy-traffic scaling limit of the particle system (see [32] and [36, §6.3]) but there are many potential conditions on the a_i, b_i that can lead to the limiting parameters satisfying the skew symmetry condition, and no candidate that we tried was adequate to guarantee reversibility.

Problem 5.2. For $N \geq 3$, formulate a necessary and sufficient condition on the a_i, b_i (and m_i) under which the elastic particle system is reversible and has a product-geometric invariant measure.

In the case $N = 3$ the problem is equivalent (see §2.5) to that of a nearest-neighbour random walk on $\mathbb{Z}_+^2$ with boundary reflections. It remains a challenging problem to obtain (necessary and/or sufficient) conditions under which a random walk on $\mathbb{Z}_+^2$ has a stationary measure that can be expressed explicitly, with particular interest on expressions as a product of geometric terms, or a finite or countable sum of such products: see e.g. [3, 4, 12, 13] and references therein. The scope of the *compensation approach* as described in [3] appears to cover the $N = 3$ case of the elastic model, but successful application of that approach to this setting remains an open problem, as far as we are aware. There are some classes of walks that have been solved, for example in [4, 12, 13], but the forms of boundary reflections assumed in [4, 12, 13] exclude application to any stable regime for the present model.

Problem 5.3. More generally, compute explicitly the invariant measure of the elastic particle system when $N \geq 3$.

A   Submartingale occupation and defocusing

For $X := (X_n)_{n \in \mathbb{Z}_+}$, a stochastic process taking values in $\mathbb{R}_+$, we define

$$\mathcal{L}_n^X(x) := \sum_{i=0}^{n} \mathbb{1}\{X_i \leq x\}, \text{ for } x \in \mathbb{R}_+, \tag{A.1}$$

the occupation time of interval $[0, x]$ up to time $n \in \mathbb{Z}_+$.

Lemma A.1. *Let* $X = (X_n)_{n \in \mathbb{Z}_+}$ *be an* $\mathbb{R}_+$*-valued process, adapted to a filtration* $(\mathcal{F}_n)_{n \in \mathbb{Z}_+}$. *Suppose that there exist* $B \in \mathbb{R}_+$ *and* $v \in (0, \infty)$ *such that, for all* $n \in \mathbb{Z}_+$,

$$\mathbb{P}[X_{n+1} - X_n \leq B] = 1; \tag{A.2}$$

$$\mathbb{E}[X_{n+1} - X_n \mid \mathcal{F}_n] \geq 0, \ a.s.; \tag{A.3}$$

$$\mathbb{E}[(X_{n+1} - X_n)^2 \mid \mathcal{F}_n] \geq v, \ a.s. \tag{A.4}$$

Then, for any $\gamma \in (0, 1/2)$ *and any* $\varepsilon \in (0, \frac{1-2\gamma}{4})$, *for all* n *sufficiently large,*

$$\mathbb{P}\left[\mathcal{L}_n^X(n^\gamma) \geq n^{\frac{1}{2}+\gamma+4\varepsilon}\right] \leq e^{-n^\varepsilon}, \ and \ \mathbb{E}\,\mathcal{L}_n^X(n^\gamma) \leq n^{\frac{1}{2}+\gamma+4\varepsilon}.$$

Remarks A.2. (i) A consequence of Lemma A.1 (taking $\gamma \approx 0$ and then $\gamma \approx 1/2$) and the Borel–Cantelli lemma is that, for every $\varepsilon > 0$ and every $B \in \mathbb{R}_+$,

$$\lim_{n\to\infty} n^{-1}\mathcal{L}_n^X\big(n^{(1/2)-\varepsilon}\big) = \lim_{n\to\infty} n^{-(1/2)-\varepsilon}\mathcal{L}_n^X(B) = 0, \text{ a.s.,} \tag{A.5}$$

which says, roughly speaking, that X is (i) for all but a vanishing proportion of time growing almost diffusively, and (ii) only of constant size for a roughly diffusively growing cumulative time. This 'diffusive' behaviour is (to within log factors) sharp in the case of one-dimensional symmetric simple random walk, and so no significantly stronger conclusion is valid in general.

(ii) If $M_0, M_1, \ldots$ is a martingale with uniformly bounded increments and $\mathbb{E}[(M_{n+1} - M_n)^2 \mid \mathcal{F}_n] \geq v$, a.s., then the hypotheses (A.2)–(A.4) are satisfied by the submartingale $X_n = |M_n|$. Thus Lemma A.1 can be viewed in relation to results on "martingale defocusing" [6, 23, 40]. In that context it is known [23] that, for example, one can construct martingales of this type for which $\mathbb{P}[|M_n| \leq B] \geq n^{-\beta}$ for $\beta \in (0, 1/2)$ that can be arbitrarily small, for *very specific* times n (indeed, the construction of the martingale in [23] depends on the specific choice of n). Our result, however, shows that in an averaged sense the majority of times conform with the intuition that such a martingale should be 'diffusive', since Lemma A.1 implies that $\sum_{i=0}^{n} \mathbb{P}[|M_i| \leq B] = O(n^{(1/2)+\varepsilon})$.

Proof of Lemma A.1. Let $x \in (0, \infty)$. Define $\lambda_0^x := 0$ and then, successively, for $k \in \mathbb{N}$,

$$\rho_k^x := \inf\{m \geq \lambda_{k-1}^x : X_m \geq 2x\}, \text{ and } \lambda_k^x := \inf\{m \geq \rho_k^x : X_m \leq x\},$$

where $\inf \emptyset := \infty$, as usual. To lighten the notation, we drop the superscript x from now on, and write simply λ_k, ρ_k. Then $0 = \lambda_0 \leq \rho_1 \leq \lambda_1 \leq \cdots$ are (possibly infinite) stopping times that partition the path of the process into *high-level* excursions over time intervals $[\rho_k, \lambda_k)$ and *low-level* excursions over time intervals $[\lambda_k, \rho_{k+1})$. Any visits by the process to $[0, x]$ must occur during low-level excursions.

We show that low-level excursions are, cumulatively, much shorter in duration than high-level excursions. First note that, by hypotheses (A.3)–(A.4),

$$\mathbb{E}[X_{n+1}^2 - X_n^2 \mid \mathcal{F}_n] = \mathbb{E}[(X_{n+1} - X_n)^2 \mid \mathcal{F}_n] + 2X_n\,\mathbb{E}[X_{n+1} - X_n \mid \mathcal{F}_n] \geq v, \text{ a.s.}$$

Consequently, Theorem 2.4.12 of [37, p. 45] shows that, for every $\ell \in \mathbb{Z}_+$, on $\{\lambda_k < \infty\}$,

$$\mathbb{P}\left[\max_{\lambda_k+\ell n \leq m \leq \lambda_k+(\ell+1)n} X_m \leq 2x \;\Big|\; \mathcal{F}_{\lambda_k+\ell n}\right] \leq \frac{(2x+B)^2}{vn},$$

by hypothesis (A.2). Hence there is a constant $r > 0$ (depending only on B and v) for which

$$\mathbb{P}[\rho_{k+1} - \lambda_k \geq (\ell+1)\lfloor rx^2 \rfloor \mid \mathcal{F}_{\lambda_k+\ell\lfloor rx^2 \rfloor}] \leq \frac{1}{e}\mathbb{1}\{\rho_{k+1} - \lambda_k \geq \ell\lfloor rx^2 \rfloor\},$$

from which we deduce that, on $\{\lambda_k < \infty\}$,

$$\mathbb{P}[\rho_{k+1} - \lambda_k \geq \ell rx^2 \mid \mathcal{F}_{\lambda_k}] \leq e^{-\ell}. \tag{A.6}$$

From (A.6) there exists $C < \infty$ (depending only on v and B) such that, for any $\varepsilon > 0$,

$$\mathbb{P}\left[\sum_{k=0}^{n-1} (\rho_{k+1} - \lambda_k)\mathbb{1}\{\lambda_k < \infty\} > Cn^{1+2\varepsilon}x^2\right] \leq \mathbb{P}\left[\bigcup_{k=0}^{n-1} \big\{(\rho_{k+1} - \lambda_k)\mathbb{1}\{\lambda_k < \infty\} > Cn^{2\varepsilon}x^2\big\}\right]$$

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$$\leq n\mathrm{e}^{-n^{2\varepsilon}}. \tag{A.7}$$

It follows from (A.6) that $\{\lambda_k < \infty\} \subseteq \{\rho_{k+1} < \infty\}$, up to sets of probability zero. Considering high-level excursions, Lemma 2.7.7 of [37, p. 76] shows that, for $k \in \mathbb{N}$, on $\{\lambda_{k-1} < \infty\}$,

$$\mathbb{P}\big[\lambda_k - \rho_k \geq 4vy^2 \mid \mathcal{F}_{\rho_k}\big] \geq \frac{1}{2}\,\mathbb{P}\Big[\max_{\rho_k \leq m \leq \lambda_k} X_m \geq 2(x+y) \,\Big|\, \mathcal{F}_{\rho_k}\Big]. \tag{A.8}$$

For $y > B$, let $\tau_k(y) := \inf\{m \geq \rho_k : X_m \geq 2(x+y)\}$; by the same argument as (A.6), one has $\{\lambda_{k-1} < \infty\} \subseteq \{\tau_k(y) < \infty\}$. Using hypothesis (A.3), the submartingale optional stopping theorem (e.g. Theorem 2.3.7 of [37, p. 33]) implies that, on $\{\lambda_{k-1} < \infty\}$,

$$\begin{aligned}
2x \leq X_{\rho_k} &\leq \mathbb{E}[X_{\lambda_k \wedge \tau_k(y)} \mid \mathcal{F}_{\rho_k}] \\
&\leq x\,\mathbb{P}[\lambda_k < \tau_k(y) \mid \mathcal{F}_{\rho_k}] + (2(x+y)+B)\,\mathbb{P}[\lambda_k > \tau_k(y) \mid \mathcal{F}_{\rho_k}],
\end{aligned}$$

since, by (A.2), $X_{\tau_k(y)} \leq 2(x+y) + B$ whenever $\tau_k(y) < \infty$. It follows that, on $\{\lambda_{k-1} < \infty\}$,

$$\mathbb{P}\Big[\max_{\rho_k \leq m \leq \lambda_k} X_m \geq 2(x+y) \,\Big|\, \mathcal{F}_{\rho_k}\Big] = \mathbb{P}[\tau_k(y) < \lambda_k \mid \mathcal{F}_{\rho_k}] \geq \frac{x}{x + 2y + B}. \tag{A.9}$$

Combining (A.8)–(A.9), we obtain

$$\mathbb{P}\big[\lambda_k - \rho_k \geq 4vy^2 \mid \mathcal{F}_{\rho_k}\big] \geq \frac{x}{2x + 4y + 2B}, \quad \text{on } \{\lambda_{k-1} < \infty\}. \tag{A.10}$$

Then, denoting $M_n := \max_{1 \leq k \leq n}(\lambda_k - \rho_k)$,

$$\begin{aligned}
\mathbb{P}\big[M_n < 4vy^2\big] &\leq \mathbb{E}\big[\mathbb{P}[M_n < 4vy^2 \mid \mathcal{F}_{\rho_n}]\mathbb{1}\{\lambda_{n-1} < \infty\}\big] \\
&= \mathbb{E}\big[\mathbb{P}[\lambda_n - \rho_n < 4vy^2 \mid \mathcal{F}_{\rho_n}]\mathbb{1}\{\lambda_{n-1} < \infty\}\mathbb{1}\{M_{n-1} < 4vy^2\}\big] \\
&\leq \Big(1 - \frac{x}{2x + 4y + 2B}\Big)\mathbb{P}\big[M_{n-1} < 4vy^2\big],
\end{aligned}$$

using (A.10) in the final step. This recursion leads to the bound

$$\mathbb{P}\Big[\sum_{k=1}^n (\lambda_k - \rho_k) \geq 4vy^2\Big] \geq \mathbb{P}\big[M_n \geq 4vy^2\big] \geq 1 - \Big(1 - \frac{x}{2x + 4y + 2B}\Big)^n. \tag{A.11}$$

Fix $\gamma \in (0, 1/2)$, $\varepsilon \in (0, \frac{1-2\gamma}{4})$, and $\alpha = \frac{1}{1-2\gamma+4\varepsilon}$. Choose $x = n^{2\alpha\gamma}$ and $y = n^\alpha/(2\sqrt{v})$. Then (A.7) and (A.11) combine to show that, for some $c > 0$ and all $n \in \mathbb{N}$ sufficiently large,

$$\mathbb{P}\Big[\sum_{k=0}^{n-1}(\rho_{k+1} - \lambda_k) \leq Cn^{1+3\alpha\varepsilon+4\alpha\gamma}, \sum_{k=1}^n (\lambda_k - \rho_k) \geq n^{2\alpha}\Big] \geq 1 - n\mathrm{e}^{-n^{3\alpha\varepsilon}} - \mathrm{e}^{-cn^{1-\alpha(1-2\gamma)}} \geq 1 - \mathrm{e}^{-n^{2\alpha\varepsilon}},$$

since $1 - \alpha(1-2\gamma) > 3\alpha\varepsilon$. But the latter event implies that $\mathcal{L}^X_{\lfloor n^{2\alpha}\rfloor}(n^{2\alpha\gamma}) \leq Cn^{1+3\alpha\varepsilon+4\alpha\gamma}$. After a change of variables, this means that, for all n large enough,

$$\mathbb{P}\Big[\mathcal{L}^X_n(n^\gamma) \leq Cn^{2\gamma+\frac{1}{2\alpha}+\frac{3\varepsilon}{2}}\Big] = \mathbb{P}\Big[\mathcal{L}^X_n(n^\gamma) \leq Cn^{\frac{1}{2}+\gamma+\frac{7\varepsilon}{2}}\Big] \geq 1 - \mathrm{e}^{-n^\varepsilon},$$

which yields the claimed result. $\qquad\square$

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