WELL POSEDNESS AND ASYMPTOTICS OF A FRACTIONAL GENERALISATION OF A SIS COMPARTMENT MODEL

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Abstract. In this paper we consider a nonlinear system of two differential equations with Caputo and Riemann–Liouville fractional derivatives. We prove the existence and uniqueness of a global solution and study its asymptotic behaviour. From a modelling perspective, our system constitutes a generalisation of the classical SIS epidemiological model, as well as of different fractional SIS models. Our results are applicable to all of these cases. In particular, we discuss how the introduction of fractional derivatives might impact the asymptotic behaviour of the solution, and compare the different epidemic outcomes resulting from the classical and fractional SIS models.

1. Introduction

In this work we aim at studying the fractional differential equation system

$$\begin{cases} {}^{C}D^{\gamma}_{0^+}I = \beta SI - \nu D^{\gamma-\alpha}_{0^+}I, & t>0, \\ {}^{C}D^{\gamma}_{0^+}S = -\beta SI + \nu D^{\gamma-\alpha}_{0^+}I, & t>0, \end{cases} \tag{1}$$

coupled with the initial conditions

$$I(0) = I_0 > 0, \qquad S(0) = S_0 > 0, \tag{2}$$

and parameters satisfying

$$\beta > 0,\ \nu > 0,\ 0 < \alpha \leq \gamma \leq 1. \tag{3}$$

Here I and S are real-valued functions of the real variable $t \in \mathbb{R}^+_0$,

$${}^{C}D^{\gamma}_{0^+}I(t) := \frac{1}{\Gamma(1-\gamma)}\frac{d}{dt}\int_0^t (t-s)^{-\gamma}[I(s) - I_0]\, ds, \quad t > 0,$$

is the Caputo fractional derivative of I of order $\gamma \in (0,1]$ and

$$D^{\gamma-\alpha}_{0^+}I(t) := \frac{1}{\Gamma(1-\gamma+\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\gamma+\alpha}I(s)\, ds, \quad t > 0,$$

is the Riemann–Liouville fractional derivative of order $\gamma - \alpha \in [0,1)$, with the understanding that $D^0_{0^+}I(t) = I(t)$ and ${}^{C}D^1_{0^+}I(t) = I'(t)$. For the sake of completeness, it is worth to mention also the Riemann–Liouville fractional integral of order $1-\gamma$ which is defined by

$$J^{1-\gamma}_{0^+}I(t) = \frac{1}{\Gamma(1-\gamma)}\int_0^t (t-s)^{-\gamma}I(s)\, ds, \quad t > 0.$$

For further details and properties of these fractional operators we refer the interested readers to the textbooks by Diethelm [7], Kilbas *et al.* [12] and Samko *et al.* [19].

Since fractional differential equations are often employed in modelling, it might be worth commenting on dimensional consistency. If (1) represents the variation of some physical quantities, then

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S and I must have the same dimensional unit M. In particular, if the variable t has dimensional unit s and α and γ are dimensionless, then β has units $\mathrm{M}^{-1}\mathrm{s}^{-\gamma}$ and ν has units $\mathrm{s}^{-\alpha}$. It is also important to mention that the system (1) constitutes a mathematical generalisation of different epidemiological models: when $\gamma = \alpha = 1$ we obtain the classical (ordinary) SIS model (see, e.g., [4, Section 10.1]; if $\gamma = 1$ and $\alpha < 1$, we obtain the fractional SIS model derived by Angstmann *et al.* in [2, Section 4.2] while if $\gamma = \alpha$, we obtain a version of the SIS model also considered by several authors (cf. [16, 9, 10]). Lastly, it might be worth to mention that Angstmann's approach was also considered by Wu *et al.* in [21] where they derive another fractional SIS model with births and deaths.

Here, in the first part of our paper, we focus on the problem of the existence of a unique global solution to the system (1), by passing to the integral formulation of the problem and using results from the Volterra integral equation theory. The required smoothness of the solution to the fractional differential system (1) results from the application of Theorem 2 below, proved by Brunner *et al.* in [6]. In the second part of our work we focus on the limits of the global solution to (1)–(3). Under certain assumptions, we show that, when $\alpha < \gamma$, the only two possible limits are $(I_\infty, S_\infty) = (0, N)$ and $(I_\infty, S_\infty) = (N, 0)$. while for $\alpha = \gamma$ the only two possible limits are $(I_\infty, S_\infty) = (0, N)$ and $(I_\infty, S_\infty) = (N - \nu/\beta, \nu/\beta)$. In Section 4 we discuss our results from an applied point of view, with reference to the classical SIS and SI models and complete our work by including, in Section 5, a collection of numerical solutions to (1)–(3), to highlight how the parameters α and β are crucial to the solution dynamics. We believe that the mathematical methods developed in this work may be useful if one is to consider even more general systems and possibly other fractional population models.

2. Well–posedness of the system

In this section we address the problem of the existence and uniqueness of a global solution to the fractional differential system (1)–(3). In addition to this, we prove that, for any initial condition (2), the solution is always positive and bounded. This is of particular interest in applications, for instance when particular cases of (1)–(3) represent a mathematical model. Indeed, systems as the SIS model in epidemiology and its variants make sense as long as both the variables S and I are non-negative. In the upcoming Theorem 1 we state the main result of this section. We will consider the function space $C^{1,\theta}(0,b]$, defined as follows: for $\theta < 1$ and $b > 0$, $C^{1,\theta}(0,b]$ is the space of continuously differentiable functions f on $(0,b]$ such that $|f'(t)| \le C_f t^{-\theta}$ for $t \in (0,b]$ ($C_f \in \mathbb{R}^+$) and $\lim_{t\to 0^+} f(t) = f(0)$. The space $C^{1,\theta}(0,b)$ ($b > 0$) is equivalently defined on the open interval $(0,b)$. Here, it is important to remark that, if $f \in C^{1,\theta}(0,b)$ then, for $0 < \delta < 1$, its Riemann–Liouville $D^{\delta}_{0^+} f$ and Caputo ${}^{C}D^{\delta}_{0^+} f$ fractional derivatives are well defined on $(0,b)$ (see, e.g., [1, Lemma 2.4]).

Theorem 1. *The fractional differential system* (1) *with initial conditions* (2) *and parameters as in* (3) *admits a unique global solution* $(I,S) \in C^1(0,+\infty) \times C^1(0,+\infty)$. *Moreover, for any* $T > 0$,

$(I,S) \in C^{1,\theta}(0,T] \times C^{1,\theta}(0,T]$, where $\theta \in [1-\alpha,1)$. Lastly, the solution satisfies the following bounds

$$0 < I(t), S(t) < N, \quad \forall t \geq 0,$$

where $N := I_0 + S_0$.

In order to prove the previous theorem, we will first present some intermediate results that describe some properties of the solutions to (1)–(3). First of all, in Lemma 1 together with Remark 1, we show that the fractional system can be uncoupled and studied as a single fractional equation. We also show that the solutions are always positive and bounded. In the subsequent Lemma 2 we establish, under certain conditions, an equivalence between the solutions of a Volterra integral equation and those of the fractional system (1)–(3). In this way, we can pass to an integral formulation of our fractional problem and apply results from the Volterra integral equation theory. We now formulate the aforementioned results.

Lemma 1. *Assume that the system* (1) *with initial condition* (2) *and parameters as in* (3) *admits a solution $(I,S) \in C^{1,\theta}(0,T) \times C^{1,\theta}(0,T)$ (here $T \in (0,\infty]$) with $\theta \in [1-\alpha,1)$. Then the following properties are satisfied:*

$$I(t) + S(t) = I_0 + S_0 =: N, \text{ for all } t \in [0,T), \tag{4}$$

$$0 < I(t) < N \text{ and } 0 < S(t) < N, \text{ for all } t \in [0,T). \tag{5}$$

Remark 1. *Observe that,* (4) *is equivalent to $S(t) = N - I(t)$ and the system* (1) *may therefore be reduced to the equivalent single differential equation*

$${}^{C}D^{\gamma}_{0+}I(t) = \beta(N - I(t))I(t) - \nu D^{\gamma-\alpha}_{0+}I(t), \; t > 0, \quad I(0) = I_0, \tag{6}$$

where $I_0 < N$.

Proof. (of Lemma 1) The property in equation (4) can be directly obtained by summing up the two equations in (1) and noticing that ${}^{C}D^{\gamma}_{0+}[I+S](t) = 0$ for all $t > 0$, with initial condition $I(0) + S(0) = I_0 + S_0$.

Let us now prove (5): The case $\alpha = 1$ forces $\gamma = 1$ whereupon we end up with the classical case. In this case, (5) is a known result, which follows by analysing the direction field of the ordinary differential equation (6).

Consider now the case $0 < \alpha < 1$. Assume first that $\alpha < \gamma < 1$. Suppose that $0 < t_* < T$ is the first zero of I. Then, $I(t) > 0$ on $[0, t_\star)$ and $I(t_\star) = 0$. Since $I \in C^{1,\theta}(0, t_\star]$, then its Riemann–Liouville fractional derivative and Caputo derivative may be represented, respectively, by (cf. the proof of [15, Theorem 3.1], with $k(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}$ therein)

$$D^{\gamma-\alpha}_{0+}I(t) = \frac{1}{\Gamma(1-\gamma+\alpha)}\left(\frac{I(t)}{t^{\gamma-\alpha}} + \int_0^t -(\gamma-\alpha)(t-s)^{\alpha-\gamma-1}[I(s) - I(t)]ds\right), \tag{7}$$

$${}^{C}D^{\gamma}_{0+}I(t) = \frac{1}{\Gamma(1-\gamma)}\left(\frac{I(t) - I_0}{t^{\gamma}} + \gamma\int_0^t (t-s)^{-\gamma-1}[I(t) - I(s)]ds\right), \tag{8}$$

for $t \in (0, t_*]$. Therefore, $D_{0+}^{\gamma-\alpha} I(t_\star) < 0$ and ${}^C D_{0+}^{\gamma} I(t_\star) < 0$, which is absurd because I solves (6). If $\gamma = 1$ or $\gamma = \alpha$, we still reach a contradiction due to the continuity of $\frac{dI}{dt}$ or ${}^C D_{0+}^{\alpha} I$ at $t_\star$, respectively. In conclusion, $I > 0$ on $[0, T)$.

Now, assume that $0 < t^\star < T$ is the first value such that $I(t^\star) = N > I_0$. Then, by (7)–(8), we get

$$D_{0+}^{\gamma-\alpha} I(t^\star) \geq \frac{1}{\Gamma(1-\gamma+\alpha)} \frac{N}{(t^\star)^{\gamma-\alpha}} > 0, \quad {}^C D_{0+}^{\gamma} I(t^\star) \geq \frac{1}{\Gamma(1-\gamma)} \frac{N - I_0}{(t^\star)^{\gamma}} > 0,$$

which again is absurd. We omit the details for when $\gamma = 1$ or $\gamma = \alpha$.

The proof is done. □

Lemma 2. *Consider the following nonlinear Volterra integral equation*

$$I(t) = I_0 + \beta \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} (N - I(s)) I(s) ds - \nu \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} I(s) ds, \ t > 0, \tag{9}$$

for $I_0 < N$. If $I \in C^{1,\theta}(0,T]$ ($T > 0$ and $\theta < 1$) solves (9)*, then it is also a solution of the differential equation* (6)*, hence $(I(t), N - I(t))$ is solution of the full system* (1) *with initial condition* (2)*.*

Proof. Since $I \in C^{1,\theta}(0,T]$, by continuity of I we have $I(0) = I_0$. Applying the Riemann–Liouville integral $J_{0+}^{1-\gamma}$ to both sides of (9) we get,

$$J_{0+}^{1-\gamma}[I - I_0](t) = \beta \int_0^t (N - I(s)) I(s) ds - \nu J_{0+}^{1-\gamma+\alpha} I(t),$$

and subsequently $\frac{d}{dt}$ one obtains (cf. [1, Lemma 2.4]) equation (6).

Now, if we define $S(t) = N - I(t)$ for $t \in [0, T]$, it follows that $S(0) = S_0$, and we immediately infer that (I, S) is the solution of system (1)–(3), which concludes the proof. □

As a consequence of Lemma 2, we can initially focus on proving the existence and uniqueness of a solution to (9) and then show that it also solves the fractional system (1)–(3). In order to accomplish it we first state the following result, due to Brunner *et al.* [6, 5].

Theorem 2. *(cf.* [5, Thm 3.1.24 and Remark 3.1.26(i)]*) Let $a > 0$ and consider the nonlinear Volterra equation*

$$u(t) = f(t) + \int_0^t K(t, s, u(s)) ds, \quad t \in [0, a]. \tag{10}$$

Assume that:

(K) the kernel $K(t, s, u)$ is continuously differentiable with respect to t, s, u for $t \in [0, a]$, $s \in [0, t)$, $u \in \mathbb{R}$, and there exists a real number $\theta \in (-\infty, 1)$ such that for $0 \leq s < t \leq a$, $u \in \mathbb{R}$, and for nonnegative integers i, j, k with $i + j + k \leq 1$, the following estimates hold:

$$\left| \left(\frac{\partial}{\partial t}\right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^j \left(\frac{\partial}{\partial u}\right)^k K(t, s, u) \right| \leq b_1(|u|)(t-s)^{-\theta-i}, \tag{11}$$

and

$$\left| \left(\frac{\partial}{\partial t}\right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^j \left(\frac{\partial}{\partial u}\right)^k \Big(K(t, s, u_1) - K(t, s, u_2)\Big) \right| \leq b_2(\max\{|u_1|, |u_2|\})|u_1 - u_2|(t-s)^{-\theta-i}, \tag{12}$$

where the functions $b_1, b_2 : [0,\infty) \to [0,\infty)$ *are monotonically increasing.*

(F) $f \in C^{1,\theta}(0,a]$.

Then, there exists a unique solution u *to* (10) *in* $C[0,a']$*, for a certain* $a' \le a$.

Moreover, if (10) *admits a solution* $u \in L^\infty(0,a)$*, then* u *belongs to the space* $C^{1,\theta}(0,a]$.

The above result can be immediately applied to our problem in order to show the existence and uniqueness of a local solution to the integral equation (6).

Proposition 1 (Local existence and uniqueness of solution)**.** *Let* $N > 0$ *and* $\theta \in [1-\alpha, 1)$*. Then the integral equation* (9) *with parameters as in* (3) *has a unique solution* $I \in C^{1,\theta}(0,a']$ *for some* $a' > 0$.

Proof. Let $a > 0$ and consider the kernel

$$K(t,s,u) = \beta\frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}(N-u)u - \nu\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}u, \quad t \in [0,a],\ s \in [0,t),\ u \in \mathbb{R}.$$

We now show that it satisfies the estimates (K) of Theorem 2, starting from the inequalities in (11):

- $i = j = k = 0$: We have,

$$\begin{aligned}|K(t,s,u)| &\le \beta\frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}(N+|u|)|u| + \nu\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|u| \\ &\le (t-s)^{-\theta}\left[\frac{\beta(t-s)^{\gamma+\theta-1}}{\Gamma(\gamma)}(N+|u|)|u| + \frac{\nu(t-s)^{\alpha+\theta-1}}{\Gamma(\alpha)}|u|\right] \\ &\le (t-s)^{-\theta}\left[\frac{\beta a^{\gamma+\theta-1}}{\Gamma(\gamma)}(N+|u|)|u| + \frac{\nu a^{\alpha+\theta-1}}{\Gamma(\alpha)}|u|\right].\end{aligned}$$

  So, with $\hat{b}_1(u) = \frac{\beta a^{\gamma+\theta-1}}{\Gamma(\gamma)}(N+u)u + \frac{\nu a^{\alpha+\theta-1}}{\Gamma(\alpha)}u$, $u \ge 0$, we get

$$|K(t,s,u)| \le (t-s)^{-\theta}\hat{b}_1(|u|).$$

- $i = 1,\ j = k = 0$: We have,

$$\begin{aligned}\left|\frac{\partial}{\partial t}K(t,s,u)\right| &= \left|\frac{\beta(\gamma-1)(t-s)^{\gamma-2}}{\Gamma(\gamma)}(N-u)u - \frac{\nu(\alpha-1)(t-s)^{\alpha-2}}{\Gamma(\alpha)}u\right| \\ &\le (t-s)^{-\theta-1}\left[\frac{\beta(1-\gamma)a^{\gamma+\theta-1}}{\Gamma(\gamma)}(N+|u|)|u| + \frac{\nu(1-\alpha)a^{\alpha+\theta-1}}{\Gamma(\alpha)}|u|\right] \\ &\le (t-s)^{-\theta-1}\hat{b}_1(|u|).\end{aligned}$$

- $i = j = 0,\ k = 1$: We have

$$\begin{aligned}\left|\frac{\partial}{\partial u}K(t,s,u)\right| &= \left|\beta\frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}(N-2u) - \nu\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right| \\ &\le (t-s)^{-\theta}\left[\frac{\beta(t-s)^{\gamma+\theta-1}}{\Gamma(\gamma)}(N+2|u|) + \frac{\nu(t-s)^{\alpha+\theta-1}}{\Gamma(\alpha)}\right] \\ &\le (t-s)^{-\theta}\left[\frac{\beta a^{\gamma+\theta-1}}{\Gamma(\gamma)}(N+2|u|) + \frac{\nu a^{\alpha+\theta-1}}{\Gamma(\alpha)}\right].\end{aligned}$$

  So, with $\bar{b}_1(u) = \frac{\beta a^{\gamma+\theta-1}}{\Gamma(\gamma)}(N+2u) + \frac{\nu a^{\alpha+\theta-1}}{\Gamma(\alpha)}$, $u \ge 0$, we get

$$\left|\frac{\partial}{\partial u}K(t,s,u)\right| \le (t-s)^{-\theta}\bar{b}_1(u).$$

The case $i = k = 0$, $j = 1$ is trivially verified since $\frac{\partial}{\partial t}K + \frac{\partial}{\partial s}K = 0$. By defining $b_1(u) := \max\{\hat{b}_1(u), \bar{b}_1(u)\}$ for $u \geq 0$, then inequalities in (11) are proved for any $\theta \in [1-\alpha, 1)$.

We now show that (12) holds for $i = j = k = 0$, leaving the other cases to the reader. We have

$$\begin{aligned}
&|K(t,s,u_1) - K(t,s,u_2)| \\
&= \left| \beta \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} [N(u_1 - u_2) + u_1^2 - u_2^2] + \nu \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (u_2 - u_1) \right| \\
&\leq (t-s)^{-\theta} |u_1 - u_2| \left| \frac{\beta(t-s)^{\gamma+\theta-1}}{\Gamma(\gamma)} (N + |u_1 + u_2|) + \frac{\nu(t-s)^{\alpha+\theta-1}}{\Gamma(\alpha)} \right| \\
&\leq (t-s)^{-\theta} |u_1 - u_2| \left[\frac{\beta a^{\gamma+\theta-1}}{\Gamma(\gamma)} (N + 2\max\{|u_1|, |u_2|\}) + \frac{\nu a^{\alpha+\theta-1}}{\Gamma(\alpha)} \right],
\end{aligned}$$

hence (12) is satisfied by defining $b_2(u) = \frac{\beta a^{\gamma+\theta-1}}{\Gamma(\gamma)}(N + 2u) + \frac{\nu a^{\alpha+\theta-1}}{\Gamma(\alpha)}$, $u \geq 0$. Lastly, assumption (F) is trivially verified with $f(t) := I_0$ and the proof is concluded. □

We may finally prove Theorem 1 stated at the beginning of this section.

Proof. (Theorem 1) Proposition 1 guarantees the existence and uniqueness of a solution I to (9), which belongs to $C^{1,\theta}(0, a']$ for a certain $a' > 0$. Using classical arguments, see e.g. [5, Section 3.1.1], it is possible to extend such a solution to its maximal interval $[0, T)$. Then (cf. Definition 3.1.3 in [5]), either $T = +\infty$ and the solution is globally defined, or $T < +\infty$ with $\lim_{t\to T^-} \sup |I(t)| = +\infty$. Let us consider the latter case. For $0 < t < \tau < T$, we get

$$\begin{aligned}
&|I(t) - I(\tau)| \\
&\quad= \Bigg| \beta \int_0^t \frac{(t-s)^{\gamma-1} - (\tau-s)^{\gamma-1}}{\Gamma(\gamma)} (N - I(s)) I(s) ds - \nu \int_0^t \frac{(t-s)^{\alpha-1} - (\tau-s)^{\alpha-1}}{\Gamma(\alpha)} I(s) ds \\
&\qquad - \beta \int_t^\tau \frac{(\tau-s)^{\gamma-1}}{\Gamma(\gamma)} (N - I(s)) I(s) ds + \nu \int_t^\tau \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} I(s) ds \Bigg| \\
&\quad\leq \beta \int_0^t \frac{(t-s)^{\gamma-1} - (\tau-s)^{\gamma-1}}{\Gamma(\gamma)} |(N - I(s)) I(s)| \, ds + \nu \int_0^t \frac{(t-s)^{\alpha-1} - (\tau-s)^{\alpha-1}}{\Gamma(\alpha)} |I(s)| ds \\
&\qquad + \beta \int_t^\tau \frac{(\tau-s)^{\gamma-1}}{\Gamma(\gamma)} |(N - I(s)) I(s)| ds + \nu \int_t^\tau \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} |I(s)| ds \\
&\quad\leq \frac{\beta N^2}{\Gamma(\gamma)} \int_0^t \left((t-s)^{\gamma-1} - (\tau-s)^{\gamma-1} \right) ds + \frac{\nu N}{\Gamma(\alpha)} \int_0^t \left((t-s)^{\alpha-1} - (\tau-s)^{\alpha-1} \right) ds \\
&\qquad + \frac{\beta N^2}{\Gamma(\gamma)} \int_t^\tau (\tau-s)^{\gamma-1} ds + \frac{\nu N}{\Gamma(\alpha)} \int_t^\tau (\tau-s)^{\alpha-1} ds \\
&\qquad = \frac{\beta N^2}{\Gamma(\gamma+1)} \left(2(\tau-t)^\gamma - \tau^\gamma + t^\gamma \right) + \frac{\nu N}{\Gamma(\alpha+1)} \left(2(\tau-t)^\alpha - \tau^\alpha + t^\alpha \right), \quad (13)
\end{aligned}$$

where the last inequality follows upon using Lemma 1. Consider any sequence $\{t_n\} \subset (0, T)$ such that $t_n < t_{n+1}$ and $\lim_{n\to\infty} t_n = T$. From (13) we deduce that the sequence $\{I_n = I(t_n)\}_{n\in\mathbb{N}} \subset \mathbb{R}$ is a Cauchy sequence. Therefore, the limit $\lim_{t\to T^-} I(t)$ exists and $\lim_{t\to T^-} \sup |I(t)| < +\infty$, which implies that (9) admits a unique global solution.

In order to prove existence and uniqueness to the fractional problem (1)–(3), it remains to ensure that the global (continuous) solution I of (9) is differentiable in $(0,+\infty)$. By Theorem 2, $I \in C^{1,\theta}(0,T]$ for any $T>0$, and the arbitrariness of $T>0$ implies that $I \in C^1(0,+\infty)$.

Finally, by Lemma 1 the lower and upper bounds for I hold globally, which finishes the proof. □

We conclude this section with an observation about the solution bounds for a particular case of (1)–(3), namely, for when $\alpha=\gamma$.

Remark 2. *Consider $\gamma=\alpha$. Then* (6) *becomes*

$$ {}^C D^{\gamma}_{0+} I(t) = \beta\left(N-\frac{\nu}{\beta}-I(t)\right)I(t),\ t>0, \quad I(0)=I_0. \tag{14}$$

Suppose that $N-\frac{\nu}{\beta}>0$. If $I_0<N-\frac{\nu}{\beta}$ then, as in the proof of Lemma 1, we conclude that $0<I(t)<N-\frac{\nu}{\beta},\ t>0$. Analogously, if $I_0>N-\frac{\nu}{\beta}$, then $N-\frac{\nu}{\beta}<I(t)<N,\ t>0$. Finally, if $I_0=N-\frac{\nu}{\beta}$, then by the uniqueness of solution we conclude that $I(t)=N-\frac{\nu}{\beta}$ for all $t\geq 0$.

3. Long time behavior of the solutions of (1)

In the previous section we showed that the system (1)–(3) admits a unique and global solution. In particular, for any positive initial data, such solution is positive and bounded. In this section we present our findings regarding to what the limit $\lim_{t\to\infty} I(t)$ is, where I is the aforementioned solution. For the benefit of the reader we treat the two cases, namely $\gamma=\alpha$ and $\gamma>\alpha$, separately.

3.1. Case $\gamma=\alpha$.

In this section we prove the following result.

Theorem 3. *Let I be the unique and global solution of* (6)*, with $0<\alpha=\gamma\leq 1$. Then there exists $L=\lim_{t\to\infty} I(t)$. In particular, if $N-\nu/\beta>0$ then $L=N-\nu/\beta$, and $L=0$ otherwise.*

Proof. When $\alpha=\gamma$, equation (6) reduces to the Caputo differential equation (14). We recall that the fractional linear initial value problem

$$ {}^C D^{\alpha}_{0+} y(t) = \lambda y(t) + k, \quad \lambda, k \in \mathbb{R}, \quad y(0)=y_0,$$

has solution given by (cf. [7, Theorem 7.2])

$$ y(t) = y_0 E_\alpha(\lambda t^\alpha) + k\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha)ds,$$

where $E_{\alpha,\beta}(x)=\sum_{k=0}^{\infty}\frac{x^\alpha}{\Gamma(\alpha k+\beta)}$ and $E_\alpha(x):=E_{\alpha,1}(x)$ ($\alpha,\beta>0$ and $x\in\mathbb{R}$) are the Mittag-Leffler functions of two and one parameters, respectively. By using some identities for these functions (cf. equality (4.4.4) and then (4.2.3) in [11]), the above solution can be expressed as

$$ y(t) = \left(y_0+\frac{k}{\lambda}\right)E_\alpha(\lambda t^\alpha) - \frac{k}{\lambda}, \quad \lambda\neq 0. \tag{15}$$

We will now proceed to prove the theorem accordingly with the sign of $N-\frac{\nu}{\beta}$:

(1) $N-\frac{\nu}{\beta}<0$: by (14), using (5) from Lemma 1, we obtain,

$$ {}^C D^{\alpha}_{0+} I(t) < \beta\left(N-\frac{\nu}{\beta}\right)I(t),$$

and, by applying the comparison result for Caputo differential equations in [14, Theorem 1],

$$I(t) < I_0 E_\alpha \left(\beta \left(N - \frac{\nu}{\beta}\right) t^\alpha\right), \quad t > 0.$$

Since $\lim_{t\to\infty} E_\alpha \left(\beta \left(N - \frac{\nu}{\beta}\right) t^\alpha\right) = 0$ (see, e.g. [17]) and $I(t) > 0$ for all $t > 0$ we conclude that $\lim_{t\to\infty} I(t) = 0$.

(2) $N - \frac{\nu}{\beta} > 0$. We consider this case by analysing the different possibilities for the initial value I_0.

When the prescribed initial condition is $I_0 = N - \frac{\nu}{\beta}$ then the result is trivial, since $I(t) = N - \frac{\nu}{\beta}$ is the constant solution (cf. Remark 2).

When $I_0 \in (0, N - \frac{\nu}{\beta})$, by Remark 2, $I(t) < N - \frac{\nu}{\beta}$, and

$${}^C D^\alpha_{0^+} I(t) = \beta \left(N - \frac{\nu}{\beta} - I(t)\right) I(t) > 0, \quad t > 0,$$

which implies $I(t) > I_0$. In particular, since

$${}^C D^\alpha_{0^+} I(t) \geq \beta \left(N - \frac{\nu}{\beta} - I(t)\right) I_0,$$

by the Caputo comparison result used in the previous case,

$$I(t) \geq \left(I_0 + \frac{\nu}{\beta} - N\right) E_\alpha(-\beta I_0 t^\alpha) + N - \frac{\nu}{\beta}.$$

Since $\lim_{t\to\infty} E_\alpha(-\beta I_0 t^\alpha) = 0$ and $I(t) < N - \frac{\nu}{\beta}$ we conclude that $\lim_{t\to\infty} I(t) = N - \frac{\nu}{\beta}$.

Similarly, we now prove that the same limit holds for the initial condition $I_0 \in \left(N - \frac{\nu}{\beta}, N\right)$. In this case, $I(t) < N$, and

$${}^C D^\alpha_{0^+} I(t) < \beta \left(N - \frac{\nu}{\beta} - I(t)\right) N,$$

which implies

$$I(t) \leq \left(I_0 + \frac{\nu}{\beta} - N\right) E_\alpha \left(-\beta N t^\alpha\right) + N - \frac{\nu}{\beta}$$

Since, by Remark 2, $I(t) > N - \nu/\beta$, also in this case, by passing to the limit, we obtain $\lim_{t\to\infty} I(t) = N - \frac{\nu}{\beta}$.

(3) It only remains to consider the case $N - \frac{\nu}{\beta} = 0$. Then (14) is

$${}^C D^\gamma_{0^+} I(t) = -\beta I^2(t), \ t > 0, \quad I(0) = I_0.$$

Here we want to prove that $\forall \varepsilon > 0 \ \exists T_\varepsilon > 0$ such that $|I(t)| < \varepsilon$. For any $A > 0$ with $A < I_0$, let us introduce the auxiliary problem

$${}^C D^\gamma_{0^+} \hat{I}(t) = \beta(A - \hat{I}(t))\hat{I}(t), \ t > 0, \quad \hat{I}(0) = I_0.$$

From the previous step, we know that $\hat{I}$ admits limit A, i.e. $\forall \hat{\varepsilon} > 0$ there exists $\hat{T} > 0$ such that $|\hat{I}(t) - A| < \hat{\varepsilon}$ for all $t > \hat{T}$. Moreover, by the same comparison theorem used above, $I(t) < \hat{I}(t)$. Since $|I(t)| < |\hat{I}(t) - A| + |A|$, the conclusion follows by considering, for all $\varepsilon > 0$ the auxiliary problem with $A = \varepsilon/2$ and $T_\varepsilon > 0$ such that $|\hat{I}(t) - \varepsilon/2| < \varepsilon/2$ for all $t > T_\epsilon$, which exists as $\hat{I}(t)$ admits limit.

The proof in complete. □

3.2. **Case $\gamma > \alpha$.** We start by showing that, if I is the solution of (6) and the limit of I at infinity exists, then it must be equal to 0 or N. To do that, we make use of the Laplace transform, which we now recall.

For a function $f : [0, \infty) \to \mathbb{R}$ its Laplace transform, denoted by $\hat{f}$, is defined by

$$\hat{f}(s) := \int_0^\infty f(r)e^{-sr}dr,$$

for the values of s such that the integral converges. It is readily seen that, if f is a bounded function then $\hat{f}$ is well-defined for $s > 0$.

Now, suppose that I solves (6) and that $L = \lim_{t\to\infty} I(t)$. Then we recall that I satisfies the integral equation (9). Applying the Laplace transform on both sides of (9) and using [19, Theorem 7.2], we obtain

$$\hat{I}(s) = \frac{I_0}{s} + \beta s^{-\gamma}\widehat{(N-I)I}(s) - \nu s^{-\alpha}\hat{I}(s). \tag{16}$$

By the Final Value Theorem (cf. [8, Theorem 34.3]) we know that $\lim_{t\to\infty} I(t) = \lim_{s\to 0^+} s\hat{I}(s)$ and $\lim_{t\to\infty}[(N-I)I](t) = \lim_{s\to 0^+} s\widehat{(N-I)I}(s)$. Therefore, multiplying in (16) by $s^{\gamma+1}$ and then taking the limits as $s \to 0^+$, we get

$$(N-L)L = 0,$$

i.e., $L = 0$ or $L = N$. We formally write this in the following

Theorem 4. *Assume that* $0 \le \alpha < \gamma \le 1$ *and* I *solves* (6) *with* $L = \lim_{t\to\infty} I(t)$*. Then,* $L \in \{0, N\}$.

Theorem 1 states that, for each $T > 0$, the solution I of (6) satisfies $|I'(t)| \le c_T t^{-\theta}$ for $t \in (0, T]$, where c_T is a constant that might depend on T. If we assume the existence of a constant $C > 0$ such that

$$|I'(t)| \le Ct^{-\theta}, \quad t > 0, \tag{17}$$

we can actually show that $\lim_{t\to\infty} I(t)$ exists. That is the content of the next result.

Theorem 5. *Let* I *be the unique and global solution of* (6)*, with* $0 \le \alpha < \gamma \le 1$*. Assume that* I *satisfies* (17)*, for* $\max\{1-\alpha, 1-\gamma+\alpha\} < \theta < 1$*. Then the limit* $\lim_{t\to\infty} I(t) := L$ *exists with* $L \in \{0, N\}$.

Proof. By applying the identity between the Riemann–Liouville and Caputo derivatives [7, Lemma 3.4], equation (6) can be expressed as

$${}^C D^{\gamma}_{0+} I(t) + \nu \, {}^C D^{\gamma-\alpha}_{0+} I(t) + \nu \frac{I_0}{\Gamma(\alpha-\gamma+1)} t^{\alpha-\gamma} = \beta(N - I(t))I(t). \tag{18}$$

Therefore, if $\gamma < 1$, by using [12, Theorem 2.1], we get

$$\begin{aligned}\beta\left|(N-I(t))I(t)\right| &= \left|\int_0^t \frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)}I'(s)ds + \int_0^t \frac{\nu(t-s)^{\alpha-\gamma}}{\Gamma(1-\gamma+\alpha)}I'(s)ds + \frac{\nu I_0 t^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}\right| \\ &\le \int_0^t \frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)}\left|I'(s)\right|ds + \int_0^t \frac{\nu(t-s)^{\alpha-\gamma}}{\Gamma(1-\gamma+\alpha)}\left|I'(s)\right|ds + \frac{\nu I_0 t^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} \\ &\le \int_0^t \frac{C(t-s)^{-\gamma}}{\Gamma(1-\gamma)}s^{-\theta}ds + \int_0^t \frac{\nu C(t-s)^{\alpha-\gamma}}{\Gamma(1-\gamma+\alpha)}s^{-\theta}ds + \frac{\nu I_0 t^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} \\ &= C^\theta_\gamma t^{1-\gamma-\theta} + C^\theta_{\gamma-\alpha}t^{1+\alpha-\gamma-\theta} + \frac{\nu I_0}{\Gamma(\alpha-\gamma+1)}t^{\alpha-\gamma}, \end{aligned} \tag{19}$$

with $C^\theta_\beta := \frac{C\Gamma(1-\theta)\Gamma(1-\beta)}{\Gamma(2-\beta-\theta)}$, where the last equality follows by [12, Property 2.1] . When $\gamma = 1$, we obtain

$$\begin{aligned}\beta\left|(N-I(t))I(t)\right| &= \left|I'(t) + \int_0^t \frac{\nu(t-s)^{\alpha-1}}{\Gamma(\alpha)}I'(s)ds + \frac{\nu I_0 t^{\alpha-1}}{\Gamma(\alpha)}\right| \\ &\le Ct^{-\theta} + \int_0^t \frac{\nu C(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\theta}ds + \frac{\nu I_0 t^{\alpha-1}}{\Gamma(\alpha)} \\ &= Ct^{-\theta} + C^\theta_{1-\alpha}t^{\alpha-\theta} + \frac{\nu I_0}{\Gamma(\alpha)}t^{\alpha-1}. \end{aligned} \tag{20}$$

All in all we see that the limit as $t \to \infty$ of the right hand side of both (19) and (20) is equal to zero, hence

$$\lim_{t\to+\infty}(N-I(t))I(t) = 0. \tag{21}$$

Now, since $I \in C[0,+\infty)$ and (recall Lemma 1) $I \in (0,N)$ for all $t \ge 0$, there are only 3 mutually exclusive possibilities:

(1) $\exists\, T > 0$ such that $I(t) > \frac{N}{2}$, $\forall t > T$. In this case, we have

$$(N-I(t))I(t) > (N-I(t))\frac{N}{2} \ge 0,\ \forall t > T.$$

Therefore, using (21), we get

$$\lim_{t\to+\infty} I(t) = N.$$

(2) $\exists\, T > 0$ such that $I(t) < \frac{N}{2}$, $\forall t > T$. In this case, we have

$$(N-I(t))I(t) > \frac{N}{2}I(t) \ge 0,\ \forall t > T.$$

Again, by using (21), we get

$$\lim_{t\to+\infty} I(t) = 0.$$

(3) There is a sequence $\{t_n\}_{n\in\mathbb{N}}$ with $0 < t_n < t_{n+1}$ and $\lim_{n\to+\infty} t_n = +\infty$ such that $I(t_n) = \frac{N}{2}$, $n \in \mathbb{N}$. In this case, $(N-I(t_n))I(t_n) = \frac{N^2}{4}$, which contradicts (21). We infer that this case actually cannot occur.

In conclusion, $\lim_{t\to\infty} I(t)$ exists and equals 0 or N. The proof is done. $\square$

While the previous result depends on inequality (17), such assumption seems to be verified by different numerical simulations, as presented in Section 5 (see Figure 4).

We can actually say more about the previous case, as reported in the following Theorem 6.

Lemma 3. [20, Theorem 4] *Let $0<\alpha<1$. Then we have the following estimate for the Mittag-Leffler function:*

$$E_\alpha(-t)\ge \frac{1}{1+\Gamma(1-\alpha)t}\quad t\ge 0.$$

Lemma 4. *Let $0<\alpha<\gamma\le 1$ and suppose that*

$${}^C D^\gamma_{0^+} I(t) > -\nu D^{\gamma-\alpha}_{0^+} I(t),\quad t>0.$$

Then, for $0<v_0<I_0=I(0)$ we have

$$I(t)>v_0E_\alpha(-\nu t^\alpha),\quad t\ge 0.$$

Proof. Let $v(t)=v_0E_\alpha(-\nu t^\alpha)$, $t\ge 0$, which is solution to ${}^C D^\gamma_{0^+}v(t)=-\nu D^{\gamma-\alpha}_{0^+}v(t)$. Put $m(t)=I(t)-v(t)$. Since $m(0)>0$, let $t^\star>0$ be the first point such that $m(t^\star)=0$ and $m(t)>0$ on $[0,t^\star)$. Then,

$${}^C D^\gamma_{0^+}m(t^\star) > -\nu D^{\gamma-\alpha}_{0^+}m(t^\star),$$

which is absurd by (7) and (8) above. Therefore, $m>0$ on $[0,\infty)$ and the proof is done. □

Lemma 5. *Suppose that $0<\alpha<\gamma\le 1$ and $\max\{1-\alpha,1-\gamma+\alpha\}<\theta<1$. Let I be the solution of (6) satisfying (17). Then,*

$$\lim_{t\to\infty}\frac{{}^C D^\gamma_{0^+}I(t)}{I(t)}=0. \tag{22}$$

If additionally $\gamma>3\alpha$, then

$$\lim_{t\to\infty}\frac{D^{\gamma-\alpha}_{0^+}I(t)}{I(t)}=0. \tag{23}$$

Proof. Let I be the solution of (6) and $0<v_0<I_0$. By Lemma 4, $I(t)>v_0E_\alpha(-\nu t^\alpha)$ for $t\ge 0$ and by Lemma 3, keeping in mind the proof of Theorem 5, we have (in the following, $C_1^\theta=C$)

$$\begin{aligned}\left|\frac{{}^C D^\gamma_{0^+}I(t)}{I(t)}\right| &\le C_\gamma^\theta\frac{t^{1-\gamma-\theta}}{v_0E_\alpha(-\nu t^\alpha)}\\ &\le \frac{C_\gamma^\theta}{v_0}t^{1-\gamma-\theta}(1+\Gamma(1-\alpha)\nu t^\alpha)\\ &= \frac{C_\gamma^\theta}{v_0}(t^{1-\gamma-\theta}+\Gamma(1-\alpha)\nu t^{1-\gamma-\theta+\alpha}).\end{aligned}$$

The definition of θ now implies that $t^{1-\gamma-\theta}\to 0$ and $t^{1-\gamma-\theta+\alpha}\to 0$ as $t\to\infty$, hence (22) is shown.

In an analogous way as before, and taking into account Lemma 3, we easily obtain

$$\left|\frac{D^{\gamma-\alpha}_{0^+}I(t)}{I(t)}\right|\le\frac{1}{v_0}\left(C^\theta_{\gamma-\alpha}t^{1+\alpha-\gamma-\theta}+\frac{I_0}{\Gamma(\alpha-\gamma+1)}t^{\alpha-\gamma}\right)(1+\Gamma(1-\alpha)\nu t^\alpha).$$

It is easy to see that the right hand side of the previous inequality has limit equal to 0 when $\gamma>3\alpha$, whence (23) holds and the proof is concluded. □

Theorem 6. *Under the conditions of Theorem 5 and with the extra hypothesis $\gamma>3\alpha$, we have $\lim_{t\to\infty}I(t)=N$.*

Proof. Since I solves (6), we get

$$\frac{{}^C D^{\gamma}_{0^+} I(t)}{I(t)} = \beta(N - I(t)) - \nu \frac{D^{\gamma-\alpha}_{0^+} I(t)}{I(t)}, \quad t > 0.$$

Now an application of Lemma 5 yields the result. □

The results presented in this section have a particular interest for the applications of the system (1)–(3), as we will discuss nextly.

4. Some modelling implications

As mentioned in the Introduction, the system (1) constitutes a generalisation of different epidemiological models with two compartments. The most famous one is the SIS model (see e.g. [18, Chapter 2]), under the particular choice $\gamma = \alpha = 1$. The function S represents the susceptible individuals to a particular disease carried by the infected ones, represented by I:

$$\begin{cases} I'(t) = \beta SI - \nu I, & t > 0, \\ S'(t) = -\beta SI + \nu I, & t > 0. \end{cases} \tag{24}$$

In this case, the system has two steady states $(I_1^*, S_1^*) = (0, N)$ and $(I_2^*, S_2^*) = (N - \nu/\beta, \nu/\beta)$. It can be easily shown that, if $N - \nu/\beta > 0$, then (I_2^*, S_2^*) is the stable steady state and (I_1^*, S_1^*) is the unstable one. On the other hand, if $N - \nu/\beta < 0$ then the stability of the two points is reversed. The system (24) models the evolution of an infection within a population of constant size N. Individuals cycle from the susceptible (S) to the infected compartment (I) and vice-versa. In the situation of $N - \nu/\beta > 0$ both the steady states (I_1^*, S_1^*) and (I_2^*, S_2^*) are physically meaningful, since $I_1^*, S_1^*, I_2^*, S_2^* \geq 0$. In particular, the first one is known as the disease-free equilibrium and it is unstable, while the second one is the endemic equilibrium, which is stable. For any initial condition $I_0 \in (0, N]$, the population will tend to the endemic steady state, i.e. a co-existence of susceptible and infected individuals.

A variant of the SIS system excludes the possibility of recovery from the disease and it is known as the SI model, where no recovery is possible:

$$\begin{cases} I'(t) = \overline{\beta} SI, & t > 0, \\ S'(t) = -\overline{\beta} SI, & t > 0. \end{cases} \tag{25}$$

It can be obtained by setting $\gamma = 1$ and $\alpha \to 0$ in (1) and defining $\overline{\beta} = \frac{\beta}{1+\nu}$. In this case, the steady states are, respectively, $(I_1^*, S_1^*) = (0, N)$ and $(I_2^*, S_2^*) = (N, 0)$, the latter being the stable one. In this model, recovery from the disease is not possible and, if the infection is present within the population, i.e. $I_0 > 0$, then it will eventually spread across all the individuals.

Other two-compartment variants of the SIS model have been proposed, for instance, some authors aimed at including memory effects into classical ODE models by an *ad hoc* replacement of the classical derivatives with fractional ones. This was done, for instance, in [9, 10, 16], where they study the particular case $\alpha = \gamma < 1$ of system (1). Here, with Theorem 3, we have shown that the replacement of the classical derivatives in (24) with Caputo fractional ones preserves the limits of the solutions to the classical SIS model. Independently of the value of fractional order $\gamma \in (0, 1]$, the endemic equilibrium $(I_2^*, S_2^*) = (N - \nu/\beta, \nu/\beta)$ is still present and, as long as $N - \nu/\beta > 0$, the population will tend to such point. On the other hand, if $N - \nu/\beta \leq 0$, the infection will eventually disappear from the population, independently of the value of $\gamma \in (0, 1]$.

A more solid modelling approach for the inclusion of memory effects in the SIS model was proposed by Angstmann and coauthors. In [2] they consider the particular case $\alpha < \gamma = 1$ of system (1) for modelling chronic disease epidemic, given by

$$\begin{cases} I'(t) = \beta SI - \nu D_{0^+}^{1-\alpha} I, & t > 0, \\ S'(t) = -\beta SI + \nu D_{0^+}^{1-\alpha} I, & t > 0, \end{cases} \tag{26}$$

Theorem 4 highlights one of the main differences with respect to the classical and to *fractionalised* SIS model. An intermediate endemic state is no longer possible, in the sense that the only two possible limit scenarios are that either the whole population will be infected or the infection will disappear. Such result is confirmed under certain decay restrictions on the derivative I' by Theorem 5 and, under the same conditions, with Theorem 6, we show that the model predicts the infection of the entire population for $\alpha < 1/3$. We believe this is an interesting aspect of the fractional SIS model proposed by Angstmann et al.: when α reduces from its classical value 1, the endemic equilibrium state immediately shifts from $N - \nu/\beta$ to N. The model interpretation is extreme: while in the classical SIS model the population will tend to the co-existence of infected and susceptible individuals, in the fractional case $\alpha < \gamma = 1$ the whole population will tend to be infected. Indeed, the recovery occurs so slowly that the fractional model by Angstmann et al. shares the same limits with the SI model where no recovery is possible.

It might be worth to mention that, when $0 < \alpha < \gamma$, $(I(t), S(t)) = (0, N)$ is the only possible constant solution to (1), which must be inevitably coupled to the initial conditions $(I_0, S_0) = (0, N)$. Indeed, only in this case, both Caputo and Riemann–Liouville fractional derivatives are zero, as well as the reaction term $\beta(N - I)I$. Since the Riemann–Liouville derivative $D_{0^+}^{1-\alpha}$ of a constant C is $\nu C t^{\alpha-1}/\Gamma(\alpha)$, $I(t) = C$ cannot be a solution to the equation for any $C \neq 0$. In particular, this implies that, if $I_0 = N$, even when $\lim_{t\to\infty} I(t) = N$, I will inevitably face an initial decline, for any $\nu > 0$. This easily follows by contradiction, as in the proof of Lemma 1. Nevertheless, when $\alpha = \gamma$, if $I_0 = N - \nu/\beta$, then $I(t) = N - \nu/\beta$ is another constant solution, as previously stated in Remark 2.

We conclude this section with a remark. While the limits are important, from a modelling point of view it must be considered also how fast the solutions will tend to such limits. A mathematical model might have an interest only within a limited amount of time, as the conditions might change (for instance with the discovery of a cure, or with genetic mutations of the pathogen). Therefore, the limits should be interpreted with caution, especially when the convergence is very slow. In the next section we report a collection of solutions to (1)–(3) for different values of the parameters, where we try to show this aspect.

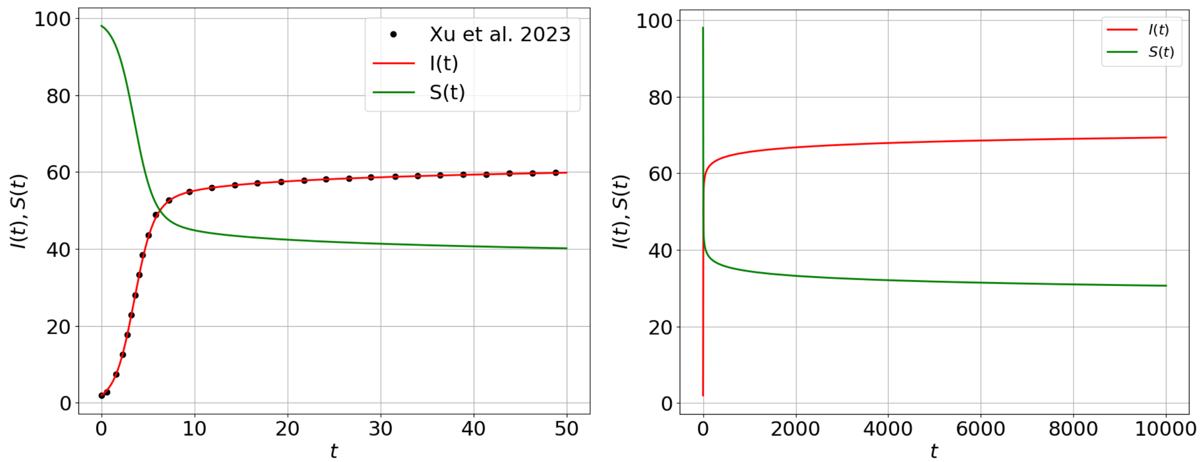

FIGURE 1. The solutions $S(t)$ and $I(t)$ of the system (1) with $\gamma = 1$, $\alpha = 0.95$, $\beta = 0.02$, $\nu = 1$, $N = 100$ and $I_0 = 2$. This parameter choice was taken from [22] and the original solution represented with the black dots. The plot on the left represents the solution up to $t = 50$, while the one on the right up to $t = 10000$.

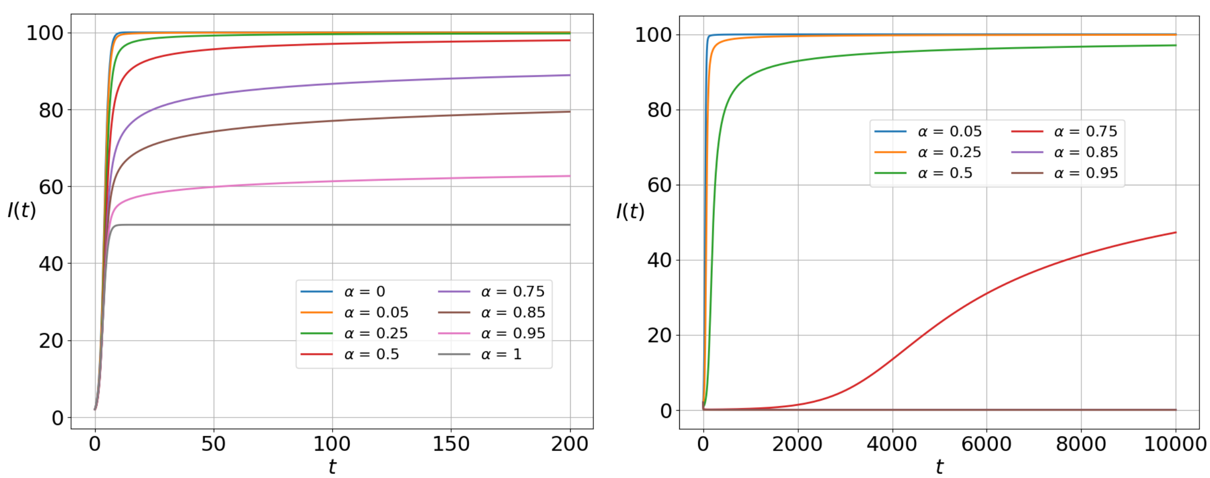

FIGURE 2. The solution $I(t)$ of the system (1) for $\gamma = 1$: on the left for $\beta = 0.02$, on the right for $\beta = 0.002$, for different values of α. The remaining parameters are $N = 100$, $I_0 = 2$, $\nu = 1$.

5. NUMERICAL EXAMPLES

In order to solve the fractional system (1)–(3), we use a numerical method based on the L1 approximation of the Caputo fractional derivative, as described in Appendix A. As a matter of testing and comparing, in Figure 1, we consider the same parameter set used in [22], where the authors applied a different numerical method based on discrete time master equations [3]. Here $\alpha = 0.95$ and $\gamma = 1$, i.e. the fractional system reduces to (26). In the image on left panel the solution $(I(t), S(t))$ is plotted up to $t = 50$, as in [22]. In this small time frame $I(t)$ seems to stabilise around some value close to 60, while on the right panel, the same solution is shown up to $t = 10000$ to highlight the fact that I continues to increase. In this case, Theorem 6 suggests that $I(t)$ might indeed tend to the value $N = 100$. To investigate the role of the parameter α in the dynamics of the

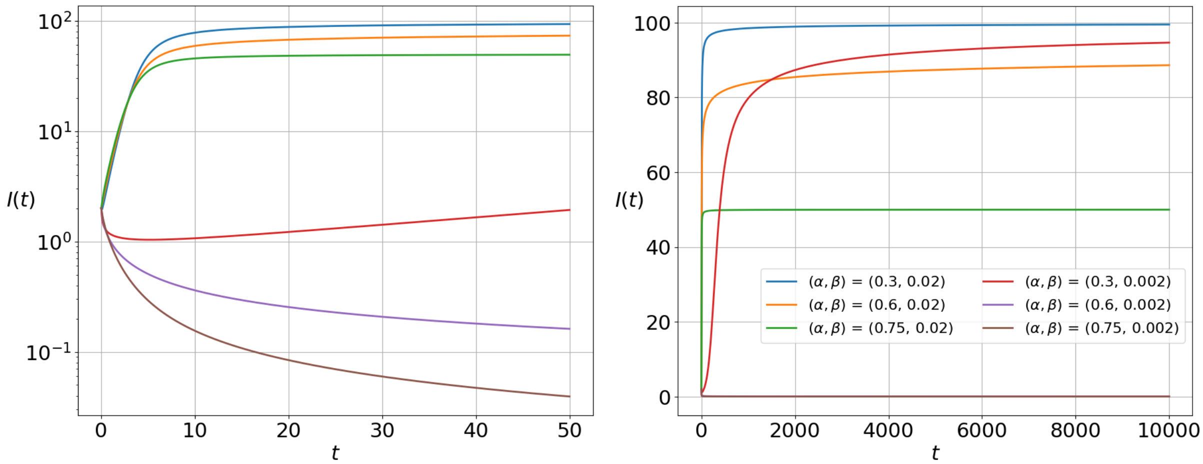

FIGURE 3. A comparison of numerical solutions to (6) for different values of $\alpha \in (0, \gamma]$ and $\beta > 0$, as indicated by the legend on the right image. Both images represent the same solutions: the left image, shows $I(t)$ up to $t = 50$ on a semi-log scale, to better capture the different initial dynamics. On the right image, the solutions are plot up to $t = 10000$, on a linear scale. Notice that, in this image, the solution corresponding to $(\alpha, \beta) = (0.6, 0.002)$ is covered by the solution with parameters $(\alpha, \beta) = (0.75, 0.002)$. The remaining parameters are $N = 100$, $\gamma = 0.75$, $I_0 = 2$, $\nu = 1$.

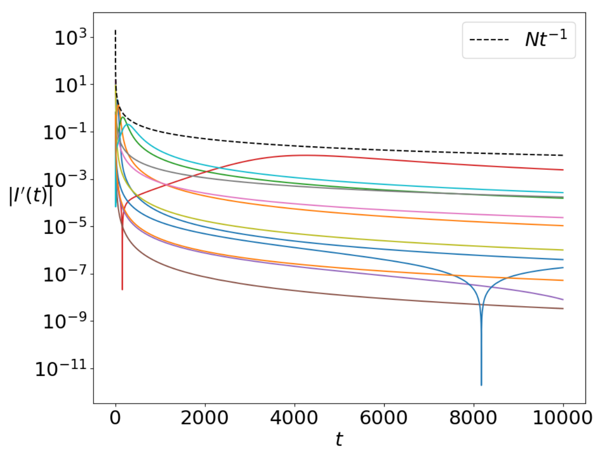

FIGURE 4. The absolute value of the numerical derivatives of all the solutions $I(t)$ reported in the right panels of Figure 2 and Figure 3, compared to the function Nt^{-1} (dashed line).

solutions to (26), in Figure 2 we report the numerical solutions $I(t)$ for different values of α. Here the left panel corresponds to the choice $\beta = 0.02$ while the right panel to $\beta = 0.002$. The higher is the fractional differentiation order $1 - \alpha$, the faster I grows. As a small remark, such dependency should not surprise. Indeed, following the model derivation in [2], the Riemann–Liouville fractional derivative $D_{0^+}^{1-\alpha} I$ results from the assumption that the survival probability in the compartment I (i.e. the probability of remaining infected) is Mittag-Leffler distributed $\sim E_\alpha(-t^\alpha)$. In particular, the smaller is the parameter α, the heavier is the tail of such distribution. As discussed in the previous section, when the recovery rate $-\nu I$ in the classical SIS model (case $\alpha = 1$) is generalised by its Riemann-Liouville derivative, this results in a jump of the limit of $I(t)$ from $N - \nu/\beta$ to N, which is the limit of the classical SI model (case $\alpha = 0$). Figures 1 and 2 show that, when α is very close to 1, the growth of $I(t)$ towards N can be very slow. Finally, in Figure 3 we report another collection of numerical solutions $I(t)$ to (1)–(3) with $\gamma = 0.75$, for different combinations of the fractional order α and the parameter β. In the left panel we plot the initial dynamics up to $t = 50$, in the right panel the dynamics up to $t = 10000$.

Since in the classical SIS model (24) the parameter β can revert the stability of $(I_2^*, S_2^*) = (N - \nu/\beta, \nu/\beta)$, in Figures 2 and 3 we consider two different values of β. In the right panel of Figure 2 $I(t)$ seems to tend to zero for $\alpha = 0.85$ and $\alpha = 0.95$, while in Figure 3 the same seems to happen for $(\alpha, \beta) = (0.6, 0.002)$ and $(\alpha, \beta) = (0.75, 0.002)$. From a modelling point of view we might interpret the results as the disappearance of the infection.

Lastly, in Figure 4 we compared the functions $|I'(t)|$ with the power function Nt^{-1}, for all solutions $I(t)$ reported for $t \in [1, 10000]$ in the previous figures, namely the right panels of Figure 2 and 3. In all of the cases, $|I'(t)| \leq Nt^{-1} < Nt^{-\theta}$ for any $\theta < 1$. Hence, at least from the numerical point of view, our assumption (17) seems reasonable, here with $C = N$.

6. Conclusion

In this work we studied the Caputo–Riemann–Liouville fractional differential system (1) with initial condition (2) and under the parameter choice (3). We were able to prove global existence and uniqueness of the solution, for any fractional differential orders $0 < \alpha \leq \gamma \leq 1$. We also prove that such solution is always positive and bounded.

We were able to identify possible limits of the solution at infinity, studying case by case. The comparison between the fractional cases with the two corresponding ordinary differential systems (the first one obtained by setting $\gamma = \alpha = 1$, the second one with $\gamma = 1$, $\alpha = 0$), helps in understanding how the influence that the inclusion of a Riemann–Liouville or a Caputo derivative might have in a mathematical model.

We remark that we imposed certain assumptions either on the existence of the limit or under power law bound of the derivative of the solution. Future works should focus on investigating under which conditions such assumptions are valid, as well as on an asymptotic analysis on the convergence rate to such limits.

From an applied point of view, we believe that our work has potential interest especially in epidemiological modelling. In addition to this, we believe that our approach can be used to study other fractional differential systems.

Appendix A. The numerical method

We solve numerically the fractional equation (6) by applying the L1 method for discretising the Caputo derivative, as described in [13, Chapter 4] and that we briefly recall in this Appendix.

Consider $h>0$ and the following homogeneous discretization of the interval $[0,T]$

$$t_j = jh, \text{ for } j=0,\dots,N \text{ where } N=\lceil T/h \rceil.$$

For any $\gamma \in (0,1]$, we have

$$\begin{aligned} {}^C D^{\gamma}_{0^+} u(t)\Big|_{t=t_j} &= \frac{1}{\Gamma(1-\gamma)} \sum_{k=0}^{j-1} \int_{t_k}^{t_{k+1}} (t_j - s)^{-\gamma} u'(t) ds \\ &\approx \frac{1}{\Gamma(1-\gamma)} \sum_{k=0}^{j-1} \frac{u(t_{k+1}) - u(t_k)}{h} \int_{t_k}^{t_{k+1}} (t_j - s)^{-\gamma} ds \end{aligned}$$

where each integral can now be explicitly calculated. Indeed, by defining

$$b_k^{\gamma} = \frac{h^{-\gamma}}{\Gamma(2-\gamma)}[(k+1)^{1-\gamma} - k^{1-\gamma}] \qquad \text{and} \qquad u_j = u(t_j), \text{ for } j=0,\dots,k,$$

the L1 discretisation of the Caputo derivative is

$$\begin{aligned} {}^C D^{\gamma}_{0^+} u(t)\Big|_{t=t_j} \approx & \frac{1}{\Gamma(1-\gamma)} \sum_{k=0}^{j-1} \frac{u_{k+1} - u_k}{h} \int_{t_k}^{t_{k+1}} (t_j - s)^{-\gamma} ds = \sum_{k=0}^{j-1} b^{\gamma}_{j-k-1}(u_{k+1} - u_k) \\ &= b_0^{\gamma} u_j - b_0^{\gamma} u_{j-1} + \sum_{k=0}^{j-2} b^{\gamma}_{j-k-1}(u_{k+1} - u_k). \end{aligned}$$

In the same way, passing from the Riemann-Liouville to the Caputo derivative, we have

$$\begin{aligned} D^{\gamma-\alpha}_{0^+} u(x_j) \approx & \frac{t^{\alpha-\gamma}}{\Gamma(1-\gamma+\alpha)} u_0 + \sum_{k=0}^{j-1} b^{\gamma-\alpha}_{j-k-1}(u_{k+1} - u_k) \\ &= b_0^{\gamma-\alpha} u_j - b_0^{\gamma-\alpha} u_{j-1} + \frac{t_j^{\alpha-\gamma}}{\Gamma(1-\gamma+\alpha)} u_0 + \sum_{k=0}^{j-2} b^{\gamma-\alpha}_{j-k-1}(u_{k+1} - u_k). \end{aligned}$$

By explicitly evaluating the nonlinear term of (6) and applying the above discretisations, we get

$$\begin{aligned} & b_0^{\alpha} u_j - b_0^{\alpha} u_{j-1} + \sum_{k=0}^{j-2} b^{\alpha}_{j-k-1}(u_{k+1} - u_k) \\ =& \beta(N - u_{j-1}) u_{j-1} - \nu b_0^{\gamma-\alpha} u_j + \nu b_0^{\gamma-\alpha} u_{j-1} - \frac{\nu (jh)^{\alpha-\gamma}}{\Gamma(1-\gamma+\alpha)} u_0 \\ & - \nu \sum_{k=0}^{j-2} b^{\gamma-\alpha}_{j-k-1}(u_{k+1} - u_k) \end{aligned}$$

and solving with respect to u_j we obtain our numerical scheme:

$$\begin{aligned} u_j =& u_{j-1} + \frac{\beta(N-u_{j-1})u_{j-1}}{(b_0^{\alpha} + \nu b_0^{\gamma-\alpha})} - \frac{\nu (jh)^{\alpha-\gamma} u_0}{\Gamma(1-\gamma+\alpha)(b_0^{\alpha} + \nu b_0^{\gamma-\alpha})} \\ & - \sum_{k=0}^{j-2} \frac{\left(b^{\alpha}_{j-k-1} + \nu b^{\gamma-\alpha}_{j-k-1}\right)}{(b_0^{\alpha} + \nu b_0^{\gamma-\alpha})} (u_{k+1} - u_k). \end{aligned}$$

The numerical scheme was coded in Python 3, and the corresponding script is available at https://github.com/davidecusseddu/FrSIS.

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