

REGULAR POLYTOPES OF RANK $n/2$ FOR TRANSITIVE GROUPS OF DEGREE n

MARIA ELISA FERNANDES AND CLAUDIO ALEXANDRE PIEDADE

ABSTRACT. Previous research established that the maximal rank of the abstract regular polytopes whose automorphism group is a transitive proper subgroup of S_n is $n/2 + 1$, with only two polytopes attaining this rank, both of which having odd ranks. In this paper, we investigate the case where the rank is equal to $n/2$ ($n \geq 14$). Our analysis reveals that reducing the rank by one results in a substantial increase in the number of regular polytopes (33 distinct families are discovered) covering all possible ranks (even and odd).

Keywords: Abstract Regular Polytopes; String C-Groups; Symmetric Groups; Alternating Groups; Permutation Groups.

2000 Math Subj. Class: 52B11, 20B35, 20B30, 05C25.

1. INTRODUCTION

Abstract polytopes are combinatorial objects that describe standard regular polytopes using their a face-lattice [13]. An abstract polytope is regular when its group of automorphisms acts regularly on the maximal chains (usually called flags). A notable feature of these structures lies in their one to one correspondence with their automorphism groups, which are string C-groups. These algebraic structures are defined not only by the group itself but also by a specified set of involutory generators, the size of which determines the rank.

The maximal rank of an abstract regular polytope whose automorphism group has degree n is $n - 1$. For $n \geq 5$, the simplex stands out as the sole polytope achieving this maximal rank [8, 9]. One permutation representation of the group of automorphism of the simplex on n points is $\langle (1, 2), (2, 3), \dots, (n - 2, n - 1) \rangle$, the polytope with Schläfli symbol $3, \dots, 3$ corresponding to the standard Coxeter group of type A_{n-1} . Indeed $n - 1$ is the maximal size of an independent set in S_n , and S_n is the unique group of degree n that admits an independent set of generators of size $n - 1$ [14].

The classification of abstract regular polytopes of ranks $n - k$ for groups of degree $n \geq 2k + 3$, was also outlined for $k \in \{1, 2, 3, 4\}$ in [10]. The automorphism group of all these high-rank abstract polytopes is S_n . The rank of the alternating groups are considerably lower - the highest rank for the alternating group A_n is $\lfloor (n - 1)/2 \rfloor$ for $n \geq 12$ [3]. The analyses of other transitive permutation groups of degree n started in [4]. Key findings indicate that a primitive group (excluding S_n and A_n) tend to possess small C-rank (the maximal size of the set of generators of a string C-group), while the maximal C-rank of an imprimitive groups is $n/2 + 1$ [4]. Moreover, only imprimitive groups with even degrees and even ranks achieve this upper bound, leaving odd degrees and ranks outside of this classification. In this paper, we bridge this gap by extending the classification down to imprimitive groups of degree $n/2$.

It emerges that transitive imprimitive groups of even degree n and rank $n/2$ fall into two categories: those with a block system with $n/2$ blocks of size 2 or those with 2 blocks of size $n/2$. Section 2.5 is dedicated to groups embedded in $C_2 \wr S_{n/2}$, where crucial general considerations when the group action on the blocks is either $S_{n/2}$ or $A_{n/2}$ are made. In particular, we provide a classification of the elementary abelian subgroups of the permutations of $G = C_2 \wr S_{n/2}$ fixing all blocks, under the premise that the group G acts on the blocks either as $S_{n/2}$ or $A_{n/2}$. Only after establishing these auxiliary results, each with the potential for application in other contexts, we attain our desired outcome which is the following.

Theorem 1.1. *Let $n/2 \geq 7$ and G be a transitive proper subgroup of S_n . If G is the automorphism group of an abstract regular polytope of rank $r \geq n/2$, then G is a string C-group having one of the permutation representation graphs of the tables displayed in Section 6.*

2. PRELIMINARIES

2.1. Independent generating sets.

Definition 2.1. Let G be a group. A set $S = \{\rho_0, \dots, \rho_{r-1}\}$ is an independent generating set of G if $\rho_i \notin \langle \rho_j \mid j \neq i \rangle$ and $G = \langle S \rangle$.

Theorem 2.2. [14, Theorem 1] The maximal size of an independent generating set for a group of degree n is $n - 1$. Moreover S_n is the only group having an independent generating set of size $n - 1$.

Theorem 2.3. [1, Theorem 2.1] Let S be an independent generating set for S_n of size $n - 1$, where $n \geq 7$. Then there is a tree T on $\{1, \dots, n\}$ such that one of the following holds:

- (a) $S = S(T)$;
- (b) for some element $s \in S(T)$, we have

$$S = \{s\} \cup \{(st)^{\epsilon(t)} : t \in S(T) \setminus \{s\}\} \text{ where } \epsilon(t) = \pm 1.$$

Conversely, each of these sets is an independent generating set for S_n .

2.2. Sggi's and permutation representation graphs.

Definition 2.4. A string group generated by involutions or, for short, a sggi is a pair $\Gamma = (G, S)$ where $G = \langle S \rangle$ with $S = \{\rho_0, \dots, \rho_{r-1}\}$ being a set of involutions that satisfy the following property, called the commuting property.

$$\forall i, j \in \{0, \dots, r-1\}, |i - j| > 1 \Rightarrow (\rho_i \rho_j)^2 = 1.$$

The dual of a sggi is obtained by reversing the sequence of generators.

Definition 2.5. Suppose that G is a permutation group of degree n and let $\Gamma = (G, \{\rho_0, \dots, \rho_{r-1}\})$ be a sggi. The permutation representation graph \mathcal{G} of Γ is an r -edge-labelled multigraph with n vertices and with an i -edge $\{a, b\}$ whenever $a\rho_i = b$ with $a \neq b$ and $i \in \{0, \dots, r-1\}$.

Notation 2.6. Let us consider the following notation.

$$\begin{aligned} I_{i_1, \dots, i_k} &:= \{0, \dots, r-1\} \setminus \{i_1, \dots, i_k\} & I^{\leq i} &:= \{0, \dots, i\} & I^{\geq i} &:= \{i, \dots, r-1\} \\ I^{< i} &:= \{0, \dots, i-1\} & I^{> i} &:= \{i+1, \dots, r-1\} \\ I_{i_1, \dots, i_k}^{< i} &:= \{0, \dots, i\} \setminus \{i_1, \dots, i_k\} & I_{i_1, \dots, i_k}^{\geq i} &:= \{i, \dots, r-1\} \setminus \{i_1, \dots, i_k\} \\ I_{i_1, \dots, i_k}^{< i} &:= \{0, \dots, i-1\} \setminus \{i_1, \dots, i_k\} & I_{i_1, \dots, i_k}^{> i} &:= \{i+1, \dots, r-1\} \setminus \{i_1, \dots, i_k\} \\ \Gamma_{i_1, \dots, i_k} &:= (G_{i_1, \dots, i_k}, \{\rho_j \mid j \in I_{i_1, \dots, i_k}\}) & \Gamma_{\{i_1, \dots, i_k\}} &:= (G_{\{i_1, \dots, i_k\}}, \{\rho_j \mid j \in \{i_1, \dots, i_k\}\}); \\ \Gamma_{< i} &:= (G_{< i}, \{\rho_0, \dots, \rho_{i-1}\}) & (i \neq 0) & \Gamma_{> i} &:= (G_{> i}, \{\rho_{i+1}, \dots, \rho_{r-1}\}) & (i \neq r-1); \end{aligned}$$

Let $\mathcal{G}_{i_1, \dots, i_k}$ (resp. $\mathcal{G}_{\{i_1, \dots, i_k\}}$) denote the permutation representation graph of Γ_{i_1, \dots, i_k} (resp. $\Gamma_{\{i_1, \dots, i_k\}}$).

Notice that when ρ_i is a k -transposition (a product of k disjoint transpositions), $\mathcal{G}_{\{i\}}$ is a matching with k edges. A consequence of the commuting property (see Definition 2.4) is that, if i and j are nonconsecutive the connected components of $\mathcal{G}_{\{i, j\}}$ with more than two vertices are $\{i, j\}$ -squares (squares with alternating labels i, j, i, j). A J -edge is a set of $|J|$ parallel edges with label-set J . Sometimes we represent these set of edges by a single edge with the label J .

Lemma 2.7. Let (G, S) be a sggi and S is an independent generating set of a primitive group G that is neither A_n nor S_n , of rank r . If $r \geq n - 3$ then G is one of the groups D_{10} , $\text{PSL}_2(5)$ or $\text{PGL}_2(5)$.

Proof. From [3, Proposition 3.3], we have that for $n \geq 8$, $r \leq n - 4$. Hence, here we will deal with the cases for $n \leq 7$. In what follows, $\text{lcs}(G)$ denotes the size of a longest chain of subgroups of G in its subgroup lattice. The following table lists all transitive primitive groups G of degree $n \leq 7$, that are neither S_n nor A_n , having a longest chain of subgroups with size $\text{lcs}(G) \geq n - 3$.

n	G	$\text{lcs}(G)$	Generated by involutions
5	D_{10}	2	yes
	$\text{AGL}_1(5)$	3	no
6	$\text{PSL}_2(5)$	4	yes
	$\text{PGL}_2(5)$	5	yes
7	$\text{PSL}_3(2)$	5	yes

We can exclude $\text{AGL}_1(5)$ since it cannot be generated by a set of involutions. Computationally it can be checked that $\text{PSL}_3(2)$ is neither a sggi of rank 4 nor 5. The remaining ones are the ones in the statement of these lemma. □

Lemma 2.8. *Suppose that $G = \langle S \rangle$ is a sggi satisfying the following conditions.*

- (a) $S = \{\rho_0, \dots, \rho_{r-1}\}$ is independent;
- (b) G_{r-1} is intransitive and
- (c) G_j is transitive for some $j \notin \{0, r-1\}$.

If j is the maximal label satisfying (c), k is the size of $G_{<j}$ -orbit and $m = n/k$, then $G_j \leq S_k \wr S_m$ and

$$r - 1 \leq (k - 1) + (m - 1).$$

Proof. As G_{r-1} is intransitive, $G_{<j}$ is intransitive. Hence $G_j \leq S_k \wr S_m$ where the blocks are the $G_{<j}$ -orbits. Particularly $G_{<j} \leq S_k$, hence the number of generators of $G_{<j}$, which is equal to j is at most $k - 1$. As j is the maximal label satisfying (c), G_i is intransitive for $i > j$. Thus, for each $i > j$ there exists a pair of $G_{<j}$ -orbits that belong to different G_i -orbits. Consider a graph whose vertices are the $G_{<j}$ -orbits having exactly one edge i between $G_{<j}$ -orbits that belong to different G_i -orbits. The graph is a forest with m vertices and $(r - 1) - j$ edges. Hence $(r - 1) - j \leq m - 1$. Consequently $r - 1 \leq (k - 1) + (m - 1)$, as wanted. □

2.3. String C-groups.

Definition 2.9. *A string C-group $\Gamma = (G, S)$ is a sggi which satisfy the following property called the intersection property.*

$$\forall J, K \subseteq \{0, \dots, r-1\}, \langle \rho_j \mid j \in J \rangle \cap \langle \rho_k \mid k \in K \rangle = \langle \rho_j \mid j \in J \cap K \rangle$$

An immediate consequence of Theorem 2.3 is the following.

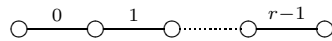
Corollary 2.10. *Let $\Gamma = (G, \{\rho_0, \dots, \rho_{r-1}\})$ be a string group of degree n generated by independent involutions. If $r = n - 1$ and $n \geq 7$ then Γ is the group of automorphisms of the $(n - 1)$ -simplex.*

Theorem 2.11. [8, Theorem 1] *For $n \geq 5$, the $(n - 1)$ -simplex is, up to isomorphism, the unique polytope of rank $n - 1$ having a group S_n as automorphism group. For $n = 4$, there are, up to isomorphism and duality, two abstract regular polyhedra whose automorphism group is S_4 , namely the hemicube and the tetrahedron. Finally, for $n = 3$, there is, up to isomorphism, a unique abstract regular polygon whose automorphism group is S_3 , namely the triangle.*

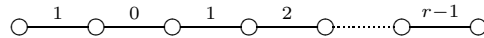
Theorem 2.12. [8, Theorem 2] *For $n \geq 7$, there exists, up to isomorphism and duality, a unique $(n - 2)$ -polytope having a group S_n as automorphism group and Schläfli symbol $\{4, 6, 3, 3, \dots, 3\}$.*

In [8] and [9] the authors give the possible permutation representation graphs of the string C-groups of Theorems 1 and 2. We list them in the following proposition.

Proposition 2.13. *The permutation representation graph of degree n of the group of automorphisms of the abstract regular polytopes of rank $r \geq n - 1$ ($n \geq 5$) is as follows.*



The permutation representation graph of degree n of the group of automorphisms of the abstract regular polytope of rank $r = n - 2$ ($n \geq 7$) is, up to duality, as follows.



Proposition 2.14. *Let $\Gamma = (G, S)$ be a sggi and $x \in \{1, 2, 3, 4\}$. If G_j is intransitive for all $j \in \{0, \dots, r-1\}$, $r = n - x$ and $n \geq 3 + 2x$, then $G \cong S_n$. Moreover if $x \in \{1, 2\}$ then Γ has, up to duality, one of the permutation representation graph given in Proposition 2.13.*

Proof. We just need to observe that each sggi G having one of the permutation representation graphs of Table 2 of [10] is isomorphic to S_n . Indeed in each case we find a transposition $(a, b) \in G$ such that the stabilizer of a is transitive of $\{1, \dots, n\} \setminus \{a\}$. Hence $G \cong S_n$.

In addition, when $x \in \{1, 2\}$ Table 2 of [10] gives only two possibilities for the permutation representation graph, precisely the ones of Proposition 2.13. □

Proposition 2.15. [4, Proposition 3.2] *If Γ is a string C-group of rank r , which is isomorphic to a primitive subgroup of S_n other than S_n or A_n , then $r < n/2$ except for the examples appearing in Table 1.*

n	G	Schäfli symbols
10	S_6	$\{3, 3, 3, 3\}$
6	A_5	$\{3, 5\}, \{5, 5\}$
6	S_5	$\{3, 3, 3\}, \{4, 5\}, \{4, 6\}, \{5, 6\}, \{6, 6\}$

TABLE 1. Primitive string C-groups of degree n and rank $r \geq n/2$.

Corollary 2.16. [6, Corollary 4.2] *If G is a finite non-abelian simple group, or more generally any finite group with no non-trivial cyclic normal subgroup, then every smooth homomorphism from the $[k, m]$ Coxeter group onto G gives rise to a regular 3-polytope with automorphism group G .*

Theorem 2.17. [3, Theorem 1.1] *The maximal rank of a string C-group for A_n is 3 if $n = 5$; 4 if $n = 9$; 5 if $n = 10$; 6 if $n = 11$ and $\lfloor \frac{n-1}{2} \rfloor$ if $n \geq 12$. Moreover, if $n = 3, 4, 6, 7$ or 8 , the group A_n does not admit a string C-group.*

Proposition 2.18. [4, Proposition 2.1] *If $\Gamma = (G, S)$ is a string C-group of rank r and G is isomorphic to a transitive imprimitive subgroup of S_n , then $r \leq n/2 + 1$. Moreover if $r = n/2 + 1$ and $n \geq 10$ then $G \cong C_2 \wr S_{n/2}$ and $n \equiv 2 \pmod{4}$. If $r = n/2 + 1$ and $n \leq 9$ then Γ is one of the string C-groups of the following table.*

n	G	Schäfli symbols
8	$2^4 : S_3 : S_3$	$\{3, 4, 4, 3\}$
6	$S_3 \times S_3$	$\{2, 3, 3\}$
6	$2^3 : S_3$	$\{2, 3, 3\}$
6	$2^3 : S_3$	$\{2, 3, 4\}$

TABLE 2. Imprimitive string C-groups of degree $n \leq 9$ with rank $r \geq n/2 + 1$.

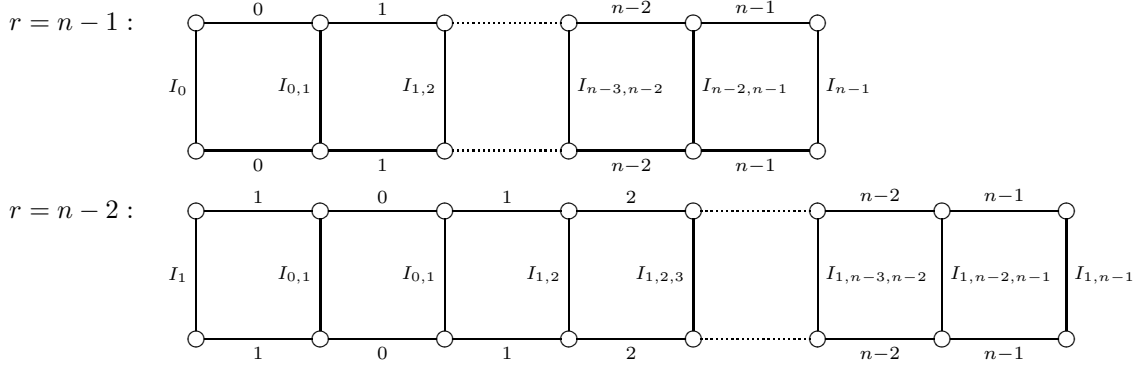
2.4. Permutation representation of string C-groups for S_n of rank $r \in \{n-2, n-1\}$ on $2n$ points.

Proposition 2.19. *Let $n \geq 7$. There exists exactly one faithful transitive permutation representation of S_n on $2n$ points.*

Proof. Let us prove that, up to conjugacy, there exist only one core-free subgroup of S_n of index $2n$ when $n \geq 7$. Suppose that H is a subgroup of S_n of index at most $2n$. By the O’Nan-Scott Theorem [17], we have one of the following possibilities for H : (a) $H \leq S_a \times S_{n-a}$, for $1 \leq a \leq n/2$; (b) $H \leq S_{n/a}^a \rtimes S_a$, for $a|n$ and $1 < a < n$; or (c) H is a primitive subgroup of S_n (different from A_n and S_n).

In case (a) $|H| \leq a!(n-a)!$, hence $\frac{n!}{a!(n-a)!} \leq 2n$, which is only possible if $a = 1$. Then $H \leq S_{n-1}$ and $|S_{n-1} : H| = 2$, which gives $H \cong A_{n-1}$. Case (b) is never possible as $\frac{n!}{a!(n/a)!^a} > 2n$, for $n \geq 7$. In case (c) using the bound for a primitive group given in [18], we have that $|H| \leq 4^n$, meaning that $\frac{n!}{4^n} \leq 2n$, which is only possible for $n \leq 11$. But then, using [19] we find no possibility for small degrees either. \square

Proposition 2.20. *If $\Gamma = (G, \{\rho_0, \dots, \rho_{r-1}\})$ is a string C-group of rank $r \in \{n-1, n-2\}$ and G is a transitive group of degree $2n$ isomorphic to S_n , then Γ has, up to duality, one the following permutation representations.*



Proof. Consider first that Γ is the string C-group of rank $n - 1$, which is known as the $(n - 1)$ -simplex. Consider the subgroup $H = \langle \rho_1\rho_2, \rho_2\rho_3, \dots, \rho_{r-3}\rho_{r-2}, \rho_{r-2}\rho_{r-1} \rangle$ of G_0 . This group is an index 2 subgroup of G_0 , known as the rotational group of Γ_0 . As $G_0 \cong S_{n-1}$, $H \cong A_{n-1}$. By Proposition 2.19 we only need to compute the Schreier coset graph with respect to H . Using the Todd–Coxeter algorithm we get the graph given in the statement of this proposition.

Now consider the case $r = n - 2$. In this case Γ is the abstract regular polytope of Theorem 2.12. Up to duality Γ is the abstract regular polytope of with Schläfli symbol $\{4, 6, 3, \dots, 3\}$. Then $G_{r-1} \cong S_{n-1}$ and the rotational subgroup H of G_{r-1} is isomorphic to A_{n-1} . Applying the Todd–Coxeter algorithm to H we obtain the second permutation representation given in the statement of this proposition. \square

2.5. Imprimitve groups with blocks of size 2 with block action isomorphic to $S_{n/2}$ or $A_{n/2}$.

Corollary 2.21. *Let $n/2 \geq 3$, Suppose that $G \leq C_2 \wr S_{n/2}$ and let $f : G \rightarrow S_{n/2}$ be the embedding of G into $S_{n/2}$. If $\text{Im}(f)$ is either $S_{n/2}$ or $A_{n/2}$ then $\text{Ker}(f)$ is either trivial or isomorphic to one of the groups:*

$$C_2, (C_2)^{n/2-1} \text{ or } (C_2)^{n/2}.$$

Proof. Let $n/2 \geq 3$ and G be either $S_{n/2}$ or $A_{n/2}$. By Lemma 2 of [12] there are only four G -modules over a field of characteristic 2. These modules correspond, respectively, to the 0-vector, the all 1's vector and the even-weight module. These possibilities are in one to one correspondence with the nontrivial possibilities for the kernel given above. \square

Lemma 2.22. *Let $n \geq 6$ and $2 < l < n/2$. Suppose that G is a transitive subgroup of $C_2 \wr A_{n/2}$ and let $f : G \rightarrow A_{n/2}$ be the embedding of G into $A_{n/2}$. If $\text{Im}(f) \cong A_{n/2}$ then the following holds.*

- (a) *If G contains a transposition fixing the blocks, then $\text{Ker}(f) \cong (C_2)^n$.*
- (b) *If G contains a 2-transposition fixing the blocks, then $(C_2)^{n-1} \leq \text{Ker}(f)$.*
- (c) *If G contains a l -transposition fixing the blocks, then $(C_2)^{n-1} \leq \text{Ker}(f)$.*

Proof. Let $\tau \in G$ be a permutation fixing the blocks. We will consider separately the cases: (a) τ is a transposition, (b) τ is a 2-transposition and (c) τ is a $(n/2 - 1)$ -transposition.

(a) Let B_1 be the block where τ acts nontrivially. Let B_i be other block. There exist a permutation $g \in G$ such that $gB_1 = B_i$. Moreover τ^g is the transposition swapping the pairs of points of B_i . Hence $\text{Ker}(f) \cong (C_2)^n$.

(b) As $A_{n/2}$ is 2-transitive, any 2-transposition fixing the blocks can be obtained by a conjugation of τ . Let B_1 and B_2 be the blocks where τ acts nontrivially. For any pair of $\{B_i, B_j\}$ there exist $g \in G$ such that $\{B_1, B_2\}g = \{B_i, B_j\}$. Hence $(C_2)^{n-1} \leq \text{Ker}(f)$.

(c) Suppose first that $n/2$ is odd. Let $\alpha \in G$ be a product of two cycles of size $n/2$ (permuting all blocks in a single cycle). Then $\tau\tau^\alpha$ is a 2-transposition fixing the blocks. Now suppose that $n/2$ is even. Let $\text{Fix}(\tau) = \{a, b\}$. Let $\beta \in G$ be a permutation that acts on the blocks as a cycle of size $n/2 - 1$ that does not fix the block $\{a, b\}$, that is $\{a, b\}\beta \neq \{a, b\}$. Then, as before, $\tau\tau^\beta$ is a 2-transposition fixing the blocks. In both cases, by (2), we get $(C_2)^{n-1} \leq \text{Ker}(f)$. \square

Theorem 2.23. *Let H be either $A_{n/2}$ or $S_{n/2}$. If G is a transitive subgroup of degree n embedded into $C_2 \wr H$, then the following statement hold.*

- (a) *The index of G in $C_2 \wr H$ is either 1, 2, $2^{n/2-1}$ or $2^{n/2}$.*

- (b) If the index of G in $C_2 \wr H$ is equal to $2^{n/2-1}$ then G contains the permutation that swaps all pairs of points belonging to the same block.
- (c) If the index of G in $C_2 \wr H$ is equal to 2 then G contains all even permutations fixing the blocks.
- (d) Let $n/2$ be even. If $|G| = 2|H|$ then G is even.

Proof. (a), (b) and (c) are immediate consequences of Corollary 2.21 and its proof.

(d) In this case G contains the permutation α swapping all pairs of points within the blocks.

Suppose first that $H = A_{n/2}$. As $n/2$ is even G contains a permutation δ that fixes exactly one block B and that permutes all the other blocks cyclically. This permutation is odd if it acts nontrivially in B . In any case δ^2 is even, indeed it is written as a product of two $(n/2 - 1)$ -cycles. Now consider any block X and the set of three blocks $\{X, X\delta^2, B\}$. There exists a permutation β that permutes these blocks cyclically, fixes another block Y and swaps the remaining blocks pair wisely. By construction β^4 is a product of two 3-cycles permuting the blocks $\{X, X\delta^2, B\}$. Hence $\langle \beta^4, \delta^2 \rangle$ acts as $A_{n/2}$ on the blocks. As $|G| = 2|A_{n/2}|$, then $G = \langle \beta^4, \delta^2, \alpha \rangle$, hence G is even, as wanted.

Now consider the case $H = S_{n/2}$. In this case G contains the permutation δ that permutes all the other blocks cyclically and δ^2 is written as a product of two $n/2$ -cycles. As in the previous case given a triple of blocks G contains a permutation, that is a product of two 3-cycles, permuting these blocks and fixing all the other points. With this we get a set of even generators for the group G , which shows that G is even. \square

3. IMPRIMITIVE STRING C-GROUPS OF RANK $r \geq n/2$

Consider a string C-group $\Gamma = (G, S)$ where G is transitive imprimitive with m blocks of imprimitivity each of size k . In what follows $r := |S|$ and let \mathcal{G} be the permutation representation graph of G .

Let L be a subset of S which is an independent generating set for the group-action on the set of blocks; C be the set of generating involutions which commute with all the involutions in L , and R the remaining set of involutions.

$$S = L \dot{\cup} C \dot{\cup} R$$

In the following proposition we resume the results obtained in Section 2 of [4].

Lemma 3.1. *If $\langle L \rangle$ acts primitively on the blocks, then $|C| \leq k - 1$.*

Proof. If the elements of C fix the blocks then $\langle C \rangle$ acts faithfully on a block. Hence $\langle C \rangle \leq S_k$ and therefore $|C| \leq k - 1$, as wanted. Consider the general case where the elements of C do not necessarily fix the blocks.

Suppose first that $m > 2$. If an element of C permutes the blocks then $\langle L \rangle$ has an imprimitive action on the blocks, a contradiction. Thus for $m > 2$, $|C| \leq k - 1$. Now suppose that $m = 2$ and let $L = \{\alpha\}$. Consider the mapping $\rho \mapsto \bar{\rho}$, where $\bar{\rho} = \rho\alpha$ if ρ swaps the blocks, or $\bar{\rho} = \rho$ if ρ fixes the blocks. This defines a one to one correspondence between C and $\bar{C} = \{\bar{\rho}_i \mid \rho_i \in C\}$. Now $\langle \bar{C} \rangle \leq S_k$. Moreover if $\bar{\rho}_i \in \langle \rho_j \mid j \neq i \rangle$ then $\rho_i \in \langle C \setminus \{\rho_i\} \rangle \langle L \rangle$, a contradiction. Thus \bar{C} is an independent set of permutations in S_k , hence $|C| = |\bar{C}| \leq k - 1$. \square

Lemma 3.2. *If $m = 4$ and the block action is isomorphic to $(C_2)^2$, then $|C| \leq n/4 - 1$. Moreover we have the following.*

- (a) If $|C| = n/4 - 1$ then $\langle L \cup C \rangle \cong S_{n/4} \times (C_2)^2$.
- (b) If $\gamma \in S$ is a central involution fixing the blocks then $|C| \leq n/8$.

Proof. Let $L = \{\alpha, \beta\}$. Now either $(\alpha\beta)^2$ is trivial or not. Let us deal with these two cases separately.

(1) Suppose that there exists a set of nonconsecutive generators $L = \{\alpha, \beta\}$ generating the block action. Now consider the mapping $\rho \mapsto \bar{\rho}$ where $\bar{\rho}$ is a permutation fixing the blocks that is obtained by undoing the block action using elements of $\langle L \rangle$. That is, $\bar{\rho} = \rho\gamma$ with $\gamma \in \langle \alpha, \beta \rangle$. As $\bar{\rho}$ centralizes $\langle L \rangle$, we conclude that $\langle \bar{\rho} \mid \rho \in C \rangle \leq S_{n/4}$ and $\{\bar{\rho} \mid \rho \in C\}$ is independent, thus $|C| \leq n/4 - 1$.

When $|C| = n/4 - 1$, $\langle \bar{\rho} \mid \rho \in C \rangle \cong S_{n/4}$, hence $\langle L \cup C \rangle \cong S_{n/4} \times (C_2)^2$.

If $\gamma \in S$ is a central involution, particularly $\gamma \in C$, then $G \leq S_{n/8} \times (C_2)^3$ with $L' = \{\alpha, \beta, \gamma\}$ generating the block action. The set C' of the elements commuting with the elements of L' is equal to $C \setminus \{\gamma\}$. As before there is a natural embedding of $C \setminus \{\gamma\}$ into $S_{n/8}$. Thus $|C \setminus \{\gamma\}| \leq n/8 - 1$, as wanted.

(2) Suppose there exist no pair $\{\alpha, \beta\}$ generating the block action with $(\alpha\beta)^2$ being trivial. Thus $L = \{\rho_i, \rho_{i+1}\}$ for some i . Then the elements of C must fix the blocks otherwise we are in case (1). Hence $|C| \leq n/4 - 1$.

Clearly if $|C| = n/4 - 1$, then $\langle C \rangle = S_{n/4}$. Finally if α centralizes $\langle C \rangle$, $|C| \leq n/8$, as wanted. \square

A $\{B_1, \dots, B_m\}$ is *maximal block system* for G if there is no other block system having a block X ($|X| \neq n$) with $B_1 \subseteq X$.

In what follows, we resume the results obtained in the proof of Proposition 2.1 of [4].

Proposition 3.3. [4] *Suppose that L , C and R are as defined above with respect to a maximal block system. Then these sets of generators satisfy the following properties.*

- (a) $\langle L \rangle$ has a primitive action on the blocks.
- (b) $|L| \leq m - 1$.
- (c) If $|L| = m - 1$ and $m \geq 5$ then the action of $\langle L \rangle$ on the blocks corresponds to the standard Coxeter generators of S_m .
- (d) If the set of labels of L is not an interval then $|L| \leq 2 \log_2 m$ and $m \geq 60$, thus $|L| < m/4 - 3$.
- (e) If the set of labels of L is not an interval then $r < n/2$.
- (f) $|C| \leq k - 1$
- (g) If the set of labels of the elements of L is an interval then $|R| \leq 2$.
- (h) If $m \neq 2$ and the set of labels of the elements of L is an interval, then $r \leq m + k - 1$.

Corollary 3.4. *If $r \geq n/2 \geq 7$ then one of the situations occurs: $k = 2$, $m = 2$.*

Proof. Suppose that $k, m \neq 2$. By Proposition 3.3 $r = |S| = |L| + |C| + |R| \leq m + k - 1$. Thus $\frac{mk}{2} \leq m + k - 1$, which is equivalent to $(m - 2)(k - 2) \leq 2$. The later inequality holds for $n \geq 14$. \square

4. CASE: IMPRIMITIVE GROUPS WITH BLOCKS OF SIZE 2.

In this case we consider that G has a maximal block system with blocks of size two, which implies that the action of G on the blocks is primitive. Consider the sets S , L , C and R as in the previous section. Let us assume that $r = |S| \geq n/2$. Then by Proposition 3.3 we have that the set of labels of L is an interval and as $m \neq 2$, $|C| \leq 1$. Furthermore, if C is nonempty then the element of C is the permutation swapping all pairs of points within a block.

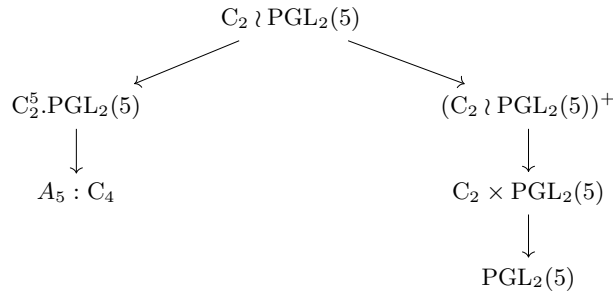
In what follows let Ψ be the sggg corresponding to the action of L on the blocks.

Proposition 4.1. *If $r \geq n/2$ then Ψ cannot be isomorphic to one of the groups D_{10} , $\text{PGL}_2(5)$ or $\text{PSL}_2(5)$.*

Proof. Suppose that Ψ is one of the groups listed in this proposition. By Theorem 2.2 $|L| \leq n/2 - 2$. In the cases $|R \cup C| = 3$, and we will assume that $C = \{\rho_0\}$ and $R = \{\rho_1, \rho_{r-1}\}$. Notice that $G_{i,j} < G_i < G$, $G_{i,j} < G_j < G$ and $G_i \neq G_j$. However G_i and G_j might be isomorphic, since the corresponding permutation representations, can give distinct subgraphs of \mathcal{G} .

Let us consider each group, $\text{PGL}_2(5)$, $\text{PSL}_2(5)$ and D_{10} , separately.

(1) $\Psi \cong \text{PGL}_2(5)$: The following diagram gives the subgroups of $C_2 \wr \text{PGL}_2(5)$ which have Ψ as the block action.



Suppose first that $|R \cup C| = 3$. As $\text{PGL}_2(5)$ is almost simple and $G_{1,r-1} = \langle L \rangle \times \langle C \rangle$, $G_{1,r-1} \not\cong \text{PGL}_2(5)$. Hence $G_{1,r-1}$ is either $A_5 : C_4$ or $C_2 \times \text{PGL}_2(5)$. Then $G_1 \cong G_{r-1} \cong C_2^5 \cdot \text{PGL}_2(5)$ or $G_1 \cong G_{r-1} \cong (C_2 \wr \text{PGL}_2(5))^+$. But these groups have a unique transitive permutation representation on 12 points that is represented as a subgraph of \mathcal{G} , which gives $G_1 = G_{r-1}$, a contradiction. Thus $|R \cup C| = 2$.

Now let $R \cup C = \{\rho_i, \rho_j\}$. To avoid the previous contradiction we must have $G_{i,j} \cong \Psi \cong \text{PGL}_2(5)$ which might be transitive or intransitive (this is the only group of the diagram above that might be intransitive). But then either $G_i \cong G_j \cong C_2 \times \text{PGL}_2(5)$ or $G_i \cong G_j \cong (C_2 \wr \text{PGL}_2(5))^+$. In the first case there is a central involution $\delta \in G_i \cap G_j = G_{i,j} \cong \text{PGL}_2(5)$, a contradiction. In the second case implies $G_i = G_j$, a contradiction.

(2) $\Psi \cong \text{PSL}_2(5)$: Consider the case where $|R \cup C| = 3$. The transitive subgroups of $C_2 \wr \text{PSL}_2(5)$ having block action Ψ are the following.

$$\text{PSL}_2(5) < C_2 \times \text{PSL}_2(5) < (C_2 \wr \text{PSL}_2(5))^+ < C_2 \wr \text{PSL}_2(5)$$

In this case we must have $G_{1,r-1} \cong C_2 \times \text{PSL}_2(5)$ and $G_1 \cong G_{r-1}$. But this gives $G_1 = G_{r-1}$, a contradiction. Then $|R \cup C| = 2$. In this case, let $R = \{\rho_i\}$ and $C = \{\rho_j\}$. This implies that $G_i \cong C_2 \times \text{PSL}_2(5)$ and G_j cannot be isomorphic to $C_2 \times \text{PSL}_2(5)$. Hence either G_j is a proper subgroup of G_i or G_i is a proper subgroup of G_j , a contradiction.

(3) $\Psi \cong D_{10}$: In this case $|R \cup C| = 3$. The transitive subgroups of $C_2 \wr D_{10}$ having block action Ψ are the following.

$$D_{10}; C_2 \times D_{10}; (C_2 \wr D_{10})^+; C_2 \wr D_{10}$$

Then $G_1 \cong C_2 \times D_{10}$. If $G_0 \cong G_{r-1} \cong (C_2 \wr D_{10})^+$ then $G_0 = G_{r-1}$, a contradiction. \square

Proposition 4.2. *Let $r \geq n/2$. If $|R \cup C| = 3$ then Ψ is isomorphic to $S_{n/2}$ or $A_{n/2}$.*

Proof. Up to duality we may assume that $C = \{\rho_0\}$, $R = \{\rho_1, \rho_{r-1}\}$ and $L = \{\rho_2, \dots, \rho_{r-2}\}$. Let α_i be the action of ρ_i on the blocks. Then $\Psi = \langle \alpha_2, \dots, \alpha_{r-2} \rangle$. Here we consider the following notation $\Psi_i := \langle \alpha_j \mid j \neq i \rangle$.

Let us prove that Ψ_i is intransitive for every $i \in \{2, \dots, r-2\}$ when $n/2 \geq 9$.

(1) Ψ_2 and Ψ_{r-2} are intransitive: Suppose that Ψ_2 is transitive. If ρ_1 swaps a pair of points inside a block, then, as ρ_1 centralizes $G_{>2}$, $\rho_1 = \rho_0$, a contradiction. Thus ρ_1 swaps a pair of blocks. Then, the transitivity of $G_{>2}$, forces ρ_1 to swap all blocks pair wisely. Moreover $\rho_0\rho_1$ is also a permutation swapping all blocks pair wisely and $\rho_0\rho_1 \in \langle L \rangle$, a contradiction. Therefore Ψ_2 is intransitive and by duality Ψ_{r-2} is also intransitive.

(2) Ψ_i is intransitive for $i \in \{3, \dots, r-3\}$: Suppose that $\Psi_i (= \Psi_{\{2, \dots, i-1\}} \times \Psi_{\{i+1, \dots, r-2\}})$ is transitive. As $\Psi_{\{2, \dots, i-1\}} \leq \Psi_{r-2}$ and $\Psi_{\{i+1, \dots, r-2\}} \leq \Psi_2$, these groups of the decomposition of Ψ_i are intransitive, by (1). Hence Ψ_i is imprimitive. Then, by Lemma 2.8, $r-4 = |L| - 1 \leq (k' - 1) + (m' - 1)$. As $|L| \geq n/2 - 3$, we have that $n/2 - 4 \leq k' + m' - 2$, which is only possible if $(k' - 1)(m' - 1) \leq 3$, which is never the case as $n/2 = k'm' \geq 9$. This proves that for $i \in \{2, \dots, r-2\}$, Ψ_i is intransitive. Now if $n/2 \geq 2 \cdot 3 + 3 = 9$, by Proposition 2.14, $\Psi \cong S_{n/2}$, as wanted.

Now suppose that $n/2 \leq 8$. As $|L| \geq n/2 - 3$, by Lemma 2.7 and Proposition 4.1 Ψ is either isomorphic to $S_{n/2}$ or to $A_{n/2}$. \square

Proposition 4.3. *If $r \geq n/2$ then $|R \cup C| < 3$ and $|L| \geq n/2 - 2$.*

Proof. Suppose that $|R \cup C| = 3$. Then by Proposition 4.2 Ψ is isomorphic to $S_{n/2}$ or $A_{n/2}$. First consider the case $\Psi \cong S_{n/2}$. Then $C = \{\rho_0\}$ and $R = \{\rho_1, \rho_{r-1}\}$. As L is a minimal set of generators generating the block action, and G is transitive, $G_{0,1} = \langle L \cup \{\rho_{r-1}\} \rangle$ is transitive. Hence the group $G_{0,1}$ is a transitive subgroup of $C_2 \wr S_{n/2}$. Then, by Theorem 2.23, G_0 and G_1 must be index 2 subgroups of $C_2 \wr S_{n/2}$, while $G_{0,1}$ is twice bigger than $S_{n/2}$. Moreover $G_{0,1}$ must contain the permutation swapping all pairs of points fixing the blocks, that is, $\rho_0 \in G_{0,1}$, a contradiction. The same argument can be applied when Ψ is isomorphic to $A_{n/2}$. \square

Proposition 4.4. *Let $r \geq n/2$. If $|R \cup C| = 2$ then $\langle L \rangle \cong S_{n/2}$, $|R| = |C| = 1$, $n/2$ is odd and $G \cong C_2 \wr S_{n/2}$.*

Proof. In this case $|L| \geq n/2 - 2$, hence Ψ is a transitive sggì of rank $n/2 - 2$ and degree $n/2$. Suppose that Ψ is neither $S_{n/2}$ nor $A_{n/2}$. Hence by Lemma 2.7 Ψ is isomorphic to one of the groups D_{10} , $\text{PSL}_2(5)$ or $\text{PGL}_2(5)$. But Proposition 4.1 excludes the possibility of Ψ being isomorphic to these groups. Hence $\langle L \rangle$ is a subgroup of $C_2 \wr H$ with H being $A_{n/2}$ or $S_{n/2}$. Let us now use Theorem 2.23 to conclude that $\langle L \rangle \cong H$. As $\langle L \rangle = G_{i,j}$ for some i and j , $\langle L \rangle$ cannot be an index 2 subgroup of $C_2 \wr H$. Suppose that $|\langle L \rangle| = 2|H|$. Then for distinct i and j the subgroups G_i and G_j must be index 2 subgroups of $C_2 \wr H$. Thus G_i and G_j , and consequently $\langle L \rangle$, contain all even permutations fixing the blocks, a contradiction. Hence $\langle L \rangle \cong H$, as wanted.

Now suppose that $H \cong A_{n/2}$. Then $\Psi \cong A_{n/2} \cong \langle L \rangle$, thus Ψ is a string C-group. Hence we can use Theorem 2.17 to conclude that $|L| \leq \frac{n/2+1}{2}$. This implies that $n/2 \leq 5$. The only alternating group of degree at most 5 that is a string C-group is A_5 . This implies that $\langle L \rangle \cong \Psi \cong A_5$. But A_5 does not have an imprimitive permutation representation on 10 points, a contradiction.

Suppose that $R \cup C = \{\rho_i, \rho_j\}$. As $G_{i,j}$ does not contain a nontrivial permutation fixing all blocks, we may consider that, $G_{i,j}$ is an index 2 subgroup of G_j and G_i is an index 2 subgroup of $C_2 \wr S_{n/2}$. Hence ρ_i commutes with all the elements of $G_{i,j}$, thus $\rho_i \in C$. Moreover as G_j is not a subgroup of G_i , ρ_i must be an odd permutation, which is only possible if $n/2$ is odd. We also conclude that $G \cong C_2 \wr S_{n/2}$

The rest follows from Theorems 2.11 and 2.12. □

Now using Proposition 4.4 it is possible to determine the possibilities for the permutation representation graph of Γ when $|R \cup C| = 2$.

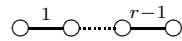
Corollary 4.5. *Let $r \geq n/2 \geq 7$. If $|R \cup C| = 2$ then $n/2$ is odd and \mathcal{G} is, up to duality, one of the graphs of Table 4.*

Proof. Using Proposition 4.4, it is possible to determine the possibilities for the permutation representation graph of G when $|R \cup C| = 2$.

Up to duality we may assume that $C = \{\rho_0\}$ where ρ_0 is the permutation swapping all pairs of points within a block, say $\rho_0 = (1, 2) \dots (n-1, n)$. Then either $R = \{\rho_1\}$ or $R = \{\rho_{r-1}\}$.

We also need to consider two possibilities, either $|L| = n/2 - 1$ or $|L| = n/2 - 2$. When $|L| = n/2 - 1$, $\langle L \rangle$ is the automorphism group of the simplex, which is self dual; when $|L| = n/2 - 2$, $\langle L \rangle$ is the automorphism group of one of the two abstract regular polytopes of rank $n/2 - 2$ for $S_{n/2}$, having one of the Schläfli symbol $\{3, \dots, 3, 6, 4\}$ or $\{4, 6, 3, \dots, 3\}$ (which are dual to each other). Finally, the possibilities for the permutation representation graph of these polytopes are determined by Propositions 2.13 and 2.20, depending on whether $\langle L \rangle$ is intransitive or transitive. If $\langle L \rangle$ is intransitive, the permutation graph of $\langle L \rangle$ is given by two copies of one of the graphs given in Proposition 2.13. The possibilities for the element of R , which must be an even permutation, are determined by the commuting property. With this we obtain the graphs listed in the Table 4. □

Proposition 4.6. *Let $r \geq n/2 \geq 7$. If $|R \cup C| = 1$ then $\Psi \cong S_{n/2}$ and the action of G on the blocks is given by the following graph.*



Proof. In this case $|L| \geq n/2 - 1$, hence Ψ is a transitive sggi of rank $n/2 - 1$ and degree $n/2$. Up to duality we may assume that $R \cup C = \{\rho_0\}$. By Corollary 2.10, we have that $\Psi \cong S_{n/2}$ and the block action graph given in the statement of this proposition. □

Proposition 4.7. *Let $r \geq n/2 \geq 7$. If $|R \cup C| = 1$ and $\langle L \rangle \cong S_{n/2}$ then \mathcal{G} is one of the graphs of Table 5 or the graphs (1) and (2) of Table 8.*

Proof. First, consider the case where $\langle L \rangle$ is intransitive. Then $\langle L \rangle$ is represented by two copies of the permutation graph of the symplex. Suppose first that $|C| = 1$. In this case $G \cong C_2 \times S_{n/2}$ and G admits another block system with exactly two blocks. The permutation representation graph of Γ is the graph (1) appearing on Table 8. If $|C| = 0$ then we get, up to duality, the permutation representation graphs (13) and (14) of Table 5.

Consider now that $\langle L \rangle$ is transitive. In this case the permutation representation of $\langle L \rangle$ is given by Proposition 2.20. If $|C| = 1$ then $G \cong C_2 \times S_{n/2}$ and G admits another block system with exactly two blocks. The permutation representation graph of Γ is the graph (2) of Table 8. If $|C| = 0$ then, we get the graphs (15) and (16) of Table 5. □

In what remains of this session assume the following.

- $r \geq n/2 \geq 7$;
- $R \cup C = \{\rho_0\}$ and
- $\langle L \rangle \not\cong S_{n/2}$.

By Proposition 4.6 the permutation representation graph of Ψ determines a natural ordering on the blocks: let B_1 be the first block (on the left) and $B_{n/2}$ be the last block (on the right). Consider the embedding $f : \langle L \rangle \rightarrow S_{n/2}$. By Corollary 2.21, either $\text{Ker}(f) \cong C_2$ or $\text{Ker}(f) \cong (C_2)^{n/2-1}$. We will represent the elements of $\alpha \in \text{Ker}(f)$ as a vector $\tilde{\alpha} \in \{0, 1\}^{n/2}$. Particularly, the central involution permuting all pairs of points within a block corresponds to the all 1's vector. In what follows consider the following notation where x^i represents a sequence of length i of x 's (x, x, \dots, x), $x \in \{0, 1\}$ and i being any nonnegative integer.

$$\begin{aligned} O &:= (0^{n/2}) \\ U &:= (1^{n/2}) \\ L_i &:= (1^i, 0^{n/2-i}) \\ R_i &:= (0^i, 1^{n/2-i}) \\ V_i &:= (1^i, 0, 0, 1^{n/2-(i+2)}) \\ T_i &:= (1^i, 0, 0, 0, 1^{n/2-(i+3)}) \end{aligned}$$

For $i \in \{1, \dots, r-1\}$ let $\rho_i = \alpha_i \beta_i$ with α_i being a permutation fixing the blocks and β_i being the permutation swapping B_i and B_{i+1} . Then thanks to the commuting property, $\tilde{\alpha}_i$ is either O , L_{i-1} , R_{i+1} or V_{i-1} .

Let $\delta_i := (\rho_i \rho_{i+1})^3$ ($i > 0$). In the following table we determine all the possibilities for $\tilde{\delta}_i$. As $\delta_i \in \text{Ker}(f)$ and $\text{Ker}(f) \cong C_2$ or $\text{Ker}(f) \cong (C_2)^{n/2-1}$, $\delta_i = (\rho_i \rho_{i+1})^3$ is either the permutation $(1, 2) \dots (n-1, n)$ or an even permutation. In the following table we determine all the possibilities for $\tilde{\delta}_i$ for all the possibilities for the pair $(\tilde{\alpha}_i, \tilde{\alpha}_{i+1})$ in some cases the result is an odd permutation, thus these cases cannot happen.

$i \in \{2, \dots, r-3\}$					$i = 1$				$i = r-2$			
$\tilde{\alpha}_i \backslash \tilde{\alpha}_{i+1}$	O	L_i	R_{i+2}	V_i	$\tilde{\alpha}_1 \backslash \tilde{\alpha}_2$	O	L_1	R_3	V_1	$\tilde{\alpha}_{r-2} \backslash \tilde{\alpha}_{r-1}$	O	L_{r-2}
O	O	L_{i+2}	R_{i+2}	U	O	O	odd	R_3	U	O	O	U
L_{i-1}	L_{i-1}	odd	T_{i-1}	R_{i-1}	O	O	odd	R_3	U	L_{r-3}	L_{r-3}	odd
R_{i+1}	R_{i-1}	T_{i-1}	odd	L_{i-1}	R_2	U	R_3	odd	O	R_{r-1}	odd	L_{r-3}
V_{i-1}	U	R_{i+2}	L_{i+2}	O						V_{r-3}	U	O

TABLE 3. Possibilities for $\tilde{\delta}_i$.

Proposition 4.8. *If $\text{Ker}(f) \cong C_2$ then \mathcal{G} is, up to duality, one of the graphs of Table 6.*

Proof. In this case $\delta_i = (1, 2)(3, 4) \dots (n-1, n)$ for some $i \in \{1, \dots, r-1\}$ and, for $j \in \{1, \dots, r-1\} \setminus \{i\}$, δ_j must be trivial. In addition, up to duality, either $\rho_0 = (1, 2)$ or $\rho_0 = (3, 4) \dots (n-1, n)$.

Since $\tilde{\delta}_i = U$ the possibilities for $(\tilde{\alpha}_i, \tilde{\alpha}_{i+1})$ are determined in Table 3. Suppose that $(\tilde{\alpha}_i, \tilde{\alpha}_{i+1}) = (V_{i-1}, O)$ and $i \neq 1$. Then, as $\tilde{\delta}_j = O$ for $j \neq i$, $\tilde{\alpha}_j = V_{j-1}$ for $j \in \{2, \dots, i-1\}$ and $\tilde{\alpha}_1 = R_2$. This gives the graphs (17) and (18) of Table 6. Analogously for when $(\tilde{\alpha}_i, \tilde{\alpha}_{i+1}) = (O, V_i)$ we get the graphs (19) and (20) of Table 6. The graphs (21)-(24) of Table 6 are obtained when $i = 1$ and $i = r-2$. \square

Lemma 4.9. *Let $i \in \{1, \dots, r-2\}$. If δ_i is a non-trivial even permutation and $\delta_i \neq (1, 2)(3, 4) \dots (n-1, n)$ then $n/2$ is odd and $\rho_0 = (1, 2)(3, 4) \dots (n-1, n)$.*

Proof. Suppose that neither δ_i nor ρ_0 is equal to the permutation $(1, 2)(3, 4) \dots (n-1, n)$. Then, by the commuting property, ρ_0 is, up to duality, one of the permutations: $(1, 2)$ or $(3, 4) \dots (n-1, n)$.

Let first consider that $i \neq 1$. Note that $(\rho_0 \rho_1)^2 = (1, 2)(3, 4)$ and $[(\rho_0 \rho_1)^2]^{\rho_1 \rho_2 \rho_1} = (3, 4)(5, 6)$. If δ_i fixes B_1 point wisely then $G_{0,1}$ contains all even permutation fixing the blocks $B_2, \dots, B_{n/2}$, particularly $(3, 4)(5, 6) \in G_{0,1}$. Hence $(3, 4)(5, 6) \in G_{0,1} \cap \langle \rho_0, \rho_1, \rho_2 \rangle = \langle \rho_2 \rangle$, a contradiction. If δ_i swaps the points of the block B_1 , then $(1, 2)(3, 4) \in G_{0,1}$. Hence $(1, 2)(3, 4) \in G_{0,1} \cap \langle \rho_0, \rho_1 \rangle$, a contradiction. Thus if δ_i is not $(1, 2)(3, 4) \dots (n-1, n)$ then $\rho_0 = (1, 2)(3, 4) \dots (n-1, n)$.

Now suppose that $i = 1$. Then G_0 contains all even permutations fixing the blocks $B_1, \dots, B_{n/2}$. Particularly $(1, 2)(3, 4) \in G_0$. If $\rho_0 = (1, 2)$ then $G_{<4}$ contains all permutations fixing the blocks B_1, B_2 and B_3 particularly $(1, 2)(3, 4) \in G_{<4}$. But $(1, 2)(3, 4) \notin \langle \rho_1, \rho_2 \rangle$, contradicting the intersection property. Now consider that $\rho_0 = (3, 4) \dots (n-1, n)$. According to Table 3, $(\tilde{\alpha}_1, \tilde{\alpha}_2) \in \{(O, R_3), (R_2, L_1)\}$, which gives

$\tilde{\delta}_1 = (0^3, 1^{n/2-3})$. Thus $n/2$ must be odd and therefore ρ_0 is even. As G_0 contains all even permutation fixing the blocks, $\rho_0 \in G_0$, a contradiction. This shows that $\rho_0 = (1, 2)(3, 4) \dots (n-1, n)$.

Finally, if $n/2$ is even, as in both cases G_0 contains all even permutations fixing the blocks, $(1, 2)(3, 4) \dots (n-1, n) \in G_0$. Hence $\rho_0 \in G_0$, a contradiction. Hence $n/2$ is odd. \square

Lemma 4.10. *Let $i, j \in \{1, \dots, r-2\}$ and $i < j$. If δ_i and δ_j are nontrivial even permutations different from $(1, 2)(3, 4) \dots (n-1, n)$ then either $j = i+1$ or there exists $k \in \{i+1, \dots, j-1\}$ such that δ_k is nontrivial.*

Proof. Suppose for a contradiction that $j \neq i+1$ and that δ_k is trivial for $k \in \{i+1, \dots, j-1\}$. Then $\langle \rho_{i+1}, \dots, \rho_j \rangle \cong S_{j-i+1}$, particularly $\langle \rho_{i+1}, \dots, \rho_j \rangle$ does not contain a nontrivial permutation fixing $\{B_{i+1}, \dots, B_{j+1}\}$. Moreover any permutation in $G_{0, i, j+1}$ that swaps a pair of points within one of the blocks $\{B_{i+1}, \dots, B_{j+1}\}$, must swap all pairs of points of these blocks.

As $\delta_j \neq (1, 2)(3, 4) \dots (n-1, n)$, either $\delta_j \delta_j^{\rho_j^{i-1}}$ or $\delta_j \delta_j^{\rho_j^{i+2}}$ is a 2-transposition fixing $\{B_{j+1}, \dots, B_{n/2}\}$. Particularly, $G_{0, i}$ contains all the 2-transpositions fixing $\{B_{i+1}, \dots, B_{j+1}\}$. Using similar argument we also conclude that $G_{0, j+1}$ contains all the 2-transpositions fixing $\{B_{i+1}, \dots, B_{j+1}\}$. But then $G_{0, j+1} \cap G_{0, i} > G_{0, i, j+1}$, a contradiction. \square

Proposition 4.11. *If $\text{Ker}(f) = C_2^{n/2-1}$ then we have the following holds.*

- (a) $|\{j \in \{1, \dots, r-2\} \mid \delta_j \neq id\}| > 1$ and $\{j \in \{1, \dots, r-2\} \mid \delta_j \neq id\}$ is an interval;
- (b) $n/2$ is odd and $\rho_0 = (1, 2)(3, 4) \dots (n-1, n)$.

Proof. As $\text{Ker}(f) = C_2^{n/2-1}$ we have that for some $i \geq 1$, δ_i is an even permutation different from $(1, 2) \dots (n-1, n)$. Suppose that δ_j is trivial for $j \in \{1, \dots, r-1\} \setminus \{i\}$. Consider first $i \neq 1, r-2$. According to Table 3 we have the following contradiction.

$$\left(\tilde{\delta}_{i-1} = O \wedge \tilde{\delta}_i \neq O \Rightarrow \tilde{\alpha}_i \in \{O, V_{i-1}\} \wedge \tilde{\alpha}_{i+1} \notin \{O, V_i\} \right) \text{ and } \left(\tilde{\delta}_{i+1} = O \Rightarrow \tilde{\alpha}_{i+1} \in \{O, V_i\} \right)$$

For $i = 1$ (and similarly when $i = r-2$) we also get the following contradiction

$$\left(\tilde{\delta}_2 = O \Rightarrow \tilde{\alpha}_2 \in \{O, V_1\} \right) \text{ and } \left(\tilde{\delta}_1 \neq O \Rightarrow (\tilde{\alpha}_1, \tilde{\alpha}_2) \in \{(R_2, L_1), (O, R_3)\} \Rightarrow \tilde{\delta}_2 \notin \{O, V_1\} \right)$$

This proves that $|\{j \in \{1, \dots, r-2\} \mid \delta_j \neq id\}| > 1$ and also shows that $\{j \mid \delta_j \neq id\}$ must be an interval. By Lemma 4.9, $n/2$ is odd and $\rho_0 = (1, 2)(3, 4) \dots (n-1, n)$. \square

Lemma 4.12. *If $x = \min\{j \in \{1, \dots, r-2\} \mid \delta_j \neq id\}$ and $h \geq x+3$ then $G_{<h}$ contains all even permutations fixing B_1, \dots, B_h .*

The dual of this lemma also holds.

Proof. Suppose first that $x > 1$. As in the previous proof we have,

$$\tilde{\delta}_{x-1} = O \wedge \tilde{\delta}_x \neq O \Rightarrow \tilde{\alpha}_x \in \{O, V_{x-1}\} \wedge \tilde{\alpha}_{x+1} \notin \{O, V_x\} \Rightarrow \tilde{\delta}_x \in \{L_{x+2}, R_{x+2}\}$$

In any case $\delta_x \delta_x^{\rho_x^{x+2}}$ is the 2-transposition swapping the points inside the blocks B_{x+2} and B_{x+3} . Hence $G_{<x+3}$ contains all even permutations fixing B_1, \dots, B_{x+3} .

Consequently for $h \geq x+3$, $G_{<h}$ contains all even permutations fixing B_1, \dots, B_h .

Suppose that $x = 1$. As, by Proposition 4.11 $\tilde{\rho}_0 = U$, we must have $\tilde{\delta}_1 = R_3$, hence $\delta_1 \delta_1^{\rho_1^2} = (3, 4)(5, 6)$ hence we get the same result as for $x > 1$. \square

Proposition 4.13. $\{j \in \{1, \dots, r-2\} \mid \delta_j \neq id\} = \{x, x+1\}$ for some $i \in \{1, \dots, r-2\}$.

Proof. Suppose that $x = \min\{j \in \{1, \dots, r-2\} \mid \delta_j \neq id\}$ and $y = \max\{j \in \{1, \dots, r-2\} \mid \delta_j \neq id\}$ and that $y > x+1$. Then, by Lemma 4.12 $G_{<x+3}$ contains all even permutations fixing B_1, \dots, B_{x+3} . But also, as $x < y-1$, $G_{>x}$ contains all even permutations fixing B_{x+1}, \dots, B_r . But then $G_{>x} \cap G_{<x+3}$ is not a dihedral group, contradicting the intersection property. Hence $y = x+1$, as wanted. \square

In what follows let x be the index determined in the previous proposition, meaning that, δ_x and δ_{x+1} are the unique nontrivial δ 's.

Proposition 4.14. *If $\text{Ker}(f) = C_2^{n/2-1}$ then \mathcal{G} is, up to duality, one of the graphs of Table 7.*

Proof. Suppose first that $x \notin \{1, r-3\}$.

As $\tilde{\delta}_{x-1} = \tilde{\delta}_{x+2} = O$, we must have $\tilde{\alpha}_x \in \{O, V_{x-1}\}$ and $\tilde{\alpha}_{x+2} \in \{O, V_{x+1}\}$. Moreover,

$$\tilde{\delta}_x \in \{L_{x+2}, R_{x+2}\} \text{ and } \tilde{\delta}_{x+1} \in \{L_x, R_x\}.$$

Let us consider separately the following cases: in case (A) x is even, in case (B) and x is odd. Notice that L_i is even weight vector if and only if i is even, while R_i is an even weight vector if and only if i is odd.

(A) In this case $\tilde{\delta}_x = L_{x+2}$ and $\tilde{\delta}_{x+1} = L_x$.

$$\tilde{\delta}_x = L_{x+2} \Rightarrow (\tilde{\alpha}_x, \tilde{\alpha}_{x+1}) \in \{(O, L_x), (V_{x-1}, R_{x+2})\}$$

If $(\tilde{\alpha}_x, \tilde{\alpha}_{x+1}) = (O, L_x)$ we get the following.

$$\begin{cases} \tilde{\alpha}_i = O, & i \neq x+1 \\ \tilde{\alpha}_{x+1} = L_x \end{cases}$$

If $(\tilde{\alpha}_x, \tilde{\alpha}_{x+1}) = (V_{x-1}, R_{x+2})$ we get the following.

$$\begin{cases} \tilde{\alpha}_i = V_{i-1}, & i \neq x+1 \\ \tilde{\alpha}_{x+1} = R_{x+2} \end{cases}$$

Then, when x is even, we get two possibilities for \mathcal{G} corresponding to graphs (25) and (26) of Table 7.

(B) In this case $\tilde{\delta}_x = R_{x+2}$ and $\tilde{\delta}_{x+1} = R_x$.

$$\tilde{\delta}_x = R_{x+2} \Rightarrow (\tilde{\alpha}_x, \tilde{\alpha}_{x+1}) \in \{(O, R_{x+2}), (V_{x-1}, L_x)\}$$

If $(\tilde{\alpha}_x, \tilde{\alpha}_{x+1}) = (O, R_{x+2})$ we get the following.

$$\begin{cases} \tilde{\alpha}_i = O, & i \neq x+1 \\ \tilde{\alpha}_{x+1} = R_{x+2} \end{cases}$$

If $(\tilde{\alpha}_x, \tilde{\alpha}_{x+1}) = (V_{x-1}, L_x)$ we get the following.

$$\begin{cases} \tilde{\alpha}_i = V_{i-1}, & i \neq x+1 \\ \tilde{\alpha}_{x+1} = L_x \end{cases}$$

Then, when x is odd, we get two possibilities for \mathcal{G} corresponding to graphs (27) and (28) of Table 7.

Suppose that $x = 1$. Then by Table 3, either $\tilde{\delta}_1 = R_3$ and $\tilde{\delta}_2 \in \{L_1, R_1, L_4, T_1, R_4\}$. As L_1 and R_4 are odd permutations these can be excluded from the set of possibilities for $\tilde{\delta}_2$. If $\tilde{\delta}_2 \in \{T_1, L_4\}$ then $\tilde{\alpha}_3 \notin \{O, V_2\}$, hence $\tilde{\delta}_3 \neq O$, a contradiction. This gives only one possibility which is, $\tilde{\delta}_2 = R_1$. Consequently $(\tilde{\alpha}_2, \tilde{\alpha}_3) \in \{(R_3, O), (L_1, V_2)\}$. If $(\tilde{\alpha}_2, \tilde{\alpha}_3) = (R_3, O)$ then

$$\begin{cases} \tilde{\alpha}_i = O, & i \neq 2 \\ \tilde{\alpha}_2 = R_3 \end{cases}$$

If $(\tilde{\alpha}_2, \tilde{\alpha}_3) = (L_1, V_2)$ then

$$\begin{cases} \tilde{\alpha}_1 = R_2 \\ \tilde{\alpha}_2 = L_1, \\ \tilde{\alpha}_i = V_{i-1}, & i \geq 3 \end{cases}$$

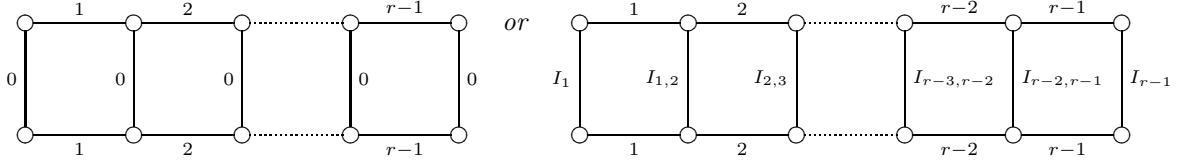
Then, when $x = 1$ (which is odd), we get two possibilities for \mathcal{G} corresponding to graphs (27) and (28) of Table 7. For $x = r-3 = n/2 - 3$ similar arguments give the possibilities (25) and (26) of Table 7. \square

5. CASE: IMPRIMITIVE GROUPS WITH TWO BLOCKS.

In what follows let $\{p_1, \dots, p_{r-1}\}$ be the Schläfli symbol of Γ .

Now we deal with the case $m = 2$, in this case L is a singleton. Let $\mathcal{B} = \{B_1, B_2\}$ denote the block system. By Proposition 2.18 we may assume that $r = n/2$.

Proposition 5.1. *Let $r = n/2$. If $|R| = 0$, then $G \cong C_2 \times S_{n/2}$ and, up to duality, we have that $p_1 = 2$ and Γ_0 is the automorphism group of a polytope of rank $(n/2 - 1)$ for $S_{n/2}$. If $n/2 \geq 7$ then the Schläfli symbol of Γ is $\{2, 3, \dots, 3\}$ and G admits the following two permutation representations,*



In the first graph the blocks are the G_0 -orbits while in the second graph each edge connects vertices in different blocks.

Proof. In this case $\langle C \rangle$ acts faithfully on the pairs of points swapped by the element of L . Hence $|C| \leq n/2 - 1$. As $r = n/2$, $|C| = n/2 - 1$.

Suppose first that $\langle C \rangle$ is intransitive. Then the permutation representation graph of $\langle C \rangle$ is given by two copies of one of the two first graphs given in Proposition 2.13. Hence, for $n/2 \geq 7$, Γ has the first permutation representation graph given in this proposition.

Let us now assume that $\langle C \rangle$ is transitive. Let $L = \{\rho_i\}$. Let, for $j \neq i$, $\alpha_j = \rho_j \rho_i^\tau$ where $\tau = 1$ if $B_1 = B_2 \rho_j$ and $\tau = 0$ if ρ_j fixes the blocks. The set $\Lambda := \{\alpha_j \mid j \in \{0, \dots, r-1\}\}$ is independent, indeed if $\alpha_k \in \langle \alpha_j \mid j \neq k \rangle$, then $\rho_k \in G_k$, a contradiction. Moreover $\langle \Lambda \rangle$ acts faithfully on the $n/2$ pairs of points that are swapped by ρ_i , and $|\Lambda| = n/2 - 1$. Hence by Corollary 2.10, as $n/2 \geq 7$, $\langle \Lambda \rangle$ is a string C-group having the first permutation representation graph given in Proposition 2.13. Particularly $\langle \Lambda \rangle \cong S_{n/2}$, which implies that $i \in \{0, r-1\}$ and the order of the product of consecutive α_j 's is 3. Consider $i = 0$ and let $j \in \{1, \dots, r-1\}$ such that $\rho_j \in C$ swaps the blocks. Suppose that the consecutive generator $\rho_k \in C$, with $k \in \{j-1, j+1\}$ does not swap the blocks. As $(\alpha_j \alpha_k)^3 = id$, by the definition of α_j , we have that $(\rho_j \rho_0 \rho_k)^3 = id$. If $(\rho_j \rho_k)^3 = id$, then we have $\rho_0 = id$, a contradiction. If $(\rho_j \rho_k)^3 \neq id$, then we have $\rho_0 = (\rho_j \rho_k)^3$, i.e. $\rho_0 \in \langle C \rangle$, a contradiction. As $\langle C \rangle$ is transitive, hence the consecutive generator must also swap the blocks, implying that all generators of C swap the blocks and the product of consecutive generators must also be 3. The case when $i = r-1$ is equivalent. This gives, up to duality, the second possibility given in the statement of this proposition. \square

Proposition 5.2. *Let $r = n/2 \geq 7$. If $\langle C \cup L \rangle$ is transitive and $|R| \neq 0$ then $\langle C \rangle$ is transitive.*

Proof. Suppose that $\langle C \cup L \rangle$ is transitive but $\langle C \rangle$ is intransitive. As $\langle C \rangle$ is a normal subgroup of $\langle C \cup L \rangle$, the $\langle C \rangle$ -orbits are swapped by the element of L . Particularly $\langle C \rangle$ must have exactly two orbits.

Notice that the $\langle C \rangle$ -orbits do not need to be B_1 and B_2 . Indeed the elements of C do not need to fix the blocks of \mathcal{B} and the elements of R do not need to preserve the $\langle C \rangle$ -orbits. Nevertheless if $L = \{\rho_i\}$ and $\delta \rho_i$ has a fixed point, then $B_1 \delta = B_2$, which implies that δ is fixed-point-free.

The group generated by C acts faithfully on the pairs of points swapped by the element of L . Thus $\langle C \rangle$ is embedded into $S_{n/2}$. Let us deal separately with the cases $\langle C \rangle \cong S_{n/2}$ and $\langle C \rangle \not\cong S_{n/2}$.

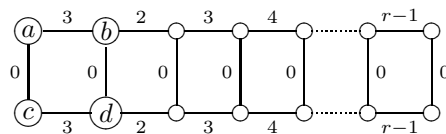
Case 1: $\langle C \rangle \cong S_{n/2}$.

In this case an element of R cannot fix the $\langle C \rangle$ -orbits, if $\delta \in R$ does fix the $\langle C \rangle$ -orbits, then $\delta \delta^{\rho_i} \in \langle C \rangle$, a contradiction.

Let us prove that in this case $|R| \neq 2$. Suppose the contrary, that $|R| = 2$. As the elements of C must be consecutive (because $\langle C \rangle \cong S_{n/2}$), up to duality, we may assume that $L = \{\rho_1\}$, $R = \{\rho_0, \rho_2\}$. In this case ρ_0 centralizes $\langle C \rangle$, this implies that $\rho_0 = \rho_1$, a contradiction. Thus $|R| = 1$ and the element of R cannot commute with all the elements of C . As $r = n/2$, we must have $|C| = n/2 - 2$.

Up to duality, we may consider $L = \{\rho_0\}$, $R = \{\rho_1\}$ and $C = \{\rho_2, \dots, \rho_{r-1}\}$.

As $n/2 \geq 7$, the permutation representation graph of $\langle C \cup L \rangle$ is, by Proposition 2.13, as follows.



Let $\{a, b, c, d\}$ be as above. If ρ_1 acts nontrivially on $\{1, \dots, n\} \setminus \{a, b, c, d\}$ then, \mathcal{G} has $n/2 - 2$ $\{0, 1\}$ -edges, which implies that $B_1 \rho_1 = B_2$. Particularly ρ_1 is fixed-point-free. Then we get that ρ_1 commutes with ρ_0 ,

contradicting the definition of C . Thus ρ_1 fixes $\{1, \dots, n\} \setminus \{a, b, c, d\}$. Now to avoid a double $\{0, 1\}$ -edge and ρ_1 fixing the $\langle C \rangle$ -orbits, let $a\rho_1 = d$, then, as ρ_1 and ρ_3 commute, $b\rho_1 = c$. But then ρ_1 and ρ_0 also commute, a contradiction.

Case 2: $\langle C \rangle \cong S_{n/2}$.

In this case, by Theorem 2.17 and Propositions 2.15 and 2.18, $|C| \leq (n/2)/2 + 1$. But as $r \leq |C| + 3$ and $r \geq n/2$, we get that $n/2 \leq 8$ (and $n/2 \geq 7$). Hence $n \in \{7, 8\}$ which implies that $|C| \leq (n/2)/2$. Consequently, $n/2 \leq |C| + 3 \leq n/4 + 3$ gives $n \leq 12$, a contradiction. \square

Lemma 5.3. *Let $r \geq n/2 - 2$, $n/2 \geq 7$ and $n \neq 16$. Suppose that $\Phi = (H, \{\alpha_0, \dots, \alpha_{r-1}\})$ is a string C -group satisfying the following*

- H_0 is transitive;
- $(\alpha_0\alpha_1)^2 = id$;
- H has a block system $\mathcal{B} = \{B_1, B_2\}$ with $B_1 = B_2\alpha_0$.

Then H has a primitive action on the $\langle \alpha_0 \rangle$ -orbits.

Proof. As $(\alpha_0\alpha_1)^2 = id$, α_0 is a central involution. As in addition H_0 is transitive, the $\langle \alpha_0 \rangle$ -orbits form a block system for H . Suppose that the action of H on the $\langle \alpha_0 \rangle$ -orbits is imprimitive. Then there exist a block system \mathcal{V} with m blocks of size k such that $H \leq S_k \wr S_m$ with $n = km$ and such that α_0 fixes the blocks. We have that k is even and $k \geq 4$. Let us also consider \mathcal{V} , with k being maximal, that is, such that the action of H on \mathcal{V} is primitive.

Now let L be a subset of $\{\alpha_0, \dots, \alpha_{r-1}\}$ generating independently the action on the m blocks. We have that $\langle L \rangle$ has a primitive action on the m blocks, hence by Proposition 3.3(d), as $r \geq n/2 - 2$, the elements of L are consecutive. Let C be the subset of $\{\alpha_0, \dots, \alpha_{r-1}\}$ that commute with all the elements of L and R be the remaining generators of H . Notice that $\alpha_0 \in C$ and $|R| \leq 2$.

Let $\bar{\alpha}_i$, for $i \neq 0$, be the action of α_i on the $\langle \alpha_0 \rangle$ -orbits. The set $\{\bar{\alpha}_i, i = 1, \dots, r-1\}$ is independent (similarly to the set Λ that was considered in the proof of Proposition 5.1). Let $\bar{L} = \{\bar{\alpha}_i | \alpha_i \in L\}$, $\bar{R} = \{\bar{\alpha}_i | \alpha_i \in R\}$ and $\bar{C} = \{\bar{\alpha}_i | \alpha_i \in C \setminus \{\alpha_0\}\}$.

We claim that $|C| \leq k/2$. Indeed if $\langle C \rangle$ fixes the blocks we have that $|\bar{C}| \leq k/2 - 1$, which implies that $|C| \leq k/2$. If an element of C swaps the blocks, then, as $\langle L \rangle$ is primitive, $m = 2$. In this case the elements of $C \setminus \{\alpha_0\}$ act independently on the $\langle L \cup \{\rho_0\} \rangle$ -orbits, which have exactly four points. Hence $|C \setminus \{\alpha_0\}| \leq n/4 - 1 = k/2 - 1$. Thus we also get what we want $|C| \leq k/2$.

Hence we have the following bound for r .

$$r = |C \cup L \cup R| \leq k/2 + m - 1 + 2.$$

Consequently, $k/2 + m + 1 \geq km/2 - 2$, which gives $(k-2)(m-1) \leq 8$.

As $n/2 \geq 7$, $n \neq 16$, k is even and $k \geq 4$ we need only to consider the following possibilities: $(k, m) = (4, 5)$; $(k, m) = (6, 3)$ or $(k, m) = (10, 2)$. Let us analyse each of them separately.

• $(k, m) = (4, 5)$: In this case $|C| = 2$ and $\langle C \rangle$ fixes the blocks. Let $C = \{\alpha_0, \alpha_1\}$. If $|L| = n/4 - 1 = 4$ then $\langle L \cup \{\bar{\alpha}_1\} \rangle \cong C_2 \times S_5$ and $\langle \bar{L} \cup \{\bar{\alpha}_1\} \cup \bar{R} \rangle \leq C_2 \wr S_5$. Thus $|R| = |\bar{R}| \leq 1$ which gives $r = |L| + |C| + |R| \leq 4 + 2 + 1 = 7 < n/2 - 2$, a contradiction.

• $(k, m) = (6, 3)$: In this case, as $r \geq 7$, we must have $|L| = 2$, $|C| = 3$ and $|R| = 2$. Moreover we may assume that $C = \{\alpha_0, \alpha_1, \alpha_2\}$, $R = \{\alpha_3, \alpha_6\}$ and $L = \{\alpha_4, \alpha_5\}$. Let $\mathcal{V} = \{V_1, V_2, V_3\}$ be the blocks system where $V_1\alpha_4 = V_2$ and $V_2\alpha_5 = V_3$. Then $\langle C \rangle$ must fix the blocks of \mathcal{V} , moreover the $\langle C \rangle$ -orbits are precisely the blocks of \mathcal{V} . Then $\langle C \rangle \cong C_2 \times S_3$, and it has two possible permutation representations determined by the transitive and the intransitive representation of S_3 on 6 points. Now $G_{3,5,6}$ has exactly two orbits, $V_1 \cup V_2$ and V_3 , one of size 12 and the other one of size 6. Thus α_6 must fix these two sets. Moreover, if $V_1\alpha_6 = V_2$, then we can redefine L , say $L = \{\alpha_5, \alpha_6\}$, and then $|R| < 2$, giving $r < n/2 - 2$, a contradiction. Hence, α_6 must fix each block of \mathcal{V} . Then α_6 must swap the blocks of \mathcal{B} which forces the equality $\alpha_6 = \alpha_0$, a contradiction.

• $(k, m) = (10, 2)$: In this case we have $|L| = 1$ and, as $r \geq n/2 - 2 = 8$, we must have $|C| = n/4 = 5$ and $|R| = 2$. Let $\mathcal{V} = \{V_1, V_2\}$ be the block system where the blocks are swapped by the element in L . In this case, $\langle C \rangle \cong C_2 \times S_5$ and $\langle C \cup L \rangle \cong C_2^2 \times S_5$. Then we may assume that all elements of C are consecutive and that the last generator of G belongs to R , that is $\alpha_7 \in R$. Furthermore α_7 commutes with all the elements of C . Particularly $\langle C \cup \{\alpha_{r-1}\} \rangle \cong C_2^2 \times S_5$. If it swaps the blocks of \mathcal{V} then we can make a different choice for the element of L giving $|R| < 2$ and $r < 8$, a contradiction. Thus α_7 fixes the blocks.

First note that $\langle C \rangle$ and ρ_7 cannot both fix the blocks of \mathcal{V} , as in that case we would get an intransitive permutation representation of $C_2^2 \times S_5$ with two orbits of size $n/2 = 10$, this is impossible. Indeed it can be checked computationally that the minimal degree of $C_2^2 \times S_5$ is greater than 10. Thus $\langle C \rangle$ must be transitive. Hence, one generator of $C \setminus \{\alpha_0\}$ swaps the blocks of \mathcal{V} . Then we can consider other elements for L giving $|C| < 5$, and therefore $r < 8$, a contradiction. \square

Corollary 5.4. *Let $r \geq n/2 \geq 7$. If $\langle L \cup C \rangle$ is transitive and $|R| \neq 0$, then $\langle L \cup C \rangle$ has a primitive action on the $\langle \rho_i \rangle$ -orbits, where $L = \{\rho_i\}$.*

Proof. As $\langle L \cup C \rangle$ is transitive, by Proposition 5.2, $\langle C \rangle$ is transitive. Suppose first that $n \neq 16$. Then by Lemma 5.3, $\langle L \cup C \rangle$ has a primitive action on the $\langle \rho_i \rangle$ -orbits, where $L = \{\rho_i\}$. Now let $n = 16$ and suppose that $\langle L \cup C \rangle$ acts imprimitively on the $\langle \rho_i \rangle$ -orbits. The action of $\langle C \rangle$ on the $\langle \rho_i \rangle$ -orbits is faithful, hence $\langle C \rangle$ is a string C-group representation of a transitive group of degree 8. Thus by Proposition 2.18 $|C| \leq 8/2 + 1 = 5$, moreover as $r \geq 16/2 = 8$ we must have $|C| = 5$ and $|R| = 2$. Hence $\langle C \rangle$ is the automorphism group of a polytope with Schläfli symbol 3, 4, 4, 3 given in Table 2. Let $C = \{\rho_0, \dots, \rho_4\}$, $R = \{\rho_5, \rho_7\}$ and $L = \{\rho_6\}$. The permutation representation of $G_{5,7}$, can be determined computationally and is as follows.

$$\begin{array}{ll} \rho_0 = (1, 10)(2, 9)(3, 12)(4, 11)(5, 16)(6, 15)(7, 14)(8, 13) & \rho_3 = (1, 9)(2, 10)(3, 13)(4, 14)(5, 15)(6, 16)(7, 11)(8, 12) \\ \rho_1 = (1, 10)(2, 9)(3, 14)(4, 13)(5, 12)(6, 11)(7, 16)(8, 15) & \rho_4 = (1, 9)(2, 10)(3, 11)(4, 12)(5, 13)(6, 14)(7, 15)(8, 16) \\ \rho_2 = (1, 3)(2, 4)(5, 7)(6, 8)(9, 11)(10, 12)(13, 15)(14, 16) & \rho_6 = (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16) \end{array}$$

Now ρ_7 is an involution commuting with all the elements of C . Computationally, it can check that there is no such involution, a contradiction. Therefore, $\langle L \cup C \rangle$ has a primitive action on the $\langle \rho_i \rangle$ -orbits. \square

Proposition 5.5. *Let $r = n/2 \geq 7$. Then $\langle C \cup L \rangle$ is transitive if and only if $|R| = 0$.*

Proof. Suppose that $|R| \neq 0$. By Proposition 5.2 $\langle C \rangle$ is transitive. Let $C = \{\rho_j \mid j \in I\}$ and $L = \{\rho_i\}$. By Corollary 5.4, $\langle L \cup C \rangle$ has a primitive action on the $\langle \rho_i \rangle$ -orbits. In addition, notice that the elements of C generate independently the action on the ρ_i -orbits. Hence if two consecutive elements of C commute then $|C| \leq \log_2(n/2) \leq n/2 - 3$, giving a contradiction. Thus I is an interval. Now let us deal separately with the cases $|R| = 2$ and $|R| = 1$.

- $|R| = 2$: We may assume that $R = \{\rho_0, \rho_2\}$, $L = \{\rho_1\}$ and $C = \{\rho_3, \dots, \rho_{r-1}\}$. Then both ρ_0 and ρ_1 centralizes $\langle C \rangle$. But $\langle C \rangle$ has a primitive action on the $\langle \rho_1 \rangle$ -orbits, hence $\rho_0 = \rho_1$, a contradiction.

- $|R| = 1$: To avoid the previous contradiction the element of R cannot centralize $\langle C \rangle$. Thus let $L = \{\rho_0\}$, $R = \{\rho_1\}$ and $C = \{\rho_2, \dots, \rho_{r-1}\}$. If $G_{1,i}$ is transitive for $i \in \{2, \dots, r-1\}$, then, $G_{1,i}$ satisfies the conditions of Lemma 5.3, hence $G_{1,i}$ has a primitive action on the ρ_0 -orbits. Moreover $C \setminus \{\rho_i\}$ generate independently the action of $G_{1,i}$ on the ρ_0 -orbits. As $|C \setminus \{\rho_i\}| = n/2 - 3$ then, by Proposition 2.7, the action on the blocks is either $S_{n/2}$ or $A_{n/2}$. Also the action of $\langle C \rangle$ on the ρ_0 -orbits must be one of these groups, giving a contradiction. Hence $G_{1,i}$ is intransitive for every $i \in \{2, \dots, r-1\}$. Let $\Phi = \{\delta_2, \dots, \delta_{r-1}\}$ where δ_i is the action of ρ_i on the ρ_0 -orbits. As Φ is a sgg of rank $n/2 - 2$ and Φ_i is intransitive for every $i \in \{2, \dots, r-1\}$, by Proposition 2.14 is, up to duality, the a string C-group having the permutation representation of given at the end of Proposition 2.13. Now if ρ_1 fixes the blocks B_1 and B_2 then $\rho_1 \rho_1^{\rho_0} \in \langle C \cup L \rangle$, a contradiction. If $B_1 = B_2 \rho_1$ then as ρ_1 commutes with ρ_i $i > 2$, but, as in Proposition 5.2, this forces ρ_1 to commute also with ρ_0 , giving a contradiction with the definition of C .

Now let $|R| = 0$. As G is a transitive group and $G = \langle C \cup L \rangle$, then $\langle C \cup L \rangle$ is intransitive. \square

Proposition 5.6. *Let $r \geq n/2 \geq 7$. If $\langle C \cup L \rangle$ is intransitive and $L = \{\rho_i\}$ then G_j is intransitive for $j \notin \{0, i, r-1\}$.*

Proof. In this case $G_j = G_{<j} \times G_{>j}$. Suppose that $\{\rho_0, \dots, \rho_{j-1}\} \not\subseteq C$ and $\{\rho_{j+1}, \dots, \rho_{r-1}\} \not\subseteq C$. Then $i - 1 \leq j - 1$ and $i + 1 \geq j + 1$, which give $j = i$, a contradiction. Hence, either $\{\rho_0, \dots, \rho_{j-1}\} \subseteq C$ or $\{\rho_{j+1}, \dots, \rho_{r-1}\} \subseteq C$. Suppose, without loss of generality that $\{\rho_0, \dots, \rho_{j-1}\} \subseteq C$. As $\langle C \cup L \rangle$ is intransitive, $G_{<j}$ is also intransitive. Therefore the $G_{<j}$ -orbits determine a block system for G_j . Now either ρ_i fix all $G_{<j}$ -orbits or swaps all of them pair-wisely. When ρ_i swaps two $G_{<j}$ -orbits, say O_1 and O_2 , then $O_1 \cup O_2$ is a block of another block system whose blocks are twice bigger. Notice that as $\{\rho_0, \dots, \rho_{j-1}, \rho_i\} \subseteq L \cup C$ $O_1 \cup O_2 \neq \{1, \dots, n\}$. Consider a maximal block system such that $G_{<j}$ fix the blocks. Then ρ_i fixes the blocks. Particularly the maximality of the blocks implies the action on the blocks is primitive. Let k' and m' be the size of a block and the number of blocks, respectively.

Consider firstly the case $m' > 2$. We have that k' is even and $k' \geq 4$. Consider the sets L' , generating independently block action; C' , the set of generators of G_j that commute with all the elements of L' ; and R' , the set of the remaining generators of G_j . We have that $|L'| \leq m' - 1$. As $\langle L' \rangle$ is primitive we may assume that the elements of L' are consecutive, hence $|R'| \leq 2$.

In this case $\langle C' \rangle$ is an imprimitive group, with two blocks, embedded into $S_{k'}$. Hence $|C'| \leq k'/2$. Consequently, when $m' \neq 2$, $m'k'/2 - 1 = r - 1 \leq (m' - 1) + k'/2 + 2$, or equivalently, $(m' - 1)(k' - 2) \leq 6$, as $n > 12$, this is only possible if $(m', k') = (4, 4)$. Now it remains to consider the cases $(m', k') = (4, 4)$ and $m' = 2$. Let us deal with them separately.

$(m', k') = (4, 4)$: In this case $|L'| = 3$, $|C'| = 2$ and $|R'| = 2$. Suppose first that $\rho_i \in C'$. In this case $j = 1$ and $C' = \{\rho_0, \rho_i\}$. Thus the elements of L' commute with ρ_i , which mean that $L' \subseteq C$. As $\langle L' \cup C' \rangle$ is transitive, we get that $\langle L \cup C \rangle$ is transitive, a contradiction. Thus $\rho_i \notin C'$. Hence $\rho_i \in R'$. Let $C' = \{\rho_0, \rho_x\}$. As ρ_i fixes the blocks (of size 4) and $B_1\rho_i = B_2$, we have that $\langle \rho_0, \rho_x, \rho_i \rangle \cong (C_2)^3$ (and ρ_0 also commutes with the elements of L'). Thus ρ_0 is a central involution in G_j . For each $l \neq j$, let $\bar{\rho}_l$ denote the action of ρ_l on the $\langle \rho_0 \rangle$ -orbits. The set $\{\bar{\rho}_x, \bar{\rho}_i\} \cup \{\bar{\rho}_y \mid \rho_y \in L'\}$ is independent and has size 5. Moreover $H := \langle \bar{\rho}_l \mid l \neq 0, j \rangle$ is a transitive subgroup of $S_2 \wr S_4$ whose block action is S_4 , while $\bar{\rho}_x$ and $\bar{\rho}_i$ fix the blocks ($\bar{\rho}_x$ is a central involution). Thus H_x and H_i are transitive subgroups of $S_2 \wr S_4$ whose block action is S_4 . As the number of blocks is even (4 blocks), we necessarily have $\bar{\rho}_x \in H_x$, a contradiction.

$m' = 2$: Let $L' = \{\rho_x\}$ and B'_1 and B'_2 be the blocks swapped by ρ_x . As ρ_i fixes B'_1 and B'_2 , $G_j \leq S_{n/4} \wr S_4$, to be precise the block action is $C_2 \times C_2$. For this embedding the block action is generated by ρ_i and ρ_x . Let $\mathcal{V} = \{V_1, V_2, V_3, V_4\}$ denote block system mentioned above with $V_1\rho_i = V_2$, $V_3\rho_i = V_4$, $V_1\rho_x = V_3$ and $V_2\rho_x = V_4$. Let $M = \{\rho_i, \rho_x\}$, D be the set of elements that commute with ρ_i and ρ_x and N be the elements of G_j that are neither in D nor in M . We have that $|D| \leq n/4 - 1$ by Lemma 3.2. In this case there are at most four generators that do not commute with both ρ_i and ρ_x , hence $|N| \leq 4$. Thus $r - 1 = |D| + |M| + |N| \leq (n/4 - 1) + 2 + 4$, giving a contradiction for $n > 24$. We need to consider $n \in \{16, 20, 24\}$.

Suppose that $M = \{\rho_i, \rho_{i+1}\}$ generates the action on the blocks $\{V_1, V_2, V_3, V_4\}$. Furthermore suppose that ρ_i and ρ_{i+1} do not commute. Let p be order of $\rho_i\rho_{i+1}$, which is even. In this case $(\rho_i\rho_{i+1})^{p/2}$ is a nontrivial permutation fixing the four blocks. Moreover this permutation commutes with both ρ_{i+1} and ρ_i . As $(\rho_i\rho_{i+1})^{p/2} \notin \langle D \rangle$, we have that $|D| \leq n/4 - 2$. Consequently $r - 1 = |D| + |M| + |N| \leq (n/4 - 2) + 2 + 2$, which gives $n \leq 12$, a contradiction. We get the same conclusion if $\{\rho_i, \rho_{i-1}\}$ generate the action on the blocks $\{V_1, V_2, V_3, V_4\}$ and $(\rho_i\rho_{i-1})^2$ is nontrivial.

Now consider that $M = \{\rho_i, \rho_x\}$ and $\rho_x \in C$. Let q be the size of a $G_{<j}$ -orbit. We have that q divides $n/2$. Let us consider all the possibilities for q .

- $q = n/2$: If $q = n/2$ then $O_1 = V_1 \cup V_2$ and $O_2 = V_3 \cup V_4$ would be the $G_{<j}$ -orbits, but then $\langle \rho_0, \dots, \rho_{j-1}, \rho_x \rangle$ would be transitive, a contradiction.

- $q = n/4$: In this case the $G_{<j}$ -orbits cannot be V_1, V_2, V_3 and V_4 , otherwise $\langle \rho_0, \dots, \rho_{j-1}, \rho_x, \rho_i \rangle$ would be transitive, a contradiction. Thus, in this case, ρ_i must fix the $G_{<j}$ -orbits. Particularly $n/4$ is even (either $n = 16$ or $n = 24$). Suppose that $q = 6$. In this case either $G_{<j} = \langle D \rangle$ or $G_{<j}$ is in the center $\langle D \rangle$. In the first case $|D| \leq 3$ and in the second case, by Lemma 3.2, $|D| \leq n/4 - 2 = 4$, hence $r - 1 \leq 4 + 2 + 4$, giving $r < n/2$, a contradiction. Thus $q = 4$. As ρ_i fix the $G_{<j}$ -orbits, $j = 2$. Moreover $\langle \rho_0, \rho_1 \rangle = C_2 \times C_2$, thus by Lemma 3.2, $r - 3 \leq 16/4 - 1$, which gives a bound below $n/2$.

- $q = n/6$ or $q = 3$: In this case $n = 24$. Consider first the case $G_{<j}$ fixing V_1, V_2, V_3 and V_4 . Recall that $G_{<j} = \langle D \rangle$ or $G_{<j}$ is in the center $\langle D \rangle$. In the first $\langle D \rangle$ is intransitive. In the second case $\langle D \rangle$ is not isomorphic to $S_{n/4}$. In both cases we get that $|D| \leq n/4 - 2$, and as $n = 24$, $r < n/2 = 12$. Thus, ρ_i fixes the $G_{<j}$ -orbits, which implies that $q = 4$. Then, as before $j = 2$ and, $G_{<2} = C_2 \times C_2$. Thus by Lemma 3.2, $r - 3 \leq 24/4 - 1 = 7 < 24/2$.

- $q \in \{n/8, n/10, n/12\}$ and $q \neq 3$: In this case $q = 2$ and $j = 1$. Thus ρ_0 is a central involution of G_1 . If $V_1\rho_0 = V_2$ (hence $V_3\rho_0 = V_4$) then if we consider $M' = \{\rho_0, \rho_x\}$ for the generators on \mathcal{V} , D' the set of generators commuting with M' and N' the generators of G_j not in $M' \cup D'$, we get $|N'| \leq 2$. In addition $\rho_0\rho_i$ fixes the blocks and commutes with all the elements of M' . Thus $|D'| < n/4 - 1$, thus $r - 1 = |D'| + |M'| + |N'| \leq (n/4 - 2) + 2 + 2$, which gives $n \leq 12$, a contradiction.

Thus for ρ_0 is a central involution fixing the blocks of \mathcal{V} . Hence $G_j \leq S_{n/8} \wr (C_2)^3$, Lemma 3.2 $r - 4 \leq n/8 - 1$, giving $r < n/2$. □

For the remaining of this section we assume the following:

- $n/2 \geq 7$;
- $\langle L \cup C \rangle$ is intransitive;
- $L = \{\rho_i\}$;
- $|R| > 0$;
- G_j is intransitive for all $j \neq i$ and $i \in \{1, \dots, r-2\}$.

To determine the remaining possibilities for \mathcal{G} , consider the graphs \mathcal{I} and \mathcal{F} given by the following construction.

Construction 5.7. Consider a graph \mathcal{I} whose vertices are the $\langle \rho_i \rangle$ -orbits denoted by $O_1, \dots, O_{n/2}$, and with a j -edge ($j \neq i$) when $a\rho_j = b$ for a and b in different $\langle \rho_i \rangle$ -orbits. The graph \mathcal{I} has the following properties.

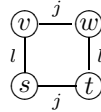
- P1: Adjacent edges, or parallel edges, of \mathcal{I} either correspond to adjacent edges, or parallel edges, of \mathcal{G} or to a pair of edges that are adjacent to a common i -edge.
- P2: If a j -edge and a l -edge are adjacent in \mathcal{I} but not in \mathcal{G} then $\{j, l\} \subseteq \{i-1, i+1\}$.
- P3: If two j -edges of \mathcal{I} are adjacent or parallel, then $j = i \pm 1$. Particularly, \mathcal{I} has a cycle with edge having the same label, if and only if, \mathcal{G} has a $\{i, j+1\}$ -cycle.

Now consider a generalization of the concept of a fracture graph introduced in [10]. Let \mathcal{F} be a spanning subgraph of \mathcal{I} with exactly one j -edge $\{O_s, O_t\}$ for each label j chosen among the j -edges connecting $\langle \rho_i \rangle$ -orbits that belong to different G_j -orbits. The number of edges of \mathcal{F} is the cardinality of the set $\{j \mid G_j \text{ is intransitive} \wedge j \neq i\}$.

Some of the key properties of fracture graphs, namely Lemmas 3.2, 3.3, 3.5 and 3.6 of [10], also hold for a graph \mathcal{F} obtained by the construction above.

Lemma 5.8. (a) Let $f = |\{j \mid G_j \text{ is intransitive} \wedge j \neq i\}|$. The graph \mathcal{F} is a forest with $n/2 - f$ connected components.

- (b) If there exist two edges $\{O_s, O_t\}$ with distinct labels j and l in \mathcal{I} , then O_s and O_t are in distinct connected components of \mathcal{F} .
- (c) If there exist two j -edges $\{O_s, O_t\}$ and $\{O_u, O_v\}$ in \mathcal{I} , then not all vertices $\{O_s, O_t, O_u, O_v\}$ are in a same connected component of \mathcal{F} .
- (d) If a cycle \mathcal{C} of \mathcal{I} contains the j -edge of \mathcal{F} , then \mathcal{C} contains another j -edge.
- (e) If O_s and O_t are vertices in the same connected component of \mathcal{F} , and $e = \{O_s, O_t\}$ is an j -edge in \mathcal{I} , then e is in \mathcal{F} .
- (f) Let O_v, O_w, O_s, O_t be vertices of an alternating square of \mathcal{I} as in the following figure.



If $\{O_v, O_w\}$ and $\{O_v, O_s\}$ are edges of \mathcal{F} then O_t is in different connected components of \mathcal{F} .

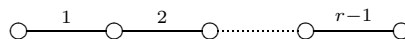
Proof. Let us use the same arguments used in Lemma 3.2 (1) of [10] to prove (a). If j is the label of an edge $\{O_s, O_t\}$ of \mathcal{F} belonging to a cycle then j is also an edge of \mathcal{G} that belongs to a cycle (either with the same number of edges or with some extra i -edges) that does not contain other j -edges. Therefore O_s and O_t are in the same G_j -orbit, a contradiction. Thus \mathcal{F} is a forest. As \mathcal{F} has $n/2$ vertices and ϵ edges, the number of connected components is given by $n/2 - \epsilon$.

To prove (b), (c), (d), (e) and (f) we just need to adapt the proofs of Lemmas 3.2 (3), 3.2(4), 3.3, 3.5 and 3.6, respectively. \square

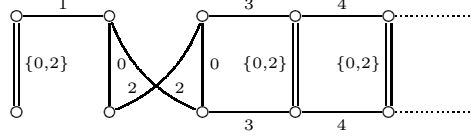
Proposition 5.9. If G_0 and G_{r-1} are intransitive, then \mathcal{G} is, up to duality, the graph (3) or (4) of Table 8.

Proof. In this case the number of edges of \mathcal{F} is precisely $n/2 - 1$. Hence by Lemma 5.8 (a) \mathcal{F} is a tree. Moreover, by Lemma 5.8 (b) and (c) $\mathcal{I} = \mathcal{F}$.

Suppose first that any pair of adjacent edges of \mathcal{F} have consecutive labels. Then, $i \in \{0, r-1\}$. Up to duality we may assume that $i = 0$ and therefore \mathcal{F} is as follows.

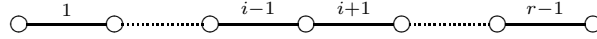


As $|R| \geq 1$, we have that $R = \{\rho_1\}$. Let O_1 and O_2 be the $\langle \rho_0 \rangle$ -orbits swapped by ρ_1 . Suppose there is a $x \neq 0$ such that $B_1 \rho_x = B_2$. First $x \neq 1$, otherwise ρ_1 and ρ_0 commute, a contradiction. In order to avoid the same contradiction, the unique permutation ρ_j (with $j \notin \{0, 1\}$) that may act nontrivially on O_1 and O_2 , is ρ_2 . This gives a unique possibility for x , which is $x = 2$. Let us then assume that $B_1 \rho_2 = B_2$, Then \mathcal{G} is as follows.



Then $\rho_2 \rho_3)^3 = \rho_0$, a contradiction. Then ρ_0 is the only generator swapping the blocks. Hence \mathcal{G} is the graph (3) of Table 8.

Now suppose that l and j are nonconsecutive labels of adjacent edges of \mathcal{F} . If $\{l, j\} \neq \{i - 1, i + 1\}$ then \mathcal{G} has an alternating $\{l, j\}$ -square, and consequently \mathcal{I} has an alternating $\{l, j\}$ -square. Then, by Lemma 5.8 (f), \mathcal{F} cannot be a tree, a contradiction. Thus $\{l, j\} = \{i - 1, i + 1\}$. This gives following possibility for \mathcal{F} .

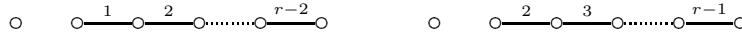


If ρ_i is the unique permutation permuting B_1 and B_2 then \mathcal{G} is the graph (4) of Table 8. Let us prove this is the only possibility. Suppose the contrary, that there exists $x \neq i$, such that $B_1 \rho_x = B_2$. Let O_s and O_t be the $\langle \rho_i \rangle$ -orbits that are merged by ρ_x . Then ρ_x is a fixed point free permutation fixing all $\langle \rho_i \rangle$ -orbits except O_s and O_t . If $x \neq i \pm 1$, as ρ_x commutes either with ρ_{i-1} or with ρ_{i+1} , then \mathcal{G} has an alternating $\{i - 1, i + 1\}$ -square. Consequently \mathcal{I} has an alternating $\{i - 1, i + 1\}$ -square. Hence, by Lemma 5.8(f) \mathcal{F} has two components components, a contradiction. If $x = i \pm 1$ then \mathcal{G} also has a pair of adjacent edges with labels $i - 1$ and $i + 1$, giving rise to the same contradiction as before. □

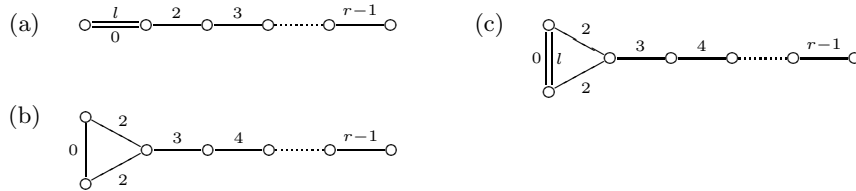
Proposition 5.10. *If G_0 is transitive and G_{r-1} is intransitive then \mathcal{G} is, up to duality, the graph (5), (6) or (7) of Table 8.*

Proof. If $i = 0$, by Proposition 5.6, G_j is intransitive for every $j \neq 0$. Then by Proposition 5.9, \mathcal{G} is the graph (3) or (4) of Table 8, but in both cases G_0 is intransitive, a contradiction. Thus $i \neq 0$. In this case \mathcal{F} is a forest with exactly two connected components. Moreover \mathcal{I} has a 0-edge $\{O_s, O_t\}$ connecting the two connected components of \mathcal{F} . Additionally, by Lemma 5.8 (c), all the edges of \mathcal{I} that do not belong to \mathcal{F} must be edges incident to O_s or to O_t .

Suppose first that any pair of adjacent edges of \mathcal{F} are consecutive. Then, up to duality, \mathcal{F} is as one of the following graphs.

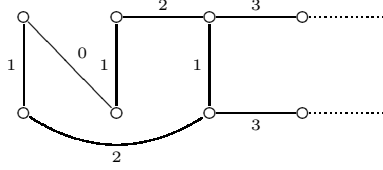


In the case on the left $i = r - 1$ and, as $|R| > 1$, ρ_{r-2} must be a transposition. But then $G_0 \cong S_{n/2} \wr C_2$, hence $\rho_0 \in G_0$, a contradiction. Thus \mathcal{F} must be the graph on the right, particularly $i = 1$. Now \mathcal{I} is one of the following graphs where the $l \in \{2, 3\}$.

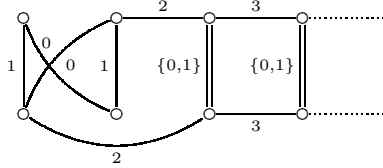


Consider firstly the graph (a). If ρ_1 is the unique permutation swapping the blocks we get the possibilities given by the graphs (4), (5) and (6) of Table 8. Suppose that, for $x \neq 1$, $B_1 \rho_x = B_2$. In any case this forces the existence of a $\{0, 2\}$ -square, giving a contradiction.

Now consider the graphs (b) and (c). In these cases \mathcal{G} contains the following graph.



But then $B_1\rho_0 = B_2$, and consequently $\rho_0 \in C$ and \mathcal{G} contains the following graph.



This implies that $(\rho_2\rho_3)^3 = \rho_0$, a contradiction.

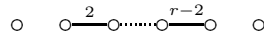
Now suppose that there is a pair of incident edges of \mathcal{F} which have nonconsecutive labels. Then their labels must be $i - 1$ and $i + 1$. Moreover $R = \{\rho_{i-1}, \rho_{i+1}\}$. In this case the elements of R are transpositions, hence $G_0 \cong S_{n/2} \wr C_2$, a contradiction. \square

Proposition 5.11. G_0 and G_{r-1} cannot be both transitive.

Proof. Suppose first that ρ_0 and ρ_{r-1} are the only permutations swapping B_1 and B_2 and let $L = \{\rho_0\}$ (meaning that $i = 0$). Then \mathcal{F} has $n/2 - 2$ edges and two connected components, which are joined in \mathcal{I} by a double $\{r - 1, l\}$ -edge for some $l \neq r - 1$. But then, this double edge must belong to a square whose vertices must belong to at least three different connected components of \mathcal{F} , a contradiction. Hence, there exists $j \notin \{0, r - 1\}$ such that $B_1\rho_j = B_2$ thus we may assume that the element of L is neither ρ_0 nor ρ_{r-1} . In this case \mathcal{F} has exactly three components.

Suppose that \mathcal{I} has a double $\{0, r - 1\}$ -edge. Then this double edge must belong to a square having two vertices in the same connected component of \mathcal{F} . Hence one edge of this square belongs to \mathcal{F} , by Lemma 5.8 (e). As $n > 8$ and $r > 6$, there is another square that is adjacent to the previous one and these two adjacent squares form a graph with 6 vertices and at most two edges in \mathcal{F} . Thus \mathcal{F} has at most four components, a contradiction.

Now suppose that \mathcal{I} has a 0-edge that is adjacent to a $(r - 1)$ -edge. Then \mathcal{I} has an alternating square whose vertices belong to different components of \mathcal{F} , a contradiction. Thus \mathcal{I} has a 0-edge and a $(r - 1)$ -edge that are not adjacent. This determine the three components of \mathcal{F} . An edge of \mathcal{F} which is adjacent to the 0-edge of \mathcal{I} must have label 1 and an edge of \mathcal{F} which is adjacent to the $(r - 1)$ -edge of \mathcal{I} must have label $r - 2$. Hence two components of \mathcal{F} are isolated vertices. Suppose first that adjacent edges of \mathcal{F} have consecutive labels. Then either $i = 1$ or $i = r - 2$. Up to duality we may assume that $i = 1$, then \mathcal{F} is as follows.



Now any edge in \mathcal{I} that is not in \mathcal{F} must be incident to one of the isolated vertices of \mathcal{F} . Hence ρ_0 is a transposition, thus $G_{r-1} \cong S_{n/2} \wr C_2$, a contradiction.

Now consider the case where \mathcal{F} has adjacent edges with nonconsecutive labels. This is only possible when a $(i - 1)$ -edge is adjacent to a $(i + 1)$ -edge, and then $R = \{\rho_{i-1}, \rho_{i+1}\}$. But in this case the elements of R are transpositions, hence $G_0 \cong S_{n/2} \wr C_2$, a contradiction. \square

6. PROOF OF THEOREM 1.1 AND LIST OF ALL POSSIBILITIES FOR \mathcal{G}

Let G be the automorphism group of an abstract regular polytope $r \geq n/2$ whose automorphism group has degree $n \geq 14$. By Theorem 2.15 G must be embedded into $S_k \wr S_m$ with $n = mk$ and Corollary 3.4 shows that either $k = 2$ or $m = 2$. In Section 4 all the possibilities for \mathcal{G} when $n/2$ blocks of size 2 were determined, while Section 5 covers all the possibilities when G has two blocks of size $n/2$. This leads to the 34 possibilities (up to duality) given in Tables (4)-(8).

Case: $ R \cup C = 2$; $\langle L \rangle \cong S_{n/2}$; $\langle L \rangle$ intransitive .			
(1)		(2)	
(3)			
(4)			
(5)			
(6)			
Case: $ R \cup C = 2$; $\langle L \rangle \cong S_{n/2}$; $\langle L \rangle$ transitive .			
(7)		(8)	
(9)			
(10)			
(11)			
(12)			

TABLE 4. $k = 2$; Corollary 4.5.

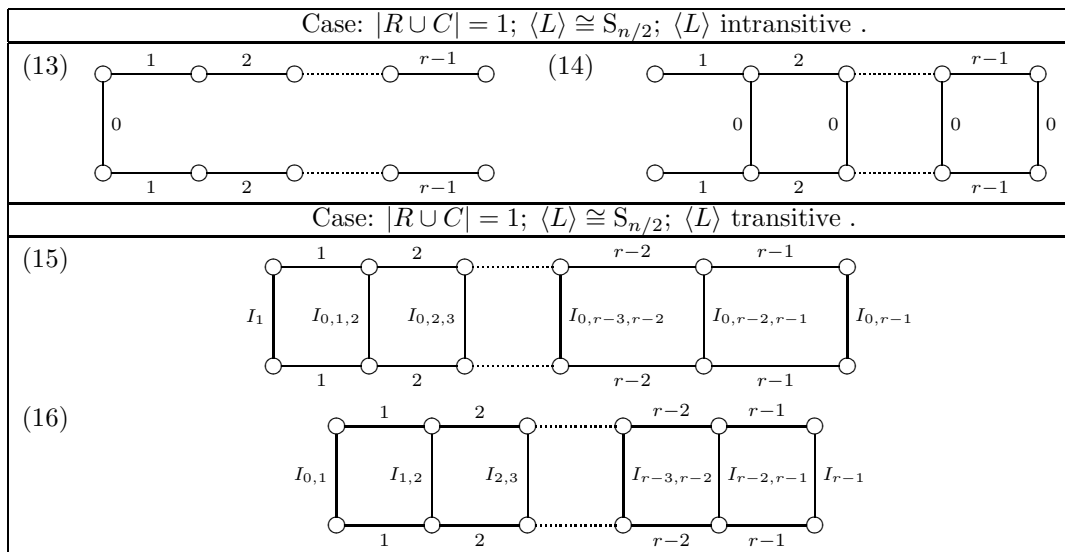


TABLE 5. $k = 2$; Proposition 4.7.

7. ACKNOWLEDGEMENTS

The author Maria Elisa Fernandes was supported by the Center for Research and Development in Mathematics and Applications (CIDMA) through the Portuguese Foundation for Science and Technology (FCT - Fundação para a Ciência e a Tecnologia), references UIDB/04106/2020 and UIDP/04106/2020. The author Claudio Alexandre Piedade was partially supported by CMUP, member of LASI, which is financed by national funds through FCT – Fundação para a Ciência e a Tecnologia, I.P., under the projects with reference UIDB/00144/2020 and UIDP/00144/2020.

REFERENCES

1. P. J. Cameron and Philippe Cara, Independent generating sets and geometries for symmetric groups, *Journal of Algebra* 258 (2); 641-650, 2002. <https://doi.org/10.1016/j.jalgebra.2015.09.040>
2. P. J. Cameron, M. E. Fernandes, and D. Leemans, The number of string C-groups of high rank, *Advances in Mathematics* 453 (2024), no. 1, 109832. doi: 10.1016/j.aim.2024.109832
3. P. J. Cameron, M. E. Fernandes, D. Leemans, and M. Mixer, Highest rank of a polytope for A_n , *Proceedings of the London Mathematical Society* 115 (2017), no. 1, 135–176. doi: 10.1112/plms.12039
4. P. J. Cameron, M. E. Fernandes, D. Leemans and M. Mixer. String C-groups as transitive subgroups of S_n . *J. Algebra* 447:468–478, 2016.
5. P. J. Cameron, Ron Solomon, Alexandre Turull, Chains of subgroups in symmetric groups, *Journal of Algebra* 127 (2); 340-352, 1989. [https://doi.org/10.1016/0021-8693\(89\)90256-1](https://doi.org/10.1016/0021-8693(89)90256-1).
6. M. Conder, D. Oliveros. The intersection condition for regular polytopes, *Journal of Combinatorial Theory, Series A* 120 (6), 1291–1304, 2013. <https://doi.org/10.1016/j.jcta.4013.03.009>.
7. J. H. Conway, R. T. Curtis, S. P. Norton, R.A. Parker, and R. A. Wilson, *Atlas of finite groups: maximal subgroups and ordinary characters for simple groups*, Clarendon Press ; Oxford University Press, Oxford [Oxfordshire] : New York, 1985. isbn: 978-0-19-853199-9
8. M. E. Fernandes and D. Leemans, Polytopes of high rank for the symmetric groups, *Adv. Math.* 228:3207–3222, 2011.
9. M. E. Fernandes, D. Leemans and M. Mixer. Corrigendum to “Polytopes of high rank for the symmetric groups”. *Adv. Math.* 238:506–508, 2013.
10. M. E. Fernandes, D. Leemans, and M. Mixer, *An extension of the classification of high rank regular polytopes*, Transactions of the American Mathematical Society 370 (2018), no. 12, 8833–8857. doi: 10.1090/tran/7425
11. D. Leemans and L. Vauthier, *An atlas of abstract regular polytopes for small groups*, Aequationes mathematicae 72 (2006), no. 3, 313–320. doi: 10.1007/s00010-006-2843-9
12. B. Mortimer, The Modular Permutation Representations of the Known Doubly Transitive Groups. *Proceedings of the London Mathematical Society*, s3–41 (1980), Issue 1, 1–20,doi:10.1112/plms/s3-41.1.1
13. P. McMullen and E. Schulte, *Abstract Regular Polytopes*, 1 ed., Cambridge University Press, December 2002. doi: 10.1017/CBO9780511546686
14. J. Whiston, Maximal independent generating sets of the symmetric group, *J. Algebra* 232 (2000), 255–268.

Case: $|R \cup C| = 1 \langle L \rangle \cong S_{n/2}; \text{Ker}(f) \cong C_2.$

(17)	
(18)	
(19)	
(20)	
(21)	
(22)	
(23)	
(24)	

TABLE 6. $k = 2$; Proposition 4.8.

15. M. I. Hartley, *An Atlas of Small Regular Abstract Polytopes*, Periodica Mathematica Hungarica **53** (2006), no. 1-2, 149–156. doi: 10.1007/s10998-006-0028-x
16. D. Leemans and L. Vauthier, *An atlas of abstract regular polytopes for small groups*, Aequationes mathematicae **72** (2006), no. 3, 313–320. doi: 10.1007/s00010-006-2843-9
17. M. Aschbacher and L. Scott, *Maximal subgroups of finite groups*, Journal of Algebra **92** (1985), no. 1, 44–80, Academic Press
18. C. E. Praeger and Jan Saxl, *On the orders of Primitive Permutation Groups*, Bulletin of the London Mathematical Society **12** (1980), no. 4, 303–307, doi:10.1112/blms/12.4.303
19. T. Connor and D. Leemans, *An atlas of subgroup lattices of finite almost simple groups*, Ars Mathematica Contemporanea **8** (2015), no. 2, 259–266, University of Primorska Press. doi:10.26493/1855-3974.455.422

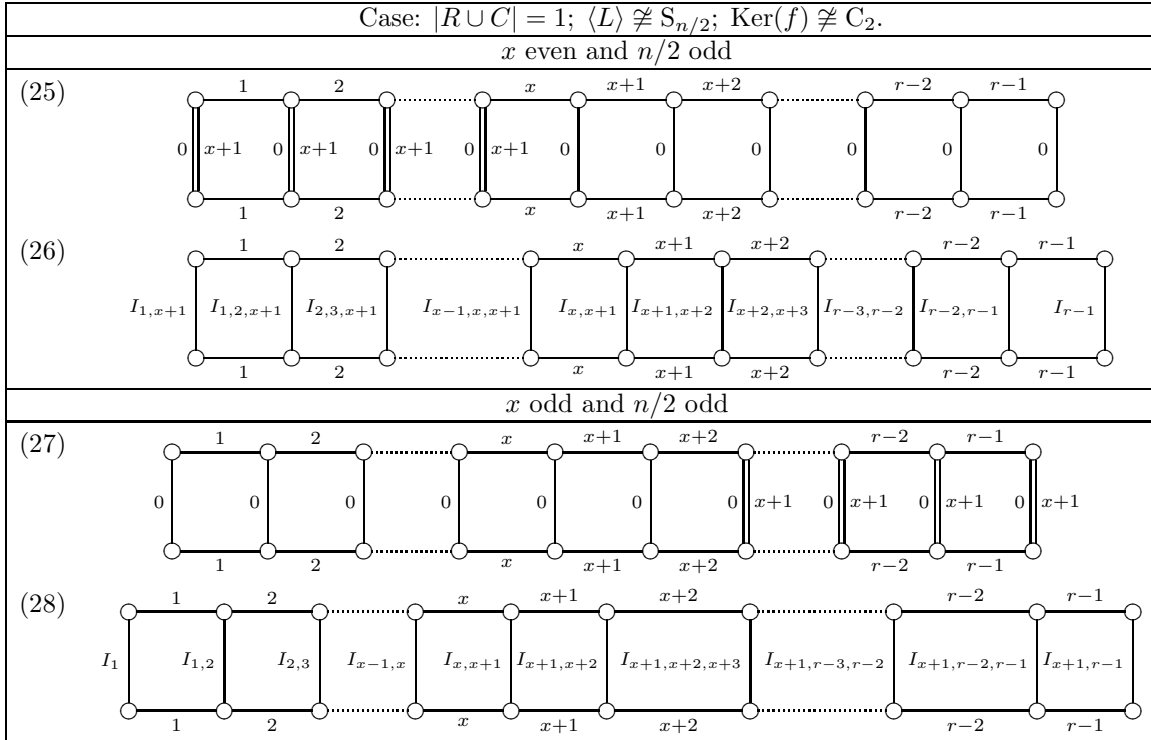


TABLE 7. $k = 2$; Proposition 4.14.

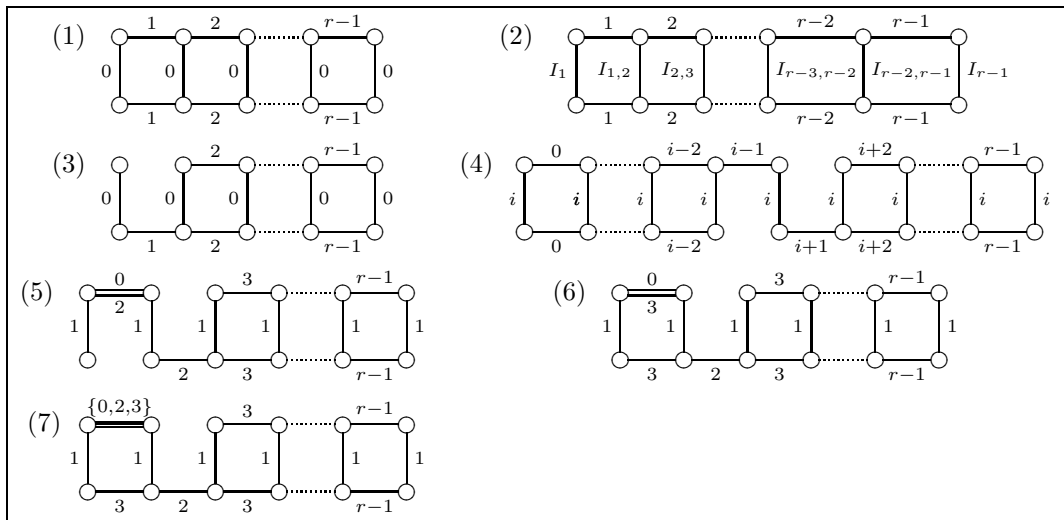


TABLE 8. $m = 2$.

MARIA ELISA FERNANDES, CENTER FOR RESEARCH AND DEVELOPMENT IN MATHEMATICS AND APPLICATIONS, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF AVEIRO, PORTUGAL
 Email address: maria.elisa@ua.pt

CLAUDIO ALEXANDRE PIEDADE, CENTRO DE MATEMÁTICA DA UNIVERSIDADE DO PORTO, UNIVERSIDADE DO PORTO, PORTUGAL
 Email address: claudio.piedade@fc.up.pt