

On extinction and survival of the Bak-Sneppen model on arbitrary graphs

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Abstract

We study the discrete Bak-Sneppen model introduced in [5]. We extend their results as well as the non-triviality result of [21] for a finite segment of \mathbb{Z}^1 with the periodic boundary condition to a large class of graphs, by using coupling between the Bak-Sneppen model and the oriented percolation in a quadrant. This allows us to avoid dealing with the so-called avalanches, thus simplifying many arguments.

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1 Introduction and main results

A classical Bak-Sneppen model, introduced in [1, 2], is defined as follows. There are $N \geq 3$ points uniformly spread over a circumference. Each point at discrete time $t = 0, 1, 2, \dots$ possesses a value $\eta_i(t) \in [0, 1]$, $i \in [N] = \{1, 2, \dots, N\}$, called *fitness*; we assume that at time zero all fitnesses are distinct. At each time t we choose an index $j = j(t)$ such that

$$\eta_j(t) = \min_{i \in [N]} \eta_i(t)$$

and replace the fitness of site j as well as that of both its nearest neighbours on the circumference by three uniform $[0, 1]$ values, drawn independently. Due to continuity of these random variables, we will a.s. never have ties and j is thus defined uniquely. Formally,

$$\eta_i(t+1) = \begin{cases} \zeta_{i-j,t}, & \text{if } i \in \{j-1, j, j+1\}; \\ \eta_i(t), & \text{otherwise} \end{cases}$$

with the contention $N+1=1$, $1-1=N$, and where $\zeta_{-1,t}, \zeta_{0,t}, \zeta_{1,t}$ are i.i.d. uniform $[0, 1]$. A long-standing and still unproven conjecture says that as $N \rightarrow \infty$, the asymptotic distribution of $\eta_i(t)$ converges to a uniform distribution over the interval $[\eta_c, 1]$ where $\eta_c \approx 0.66$.

A *discrete* version of the Bak-Sneppen model introduced in [5] and later studied by [20, 21, 24] allows only two values of fitnesses, namely $\eta_i(t) \in \{0, 1\}$, and the fitnesses are drawn from *Bernoulli*(p) distribution. Of course, in this case, the minimum value may be no longer uniquely defined, and thus one draws $j(t)$ with equal probability for all those j such that $\eta_j(t) = \min_i \eta_i(t)$

(which is typically = 0, except the case when all η_i s are ones). It is easy to see that when $0 < p < 1$ we obtain an ergodic Markov chain with 2^N states. Let

$$\pi(\eta : \eta_i = 1) = \pi_{N,p}(\eta : \eta_i = 1) = \lim_{t \rightarrow \infty} \mathbb{P}(\eta_i(t) = 1)$$

be the probability¹ that site i is in state 1 under the stationary measure. We are interested in describing $\pi_{N,p}(\eta : \eta_i = 1)$ for very large N .

By comparison with e.g. branching process, it is quite easy to see that if $p \geq 2/3$ then $\pi_{N,p}(\eta : \eta_i = 1) \rightarrow 1$ as $N \rightarrow \infty$. It is much harder, however, to show that the same does not happen for *all* $p > 0$. Eventually, it was shown in [21] that for small enough p one has $\limsup_{N \rightarrow \infty} \pi_{N,p}(\eta : \eta_i = 1) < 1$; the proof in [21] is quite complicated. The purpose of the current paper is to provide an alternative, and possibly simpler, proof, based on the comparison with oriented percolation, as well as to extend the result to more general graphs.

Indeed, it is very easy to define the discrete Bak-Sneppen model on an arbitrary finite connected graph G without loops, by replacing one of the zero fitness by a Bernoulli variable together with fitnesses of all the neighbours in the graph (in the original problem, the underlying graph G is just a circle graph with N vertices). We can even extend this model to *infinite graphs with uniformly bounded degree*. To achieve this, we switch to a continuous-time model, assuming that each vertex x has its own independent Poisson clock with a rate of 1. When such a clock rings, one of the two things happens. If the fitness of x is 1, nothing happens (except when there are no zero fitnesses anywhere on the graph). If the fitness of x is 0, the fitness of x and each of its neighbours $y \sim x$ is replaced by an independent Bernoulli(p) variable. In case of a finite graph, the embedded Markov chain coincides with the Bak-Sneppen model described above. *For convenience, from now on we will work only with the continuous time version of the model even in case of finite graphs.*

While one of the first conjectures regarding the distribution of fitness in the continuous model remains open, there are numerous results for related models. In [12] the authors studied the model without geometric interactions. The Bak-Sneppen evolution model on a regular hypercubic lattice in high dimensions was studied in [7]. A local barycentric version of the process was studied in [17]. The hydrodynamic limit of the Bak-Sneppen branching diffusion was obtained in [9, 10]. For other recent and related results on the Bak-Sneppen model see [4, 6, 16, 11, 20] and references therein.

Let G be a finite connected graph without loops or repeated edges. Let d be the maximum degree of a vertex. For $x \in G$, we refer to $x \cup \{y : y \sim x\}$ as the *neighbourhood* of x .

Definition 1.1. *We say that (x_1, \dots, x_n) is a self-avoiding path, if $x_j \sim x_{j+1}$ for all $j = 1, \dots, n-1$, and all sites in the path are distinct. Further, we say that a self-avoiding path (x_1, \dots, x_n) is a chain, if the neighbourhoods of x_i and x_j do not intersect whenever $j - i \geq 3$.*

Let $\ell(G)$ be the length of the longest chain in G ; also, let $\ell_x(G)$ be the length of the longest chain containing x . Note that, for uniformly bounded-degree graphs, it is straightforward to obtain that $\ell_x(G)$ grows at least logarithmically in $|G|$. Also, observe that the shortest (in the sense of the graph distance) path between two distinct sites x and y is a chain.²

Let π denote the stationary distribution for our (continuous-time) Markov chain. We now state our first result, which says that ones become “locally improbable” if the parameter p is small.

Theorem 1.2. *For a given d there exists a function $g_d : [0, 1] \rightarrow [0, 1]$ with the property $g_d(p) \rightarrow 0$ as $p \downarrow 0$, such that for any finite connected graph G of maximal degree d and for any $x \in G$ we have (for the Bak-Sneppen with parameter p)*

$$\pi(\eta : \eta_x = 1) < g_d(p).$$

Next, we state a “concentration of measure” result, which says that, under some additional conditions, in the small- p regime, the stationary measure of the set of configurations with “too many” 1s converges to zero very quickly as the graph size goes to infinity. For a configuration η , let $\bar{\eta} = |G|^{-1} \sum_{x \in G} \eta_x$ be the proportion of 1s in the configuration.

¹By symmetry, it does not depend on i

²Indeed, if it were not a chain, it is not difficult to see that one would be able to construct a shorter path.

Theorem 1.3. Assume that G (of maximal degree d) can be covered by K chains, each of size at least ℓ . Then there exists a function $\tilde{g}_d : [0, 1] \rightarrow [0, 1]$ with the property $\tilde{g}_d(p) \rightarrow 0$ as $p \downarrow 0$, such that

$$\pi(\eta : \bar{\eta}_x \geq \tilde{g}_d(p)) \leq c_1 K e^{-c_2 \ell}.$$

While Theorem 1.3 applies to many graphs, e.g., finite parts of integer lattices, discrete tori, and so on, it does not apply to many other families of graphs (e.g. trees). It would be interesting to determine what can be done in the case of more general graphs. For now, we would like to leave it as an open problem.

Next, we present the complementary statement for the extinction of zeros, which generalizes the results obtained in [5, 24].

Theorem 1.4. Let d be the maximal degree of G . There is $q_0 = q_0(d) > \frac{1}{d+1}$ (see (22)) such that if $q := 1 - p \leq q_0$, then the zeros “become extinct”, in the sense that

$$\pi(\eta : \text{there are more than } k \text{ zeros}) \leq c_1 e^{-c_2 k} \tag{1}$$

for some positive constants c_1, c_2 depending on d .

Remark 1.5. A simple comparison of the number of zeros in the Bak-Sneppen model with the position of a random walk with negative drift on \mathbb{Z}_+ gives the result of Theorem 1.4 for $q < \frac{1}{d+1}$; however, the statement of Theorem 1.4 is non-trivial.

Remark 1.6. For one- ($d = 2$) and two-dimensional ($d = 4$) tori Theorem 1.4 gives

$$q_0 = \frac{7}{3} - \frac{2\sqrt{19}}{3} \sin\left(\frac{\arctan\left(\frac{9\sqrt{107}}{137}\right)}{3} + \frac{\pi}{6}\right) \approx 0.412 > \frac{1}{3}$$

and $q_0 \approx 0.2145549758 > 1/5$ respectively.

2 Proofs

From now on, we will denote $q := 1 - p$.

We start by recalling a general (but elementary) fact about non-reversible Markov chains, in discrete or continuous time. We denote by $P_{xy}^{(t)}$ the transition probability in time t from x to y , and let π denote the stationary measure. When dealing with the stationary measure of finite non-reversible Markov chains, the following global balance equation is often useful:

$$\sum_{\substack{x \in A \\ y \in A^c}} \pi_x P_{xy}^{(t)} = \sum_{\substack{x \in A \\ y \in A^c}} \pi_y P_{yx}^{(t)} \tag{2}$$

(we leave the above formula without proof; informally, in the stationary regime, the flow that goes out of A should be equal to the flow into A). One interesting implication of (2) is the following

Lemma 2.1. Assume that

$$\sum_{y \in A^c} P_{xy}^{(t)} \geq c > 0 \quad \text{for all } x \in A,$$

and

$$\sum_{x \in A} P_{yx}^{(t)} < \varepsilon \quad \text{for all } y \in A^c.$$

Then $\pi(A) < \varepsilon/c$.

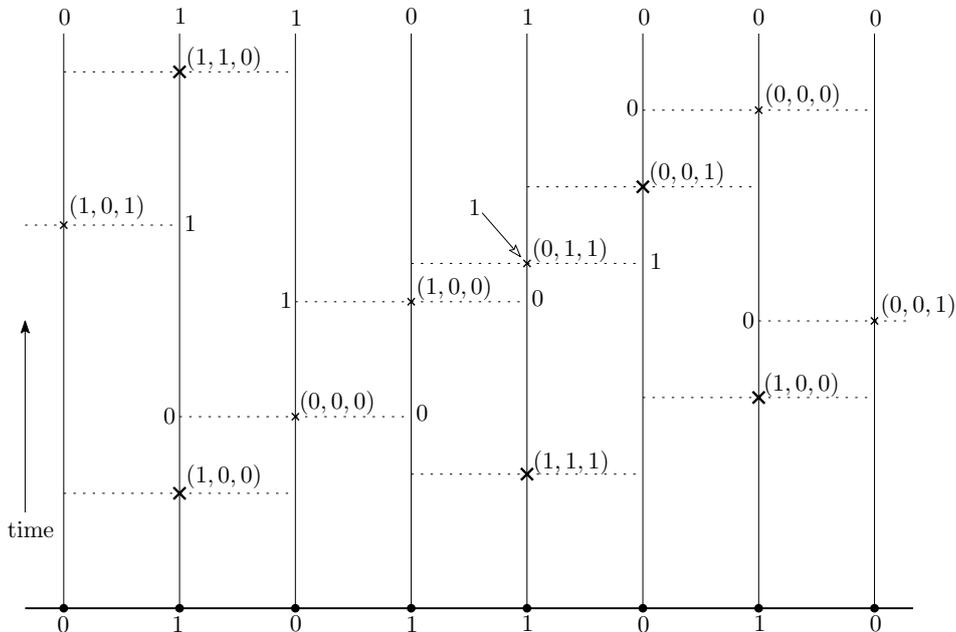


Figure 1: The graphical construction: each site has a marked Poisson clock of rate one, with the marks being the proposed neighbourhood changes. When the clock rings and the current state is 1, nothing happens; if the current state is 0, the changes take place. The rings that are “muted” (because the current state is 1) are indicated by larger crosses on the picture.

Proof. Indeed, using (2), we can write

$$\pi(A) = \sum_{x \in A} \pi_x \leq c^{-1} \sum_{\substack{x \in A \\ y \in A^c}} \pi_x P_{xy}^{(t)} = c^{-1} \sum_{\substack{x \in A \\ y \in A^c}} \pi_y P_{yx}^{(t)} < c^{-1} \varepsilon \sum_{y \in A^c} \pi_y \leq c^{-1} \varepsilon.$$

□

The above result formalizes the intuitively clear idea that if it is easy to leave the set A , but difficult to reach it when starting from outside, then the stationary probability of being in A should be small. This approach, for example, was employed by one of the authors in [23].

2.1 Graphical representation and blocks

Let us describe the process using the following graphical construction (see Figure 1, which illustrates the construction on \mathbb{Z} , though it applies to any graph). It is important to note that on any (possibly infinite) graph with uniformly bounded degree, this construction guarantees the existence of the process via a standard argument. Specifically, it suffices to establish the existence of the process on a short time interval $[0, \delta]$ for some small $\delta > 0$, and then extend it iteratively. If δ is small enough, it is straightforward to show that the cluster of sites with which a given site interacts – either directly or indirectly (as illustrated in the figure) – is almost surely finite. Within such finite clusters, the clock rings can be completely ordered, and the graphical construction therefore defines the process uniquely.

This construction applies both to infinite graphs with infinitely many zeros in the initial configuration and to finite graphs, but in the latter case, only up to the time when all states become ones; beyond that point, the process must be restarted.

Our method of analyzing the Bak-Sneppen model in the small- p regime is based on its comparison with the oriented percolation model. For that, we introduce “blocks”, as follows. First, for an $x \in G$ and $k \in \mathbb{Z}_+$ we say that $\{x\} \times ((k-1)L, kL]$ is an x -stick (more specifically, a level- k stick based at x). Two same-level sticks are *neighbours* if their bases are neighbours in G . Let A be a subset of the union of $\{x\}$ and its neighbours on G , thus $|A| \leq d+1$. We say that an x -stick is

- *A-good*, if all the clocks' rings on it only produced zero marks on the vertices of A (in particular, if there were no rings at all, then the stick is good).

Note that all sticks are good or not independently of each other, regardless of which neighbourhoods are included. Hence,

$$\mathbb{P}[\text{a stick is } A\text{-good}] \geq \sum_{k=0}^{\infty} \frac{L^k e^{-L}}{k!} q^{|A|k} = e^{-L(1-q^{|A|})}. \quad (3)$$

Further, if $x \sim y$ and $k \in \mathbb{Z}_+$, then $\{x, y\} \times ((k-1)L, kL]$ is a (level- k) block. We declare this block *nice*, if

- the clocks in both x - and y -sticks rang at least once;
- both x - and y -sticks are $\{x, y\}$ good;
- for all v such that $v \sim x$ we have v -stick is $\{x\}$ -good and for all w such that $w \sim y$ we have w -stick is $\{y\}$ -good (if there is v which is a neighbour of both x and y , we require v -stick to be $\{x, y\}$ -good).

Further, two same-level blocks $\{x, y\} \times ((k-1)L, kL]$ and $\{x', y'\} \times ((k-1)L, kL]$ are *separated*, if

$$\{z \in G : z \sim x \text{ or } z \sim y\} \cap \{z' \in G : z' \sim x' \text{ or } z' \sim y'\} = \emptyset$$

(i.e., their neighbourhoods do not intersect). An important observation is that if a set of same-level blocks has the property that any two blocks from there are separated, then these blocks are nice (or not) independently.

Note that if we know that (x_1, \dots, x_n) is a chain, this implies that the same-level blocks based on $\{x_{i-1}, x_i\}$ and $\{x_j, x_{j+1}\}$ are separated whenever $j - i \geq 3$.

Now, a straightforward but crucial observation is:

Proposition 2.2. *If a block is nice and has at least one zero at its bottom, then it will have both zeros at its top.*

To compute the probability of a block *being nice*, observe that

$$\mathbb{P}[x\text{-stick is good and had at least one ring}] = \sum_{k=1}^{\infty} \frac{L^k e^{-L}}{k!} q^{2k} = e^{-L(1-q^2)} - e^{-L}$$

and the same formula holds for y -stick. Since the sites of the block have at most $2(d-1)$ neighbours in G ,

$$\mathbb{P}[\text{a block is nice}] \geq e^{-2L(d-1)p} \times (e^{-L(1-q^2)} - e^{-L})^2. \quad (4)$$

Indeed, let n_2 (n_1 resp.) be the number of vertices which are adjacent to both x and y (to one of them, resp.) Then the probability that all those neighbouring sticks are good is

$$\begin{aligned} (e^{-Lp})^{n_1} \times (e^{-L(1-q^2)})^{n_2} &= \exp(-Lp((n_1 + 2n_2) - n_2p)) \\ &\geq \exp(-L(2n_2 + n_1)p) \geq \exp(-2L(d-1)p) \end{aligned}$$

since $n_1 + 2n_2 \leq 2(d-1)$ as there are at most $2(d-1)$ edges originating from x and y to the “outside world”.

We find that the above expression is maximised with respect to L when L equals

$$\widehat{L}(p, d) := \frac{1}{q^2} \ln \left(1 + \frac{q^2}{(q+d)p} \right) = (1 + O(p)) \ln \frac{1}{(d+1)p} \quad \text{as } p \rightarrow 0, \quad (5)$$

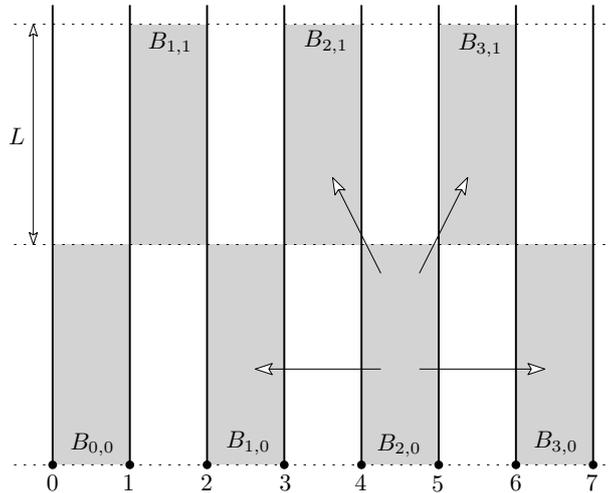


Figure 2: For the block $B_{2,0}$, blocks $B_{1,0}$ and $B_{3,0}$ are neighbors (shown by the horizontal arrows), and $B_{2,1}$ and $B_{3,1}$ are descendants (shown by the diagonal arrows).

(so that a smaller p corresponds to a larger L), then

$$\mathbb{P}[\text{a block is nice}] \geq 1 - 2(d+1)p \ln p^{-1} + O(p) \rightarrow 1 \quad \text{as } p \rightarrow 0. \quad (6)$$

Next, to build a coupling with an oriented percolation process, we consider blocks on top of a chain. One can naturally map a chain onto a (one-dimensional) segment; so we now work with the following notation. Let us divide the space-time $\mathbb{Z} \times \mathbb{R}_+$ into blocks $(B_{k,n}, k \in \mathbb{Z}, n \in \mathbb{Z}_+)$, where

$$B_{k,n} = \begin{cases} \{2k, 2k+1\} \times [nL, (n+1)L), & \text{if } n \text{ is even,} \\ \{2k-1, 2k\} \times [nL, (n+1)L), & \text{if } n \text{ is odd} \end{cases}$$

($L > 0$ is a parameter to be chosen later). For a given block $B_{k,n}$, we refer to $B_{k-1,n}$ and $B_{k+1,n}$ as its *neighbors*, and to

- $B_{k,n+1}$ and $B_{k+1,n+1}$, if n is even
- $B_{k-1,n+1}$ and $B_{k,n+1}$, if n is odd

as its *descendants*. See Figure 2.

Now, we observe that each horizontal line of blocks is a one-dependent field (i.e., the status of a block does not depend on the statuses of blocks which are not its neighbours). Therefore, we can use the stochastic domination technique of [19] (see also Theorem 2.1 in [22]). Indeed, Theorem 1.3 of [19] implies that, due to (6), the field of nice blocks dominates an independent Bernoulli field with success probability of at least $1 - 3\sqrt{2(d+1)p \ln p^{-1}}$ (the factor 3 is here for the sake of cleanness; anything strictly larger than $2\sqrt{2}$ would do). As a result, we have oriented percolation there, dominated by the original model, while the percolation model can be made as supercritical as we want (and so it is “well-behaved”, so there should be enough zeros everywhere).

Consequently, to work with connected finite graphs, we need to understand the oriented percolation process restricted to the segment $\{1, \dots, N\}$, with additional regenerations when it “dies out”. We will discuss this further in Section 2.3, but first let us present an alternative construction.

2.2 An alternative construction that does not use stochastic domination

In this section, we present an alternative definition of blocks, referred to as “4-blocks”, which also allows us to show that the Bak–Sneppen process dominates oriented percolation in the small- p regime. The key difference is that the states of the blocks are constructed to be independent, eliminating

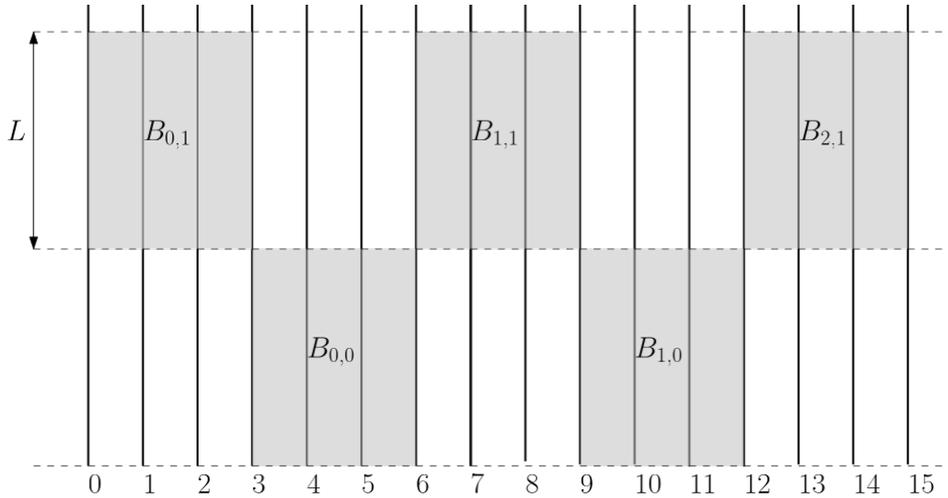


Figure 3: Block constructions; the blocks are shaded, $B = [4k, 4k + 3] \times [jL, (j + 1)L]$ for k, j such that $k + j$ is even.

the need for stochastic domination arguments. This comes at the cost of a smaller probability that a block is “nice”. We believe this construction is nevertheless of interest and worth including, even though – to keep the notation and arguments simpler – we do not use it in the remainder of the paper.

Assume that x_1, \dots, x_m is a chain and $m \geq 4$. A *4-block*, started at time t , is the union of four neighbouring sticks on that chain, i.e., $\{x_{k_0}, x_{k_0+1}, x_{k_0+2}, x_{k_0+3}\} \times [t, t + L]$, where $1 \leq k_0 \leq m - 3$. A 4-block is called *nice* if, within time $[t, t + L]$,

- for each $j = 0, 1, 2, 3$, the clock on $\{x_{k_0+j}\} \times [t, t + L]$ ringed at least once;
- x_{k_0} -stick is $\{x_{k_0}, x_{k_0+1}\}$ -good;
- x_{k_0+1} -stick is $\{x_{k_0}, x_{k_0+1}, x_{k_0+2}\}$ -good;
- x_{k_0+2} -stick is $\{x_{k_0+1}, x_{k_0+2}, x_{k_0+3}\}$ -good;
- x_{k_0+3} -stick is $\{x_{k_0+2}, x_{k_0+3}\}$ -good;
- for each site w adjacent to *some* of $x_{k_0}, x_{k_0+1}, x_{k_0+2}, x_{k_0+3}$, the w -stick is A_w -good, where A_w is the subset of those of $x_{k_0}, \dots, x_{k_0+3}$ which are adjacent to w ;
- the *propagation event* occurs: in the interval $[t, t + L]$, there is at least one time-wise increasing sequence of rings at $x_{k_0}, x_{k_0+1}, x_{k_0+2}$ (in this order), and at least one time-wise increasing sequence of rings at $x_{k_0+3}, x_{k_0+2}, x_{k_0+1}$ (see Figure 4).

Now, it is straightforward to observe that

- blocks are nice (or not) independently of each other;
- if at the beginning of the time of the block, one of its “extreme” two sites (i.e., x_{k_0}, x_{k_0+3}) had zero and the block is nice, then at the end *both* x_{k_0} and x_{k_0+3} will have zero at the end time.

This permits us to make a direct comparison with the oriented percolation, as shown in Figure 3.

Let us estimate the probability that the block is nice. A relatively easy lower bound is obtained by noting that if x_{k_0} and x_{k_0+3} each rings at least once in time $[0, L/3]$, while x_{k_0+1} and x_{k_0+2} each rings both in $[L/3, 2L/3]$ and in $[2L/3, L]$ (we set w.l.o.g. $t = 0$), then the block must be nice. This

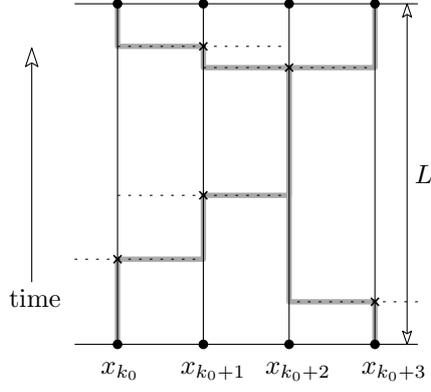


Figure 4: On the definition of the propagation event.

gives

$$\begin{aligned}
& \mathbb{P}[\text{a 4-block is nice}] \\
& \geq e^{-(4d-2)L} \cdot e^{(4d-6)Lq} \cdot \left[e^{Lq^2} - e^{2Lq^2/3} \right]^2 \cdot \left[e^{Lq^3} - 2e^{2Lq^3/3} + e^{Lq^3/3} \right]^2 \\
& = e^{\left(\frac{2q^3}{3} + \frac{4q^2}{3} + (4d-6)q - (4d-2) \right)L} \cdot \left(e^{\frac{Lq^2}{3}} - 1 \right)^2 \cdot \left(e^{\frac{Lq^3}{3}} - 1 \right)^4 =: \Theta(L, p).
\end{aligned} \tag{7}$$

Indeed, there are 4 vertices in a block; let n_j , $j = 1, 2, 3, 4$, denote the number of vertices in the neighbourhood of the block adjacent to exactly j vertices of the block, then

$$n_1 + 2n_2 + 3n_3 + 4n_4 \leq (d-1) + (d-2) + (d-2) + (d-1) = 4d-6$$

as this is the maximum possible number of edges originating from the block. Hence, the probability that all the sticks in the neighbourhood of a block are good is

$$\prod_{j=1}^4 \left[\sum_{k=0}^{\infty} \frac{L^k e^{-L}}{k!} \cdot q^{jk} \right]^{n_j} = \exp \left(-L [n_1(1-q) + n_2(1-q^2) + n_3(1-q^3) + n_4(1-q^4)] \right)$$

by (3). At the same time,

$$\begin{aligned}
& n_1(1-q) + n_2(1-q^2) + n_3(1-q^3) + n_4(1-q^4) = (1-q)(n_1 + 2n_2 + 3n_3 + 4n_4) \\
& - (1-q)^2(n_2 + n_3(2+q) + n_4(q^2+3)) \leq (1-q)(4d-6)
\end{aligned}$$

The probability that x_{k_0} rang at least once in $[0, L/3]$ and it was $\{x_{k_0}, x_{k_0+1}\}$ -good is

$$\sum_{k=1}^{\infty} \frac{L^k e^{-L}}{k!} \cdot q^{2k} \left[1 - \left(\frac{2}{3} \right)^k \right] = e^{-L} \left[e^{Lq^2} - e^{2Lq^2/3} \right]$$

(with the same expression for x_{k_0+3}) as conditionally on having k rings, their times are i.i.d. uniformly distributed in $[0, L]$. By similar arguments, the probability that x_{k_0+1} rang at least once in each $[L/3, 2L/3]$ and $[2L/3, L]$, and it was $\{x_{k_0}, x_{k_0+1}, x_{k_0+2}\}$ -good

$$\sum_{k=2}^{\infty} \frac{L^k e^{-L}}{k!} \cdot q^{3k} \left[1 - 2 \left(\frac{2}{3} \right)^k + \left(\frac{1}{3} \right)^k \right] = e^{-L} \left[e^{Lq^3} - 2e^{2Lq^3/3} + e^{Lq^3/3} \right]$$

(and the same expression for x_{k_0+2}).

To maximize the expression for $\Theta(L, p)$ in the regime $p \rightarrow 0$, we take

$$\tilde{L} = \tilde{L}(p, d) = 3 \ln \frac{1}{2pd},$$

which gives

$$\Theta(\tilde{L}, p) = 1 - 12(d+1)p \ln p^{-1} + O(p) \quad \text{as } p \rightarrow 0. \tag{8}$$

2.3 Oriented percolation on a strip

In this section, we study properties of the oriented (bond) percolation process restricted to a strip; i.e., the process lives on the sites with the even sum of the coordinates, and all sites with horizontal coordinate outside the finite segment $[0, 2N]$ are deleted. The oriented edges are in the upper diagonal directions, i.e., from (m, n) to $(m - 1, n + 1)$ and $(m + 1, n + 1)$. We are interested in the regime where the probability θ that a bond is open is close to 1.

Note that [14] deals only with site percolation, but due to the natural limits between site and bond percolation, this is not a problem (see, e.g., [8, Theorem 1.33] and its proof, in particular, equation (1.35)).

Let $\{x \rightarrow y\}$ be the event that x is connected to y by an oriented path. Let H_n be the set of sites (on the strip) with the vertical coordinate equal to n , then $|H_n| = N$ or $= N + 1$ depending on the parity of n . For $B \subset H_0$, we denote by $\xi_n^B \subset H_n$ the set of sites on level n connected to at least one site of B by an open (oriented) path. For $h \in (0, 1]$, we say that $A \in H_n$ is h -good if $|A|/|H_n| \geq h$.

Proposition 2.3. *Let $K_0 > 3$ be a fixed constant. Assume that $\theta > \frac{8}{9}$, and let $K \in [3, K_0]$. Then there exist constants $C = C(K_0) > 0$ and $C' = C'(K_0) > 0$ such that the following hold.*

(i) *For any $x \in H_0$ and $y \in H_{KN}$ we have $\mathbb{P}[x \rightarrow y] \geq 1 - C(1 - \theta)$.*

(ii) *Let $h \in (0, 1)$ be such that $\frac{h}{2} \ln \frac{1}{1-\theta} > (1 - h) \ln 3$. Then for any $x \in H_0$*

$$\mathbb{P}[\xi_{KN}^x \text{ is } (1 - h)\text{-good}] \geq 1 - C \exp\left(-\frac{1}{2} \ln \frac{1}{9(1-\theta)}\right) - C' N^2 \exp\left((1 - h)N \ln 3 - \frac{hN}{2} \ln \frac{1}{1-\theta}\right).$$

(iii) *For any $B \subset H_0$ with $|B| \geq N/2$, we have*

$$\mathbb{P}[\xi_{KN}^B \text{ is } (1 - h)\text{-good}] \geq 1 - 2C' N^2 \exp\left((1 - h)N \ln 3 - \frac{hN}{2} \ln \frac{1}{1-\theta}\right).$$

Remark 2.4. *While the results of Proposition 2.3 apply only for $\theta > 8/9$, we believe that they can be extended to all $\theta > \theta_{cr}$ where θ_{cr} is the critical percolation threshold for two-dimensional oriented bond percolation. This would give “the survival of zeros” in the Bak-Sneppen model on a circle graph of size $N \gg 1$ (one-dimensional tori) for $p < 0.0015$, since in this case $\Theta(14, 0.0015) = 0.73 \dots > 0.726 \dots$ (see (8)) which is a proven lower bound for θ_{cr} .*

Proof of Proposition 2.3. All these statements essentially follow from the usual “contour arguments” (but, still, some care has to be taken). The basic idea is depicted in Figure 5: the grey contours prevent the cluster of $\{x_1, x_2, x_3\}$ from reaching the upper part or prevent $\{y_1, y_2, y_3\}$ from being reached from below (for now, ignore what is happening around $\{y_4, y_5\}$ in that picture). These contours start and end on the boundary, and the first coordinate of the starting point has to be strictly less than the first coordinate of the endpoint. In these contours, the dashed arrows (i.e., those that decrease the first coordinate) are “free”, but the solid arrows (i.e., those that increase the first coordinate) have a cost of $(1 - \theta)$ elevated to the number of transversal arrows it cuts. See e.g. the discussion in Section 10 of [14]. In what follows, we refer to the number of steps between integer sites whose coordinate sums are odd as to the *length* of the contour.

Then, it becomes a standard contour-counting argument: the probability that there exists a “separating” closed contour is trivially bounded above by the expected number of such contours. When dealing with contours of a fixed size, say k , first, we need to figure out where that contour may start, to be able to cut out what has to be cut out, and note that on each step the path of the contour may take *at most three* possible directions. Each step that increases the first coordinate incurs a cost, and the contour must, overall, make progress in that direction. Therefore, the expected number of such contours is at most $3^k(1 - \theta)^{k/2}$ times the number of possible starting points. This argument carries through in a straightforward manner in the case of (i), so we omit the details.

When addressing parts (ii) and (iii), an additional difficulty arises: if $B \in H_0$ (or the “target set” is on the level KN) contains sites that are well-separated, then these sites may be enclosed by

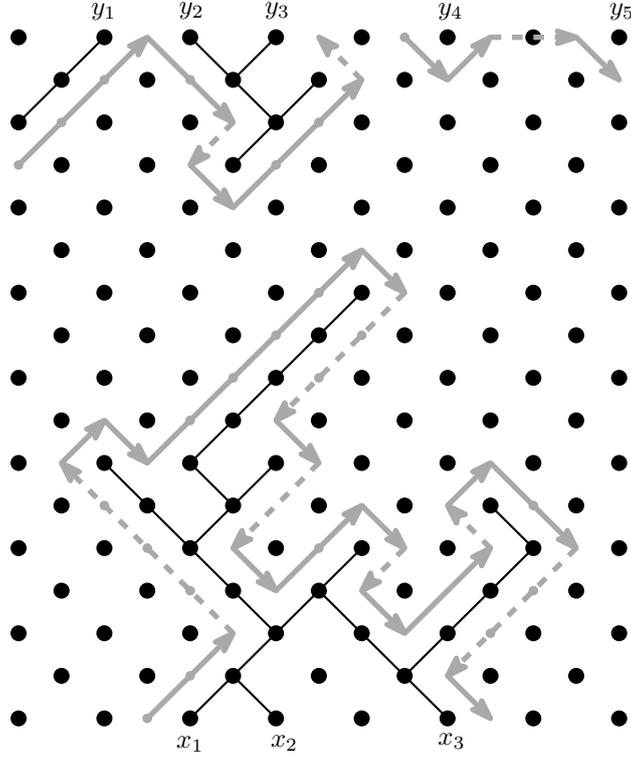


Figure 5: Oriented percolation on a strip in the upwards direction, and separating contours. Solid parts of the contours are “paid”, and dashed parts are “free”.

a collection of small disjoint contours, making them cumbersome to handle directly. To address this, we now modify the notion of a separating contour to include configurations such as the one shown in the top right part of Figure 5 (which cuts out the set $\{y_4, y_5\}$ from the rest of the picture): according to the modified definition, the separating contour is allowed to take a “free” horizontal step (at the top or bottom boundary) through the points which are outside of the target set, precisely as shown in the picture.

Now, let us prove (ii). We need to find an upper bound on the expected number of contours that separate $x \in H_0$ from a subset of H_{KN} of cardinality at least hN . First, if the contour is of size 1 (cutting away the site at the “corner” from the rest), then it has cost $(1 - \theta)$. Secondly, if the length $k \geq 2$ of such a contour is less than hN , the contour has to be in the lower part of the picture, cutting out x from the rest. There are at most k candidates for the initial point of the contour, so the expected number of cutting contours of length less than hN is bounded above by (recall that we assumed that $\theta \geq \frac{8}{9}$, so that $3(1 - \theta)^{1/2} < 1$)

$$(1 - \theta) + \sum_{k=2}^{hN} k 3^k (1 - \theta)^{k/2} \leq C(1 - \theta).$$

Next, we have to deal with longer contours, those of length at least hN . This contour can be a “special one” (with strictly horizontal “free” edges, as the one in the top right part of Figure 5), so we are no longer able to say that at least half of its steps must have a cost. However, the number of such horizontal “free” edges can be at most $(1 - h)N$. Therefore, such a contour has to contain at least $g_N(k) := \frac{hN}{2} + \frac{1}{2}(k - (1 - h)N)^+$ “paid” edges. Also, the number of possible starting points is now $O(N)$. We then obtain that the expected number of cutting contours of length at least hN is bounded above by

$$\begin{aligned} \sum_{k \geq hN}^{(1-h)N} cN 3^k (1 - \theta)^{g_N(k)} &= \sum_{k=hN}^{(1-h)N} cN 3^k (1 - \theta)^{\frac{hN}{2}} + \sum_{k > hN} cN 3^k (1 - \theta)^{\frac{k}{2} - \frac{k}{2}(1-2h)N} \\ &\leq c' N^2 \exp\left((1 - h)N \ln 3 - \frac{hN}{2} \ln \frac{1}{1 - \theta}\right). \end{aligned}$$

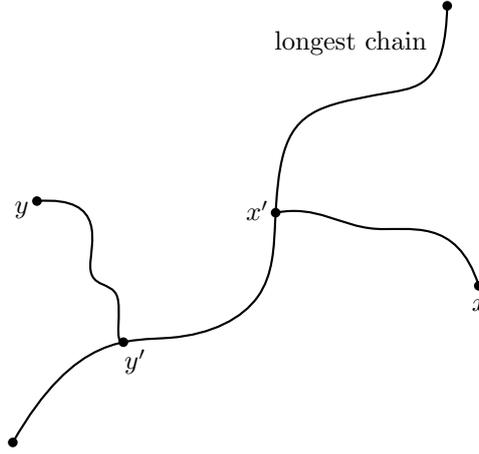


Figure 6: On the proof of Theorem 1.2: propagating a zero to x .

Gathering the pieces, we obtain the statement in (ii). The proof of (iii) is analogous. \square

3 Proofs of the main results

Proof of Theorems 1.2 and 1.3. In both cases, the idea is to observe that, as discussed in Sections 2.1–2.2, for small p and on a given chain, the model dominates an oriented percolation process; then, the results of Section 2.3 can be applied, also with the help of Lemma 2.1.

In the case of Theorem 1.2, we apply Lemma 2.1 with $A = \{\eta : \eta_x = 1\}$ and $t = 9\widehat{L}(p, d)\ell(G) + 1$. To do that, we need to prove that

- (i) if $\eta \notin A$ (i.e., $\eta_x = 0$) then with probability close to 1 after time $9\widehat{L}(p, d)\ell(G) + 1$ we will still have $\eta \notin A$;
- (ii) if $\eta \in A$ (i.e., $\eta_x = 1$) then with at least a constant probability after time $9\widehat{L}(p, d)\ell(G) + 1$ we will have $\eta \notin A$ (i.e., $\eta_x = 0$).

Both of these facts are consequences of Proposition 2.3 (I). Indeed, first, let x' be the closest to x site on the longest chain (see Figure 6), and let $d_1 \leq \ell(G)$ be the graph distance from x to x' ; let us write $d_1 = \gamma\ell(G)$ for some $\gamma \leq 1$. Then, in time $3\gamma\widehat{L}(p, d)\ell(G)$ we will have a 0 at x' with high probability by Proposition 2.3 (i); we will also have a 0 at x' with high probability at time $(9 - 3\gamma)\widehat{L}(p, d)\ell(G) + 1$ (again by Proposition 2.3 (i), but this time applied to the main chain). Then, we can send a 0 back to x in time $3\gamma\widehat{L}(p, d)\ell(G)$.

To show (ii), note that in the “worst case” (ones are everywhere) we will still have that, first, with a uniformly positive probability in time 1 some of these ones will change to zeros, say, at site y . Then (see Figure 6) a similar argument will work. Indeed, let y' be the closest site to y on the main chain and let $\gamma'\ell(G)$ (where $\gamma' \leq 1$) be the graph distance between y and y' . In time $3\gamma'\widehat{L}(p, d)\ell(G)$, zeros will propagate to the longest chain (i.e., to y'). Then, after waiting for $(9 - 3\gamma' - 3\gamma)\widehat{L}(p, d)\ell(G)$ units of time, with high probability we will have a zero at x' at time $1 + (9 - 3\gamma)\widehat{L}(p, d)\ell(G)$, and then we will have a zero at site x itself at time $1 + 9\widehat{L}(p, d)\ell(G)$.

The proof of Theorem 1.3 proceeds analogously: we apply Lemma 2.1 with Proposition 2.3 (ii)–(iii) on each of those “large chains”, and then use the union bound. \square

For the proof of Theorem 1.4 we need the following technical result. While it is probably known, for the sake of completeness, we present its proof as well.

Lemma 3.1. *Let X_n be an ergodic Markov chain on a finite state space \mathcal{S} with the stationary distribution π ; assume that 0 is one of the states. Suppose there is a non-negative function $f : \mathcal{S} \rightarrow \mathbb{R}$ such that*

$$\begin{aligned} f(0) &= 0; \\ \max_{x \in \mathcal{S}: p_{0x} > 0} f(x) &= c_0; \\ |f(X_{n+1}) - f(X_n)| &\leq C; \\ \mathbb{E}(f(X_{n+1}) - f(X_n) \mid X_n = x) &< -\varepsilon, \quad x \neq 0, \end{aligned} \tag{9}$$

for some $\varepsilon > 0$, $c_0 > 0$, and $C > 0$. Then for some $c_1, c_2 > 0$ depending on c_0, C, ε only,

$$\pi(A_k) \leq c_1 e^{-c_2 k}, \quad \text{where } A_k = \{x \in \mathcal{S} : f(x) > k\}, \quad k > 0. \tag{10}$$

Proof. By the cycle formula (see e.g. Proposition 1.14 in [18])

$$\pi(x) = \pi(0) \mathbb{E}_0 \sum_{n=0}^{T_0-1} 1_{X_n=x},$$

where \mathbb{E}_0 is the expectation under $X_0 = 0$ and $T_0 = \inf\{n \geq 1 : X_n = 0\}$. Hence

$$\pi(A_k) = \pi(0) \mathbb{E}_0 \sum_{n=0}^{T_0-1} 1_{X_n \in A_k}.$$

Let $b \geq k$ and define the stopping time

$$T_{0,b} = \inf\{n > 0 : X_n = 0 \text{ or } X_n \in A_b\} = \inf\{n > 0 : X_n = 0 \text{ or } X_n > b\}.$$

Under the assumption of the Lemma, see Theorem 2.17 in [15] or Lemma 4 in [24], it holds that $\mathbb{E}(e^{c_1 f(X_n)} \mid X_n = x) \leq e^{c_1 f(x)}$ for $x \neq 0$ for some $c_1 > 0$. Then $e^{hf(X_{n \wedge T})}$ is a supermartingale and by the optional stopping theorem

$$\mathbb{P}_x(X_{T_{0,b}} = 0) + \mathbb{P}_x(X_{T_{0,b}} \in A_b) e^{c_1 b} \leq \mathbb{E}_x e^{c_1 f(X_{T_{0,b}})} \leq \mathbb{E}_x e^{c_1 f(X_0)} = e^{c_1 f(x)} \leq e^{c_0 c_1}$$

for every x such that $p_{0x} > 0$. At the same time,

$$\mathbb{P}_0(X_{T_{0,b}} \in A_b) = \sum_{x \in \mathcal{S}} p_{0x} \mathbb{P}_x(X_{T_{0,b}} \in A_b) \leq \max_{x \in \mathcal{S}: p_{0x} > 0} \mathbb{P}_x(X_{T_{0,b}} \in A_b)$$

(note $\mathbb{P}_x(X_{T_{0,b}} \in A_b) = 0$ if $x = 0$) yielding

$$\mathbb{P}_0(X_{T_{0,b}} \in A_b) \leq e^{c_0 c_1} e^{-c_1 b}$$

Next, note that the conditions of the lemma imply³

$$\mathbb{P}(f(X_{n+1}) - f(X_n) \leq -\varepsilon/2 \mid X_n \neq 0) \geq \frac{\varepsilon}{2C - \varepsilon} =: \delta_1 > 0.$$

Let $h = \varepsilon/4$ and $B_i = [k + ih, k + (i+1)h)$, $i = 0, 1, 2, \dots$. Every time $f(X_n) \in B_i$, with probability at least δ we have $f(X_{n+1}) \leq k + ih - h$. Then, with probability at least $\delta_2 > 0$ (independent of k, i) the process X_n will reach 0 before $f(X_n)$ ever reaching level $k + ih$. Indeed, since $e^{c_1 f(X_n)}$ is a supermartingale, and denoting the stopping time $\tau = T_{0, k+ih}$ to be the moment when the walk either hits $[k + ih, \infty)$ or 0, by the optional stopping theorem we obtain

$$(1 - \mathbb{P}(f(X_\tau) \geq k)) + \mathbb{P}(f(X_\tau) \geq k) e^{c_1(k+ih)} \leq \mathbb{E}(e^{c_1 f(X_\tau)}) \leq e^{c_1 f(X_0)} \leq e^{c_1(k+(i-1)h)}$$

³Let $Z = f(X_{n+1}) - f(X_n)$. By Markov inequality, $\mathbb{P}(Z \leq -\varepsilon/2) = 1 - \mathbb{P}(C + Z > C - \varepsilon/2) \geq 1 - \frac{\mathbb{E}(C+Z)}{C - \varepsilon/2} \geq 1 - \frac{C - \varepsilon}{C - \varepsilon/2} = \frac{\varepsilon/2}{C - \varepsilon/2}$ since $\mathbb{E}Z \leq -\varepsilon$ and $Z + C \geq 0$.

when $f(X_0) \leq k + ih - h$; whence

$$\mathbb{P}(f(X_\tau) \geq k) \leq \frac{e^{c_1(k+(i-1)h)} - 1}{e^{c_1(k+ih)} - 1} \leq e^{-c_1h} =: \delta_2.$$

Hence, the number of such returns is uniformly bounded by a geometric random variable with finite expectation $(\delta_1\delta_2)^{-1}$. As a result,

$$\begin{aligned} \mathbb{E}_0 \sum_{n=0}^{T_0-1} 1_{X_n \in B_i} &\leq \mathbb{E}_0 \left(\sum_{n=0}^{T_0-1} 1_{X_n \in B_i} \mid \text{reach } A_{k+ih} \text{ before } 0 \right) \mathbb{P}(\text{reach } A_{k+ih} \text{ before } 0) \\ &\leq (\delta_1\delta_2)^{-1} \times e^{c_0c_1} e^{-c_1(k+ih)}. \end{aligned}$$

Therefore, we can write

$$\begin{aligned} \pi(A_k) &\leq \mathbb{E}_0 \sum_{n=0}^{T_0-1} 1_{X_n \in A_k} = \mathbb{E}_0 \sum_{i=0}^{\infty} \sum_{n=0}^{T_0-1} 1_{X_n \in B_i} \\ &\leq \sum_{i=0}^{\infty} (\delta_1\delta_2)^{-1} e^{c_0c_1} e^{-c_1(k+ih)} = \frac{(\delta_1\delta_2)^{-1} e^{c_0c_1}}{1 - e^{-c_1h}} e^{-c_1k}, \end{aligned}$$

thus concluding the proof of Lemma 3.1. \square

Now, we are ready to prove the “extinction of zeros” result.

Proof of Theorem 1.4. The proof will proceed in two parts. First, we present a relatively easy argument showing that there exists $q_0 > \frac{1}{d+1}$ such that the statement of the Theorem holds for all $q < q_0$. Second, we provide a more refined argument for constant degree graphs, yielding an improved estimate for q_0 .

Let us think of zeros as “particles”. A *type 1* particle is a particle which is “alone” (has no neighbours), and a *type 2* particle is a particle which has at least one neighbour. Let (with discrete time) $N_k^{(j)}$ be the number of particles of type j at time k , and denote $\zeta_k = (N_k^{(1)}, N_k^{(2)})$. Finally, let $N_k = N_k^{(1)} + N_k^{(2)}$ be the total number of zeros. We will now prove that there exists $q_0 > \frac{1}{d+1}$ such that for $q \in [\frac{1}{d+1}, q_0)$ and $(n_1, n_2) \neq (0, 0)$ we have

$$\Delta(n_1, n_2) := \mathbb{E}(f(\zeta_{k+1}) - f(\zeta_k) \mid \zeta_k = (n_1, n_2)) < -\varepsilon \quad (11)$$

for some $\varepsilon = \varepsilon(q) > 0$, where

$$f(n_1, n_2) = n_1 + (1-h)n_2 = (n_1 + n_2) - hn_2$$

with some $h = h(q) \in (0, 1)$ to be chosen later. Intuitively, in this function, type 2 particles have a smaller “weight” than those of type 1. Once we establish (11), this will imply what we want by Lemma 3.1.

From now on, denote by v the site where the chosen particle is located. The change in N_k can be the result of replacing a single zero (a *type 1* particle) or a non-single zero (*type 2*). Hence

$$\mathbb{E}(N_{k+1} - N_k \mid \zeta_k = (n_1, n_2)) \leq (d+1)q - 1 - \frac{n_2}{n_1 + n_2} \quad (12)$$

since when a zero particle at site v is replaced, a new zero is placed at v and each of its (up to d) neighboring sites with probability q . This replacement mechanism ensures that at least one zero is removed (specifically at v), and potentially more zeros are eliminated (whenever a type 2 particle is replaced).

First, observe that if we replace a type 1 particle, we can only possibly increase the number of particles of type 2;

$$\mathbb{E}(N_{k+1}^{(2)} - N_k^{(2)} \mid \zeta_k = (n_1, n_2), v \text{ is type 1}) \geq 2q^2 \quad (13)$$

(the probability that both v and its neighbour are zeros).

Now, suppose that we replace a type 2 particle. Let us get the lower bound for the change in the quantity of type 2 particles. When we replace v and all its neighbours by one, this can affect v , its neighbours, and the neighbours of these neighbours. Hence

$$\mathbb{E}(N_{k+1}^{(2)} - N_k^{(2)} \mid \zeta_k = (n_1, n_2), v \text{ is type 2}) \geq -(1 + d + d(d-1)) = -(1 + d^2). \quad (14)$$

Combining, we get

$$\begin{aligned} \mathbb{E}(N_{k+1}^{(2)} - N_k^{(2)} \mid \zeta_k = (n_1, n_2)) &\geq 2q^2 \frac{n_1}{n_1 + n_2} - (1 + d^2) \frac{n_2}{n_1 + n_2} \\ &= 2q^2 - (1 + d^2 + 2q^2) \frac{n_2}{n_1 + n_2}, \end{aligned}$$

so

$$\begin{aligned} \Delta(n_1, n_2) &= \mathbb{E}(N_{k+1} - N_k \mid \zeta_k = (n_1, n_2)) - h \mathbb{E}(N_{k+1}^{(2)} - N_k^{(2)} \mid \zeta_k = (n_1, n_2)) \\ &\leq (d+1)q - 1 - \frac{n_2}{n_1 + n_2} - h \left(2q^2 - (1 + d^2 + 2q^2) \frac{n_2}{n_1 + n_2} \right) \\ &= (d+1)q - 1 - 2hq^2 - \frac{n_2}{n_1 + n_2} (1 - h(1 + d^2 + 2q^2)). \end{aligned}$$

By setting $h = \frac{1}{d^2+3}$, we ensure that the expression in the square brackets is non-negative and thus $\Delta \leq (d+1)q - 1 - 2hq^2 < 0$ as long as

$$q < q_0 := \frac{1}{d+1 - 4h(d+1 + \sqrt{(d+1)^2 - 8h})^{-1}}$$

where trivially $q_0 > \frac{1}{d+1}$.

We now proceed with a sharper estimate of the change in (11) for the case of constant degree graphs. Recall that v denotes the particle to be updated at time k . We introduce the following notations for an update that occurred at time k , noting that the entire neighbourhood of v is updated.

- Let $M^{(k)}$ be the number of particles that were neighbours to v .
- Let $W^{(k)} = 1$ ($W^{(k)} = 1 - h$ respectively) if v was of type 1 (type 2 respectively), that is, $W^{(k)}$ is the weight of the updating particle in our Lyapunov function. Note that $W^{(k)} = 1$ if and only if $M^{(k)} = 0$.
- For $m = 1, 2$, let $X_m^{(k)}$ be the number of *new* type m particles created in the neighbourhood (including v itself) of v .
- We denote by $Z^{(k)}$ the number of “transitions” of particles outside of the immediate neighbourhood of v from type 2 to type 1 which were induced by the update (i.e., it is the number of the “external” particles which had a neighbour before the update, but became lonely after it); such transitions may only occur when v is type 2. The reverse transitions from type 1 to type 2 may occur as well; however, we do not care about those since such a transition decreases the “weight” of that particle, and we only need an *upper bound* in (11).

Then

$$\begin{aligned} f(\zeta_{k+1}) - f(\zeta_k) &\leq X_1^{(k)} + (1-h)X_2^{(k)} + hZ^{(k)} - (1-h)M^{(k)} - W^{(k)} \\ &= X_1^{(k)} + X_2^{(k)} - hX_2^{(k)} + hZ^{(k)} - (1-h)M^{(k)} - W^{(k)}. \end{aligned} \quad (15)$$

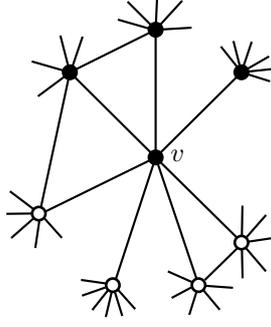


Figure 7: An example: there is a type 2 particle at site v of a graph of constant degree $d = 7$, and $m = 3$ neighbours of v contain particles.

We denote by $\mathbb{E}_{v,\eta}$ the expectation given that the current configuration is η and the update occurs at v . Whether v is type 1 or 2, the total (i.e., of both types) expected size of its progeny is $q(d+1)$:

$$\mathbb{E}_{v,\eta}(X_1^{(k)} + X_2^{(k)}) = q(d+1). \quad (16)$$

Next, the expected size of type-2 progeny is at least $dq^2 + q(1 - (1-q)^d)$ as the probability that a neighbour of v will be of type 2 after the update is at least q^2 , and the probability that v will be of type 2 after the update is exactly $q(1 - (1-q)^d)$. Hence, when v is type 1,

$$\mathbb{E}_{v,\eta}X_2^{(k)} \geq dq^2 + q(1 - (1-q)^d). \quad (17)$$

Also, when v is type 1, $M^{(k)} = 0$ and, as noted above, $Z^{(k)} = 0$. Hence, from (15)–(17) we have

$$\mathbb{E}(f(\zeta_{k+1}) - f(\zeta_k) \mid \zeta_k, v \text{ is type 1}) \leq q(d+1) - h(dq^2 + q(1 - (1-q)^d)) - 1. \quad (18)$$

Next, assume that v is type 2, and exactly $M^{(k)} = m(\geq 1)$ neighbours of v have particles (see e.g. Figure 7). Then

$$\mathbb{E}_{v,\eta}Z^{(k)} \leq m(1-q)(d-1) \quad (19)$$

(as type 2 to type 1 transitions can occur only at the sites next to a neighbouring particle of v). Assume also that

$$h < \frac{1}{q+d-dq}. \quad (20)$$

Then, (15) implies (recall that $m \geq 1$)

$$\begin{aligned} & \mathbb{E}(f(\zeta_{k+1}) - f(\zeta_k) \mid \zeta_k, v \text{ is type-2}) \\ & \leq q(d+1) - h(q^2d + q(1 - (1-q)^d)) + hm(1-q)(d-1) - (1-h)(m+1) \\ & = q(d+1) - h(q^2d + q(1 - (1-q)^d)) - m(1 - h(d+q-dq)) - 1 + h \end{aligned}$$

(using that $1 - h - hq - h(d-1) > 0$ by (20))

$$\begin{aligned} & \leq q(d+1) - h(q^2d + q(1 - (1-q)^d)) - (1 - h(d+q-dq)) - 1 + h \\ & = q(d+1) - 2 + h(1 + d(1-q-q^2) + q(1-q)^d). \end{aligned} \quad (21)$$

Now, $\mathbb{E}(f(\zeta_{k+1}) - f(\zeta_k) \mid \zeta_k = (n_1, n_2) \neq (0,0))$ is bounded from above by a linear combination of the right-hand sides of (18) and (21); therefore, we need to choose $h \in (0, 1)$ in such a way that both expressions are strictly negative. That is, we need that

$$h > T_1(q, d) := \frac{q(d+1) - 1}{dq^2 + q(1 - (1-q)^d)} \quad \text{and}$$

$$h < T_2(q, d) := \frac{2 - q(d + 1)}{1 + d(1 - q - q^2) + q(1 - q)^d},$$

together with (20).

Note that $T_1(\frac{1}{d+1}, d) = 0$, while $T_1(q, d)$ increases in q at least in the interval $[\frac{1}{d+1}, \frac{2}{d+1}]$; on the other hand, $T_2(q, d)$ and the right-hand side of (20) are all positive at $q = \frac{1}{d+1}$. Define

$$q_0 = q_0(d) = \sup \left\{ q > \frac{1}{d+1} : T_1(q, d) < \min \left(1, T_2(q, d), \frac{1}{q+d-qd} \right) \right\} \quad (22)$$

Then for each $q \in [\frac{1}{d+1}, q_0)$ there exists $h \in (0, 1)$ such that (11) holds. This concludes the proof of Theorem 1.4. \square

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