An analogue of a formula of Popov

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Abstract

Let $r_k(n)$ denote the number of representations of the positive integer n as the sum of k squares. We prove a new summation formula involving $r_k(n)$ and the Bessel functions of the first kind, which constitutes an analogue of a result due to the Russian mathematician A. I. Popov.

1 Introduction

The insights contained in Ramanujan's lost notebook [18] still inspire a lot of new mathematical ideas. Recently, the discovery of some overlooked works from the Russian school, namely from the mathematician N. S. Koshliakov [14] has produced a similar effect on mathematical research (cf. [5, 10, 20]). Belonging to the same tradition as Koshliakov, the mathematician and linguist Alexander Ivanovich Popov (1899-1973) made his contribution to Mathematics by providing new interesting summation formulas.

In a fairly unknown paper [16], Popov states the following beautiful result. If $\mathrm{Re}(x) > 0$ and $z \in \mathbb{C}$, then

$$\frac{z^{\frac{k}{2}-1}\pi^{\frac{k}{4}-\frac{1}{2}}x^{\frac{k}{4}}}{2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)}\,e^{z^2/8}+\sqrt{x}\,e^{z^2/8}\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}\,e^{-\pi n x}\,J_{\frac{k}{2}-1}(\sqrt{\pi n x}z)$$
$$=\frac{z^{\frac{k}{2}-1}\pi^{\frac{k}{4}-\frac{1}{2}}x^{-\frac{k}{4}}}{2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)}\,e^{-z^2/8}+\frac{e^{-z^2/8}}{\sqrt{x}}\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}\,e^{-\frac{\pi n}{x}}\,I_{\frac{k}{2}-1}\left(\sqrt{\frac{\pi n}{x}}z\right), \tag{1.1}$$

where $r_k(n)$ denotes the number of representations of the positive integer n as a sum of k squares and, as usual, $J_\nu(z)$ and $I_\nu(z)$ respectively denote the Bessel and modified Bessel functions of the first kind. A couple of reasons why this identity is fascinating are already provided by Berndt, Dixit, Kim and Zaharescu in [[6], pp. 3795-3796]. For the purposes of our discussion, we shall enumerate them.

1. If we construct the Dirichlet series attached to $r_k(n)$,

$$\zeta_k(s)=\sum_{n=1}^{\infty}\frac{r_k(n)}{n^s},\quad \mathrm{Re}(s)>\frac{k}{2}, \tag{1.2}$$

then $\zeta_k(s)$ can be continued to the complex plane as a meromorphic function possessing only a simple pole at $s=\frac{k}{2}$ with residue $\pi^{k/2}/\Gamma(k/2)$. Moreover, it satisfies the functional equation

$$\pi^{-s}\Gamma(s)\,\zeta_k(s)=\pi^{s-\frac{k}{2}}\Gamma\left(\frac{k}{2}-s\right)\zeta_k\left(\frac{k}{2}-s\right). \tag{1.3}$$

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Note that, when $k=1$, $r_1(n)=2$ if and only if n is a perfect square and zero otherwise. Therefore, (1.2) reduces to

$$\zeta_1(s):=\sum_{n=1}^{\infty}\frac{r_1(n)}{n^s}=2\sum_{n=1}^{\infty}\frac{1}{n^{2s}}=2\zeta(2s),\quad \text{Re}(s)>\frac{1}{2}. \tag{1.4}$$

Furthermore, (1.3) with $k=1$ gives the functional equation for Riemann's $\zeta-$function

$$\pi^{-s}\Gamma\left(s\right)\zeta(2s)=\pi^{s-\frac{1}{2}}\Gamma\left(\frac{1}{2}-s\right)\zeta\left(1-2s\right). \tag{1.5}$$

The first point highlighted in [6] is that the powers of n in the denominators of both sides of (1.1) are remindful of the functional equation (1.3).

2. Riemann's second proof of the functional equation for $\zeta(s)$, (1.5), employs the transformation formula for Jacobi's $\theta-$function,

$$\theta(x):=\sum_{n\in\mathbb{Z}}e^{-\pi n^2 x}=\frac{1}{\sqrt{x}}\sum_{n\in\mathbb{Z}}e^{-\frac{\pi n^2}{x}}:=\frac{1}{\sqrt{x}}\theta\left(\frac{1}{x}\right),\quad \text{Re}(x)>0. \tag{1.6}$$

The theta transformation formula associated to the Dirichlet series $\zeta_k(s)$ can be obtained by taking the k^{th} power on both sides of (1.6). This results in the transformation

$$\sum_{n=0}^{\infty}r_k(n)\,e^{-\pi n x}=x^{-\frac{k}{2}}\sum_{n=0}^{\infty}r_k(n)\,e^{-\frac{\pi n}{x}},\quad \text{Re}(x)>0. \tag{1.7}$$

Of course, the exponential factors on both sides of (1.1) remind us the theta transformation formula (1.7). In fact, (1.7) is a particular case of (1.1) when we let $z\to 0$, due to the limiting relations for the Bessel functions [[15], p. 223, eq. (10.7.3)]

$$\lim_{y\to 0}y^{-\nu}J_{\nu}(y)=\lim_{y\to 0}y^{-\nu}I_{\nu}(y)=\frac{2^{-\nu}}{\Gamma(\nu+1)}.$$

3. Chandrasekharan and Narasimhan [[9], p. 19, eq. (65)] proved yet another equivalent identity to (1.3) and (1.7). If $x>0$ and $q>\frac{k-1}{2}$, then

$$\frac{1}{\Gamma(q+1)}\sum_{0\leq n\leq x}{}^{'}r_k(n)\,(x-n)^q=\frac{\pi^{\frac{k}{2}}x^{\frac{k}{2}+q}}{\Gamma\left(q+1+\frac{k}{2}\right)}+\pi^{-q}\sum_{n=1}^{\infty}r_k(n)\left(\frac{x}{n}\right)^{\frac{k}{4}+\frac{q}{2}}J_{\frac{k}{2}+q}\left(2\pi\sqrt{nx}\right), \tag{1.8}$$

where the Bessel series on the right-hand side converges absolutely. The prime on the summation sign indicates that, if $q=0$ and x is an integer, then the last contribution in this Riesz sum is just $\frac{1}{2}r_k(x)$. The appearance of the Bessel functions in (1.1) reminds us of (1.8).

Berndt, Dixit, Kim and Zaharescu emphasized the importance of Popov's result by explaining its connection with the formulas stated in the previous three items. We would like to add another item to the list, which is perhaps more directly related to the second item above. When $k=1$, (1.1) gives the identity

$$x^{\frac{1}{4}}\,e^{z^2/8}\left\{1+2\sum_{n=1}^{\infty}e^{-\pi n^2 x}\,\cos\left(\sqrt{\pi x}\,nz\right)\right\}=x^{-\frac{1}{4}}\,e^{-z^2/8}\left\{1+2\sum_{n=1}^{\infty}e^{-\frac{\pi n^2}{x}}\,\cosh\left(\sqrt{\frac{\pi}{x}}\,nz\right)\right\},$$

which is mainly due to the particular cases of the Bessel functions [[22], p. 248],

$$J_{-\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\,\cos(x),\quad I_{-\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\,\cosh(x).$$

Aiming to study the zeros of shifted combinations of Dirichlet series [19], Yakubovich and the author of this note have discovered, independently of Popov, an extension of (1.1) to a class of Hecke Dirichlet series.[1] The main idea in [19] is that (1.1) can be achieved through an integral representation of the form[2]

$$\begin{aligned}
&\sqrt{x}\,e^{z^2/8}\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}\,e^{-\pi n x}\,J_{\frac{k}{2}-1}(\sqrt{\pi n x}\,z)-\frac{\pi^{\frac{k}{4}-\frac{1}{2}}z^{\frac{k}{2}-1}e^{-z^2/8}}{2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)x^{\frac{k}{4}}}\\
=&\frac{e^{-z^2/8}}{\sqrt{x}}\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}\,e^{-\frac{\pi n}{x}}\,I_{\frac{k}{2}-1}\left(\sqrt{\frac{\pi n}{x}}z\right)-\frac{\pi^{\frac{k}{4}-\frac{1}{2}}z^{\frac{k}{2}-1}x^{\frac{k}{4}}\,e^{z^2/8}}{2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)}\\
=&\left(\frac{\sqrt{\pi}z}{2}\right)^{\frac{k}{2}-1}\frac{e^{z^2/8}}{2\pi\Gamma\left(\frac{k}{2}\right)}\int_{-\infty}^{\infty}\pi^{-\frac{k}{4}-it}\Gamma\left(\frac{k}{4}+it\right)\zeta_k\left(\frac{k}{4}+it\right)\,{}_1F_1\left(\frac{k}{4}+it;\,\frac{k}{2};\,-\frac{z^2}{4}\right)\,x^{-it}\,dt.
\end{aligned}\qquad(1.9)$$

The main advantage of the representation (1.9) is that Popov's formula can be now expressed as a corollary of the symmetries of an integral involving the important Dirichlet series $\zeta_k(s)$. When it comes to study analogues of a summation formula and the connection between it and the behavior of the associated Dirichlet series, it is very important to keep a certain symmetry in the analytic structure of both sides of the identity, which is helpful to preserve the "modular shape" of it.

Looking at Popov's formula (1.1), one may see that the indices of the Bessel functions appearing at both sides of it depend on k. Therefore, in our search for a new analogue of (1.1), it is not unreasonable to start with the study of a more general series of the form

$$\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{\nu}{2}}}\,e^{-\pi n x}\,J_{\nu}(\sqrt{\pi n x}z),\qquad(1.10)$$

where ν is some complex parameter which is independent of k. Getting a formula for the series (1.10) would provide a generalization of Popov's formula and, hopefully, furnish a beautiful analogue of it. The study of this series, however, ultimately leads to a huge disappointment.

In fact, as it can be seen in [21], for any $\mathrm{Re}(\nu)>-1$ and $\mathrm{Re}(x)>0,\ z\in\mathbb{C}$, the following summation formula holds

$$\begin{aligned}
&\frac{x^{\frac{\nu+1}{2}}z^{\nu}\pi^{\frac{\nu}{2}}}{2^{\nu}\Gamma(\nu+1)}e^{z^2/8}+\sqrt{x}e^{z^2/8}\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{\nu}{2}}}\,e^{-\pi x n}\,J_{\nu}(\sqrt{\pi n x}z)=\frac{\pi^{\frac{\nu}{2}}x^{\frac{\nu-k+1}{2}}z^{\nu}}{2^{\nu}\Gamma(\nu+1)}e^{z^2/8}\,{}_1F_1\left(\frac{k}{2};\,\nu+1;\,-\frac{z^2}{4}\right)\\
&+\frac{x^{\frac{\nu-k+1}{2}}z^{\nu}\pi^{\frac{\nu}{2}}}{2^{\nu}\Gamma(\nu+1)}\,e^{-z^2/8}\sum_{n=1}^{\infty}r_k(n)e^{-\frac{\pi n}{x}}\,\Phi_3\left(1-\frac{k}{2}+\nu;\,\nu+1;\,\frac{z^2}{4},\,\frac{\pi z^2 n}{4x}\right),
\end{aligned}\qquad(1.11)$$

where $\Phi_3(b;c;w,u)$ is the usual Humbert function,

$$\Phi_3(b;c;\,w,u)=\sum_{k,m=0}^{\infty}\frac{(b)_k}{(c)_{k+m}}\frac{w^k u^m}{k!\,m!}.$$

Why is (1.11) not as beautiful as (1.1)? One reason for this is perhaps the generality of (1.11), as (1.11) implies Popov's formula when $\nu=\frac{k}{2}-1$. The symmetries of (1.1) are lost once we pick a general index for the Bessel function $J_{\nu}(z)$.

[1] Such extension was stated for the first time by Berndt in [2]. Berndt used a generalized version of Voronoï's summation formula.
[2] When $k=1$, (1.9) gives a formula due to Dixit [[11], p. 374, eq. (1.13)].

Besides the loss of symmetry, another reason why (1.11) is not appealing at all is that the analytical structures of both sides of it are drastically different. We have started the summation formula at the left with the Bessel function of the first kind, $J_{\nu}(z)$ (which, in hypergeometric terms, is nothing but a function of the form ${}_0F_1$), but we ended up on the right with a confluent hypergeometric function of two variables!

Although this first attempt fails in bringing an interesting analogue of (1.1), we note that, if we slightly change the argument and indices of the Bessel functions in (1.1), we can achieve a new analogue of Popov's result. This is stated as our main result.

Theorem 1.1. *If $x > y > 0$, then we have the transformation formula*

$$\frac{(\pi y)^{\frac{k}{4}-\frac{1}{2}}}{2^{\frac{k}{4}-\frac{1}{2}}\Gamma\left(\frac{k}{4}+\frac{1}{2}\right)}+\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}\,e^{-\pi n x}J_{\frac{k}{4}-\frac{1}{2}}\left(\pi n y\right)$$
$$=\frac{(\pi y)^{\frac{k}{4}-\frac{1}{2}}}{2^{\frac{k}{4}-\frac{1}{2}}\Gamma\left(\frac{k}{4}+\frac{1}{2}\right)(x^2+y^2)^{\frac{k}{4}}}+\frac{1}{\sqrt{x^2+y^2}}\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}\,e^{-\frac{\pi n x}{x^2+y^2}}J_{\frac{k}{4}-\frac{1}{2}}\left(\frac{\pi n y}{x^2+y^2}\right). \tag{1.12}$$

Moreover, under the same conditions,

$$\frac{(\pi y)^{\frac{k}{4}-\frac{1}{2}}}{2^{\frac{k}{4}-\frac{1}{2}}\Gamma\left(\frac{k}{4}+\frac{1}{2}\right)}+\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}\,e^{-\pi n x}I_{\frac{k}{4}-\frac{1}{2}}\left(\pi n y\right)$$
$$=\frac{(\pi y)^{\frac{k}{4}-\frac{1}{2}}}{2^{\frac{k}{4}-\frac{1}{2}}\Gamma\left(\frac{k}{4}+\frac{1}{2}\right)(x^2-y^2)^{\frac{k}{4}}}+\frac{1}{\sqrt{x^2-y^2}}\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}\,e^{-\frac{\pi n x}{x^2-y^2}}I_{\frac{k}{4}-\frac{1}{2}}\left(\frac{\pi n y}{x^2-y^2}\right). \tag{1.13}$$

The reasons for the interest of (1.12) and (1.13) are almost the same as the reasons why (1.1) is fascinating. Due to the appearance of the Bessel functions $J_{\nu}(z)$ and $I_{\nu}(z)$ and the exponential factors, as well as the same powers on the denominators of both sides of (1.12), our identity matches the criteria mentioned in items 1-3. Moreover, if we define a generalized $\theta-$function in the following way

$$\Theta_k\left(x,y\right):=\frac{(\pi y)^{\frac{k}{4}-\frac{1}{2}}}{2^{\frac{k}{4}-\frac{1}{2}}\Gamma\left(\frac{k}{4}+\frac{1}{2}\right)}+\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}\,e^{-\pi n x}J_{\frac{k}{4}-\frac{1}{2}}\left(\pi n y\right),$$

then (1.12) implies the transformation formula

$$\Theta_k\left(x,y\right)=\frac{1}{\sqrt{x^2+y^2}}\Theta_k\left(\frac{x}{x^2+y^2},\frac{y}{x^2+y^2}\right), \tag{1.14}$$

which also reduces to (1.6) in the limit $y\rightarrow 0^{+}$.

This paper is organized as follows. In section 2 we give some auxiliary results to prove (1.12), which mainly concern the behavior of Gauss' hypergeometric function ${}_2F_1\left(a,b;c;z\right)$. In section 3, we prove rigorously (1.12) by using a variant of our approach to get (1.1) given in [19]. Lastly, the final section 4 is devoted to a proof of a generalization of the Ramanujan-Guinand formula [7], which employs the same method as the proof of our Theorem 1.1.

Before moving to the next section, let us remark that (1.12) and (1.13) contain some interesting analogues and particular cases. First, if we take $k=4$ in (1.12), we obtain the curious identity

$$2\pi y+\sum_{n=1}^{\infty}\frac{r_4(n)}{n}\left\{e^{-\pi n(x-y)}-e^{-\pi n(x+y)}\right\}=\frac{2\pi y}{x^2-y^2}+\sum_{n=1}^{\infty}\frac{r_4(n)}{n}\left\{e^{-\frac{\pi n}{x+y}}-e^{-\frac{\pi n}{x-y}}\right\},$$

valid for $x > y > 0$.

It is also no surprise that we can extend our formulas (1.12) and (1.13) to a more general class of Dirichlet series satisfying Hecke's functional equation in the same spirit as [3, 4, 9] . We list some interesting examples.

1. Let $k \geq 3$ be an odd positive integer and consider the divisor function $\sigma_k(n)$. Then the Dirichlet series associated to it, $\zeta(s)\zeta(s-k)$, will satisfy a functional equation similar to (1.3). Adapting the proof of Theorem 1.1 below, one can show the following identity

$$-\frac{B_{k+1}(\pi y)^{\frac{k}{2}}}{2(k+1)\Gamma\left(\frac{k}{2}+1\right)}+\sum_{n=1}^{\infty}\frac{\sigma_k(n)}{n^{\frac{k}{2}}}\,e^{-2\pi n x}J_{\frac{k}{2}}\left(2\pi n y\right)$$
$$=\frac{(-1)^{\frac{k-1}{2}}B_{k+1}\left(\pi y\right)^{\frac{k}{2}}}{2(k+1)\Gamma\left(\frac{k}{2}+1\right)\left(x^2+y^2\right)^{\frac{k+1}{2}}}+\frac{(-1)^{\frac{k+1}{2}}}{\sqrt{x^2+y^2}}\sum_{n=1}^{\infty}\frac{\sigma_k(n)}{n^{\frac{k}{2}}}e^{-\frac{2\pi n x}{x^2+y^2}}\,J_{\frac{k}{2}}\left(\frac{2\pi n y}{x^2+y^2}\right),$$

which is valid for $x > y > 0$.

2. For $z \in \mathbb{H} := \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$, let $f(z)$ be a holomorphic cusp form with weight $k \geq 12$ for the full modular group with Fourier coefficients $a_f(n)$. The L-function associated to $f(z)$, $L_f(s)$, satisfies an analogue of the functional equation (1.3). Using the same formalism as in this paper, we may get the formula

$$\sum_{n=1}^{\infty}\frac{a_f(n)}{n^{\frac{k-1}{2}}}\,e^{-2\pi n x}J_{\frac{k-1}{2}}\left(2\pi n y\right)=\frac{(-1)^{k/2}}{\sqrt{x^2+y^2}}\sum_{n=1}^{\infty}\frac{a_f(n)}{n^{\frac{k-1}{2}}}\,e^{-\frac{2\pi n x}{x^2+y^2}}J_{\frac{k-1}{2}}\left(\frac{2\pi n y}{x^2+y^2}\right),\quad x>y>0.$$

In particular, the following particular case for Ramanujan's $\tau-$function takes place

$$\sum_{n=1}^{\infty}\frac{\tau(n)}{n^{\frac{11}{2}}}\,e^{-2\pi n x}J_{\frac{11}{2}}\left(2\pi n y\right)=\frac{1}{\sqrt{x^2+y^2}}\sum_{n=1}^{\infty}\frac{\tau(n)}{n^{\frac{11}{2}}}\,e^{-\frac{2\pi n x}{x^2+y^2}}J_{\frac{11}{2}}\left(\frac{2\pi n y}{x^2+y^2}\right),\quad x>y>0.$$

3. Let χ be a nonprincipal, primitive and even Dirichlet character modulo q. Its $L-$function, $L(2s,\chi)$, satisfies an analogue of the functional equation (1.3). Thus, for $x > y > 0$, the following formula holds

$$\sum_{n=1}^{\infty}\chi(n)\sqrt{n}\,e^{-\frac{\pi n^2}{q}x}J_{-\frac{1}{4}}\left(\frac{\pi n^2 y}{q}\right)=\frac{G(\chi)}{\sqrt{q\left(x^2+y^2\right)}}\sum_{n=1}^{\infty}\overline{\chi}(n)\sqrt{n}\,e^{-\frac{\pi n^2 x}{q(x^2+y^2)}}J_{-\frac{1}{4}}\left(\frac{\pi n^2 y}{q\left(x^2+y^2\right)}\right),$$

where $G(\chi)$ denotes the Gauss sum

$$G(\chi):=\sum_{\ell=1}^{q-1}\chi(\ell)\,e^{\frac{2\pi i\ell}{q}}.$$

Analogously, if χ is a nonprincipal, primitive and odd Dirichlet character modulo q, then the associated $L-$function, $L(2s-1,\chi)$, also satisfies a functional equation similar to (1.3). This gives the identity

$$\sum_{n=1}^{\infty}\sqrt{n}\,\chi(n)\,e^{-\frac{\pi n^2 x}{q}}J_{\frac{1}{4}}\left(\frac{\pi n^2 y}{q}\right)=-\frac{iG(\chi)}{\sqrt{q\left(x^2+y^2\right)}}\sum_{n=1}^{\infty}\sqrt{n}\,\overline{\chi}(n)\,e^{-\frac{\pi n^2 x}{q(x^2+y^2)}}J_{\frac{1}{4}}\left(\frac{\pi n^2 y}{q\left(x^2+y^2\right)}\right),$$

which holds for $x > y > 0$.

2 Preliminary results

For $|z| < 1$, the hypergeometric function ${}_2F_1(a,b;c;z)$ is defined by the Gauss series,

$${}_2F_1\left(a,b;c;z\right)=\sum_{\ell=0}^{\infty}\frac{(a)_\ell(b)_\ell}{(c)_\ell\,\ell!}\,z^\ell=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{\ell=0}^{\infty}\frac{\Gamma(a+\ell)\Gamma(b+\ell)}{\Gamma(c+\ell)\,\ell!}\,z^\ell, \tag{2.1}$$

and defined elsewhere by analytic continuation. For example, the analytic continuation can be given via Slater's theorem [26]. There is, however, another method of continuing the series (2.1) via Euler's integral representation: if $\text{Re}(c) > \text{Re}(b) > 0$ and $|\arg(1-z)| < \pi$, then ${}_2F_1(a,b;c;z)$ satisfies the formula

$$ {}_2F_1\left(a,b;c;z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt, \tag{2.2} $$

where we have chosen the principal branch $(1-tz)^{-a} = e^{-a\log(1-tz)}$, with $\log(1-tz)$ being real for $z \in [0,1]$.

There are several applications of Euler's formula. Using it, one is able to derive the two most famous transformation formulas for hypergeometric functions,

$$ {}_2F_1\left(a,b;c;z\right) = (1-z)^{-a}\, {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right), \tag{2.3} $$

$$ {}_2F_1\left(a,b;c;z\right) = (1-z)^{c-a-b}\, {}_2F_1\left(c-a, c-b; c; z\right), \tag{2.4} $$

respectively known as Pfaff's and Euler's formulas[3]. It is also known that, for any $c \neq 0, -1, -2, \dots$ and fixed z, the function ${}_2F_1\left(a,b;c;z\right)$ is an entire function of a, b and c [[15], p. 384, 15.2(ii)].

Just like (2.2), there are other interesting integral representation of the hypergeometric function. The most important formula in this paper is the Mellin transform [[8], p. 155, eq. (3.10.3)],

$$ \int_0^\infty x^{s-1}e^{-\alpha x}J_\nu(\beta x)\,dx = \frac{\beta^\nu}{2^\nu \alpha^{s+\nu}}\frac{\Gamma(s+\nu)}{\Gamma(\nu+1)}\,{}_2F_1\left(\frac{s+\nu}{2}, \frac{s+\nu+1}{2}; \nu+1; -\frac{\beta^2}{\alpha^2}\right), $$

valid for $\text{Re}(s) > -\text{Re}(\nu)$ and $\text{Re}(\alpha) > |\text{Im}(\beta)|$. Taking $\nu = \frac{k}{4} - \frac{1}{2}$ in the above representation and assuming that $\alpha > \beta > 0$, we see that the formula

$$ \int_0^\infty x^{s-1}e^{-\alpha x}J_{\frac{k}{4}-\frac{1}{2}}(\beta x)\,dx = \left(\frac{\beta}{2\alpha}\right)^{\frac{k}{4}-\frac{1}{2}} \alpha^{-s}\frac{\Gamma\left(s+\frac{k}{4}-\frac{1}{2}\right)}{\Gamma\left(\frac{k}{4}+\frac{1}{2}\right)}\,{}_2F_1\left(\frac{s}{2}+\frac{k}{8}-\frac{1}{4}, \frac{s}{2}+\frac{k}{8}+\frac{1}{4}; \frac{k}{4}+\frac{1}{2}; -\frac{\beta^2}{\alpha^2}\right) \tag{2.5} $$

must hold for any $\text{Re}(s) > \frac{1}{2} - \frac{k}{4}$. Since $k \geq 1$ by hypothesis, (2.5) must be always valid for $\text{Re}(s) > \frac{1}{4}$. This suggests the following inversion formula, which holds for $\sigma > \frac{1}{4}$ and $\alpha > \beta > 0$,

$$ e^{-\alpha x}J_{\frac{k}{4}-\frac{1}{2}}(\beta x) = \frac{(\beta/2\alpha)^{\frac{k}{4}-\frac{1}{2}}}{2\pi i\,\Gamma\left(\frac{k}{4}+\frac{1}{2}\right)}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma\left(s+\frac{k}{4}-\frac{1}{2}\right)\,{}_2F_1\left(\frac{4s+k-2}{8}, \frac{4s+k+2}{8}; \frac{k}{4}+\frac{1}{2}; -\frac{\beta^2}{\alpha^2}\right)(\alpha x)^{-s}ds. \tag{2.6} $$

Since it will be instructive later, let us check that Mellin's inversion theorem is valid for our pair of functions and prove (2.6). Of course, for $\alpha > \beta > 0$, the function $f(x) = e^{-\alpha x}J_\nu(\beta x)$ is continuous and, for any $\sigma > -\nu$, $x^{\sigma-1}f(x) \in L_1(\mathbb{R}_+)$. Hence, in order to check the conditions of Mellin's inversion theorem, one just needs to see that

$$ \Gamma\left(\sigma+\frac{k}{4}-\frac{1}{2}+it\right)\,{}_2F_1\left(\frac{4\sigma+k-2}{8}+\frac{it}{2}, \frac{4\sigma+k+2}{8}+\frac{it}{2}; \frac{k}{4}+\frac{1}{2}; -\frac{\beta^2}{\alpha^2}\right) \in L_1\left(\sigma-i\infty, \sigma+i\infty\right). \tag{2.7} $$

A way to verify (2.7) is by invoking an asymptotic expansion due to Watson [24] (cf. [[12], Vol 1, p. 77]). However, since Watson's analysis is much stronger and difficult than what we need in this note, we shall proceed with a simpler idea, following closely an analogous argument given in [[26], p. 22].

[3]Alternatively, under the terminology of [26], they are also called Boltz formula and self-transformation formula.

Lemma 2.1. *For $\alpha > \beta > 0$ and $\sigma, \nu \in \mathbb{R}$ are such that $\nu > -1$ and $\sigma > -\nu$, then the following asymptotic expansion takes place*

$$
\begin{aligned}
&{}_2F_1\left(\frac{\sigma+\nu}{2}+\frac{it}{2}, \frac{\sigma+\nu+1}{2}+\frac{it}{2}; \nu+1; -\frac{\beta^2}{\alpha^2}\right) \\
&= \Gamma(\nu+1)\left(\frac{\beta t}{2\alpha}\right)^{-\nu} I_\nu\left(\frac{\beta t}{\alpha}\right)\left\{1+O\left(\frac{1}{t}\right)\right\}, \qquad t\to\infty.
\end{aligned} \tag{2.8}
$$

Proof. First, let us note that we have the uniform estimate for $\beta/\alpha < x_0 < 1$,

$$
\begin{aligned}
&\left|{}_2F_1\left(\frac{\sigma+\nu}{2}+\frac{it}{2}, \frac{\sigma+\nu+1}{2}+\frac{it}{2}; \nu+1; -\frac{\beta^2}{\alpha^2}\right)\right| \\
&\le \frac{\sqrt{\pi}\, 2^{1-\sigma-\nu}\Gamma(\nu+1)}{\left|\Gamma\left(\frac{\sigma+\nu}{2}+\frac{it}{2}\right)\Gamma\left(\frac{\sigma+\nu+1}{2}+\frac{it}{2}\right)\right|}\sum_{\ell=0}^{\infty}\frac{\Gamma(\sigma+\nu+2\ell)}{\Gamma(\nu+1+\ell)\,\ell!}\left(\frac{x_0}{2}\right)^{2\ell},
\end{aligned}
$$

with the latter series being convergent due to the ratio test. Hence,

$$
{}_2F_1\left(\frac{\sigma+\nu}{2}+\frac{it}{2}, \frac{\sigma+\nu+1}{2}+\frac{it}{2}; \nu+1; -\frac{\beta^2}{\alpha^2}\right) = \lim_{N\to\infty}\sum_{\ell=0}^{N}\frac{(-1)^\ell\,(\sigma+\nu+it)_{2\ell}}{(\nu+1)_\ell\,\ell!}\left(\frac{\beta}{2\alpha}\right)^{2\ell}.
$$

We shall consider the asymptotic expansion (with respect to the parameter t) of the previous finite sum over ℓ. From an exact version of Stirling's formula, we know that

$$
(\sigma+\nu+it)_{2\ell} = (-1)^\ell t^{2\ell}\left(1+O\left(\frac{1}{t}\right)\right), \quad t\to\infty, \tag{2.9}
$$

where the remainder is uniform with respect to $0 \le \ell \le N$ [[26], p. 20, eq. (1.145)]. Hence, as $t\to\infty$,

$$
\sum_{\ell=0}^{N}\frac{(-1)^\ell\,(\sigma+\nu+it)_{2\ell}}{(\nu+1)_\ell\,\ell!}\left(\frac{\beta}{2\alpha}\right)^{2\ell} = \sum_{\ell=0}^{N}\frac{1}{(\nu+1)_\ell\,\ell!}\left(\frac{\beta t}{2\alpha}\right)^{2\ell} + O\left(\frac{1}{t}\sum_{\ell=0}^{N}\frac{1}{(\nu+1)_\ell\,\ell!}\left(\frac{\beta t}{2\alpha}\right)^{2\ell}\right).
$$

Clearly, when $N\to\infty$, the main term in the above expression can be identified with the modified Bessel function of the first kind, $I_\nu(x)$. Thus, as $t\to\infty$, we are able to get

$$
\begin{aligned}
{}_2F_1\left(\frac{\sigma+\nu}{2}+\frac{it}{2}, \frac{\sigma+\nu+1}{2}+\frac{it}{2}; \nu+1; -\frac{\beta^2}{\alpha^2}\right) &= \left\{1+O\left(\frac{1}{t}\right)\right\}\lim_{N\to\infty}\sum_{\ell=0}^{N}\frac{1}{(\nu+1)_\ell\,\ell!}\left(\frac{\beta t}{2\alpha}\right)^{2\ell} \\
&= \Gamma(\nu+1)\left(\frac{\beta t}{2\alpha}\right)^{-\nu} I_\nu\left(\frac{\beta t}{\alpha}\right)\left\{1+O\left(\frac{1}{t}\right)\right\}.
\end{aligned}
$$

□

Take $\nu := \frac{k}{4}-\frac{1}{2}$ in the asymptotic expansion (2.8). Since $\nu > -\frac{1}{2}$, we may use the Poisson integral [[15], p. 252, eq. (10.32.2)],

$$
\left(\frac{z}{2}\right)^{-\nu} I_\nu(z) = \frac{1}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}\int_{-1}^{1}(1-t^2)^{\nu-\frac{1}{2}}e^{zt}dt, \quad \nu > -\frac{1}{2}, \ \ z\in\mathbb{C},
$$

to reach the elementary bound

$$
\left|\left(\frac{z}{2}\right)^{-\nu} I_\nu(z)\right| \le \frac{e^{|\mathrm{Re}(z)|}}{\Gamma(\nu+1)}. \tag{2.10}
$$

Using Stirling's formula, (2.8) and (2.10) with $\nu=\frac{k}{4}-\frac{1}{2}$, one can easily derive the bound

$$\begin{aligned}&\left|\Gamma\left(\sigma+\frac{k}{4}-\frac{1}{2}+it\right)\,{}_2F_1\left(\frac{4\sigma+k-2}{8}+\frac{it}{2},\,\frac{4\sigma+k+2}{8}+\frac{it}{2};\,\frac{k}{4}+\frac{1}{2};-\frac{\beta^2}{\alpha^2}\right)\right|\\&=(2\pi)^{\frac{1}{2}}\Gamma\left(\frac{k}{4}+\frac{1}{2}\right)\,|t|^{\sigma+\frac{k}{4}-1}e^{-\frac{\pi}{2}|t|}\left|\left(\frac{\beta t}{2\alpha}\right)^{-\frac{k}{4}+\frac{1}{2}}I_{\frac{k}{4}-\frac{1}{2}}\left(\frac{\beta t}{\alpha}\right)\right|\left\{1+O\left(\frac{1}{|t|}\right)\right\}\\&\qquad\le(2\pi)^{\frac{1}{2}}|t|^{\sigma+\frac{k}{4}-1}\exp\left(-\left(\frac{\pi}{2}-\frac{\beta}{\alpha}\right)|t|\right)\left\{1+O\left(\frac{1}{|t|}\right)\right\},\quad |t|\to\infty,\end{aligned}\tag{2.11}$$

which shows (2.7), because $\beta/\alpha<1$ by hypothesis. This inequality will be very useful in the sequel.

3 Proof of Theorem 1.1

It suffices to prove the formula (1.12), because the proof of (1.13) is totally analogous and the only difficulty required for this is just to check that all the steps below are also valid then y is replaced by iy in (1.12). Also, due to the inequality (2.10), the condition $x>y>0$ ensures that both series on (1.13) are absolutely convergent.

We start by transforming the infinite series on the left-hand side of (1.12). Using the integral representation (2.6) with $\alpha=\pi x$, $\beta=\pi y$ and $x=n$ and choosing $\sigma>\max\left\{\frac{k}{4}+\frac{1}{2},\,\frac{3k}{4}-\frac{1}{2}\right\}$ in (2.6), we find that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}\,e^{-\pi nx}J_{\frac{k}{4}-\frac{1}{2}}\left(\pi ny\right)&=\frac{y^{\frac{k}{4}-\frac{1}{2}}}{(2x)^{\frac{k}{4}-\frac{1}{2}}\Gamma\left(\frac{k}{4}+\frac{1}{2}\right)}\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}\\\times\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma\left(s+\frac{k}{4}-\frac{1}{2}\right)\,{}_2F_1&\left(\frac{4s+k-2}{8},\,\frac{4s+k+2}{8};\,\frac{k}{4}+\frac{1}{2};-\frac{y^2}{x^2}\right)(\pi xn)^{-s}\,ds.\end{aligned}$$

By our choice of $\sigma>\frac{k}{4}+\frac{1}{2}$ and by the fact that the Dirichlet series defining $\zeta_k(s)$, (1.2), converges absolutely in the region $\mathrm{Re}(s)>\frac{k}{2}$, the interchange of the orders of integration and summation is possible due to absolute convergence. Thus, if $\sigma>\frac{k}{4}+\frac{1}{2}$, we have the integral representation

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}\,e^{-\pi nx}J_{\frac{k}{4}-\frac{1}{2}}\left(\pi ny\right)&=\frac{y^{\frac{k}{4}-\frac{1}{2}}}{(2x)^{\frac{k}{4}-\frac{1}{2}}\,\Gamma\left(\frac{k}{4}+\frac{1}{2}\right)}\\\times\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma\left(s+\frac{k}{4}-\frac{1}{2}\right)\zeta_k\left(s+\frac{k}{4}-\frac{1}{2}\right){}_2F_1&\left(\frac{4s+k-2}{8},\,\frac{4s+k+2}{8};\,\frac{k}{4}+\frac{1}{2};-\frac{y^2}{x^2}\right)(\pi x)^{-s}ds.\end{aligned}\tag{3.1}$$

We shift the line of integration in (3.1) from $\mathrm{Re}(s)=\sigma$ to $\mathrm{Re}(s)=\frac{k}{2}-\sigma$ and apply Cauchy's residue theorem by performing an integration over the positively oriented rectangular contour $\left[\sigma\pm iT,\,\frac{k}{2}-\sigma\pm iT\right]$, $T>0$. By the Phragmén-Lindelöf principle [23], we know that, for any $\delta>0$, $\zeta_k(\sigma+it)\ll_{\delta}|t|^{A(\sigma)+\delta}$, where

$$A(\sigma)=\begin{cases}0 & \sigma>\frac{k}{2}+\delta\\ \frac{k}{2}-\sigma & -\delta\le\sigma\le\frac{k}{2}+\delta\\ \frac{k}{2}-2\sigma & \sigma<-\delta\end{cases}.\tag{3.2}$$

Thus, we can easily find the estimate for the contribution of the horizontal segments,

$$\int_{\sigma\pm iT}^{\frac{k}{2}-\sigma\pm iT}\left|\Gamma\left(s+\frac{k}{4}-\frac{1}{2}\right)\zeta_k\left(s+\frac{k}{4}-\frac{1}{2}\right){}_2F_1\left(\frac{4s+k-2}{8},\frac{4s+k+2}{8};\frac{k}{4}+\frac{1}{2};-\frac{y^2}{x^2}\right)(\pi x)^{-s}\right||ds|$$

$$\ll_{\delta,\sigma,x}(2\pi)^{\frac{1}{2}}T^{\sigma+A\left(\sigma+\frac{k}{4}-\frac{1}{2}\right)+\frac{k}{4}-1}\exp\left[-\left(\frac{\pi}{2}-\frac{y}{x}\right)T\right],$$

which tends to zero as $T\to\infty$. Therefore, by Cauchy's theorem,

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma\left(s+\frac{k}{4}-\frac{1}{2}\right)\zeta_k\left(s+\frac{k}{4}-\frac{1}{2}\right){}_2F_1\left(\frac{4s+k-2}{8},\frac{4s+k+2}{8};\frac{k}{4}+\frac{1}{2};-\frac{y^2}{x^2}\right)(\pi x)^{-s}ds$$

$$=\frac{1}{2\pi i}\int_{\frac{k}{2}-\sigma-i\infty}^{\frac{k}{2}-\sigma+i\infty}\Gamma\left(s+\frac{k}{4}-\frac{1}{2}\right)\zeta_k\left(s+\frac{k}{4}-\frac{1}{2}\right){}_2F_1\left(\frac{4s+k-2}{8},\frac{4s+k+2}{8};\frac{k}{4}+\frac{1}{2};-\frac{y^2}{x^2}\right)(\pi x)^{-s}ds$$

$$+\sum_{\rho}\text{Res}_{s=\rho}\left\{\Gamma\left(s+\frac{k}{4}-\frac{1}{2}\right)\zeta_k\left(s+\frac{k}{4}-\frac{1}{2}\right){}_2F_1\left(\frac{4s+k-2}{8},\frac{4s+k+2}{8};\frac{k}{4}+\frac{1}{2};-\frac{y^2}{x^2}\right)(\pi x)^{-s}\right\},\tag{3.3}$$

where the last sum denotes the contribution from the residues of the integrand inside the contour. The hypergeometric function appearing in the previous integral is an entire function of s, so the possible poles come from the factors involving the Gamma function and the Dirichlet series $\zeta_k\left(s+\frac{k}{4}-\frac{1}{2}\right)$. Since $\zeta_k(s)$ has trivial zeros at the points $s\in\mathbb{Z}^-$, the only possible poles of the product $\Gamma\left(s+\frac{k}{4}-\frac{1}{2}\right)\zeta_k\left(s+\frac{k}{4}-\frac{1}{2}\right)$ inside the rectangular contour are located at $\rho=\frac{k}{4}+\frac{1}{2}$ and $\rho=\frac{1}{2}-\frac{k}{4}$. Routine calculations show that[4]

$$\text{Res}_{s=\frac{k}{4}+\frac{1}{2}}\left\{\mathcal{J}_{x,y}(s)\right\}=\pi^{\frac{k}{4}-\frac{1}{2}}x^{-\frac{k}{4}-\frac{1}{2}}{}_2F_1\left(\frac{k}{4},\frac{k}{4}+\frac{1}{2};\frac{k}{4}+\frac{1}{2};-\frac{y^2}{x^2}\right)=\frac{(\pi x)^{\frac{k}{4}-\frac{1}{2}}}{(x^2+y^2)^{\frac{k}{4}}},\tag{3.4}$$

$$\text{Res}_{s=\frac{1}{2}-\frac{k}{4}}\left\{\mathcal{J}_{x,y}(s)\right\}=-(\pi x)^{\frac{k}{4}-\frac{1}{2}}.\tag{3.5}$$

Using (3.4), (3.5) and finally rearranging the terms in (3.3), we are able to obtain the representation

$$\frac{(\pi y)^{\frac{k}{4}-\frac{1}{2}}}{2^{\frac{k}{4}-\frac{1}{2}}\Gamma\left(\frac{k}{4}+\frac{1}{2}\right)}+\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}e^{-\pi nx}J_{\frac{k}{4}-\frac{1}{2}}(\pi ny)=\frac{(\pi y)^{\frac{k}{4}-\frac{1}{2}}}{2^{\frac{k}{4}-\frac{1}{2}}\Gamma\left(\frac{k}{4}+\frac{1}{2}\right)(x^2+y^2)^{\frac{k}{4}}}+\frac{y^{\frac{k}{4}-\frac{1}{2}}}{(2x)^{\frac{k}{4}-\frac{1}{2}}\Gamma\left(\frac{k}{4}+\frac{1}{2}\right)}$$

$$\times\frac{1}{2\pi i}\int_{\frac{k}{2}-\sigma-i\infty}^{\frac{k}{2}-\sigma+i\infty}\Gamma\left(s+\frac{k}{4}-\frac{1}{2}\right)\zeta_k\left(s+\frac{k}{4}-\frac{1}{2}\right){}_2F_1\left(\frac{4s+k-2}{8},\frac{4s+k+2}{8};\frac{k}{4}+\frac{1}{2};-\frac{y^2}{x^2}\right)(\pi x)^{-s}ds.\tag{3.6}$$

Note that the left-hand side and the first term on the right-hand side of (3.6) are already the first three terms given in (1.12). Thus, we just need to see that the integral in (3.6) reduces to the infinite series on the right of (1.12). This will be the final technical procedure in our proof.

Invoking the functional equation for $\zeta_k(s)$, (1.3), together with Euler's transformation formula (2.4), one can

[4] Recall that the residue of $\zeta_k(s)$ at $s=\frac{k}{2}$ is $\pi^{k/2}/\Gamma(k/2)$ and $\zeta_k(0)=-1$.

check that the integral on the right-hand side of (3.6) is equal to

$$=\frac{\sqrt{x^2+y^2}}{\pi x}\int_{\frac{k}{2}-\sigma-i\infty}^{\frac{k}{2}-\sigma+i\infty}\Gamma\left(\frac{k}{4}+\frac{1}{2}-s\right)\zeta_k\left(\frac{k}{4}+\frac{1}{2}-s\right)\,{}_2F_1\left(\frac{k+6-4s}{8},\frac{k+2-4s}{8};\frac{k}{4}+\frac{1}{2};-\frac{y^2}{x^2}\right)\left(\frac{\pi x}{x^2+y^2}\right)^{s}ds$$

$$=\frac{(\pi x)^{\frac{k}{2}-1}}{(x^2+y^2)^{\frac{k-1}{2}}}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma\left(s-\frac{k}{4}+\frac{1}{2}\right)\zeta_k\left(s-\frac{k}{4}+\frac{1}{2}\right)\,{}_2F_1\left(\frac{4s+6-k}{8},\frac{4s+2-k}{8};\frac{k}{4}+\frac{1}{2};-\frac{y^2}{x^2}\right)\left(\frac{\pi x}{x^2+y^2}\right)^{-s}ds.$$

Since we have chosen $\sigma>\max\left\{\frac{k}{4}+\frac{1}{2},\frac{3k}{4}-\frac{1}{2}\right\}$, the Dirichlet series $\zeta_k\left(s-\frac{k}{4}+\frac{1}{2}\right)$ is absolutely convergent on the line $\text{Re}(s)=\sigma$. Hence, by absolute convergent once more,

$$\begin{aligned}&\frac{(\pi x)^{\frac{k}{2}-1}}{(x^2+y^2)^{\frac{k-1}{2}}}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma\left(s-\frac{k}{4}+\frac{1}{2}\right)\zeta_k\left(s-\frac{k}{4}+\frac{1}{2}\right)\,{}_2F_1\left(\frac{4s+6-k}{8},\frac{4s+2-k}{8};\frac{k}{4}+\frac{1}{2};-\frac{y^2}{x^2}\right)\left(\frac{\pi x}{x^2+y^2}\right)^{-s}ds\\
&=\frac{(\pi x)^{\frac{k}{2}-1}}{(x^2+y^2)^{\frac{k-1}{2}}}\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{1}{2}-\frac{k}{4}}}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma\left(s-\frac{k}{4}+\frac{1}{2}\right)\,{}_2F_1\left(\frac{4s+6-k}{8},\frac{4s+2-k}{8};\frac{k}{4}+\frac{1}{2};-\frac{y^2}{x^2}\right)\left(\frac{\pi xn}{x^2+y^2}\right)^{-s}ds\\
&=\frac{1}{\sqrt{x^2+y^2}}\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}\int_{\sigma+1-\frac{k}{2}-i\infty}^{\sigma+1-\frac{k}{2}+i\infty}\Gamma\left(z+\frac{k}{4}-\frac{1}{2}\right)\,{}_2F_1\left(\frac{4z+k+2}{8},\frac{4z+k-2}{8};\frac{k}{4}+\frac{1}{2};-\frac{y^2}{x^2}\right)\left(\frac{\pi xn}{x^2+y^2}\right)^{-z}dz.\end{aligned}\tag{3.7}$$

By hypothesis, $\sigma+1-\frac{k}{2}>\frac{1}{4}$, so we can invoke representation (2.6) in the last step with $\alpha=\frac{\pi x}{x^2+y^2}$, $\beta=\frac{\pi y}{x^2+y^2}$ and $x=n$, which yields

$$\begin{aligned}&\int_{\sigma+1-\frac{k}{2}-i\infty}^{\sigma+1-\frac{k}{2}+i\infty}\Gamma\left(z+\frac{k}{4}-\frac{1}{2}\right)\,{}_2F_1\left(\frac{4z+k+2}{8},\frac{4z+k-2}{8};\frac{k}{4}+\frac{1}{2};-\frac{y^2}{x^2}\right)\left(\frac{\pi xn}{x^2+y^2}\right)^{-z}dz\\
&=\frac{2\pi i\,(2x)^{\frac{k}{4}-\frac{1}{2}}\Gamma\left(\frac{k}{4}+\frac{1}{2}\right)}{y^{\frac{k}{4}-\frac{1}{2}}}\exp\left(-\frac{\pi nx}{x^2+y^2}\right)J_{\frac{k}{4}-\frac{1}{2}}\left(\frac{\pi ny}{x^2+y^2}\right).\end{aligned}\tag{3.8}$$

Returning to (3.6) and (3.7), the use of the integral representation (3.8) shows at last that (3.6) results in the transformation formula

$$\begin{aligned}&\frac{(\pi y)^{\frac{k}{4}-\frac{1}{2}}}{2^{\frac{k}{4}-\frac{1}{2}}\Gamma\left(\frac{k}{4}+\frac{1}{2}\right)}+\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}\,e^{-\pi nx}J_{\frac{k}{4}-\frac{1}{2}}(\pi ny)\\
&=\frac{(\pi y)^{\frac{k}{4}-\frac{1}{2}}}{2^{\frac{k}{4}-\frac{1}{2}}\Gamma\left(\frac{k}{4}+\frac{1}{2}\right)(x^2+y^2)^{\frac{k}{4}}}+\frac{1}{\sqrt{x^2+y^2}}\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}\,e^{-\frac{\pi nx}{x^2+y^2}}J_{\frac{k}{4}-\frac{1}{2}}\left(\frac{\pi ny}{x^2+y^2}\right),\end{aligned}$$

which is precisely what we have claimed. ■

4 Concluding Remarks: A generalization of the Ramanujan-Guinand formula

On page 253 of his Lost Notebook [1, 7, 18], Ramanujan states the following formula, quoted from [[1], p. 94].

Entry 3.3.1. (p. 253): Let $\sigma_k(n)=\sum_{d|n} d^k$ and let $K_\nu(z)$ be the modified Bessel function of the second kind. If α and β are two positive numbers such that $\alpha\beta=\pi^2$ and if ν is any complex number, then

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\sigma_{-\nu}(n)\,n^{\nu/2}K_{\frac{\nu}{2}}(2n\alpha)-\sqrt{\beta}\sum_{n=1}^{\infty}\sigma_{-\nu}(n)\,n^{\nu/2}K_{\frac{\nu}{2}}(2n\beta)$$
$$=\frac{1}{4}\Gamma\left(-\frac{\nu}{2}\right)\zeta(-\nu)\left\{\beta^{(1+\nu)/2}-\alpha^{(1+\nu)/2}\right\}+\frac{1}{4}\Gamma\left(\frac{\nu}{2}\right)\zeta(\nu)\left\{\beta^{(1-\nu)/2}-\alpha^{(1-\nu)/2}\right\}. \tag{4.1}$$

This formula was rediscovered by Guinand in 1955 [13], who employed Watson's formula [25] in order to derive it. Using an idea similar to the one employed by Guinand (but working with a general analogue of Watson's formula), we have proved in [21] the following generalization of (4.1).

Theorem 4.1. *Let $r_k(n)$ denote the number of ways in which n can be written as the sum of k squares. If $x,y>0$ and if ν is any complex number, then the following formula holds*

$$2\Gamma\left(\frac{k}{2}\right)(\pi y)^{1-\frac{k}{2}}\sum_{m,n=1}^{\infty}r_k(m)\,r_k(n)\,\left(\frac{m}{n}\right)^{\frac{\nu}{2}}(mn)^{\frac{1}{2}-\frac{k}{4}}\,J_{\frac{k}{2}-1}\left(2\pi\sqrt{m\,n}\,y\right)\,K_\nu\left(2\pi\sqrt{m\,n}\,x\right)$$
$$-\frac{2\Gamma\left(\frac{k}{2}\right)(\pi y)^{1-\frac{k}{2}}}{x^2+y^2}\sum_{m,n=1}^{\infty}r_k(m)\,r_k(n)\,\left(\frac{m}{n}\right)^{\frac{\nu}{2}}(mn)^{\frac{1}{2}-\frac{k}{4}}\,J_{\frac{k}{2}-1}\left(\frac{2\pi\sqrt{m\,n}\,y}{x^2+y^2}\right)\,K_\nu\left(\frac{2\pi\sqrt{m\,n}\,x}{x^2+y^2}\right)$$
$$=x^{-\nu}\eta_k(\nu)\left\{\frac{1}{(x^2+y^2)^{\frac{k}{2}-\nu}}-1\right\}+x^{\nu}\eta_k(-\nu)\left\{\frac{1}{(x^2+y^2)^{\frac{k}{2}+\nu}}-1\right\}, \tag{4.2}$$

where $\eta_k(s)$ denotes the completed Dirichlet series,

$$\eta_k(s)=\pi^{-s}\Gamma(s)\,\zeta_k(s).$$

Moreover, if ν is any complex number and $x>y>0$, then the analogous formula is valid

$$2\Gamma\left(\frac{k}{2}\right)(\pi y)^{1-\frac{k}{2}}\sum_{m,n=1}^{\infty}r_k(m)\,r_k(n)\,\left(\frac{m}{n}\right)^{\frac{\nu}{2}}(mn)^{\frac{1}{2}-\frac{k}{4}}\,I_{\frac{k}{2}-1}\left(2\pi\sqrt{m\,n}\,y\right)\,K_\nu\left(2\pi\sqrt{m\,n}\,x\right)$$
$$-\frac{2\Gamma\left(\frac{k}{2}\right)(\pi y)^{1-\frac{k}{2}}}{x^2-y^2}\sum_{m,n=1}^{\infty}r_k(m)\,r_k(n)\,\left(\frac{m}{n}\right)^{\frac{\nu}{2}}(mn)^{\frac{1}{2}-\frac{k}{4}}\,I_{\frac{k}{2}-1}\left(\frac{2\pi\sqrt{m\,n}\,y}{x^2-y^2}\right)\,K_\nu\left(\frac{2\pi\sqrt{m\,n}\,x}{x^2-y^2}\right)$$
$$=x^{-\nu}\eta_k(\nu)\left\{\frac{1}{(x^2-y^2)^{\frac{k}{2}-\nu}}-1\right\}+x^{\nu}\eta_k(-\nu)\left\{\frac{1}{(x^2-y^2)^{\frac{k}{2}+\nu}}-1\right\}. \tag{4.3}$$

At first glance, (4.2) and (4.3) do not seem to be related to (4.1), despite the fact that both formulas share the Modified Bessel function, $K_\nu(z)$, as an element in their infinite series. In the next few lines, we shall argue why (4.2) and (4.3) are, in fact, generalizations of (4.1). Recalling that $\zeta_1(s)=2\zeta(2s)$ (see (1.4) above) and that $r_1(n)=2$ iff n is a perfect square and 0 otherwise, we see that a particular case of (4.2) is

$$8\pi\sqrt{y}\sum_{m,n=1}^{\infty}\left(\frac{m}{n}\right)^{\nu}\sqrt{m\,n}\,J_{-\frac{1}{2}}\left(2\pi m\,n\,y\right)\,K_\nu\left(2\pi m\,n\,x\right)$$
$$-\frac{8\pi\sqrt{y}}{x^2+y^2}\sum_{m,n=1}^{\infty}\left(\frac{m}{n}\right)^{\nu}\sqrt{m\,n}\,J_{-\frac{1}{2}}\left(\frac{2\pi m\,n\,y}{x^2+y^2}\right)\,K_\nu\left(\frac{2\pi m\,n\,x}{x^2+y^2}\right)$$
$$=2(\pi x)^{-\nu}\,\Gamma(\nu)\zeta\left(2\nu\right)\left\{\frac{1}{(x^2+y^2)^{\frac{1}{2}-\nu}}-1\right\}+2(\pi x)^{\nu}\,\Gamma(-\nu)\zeta\left(-2\nu\right)\left\{\frac{1}{(x^2+y^2)^{\frac{1}{2}+\nu}}-1\right\}. \tag{4.4}$$

However, this expression can be simplified further. Indeed, recalling the reduction formula for the Bessel function [[22], p. 248]

$$J_{-\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\,\cos(x),$$

and making the substitution of ν by $\nu/2$ in (4.4), we can now rewrite (4.4) in the following manner

$$\sum_{m,n=1}^{\infty}\left(\frac{m}{n}\right)^{\nu/2}\cos\left(2\pi m\,n\,y\right)\,K_{\nu/2}\left(2\pi m\,n\,x\right)-\frac{1}{\sqrt{x^{2}+y^{2}}}\sum_{m,n=1}^{\infty}\left(\frac{m}{n}\right)^{\nu/2}\cos\left(\frac{2\pi m\,n\,y}{x^{2}+y^{2}}\right)\,K_{\nu/2}\left(\frac{2\pi m\,n\,x}{x^{2}+y^{2}}\right)$$
$$=\frac{1}{4}(\pi x)^{-\frac{\nu}{2}}\,\Gamma\left(\frac{\nu}{2}\right)\,\zeta\left(\nu\right)\left\{\frac{1}{\left(x^{2}+y^{2}\right)^{\frac{1-\nu}{2}}}-1\right\}+\frac{1}{4}(\pi x)^{\frac{\nu}{2}}\,\Gamma\left(-\frac{\nu}{2}\right)\,\zeta\left(-\nu\right)\left\{\frac{1}{\left(x^{2}+y^{2}\right)^{\frac{1+\nu}{2}}}-1\right\}. \tag{4.5}$$

The main formal difference between (4.1) and the newly derived formula (4.5) is that (4.1) is expressed via two single infinite series, whilst the infinite series in (4.5) contain two variables of summation. A way to circumvent this is by recalling the definition of the divisor function

$$\sigma_{z}(n)=\sum_{d|n}d^{z},$$

which helps us to see that (4.5) can be rewritten as

$$\sum_{n=1}^{\infty}\sigma_{-\nu}(n)\,n^{\nu/2}\cos\left(2\pi ny\right)\,K_{\nu/2}\left(2\pi n\,x\right)-\frac{1}{\sqrt{x^{2}+y^{2}}}\sum_{n=1}^{\infty}\sigma_{-\nu}(n)\,n^{\nu/2}\,\cos\left(\frac{2\pi n\,y}{x^{2}+y^{2}}\right)\,K_{\nu/2}\left(\frac{2\pi n\,x}{x^{2}+y^{2}}\right)$$
$$=\frac{1}{4}(\pi x)^{-\frac{\nu}{2}}\,\Gamma\left(\frac{\nu}{2}\right)\zeta\left(\nu\right)\left\{\frac{1}{\left(x^{2}+y^{2}\right)^{\frac{1-\nu}{2}}}-1\right\}+\frac{1}{4}(\pi x)^{\frac{\nu}{2}}\Gamma\left(-\frac{\nu}{2}\right)\,\zeta(-\nu)\left\{\frac{1}{\left(x^{2}+y^{2}\right)^{\frac{1+\nu}{2}}}-1\right\}. \tag{4.6}$$

Let us note now that (4.6) is very similar to (4.1), being even more general than it. If we replace x by α/π and let $y=0$ in (4.6), we can recover the Ramanujan-Guinand formula in the same form as (4.1).

Having argued that (4.2) (respectively (4.3)) constitutes a generalization of the beautiful result of Ramanujan, why do we refer to it in this note about Popov's formula? First, as we have shown in [21], (4.2) is an indirect consequence of (1.1). Second, the transformation formulas (4.2) and (4.3) are clearly remindful of (1.12) and (1.13), because the roles of x and y in these formulas are transformed in similar ways. In fact, for $\nu\in\mathbb{C}$, $k\in\mathbb{N}$ and $x,y>0$, if we set the function

$$\Psi_{k}\left(\nu;\,x,y\right):=x^{-\nu}\eta_{k}(\nu)+x^{\nu}\eta_{k}(-\nu)$$
$$+2\Gamma\left(\frac{k}{2}\right)\left(\pi y\right)^{1-\frac{k}{2}}\sum_{m,n=1}^{\infty}r_{k}(m)\,r_{k}(n)\,\left(\frac{m}{n}\right)^{\frac{\nu}{2}}\left(m\,n\right)^{\frac{1}{2}-\frac{k}{4}}\,J_{\frac{k}{2}-1}\left(2\pi\sqrt{m\,n}\,y\right)\,K_{\nu}\left(2\pi\sqrt{m\,n}\,x\right),$$

then (4.2) is equivalent to the transformation

$$\Psi_{k}\left(\nu;\,x,y\right)=\frac{1}{\left(x^{2}+y^{2}\right)^{\frac{k}{2}}}\,\Psi_{k}\left(\nu;\,\frac{x}{x^{2}+y^{2}},\,\frac{y}{x^{2}+y^{2}}\right),$$

which is very similar to (1.14), provided at the introduction of this paper. Besides, the proof of (4.2) (resp. (4.3)) here presented uses exactly the same argument as the proof of (1.12) (resp. (1.13)). Thus, in order to conclude our paper, we present a short proof of the formulas (4.2) and (4.3).

Just like before, we start with a Mellin representation [[8], p. 224, eq. (3.14.12.1)]

$$\int_0^\infty x^{s-1} J_\mu(\beta x)\, K_\nu(\alpha x)\, dx = \frac{2^{s-2}\beta^\mu}{\alpha^{s+\mu}\Gamma(\mu+1)}\Gamma\left(\frac{s+\mu-\nu}{2}\right)\Gamma\left(\frac{s+\mu+\nu}{2}\right)\,{}_2F_1\left(\frac{s+\mu-\nu}{2},\frac{s+\mu+\nu}{2};\mu+1;-\frac{\beta^2}{\alpha^2}\right), \tag{4.7}$$

which is valid for $\mathrm{Re}(s+\mu) > |\mathrm{Re}(\nu)|$ and $\mathrm{Re}(\alpha) > |\mathrm{Im}(\beta)|$. Taking $\mu = \frac{k}{2}-1 > -1$ and assuming that $\alpha > \beta > 0$, we find the Mellin inverse of (4.7),

$$J_{\frac{k}{2}-1}(\beta x)\, K_\nu(\alpha x) = \frac{1}{8\pi i \Gamma\left(\frac{k}{2}\right)}\left(\frac{\beta}{\alpha}\right)^{\frac{k}{2}-1}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma\left(\frac{s+k/2-1-\nu}{2}\right)\Gamma\left(\frac{s+k/2-1+\nu}{2}\right)$$
$$\times\ {}_2F_1\left(\frac{s+k/2-1-\nu}{2},\frac{s+k/2-1+\nu}{2};\frac{k}{2};-\frac{\beta^2}{\alpha^2}\right)\left(\frac{\alpha x}{2}\right)^{-s} ds, \tag{4.8}$$

where $\sigma := \mathrm{Re}(s) > |\mathrm{Re}(\nu)| + 1 - \frac{k}{2}$.

The conditions for the application of the Mellin inversion formula (4.8) are met once we prove an estimate analogous to (2.8). This is done in the next lemma.

Lemma 4.1. *Let $\mu > -1$ and $\sigma + \mu > |Re(\nu)|$. Also, suppose that $x > y > 0$. Then there exists some sufficiently large τ_0 such that, for any $t \geq \tau_0$, the following inequality holds*

$$\left|\frac{\Gamma\left(\frac{\sigma+\mu-\nu}{2}+\frac{it}{2}\right)\Gamma\left(\frac{\sigma+\mu+\nu}{2}+\frac{it}{2}\right)}{\Gamma(\mu+1)}\,{}_2F_1\left(\frac{\sigma+\mu-\nu}{2}+\frac{it}{2},\frac{\sigma+\mu+\nu}{2}+\frac{it}{2};\ \mu+1;\ -\frac{\beta^2}{\alpha^2}\right)\right|$$
$$\leq 4\pi\left(\frac{t}{2}\right)^{\sigma+\mu-1}\exp\left\{-\left(\frac{\pi}{2}-\frac{\beta}{\alpha}\right)t\right\}. \tag{4.9}$$

Proof. For $0 < \beta/\alpha \leq X_0 < 1$, the use of the hypothesis $\sigma+\mu > |\mathrm{Re}(\nu)|$ shows the uniform bound

$$\left|\frac{\Gamma\left(\frac{\sigma+\mu-\nu}{2}+\frac{it}{2}\right)\Gamma\left(\frac{\sigma+\mu+\nu}{2}+\frac{it}{2}\right)}{\Gamma(\mu+1)}\,{}_2F_1\left(\frac{\sigma+\mu-\nu}{2}+\frac{it}{2},\frac{\sigma+\mu+\nu}{2}+\frac{it}{2};\ \mu+1;\ -\frac{\beta^2}{\alpha^2}\right)\right|$$
$$\leq \sum_{\ell=0}^\infty \frac{\Gamma\left(\frac{\sigma+\mu-\mathrm{Re}(\nu)}{2}+\ell\right)\Gamma\left(\frac{\sigma+\mu+\mathrm{Re}(\nu)}{2}+\ell\right)}{\Gamma(\mu+1+\ell)\,\ell!}X_0^{2\ell},$$

where the latter series is convergent due to the ratio test. Thus, we have that

$$\frac{\Gamma\left(\frac{\sigma+\mu-\nu}{2}+\frac{it}{2}\right)\Gamma\left(\frac{\sigma+\mu+\nu}{2}+\frac{it}{2}\right)}{\Gamma(\mu+1)}\,{}_2F_1\left(\frac{\sigma+\mu-\nu}{2}+\frac{it}{2},\frac{\sigma+\mu+\nu}{2}+\frac{it}{2};\ \mu+1;\ -\frac{\beta^2}{\alpha^2}\right)$$
$$= \frac{\Gamma\left(\frac{\sigma+\mu-\nu}{2}+\frac{it}{2}\right)\Gamma\left(\frac{\sigma+\mu+\nu}{2}+\frac{it}{2}\right)}{\Gamma(\mu+1)}\lim_{N\to\infty}\sum_{\ell=0}^N\frac{(-1)^\ell\left(\frac{\sigma+\mu-\nu}{2}+\frac{it}{2}\right)_\ell\left(\frac{\sigma+\mu+\nu}{2}+\frac{it}{2}\right)_\ell}{(\mu+1)_\ell\,\ell!}\left(\frac{\beta}{\alpha}\right)^{2\ell}.$$

Using the fact that the remainder in Stirling's formula can be uniformly estimated with respect to the index ℓ (cf. [[26], p. 20, eq. (1.145)]), we see that

$$\frac{\Gamma\left(\frac{\sigma+\mu-\nu}{2}+\frac{it}{2}\right)\Gamma\left(\frac{\sigma+\mu+\nu}{2}+\frac{it}{2}\right)}{\Gamma(\mu+1)}\sum_{\ell=0}^N\frac{(-1)^\ell\left(\frac{\sigma+\mu-\nu}{2}+\frac{it}{2}\right)_\ell\left(\frac{\sigma+\mu+\nu}{2}+\frac{it}{2}\right)_\ell}{(\mu+1)_\ell\,\ell!}\left(\frac{\beta}{\alpha}\right)^{2\ell}$$
$$= \frac{\Gamma\left(\frac{\sigma+\mu-\nu}{2}+\frac{it}{2}\right)\Gamma\left(\frac{\sigma+\mu+\nu}{2}+\frac{it}{2}\right)}{\Gamma(\mu+1)}\sum_{\ell=0}^N\frac{1}{(\mu+1)_\ell\,\ell!}\left(\frac{\beta t}{2\alpha}\right)^{2\ell}\left\{1+O\left(\frac{1}{t}\right)\right\}$$
$$= \frac{2\pi}{\Gamma(\mu+1)}\left(\frac{t}{2}\right)^{\sigma+\mu-1}e^{-\frac{\pi}{2}t}\sum_{\ell=0}^N\frac{1}{(\mu+1)_\ell\,\ell!}\left(\frac{\beta t}{2\alpha}\right)^{2\ell}\left\{1+O\left(\frac{1}{t}\right)\right\}, \tag{4.10}$$

where the last step is just an application of Stirling's formula for the product $\Gamma\left(\frac{\sigma+\mu-\nu}{2}+\frac{it}{2}\right)\Gamma\left(\frac{\sigma+\mu+\nu}{2}+\frac{it}{2}\right)$. Recalling the power series for the modified Bessel function $I_\nu(z)$, (4.10) shows that

$$\frac{\Gamma\left(\frac{\sigma+\mu-\nu}{2}+\frac{it}{2}\right)\Gamma\left(\frac{\sigma+\mu+\nu}{2}+\frac{it}{2}\right)}{\Gamma(\mu+1)}\,{}_2F_1\left(\frac{\sigma+\mu-\nu}{2}+\frac{it}{2},\frac{\sigma+\mu+\nu}{2}+\frac{it}{2};\,\mu+1;\,-\frac{y^2}{x^2}\right)$$
$$=2\pi\left(\frac{t}{2}\right)^{\sigma+\mu-1}\left(\frac{\beta t}{2\alpha}\right)^{-\mu}I_\mu\left(\frac{\beta t}{\alpha}\right)e^{-\frac{\pi}{2}t}\left\{1+O\left(\frac{1}{t}\right)\right\},\quad t\to\infty.$$

Finally, appealing to (2.10) we find the desired inequality (4.9), which proves Lemma 4.1. □

By the previous lemma, one concludes that

$$\frac{\Gamma\left(\frac{\sigma+\mu-\nu}{2}+\frac{it}{2}\right)\Gamma\left(\frac{\sigma+\mu+\nu}{2}+\frac{it}{2}\right)}{\Gamma(\mu+1)}\,{}_2F_1\left(\frac{\sigma+\mu-\nu}{2}+\frac{it}{2},\frac{\sigma+\mu+\nu}{2}+\frac{it}{2};\,\mu+1;\,-\frac{y^2}{x^2}\right)\in L_1\left(\sigma-i\infty,\sigma+i\infty\right),$$

whenever $\sigma+\mu>|\text{Re}(\nu)|$, $\mu>-1$ and $\beta>\alpha>0$. Therefore, the integral representation (4.8) must hold under analogous conditions. We are now ready to give a new proof of Theorem 4.1 using the main ideas of this paper.

Proof of Theorem 4.1. As in the proof of Theorem 1.1, it suffices for us to show the first formula (4.2), as the arguments here presented can be easily modified to give (4.3). Throughout this argument, we shall assume that $\text{Re}(\nu)>0$ and, for simplicity, that $\nu\neq\frac{k}{2}$. It is enough to prove (4.2) under these simpler conditions because (4.2) is invariant under the reflection $\nu\leftrightarrow-\nu$ and the extension to the point $\nu=\frac{k}{2}$ can be easily provided by analytic continuation. Also, for simplicity in our notation, let

$$\eta_k(s):=\pi^{-s}\Gamma(s)\,\zeta_k\left(s\right). \tag{4.11}$$

In this proof, we shall assume that $\sigma>\max\left\{\frac{k}{2}+1+\text{Re}(\nu),k-1+\text{Re}(\nu)\right\}$. By absolute convergence of the Dirichlet series defining $\zeta_k(s)$ in the region $\text{Re}(s)>\frac{k}{2}$, together with the Mellin representation (4.8), we have

$$2\Gamma\left(\frac{k}{2}\right)(\pi y)^{1-\frac{k}{2}}\sum_{m,n=1}^{\infty}r_k(m)\,r_k(n)\,\left(\frac{m}{n}\right)^{-\frac{\nu}{2}}(m\,n)^{\frac{1}{2}-\frac{k}{4}}J_{\frac{k}{2}-1}\left(2\pi\sqrt{mn}\,y\right)\,K_\nu\left(2\pi\sqrt{mn}\,x\right)$$
$$=\frac{x^{1-\frac{k}{2}}}{4\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\eta_k\left(\frac{s-\nu-1}{2}+\frac{k}{4}\right)\eta_k\left(\frac{s+\nu-1}{2}+\frac{k}{4}\right)$$
$$\times\,{}_2F_1\left(\frac{s-\nu-1}{2}+\frac{k}{4},\frac{s+\nu-1}{2}+\frac{k}{4};\,\frac{k}{2};-\frac{y^2}{x^2}\right)x^{-s}\,ds. \tag{4.12}$$

Considering the integral in (4.12), we shift its line of integration from $\text{Re}(s)=\sigma$ to $\text{Re}(s)=\frac{k}{2}-\sigma$ by integrating along a positively oriented rectangular contour with vertices $\sigma\pm iT$ and $\frac{k}{2}-\sigma\pm iT$. Using Lemma 4.1 and the convex estimates for $\zeta_k(s)$, (3.2), the integrals along the horizontal segments, $\left[\frac{k}{2}-\sigma\pm iT,\,\sigma\pm iT\right]$ tend to zero as $T\to\infty$.

Since $\nu\neq\frac{k}{2}$ and $\text{Re}(\nu)>0$ by hypothesis, we find that the integrand in (4.12) contains four simple poles inside the rectangular contour $\left[\sigma\pm iT,\,\frac{k}{2}-\sigma\pm iT\right]$. In fact, these poles are located at the points

$$\rho_1=1+\nu-\frac{k}{2},\;\rho_2=\frac{k}{2}+1+\nu,\;\rho_3=1-\frac{k}{2}-\nu,\;\rho_4=\frac{k}{2}+1-\nu,$$

and their simplicity is due to the conditions $\nu \neq \frac{k}{2}$ and $\text{Re}(\nu) > 0$. Let $\mathscr{R}_j(x)$, $j = 1, ..., 4$, denote the respective residues of ρ_j, $j = 1, ..., 4$: from simple calculations, we have that

$$\mathscr{R}_1(x) = -2\eta_k\left(\nu\right) x^{\frac{k}{2}-\nu-1},$$

$$\mathscr{R}_2(x) = 2\eta_k\left(\frac{k}{2}+\nu\right)\left(\frac{x}{2}\right)^{-\frac{k}{2}-1-\nu} {}_2F_1\left(\frac{k}{2}, \frac{k}{2}+\nu; \frac{k}{2}; -\frac{y^2}{x^2}\right) = 2\eta_k\left(\frac{k}{2}+\nu\right) x^{\frac{k}{2}+\nu-1}\left(x^2+y^2\right)^{-\frac{k}{2}-\nu}$$
$$= 2\eta_k\left(-\nu\right) x^{\frac{k}{2}+\nu-1}\left(x^2+y^2\right)^{-\frac{k}{2}-\nu},$$

where the last step is just an application of the functional equation of $\zeta_k(s)$, (1.3). Analogously,

$$\mathscr{R}_3(x) = -2\eta_k\left(-\nu\right) x^{\frac{k}{2}+\nu-1}$$

and

$$\mathscr{R}_4(x) = 2\eta_k\left(\frac{k}{2}-\nu\right) x^{\frac{k}{2}-\nu-1}\left(x^2+y^2\right)^{-\frac{k}{2}+\nu} = 2\eta_k\left(\nu\right) x^{\frac{k}{2}-\nu-1}\left(x^2+y^2\right)^{-\frac{k}{2}+\nu}.$$

Using Cauchy's residue theorem and the calculations performed in the previous lines, we find that (4.12) implies the formula

$$2\Gamma\left(\frac{k}{2}\right)(\pi y)^{1-\frac{k}{2}}\sum_{m,n=1}^{\infty} r_k(m)\, r_k(n)\left(\frac{m}{n}\right)^{-\frac{\nu}{2}}(mn)^{\frac{1}{2}-\frac{k}{4}} J_{\frac{k}{2}-1}\left(2\pi\sqrt{mn}\,y\right) K_\nu\left(2\pi\sqrt{mn}\,x\right)$$
$$= \frac{x^{1-\frac{k}{2}}}{4\pi i}\int_{\frac{k}{2}-\sigma-i\infty}^{\frac{k}{2}-\sigma+i\infty}\eta_k\left(\frac{s-\nu-1}{2}+\frac{k}{4}\right)\eta_k\left(\frac{s+\nu-1}{2}+\frac{k}{4}\right)$$
$$\times {}_2F_1\left(\frac{s-\nu-1}{2}+\frac{k}{4}, \frac{s+\nu-1}{2}+\frac{k}{4}; \frac{k}{2}; -\frac{y^2}{x^2}\right) x^{-s}\, ds$$
$$+\eta_k(\nu)\, x^{-\nu}\left\{\left(x^2+y^2\right)^{\nu-\frac{k}{2}}-1\right\} + \eta_k(-\nu)\, x^{\nu}\left\{\left(x^2+y^2\right)^{-\nu-\frac{k}{2}}-1\right\}, \tag{4.13}$$

which is now in the right shape for us to perform the final transformations. Using the functional equation for $\eta_k(s)$ and Euler's formula (2.4), the integral on the right-hand side of (4.13) can be written as

$$\int_{\frac{k}{2}-\sigma-i\infty}^{\frac{k}{2}-\sigma+i\infty}\eta_k\left(\frac{s-\nu-1}{2}+\frac{k}{4}\right)\eta_k\left(\frac{s+\nu-1}{2}+\frac{k}{4}\right) {}_2F_1\left(\frac{s-\nu-1}{2}+\frac{k}{4}, \frac{s+\nu-1}{2}+\frac{k}{4}; \frac{k}{2}; -\frac{y^2}{x^2}\right) x^{-s}\, ds$$
$$= x^{-\frac{k}{2}}\left(\frac{x^2+y^2}{x^2}\right)^{1-\frac{k}{2}}\int_{\sigma-i\infty}^{\sigma+i\infty}\eta_k\left(\frac{s+\nu+1}{2}\right)\eta_k\left(\frac{s-\nu+1}{2}\right) {}_2F_1\left(\frac{s+\nu+1}{2}, \frac{s-\nu+1}{2}; \frac{k}{2}; -\frac{y^2}{x^2}\right)\left(\frac{x}{x^2+y^2}\right)^{-s} ds$$
$$= \frac{8\pi i\, x^{\frac{k}{2}-1}}{x^2+y^2}\Gamma\left(\frac{k}{2}\right)(\pi y)^{1-\frac{k}{2}}\sum_{m,n=1}^{\infty} r_k(m)\, r_k(n)\left(\frac{m}{n}\right)^{-\frac{\nu}{2}}(m\,n)^{\frac{1}{2}-\frac{k}{4}} J_{\frac{k}{2}-1}\left(\frac{2\pi\sqrt{mn}\,y}{x^2+y^2}\right) K_\nu\left(\frac{2\pi\sqrt{mn}\,x}{x^2+y^2}\right), \tag{4.14}$$

where in the last step we have used (4.12) with x and y respectively replaced by $x/(x^2+y^2)$ and $y/(x^2+y^2)$. Note that this last step is actually valid because, by our initial choice of σ, we know that $\sigma > k-1+\text{Re}(\nu)$. Using (4.14) in (4.13) yields the desired identity (4.2). □

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References

[1] G. E. Andrews and B. C. Berndt, Ramanujan's Lost Notebook, Part IV, Springer-Verlag, New York, 2013.

[2] B. C. Berndt, Identities involving the coefficients of a class of Dirichlet series. V., *Trans. Amer. Math. Soc.*, **160** (1971), 139–156.

[3] B. C. Berndt, A. Dixit, R. Gupta, A. Zaharescu, A Class of Identities Associated with Dirichlet Series Satisfying Hecke's Functional Equation, *Proc. Amer. Math. Society*, **150** (2022), 4785–4799.

[4] B. C. Berndt, A. Dixit, R. Gupta, and A. Zaharescu, Two General Series Identities Involving Modified Bessel Functions and a Class of Arithmetical Functions, *Canad. J. Math.* (2022), 1—31.

[5] B. Berndt, A. Dixit, R. Gupta, A. Zaharescu, Ramanujan and Koshliakov Meet Abel and Plana, preprint available on arxiv with reference arXiv:2112.09819.

[6] B. C. Berndt, A. Dixit, S. Kim and A. Zaharescu, On a theorem of A. I. Popov on sums of squares, *Proc. Amer. Math. Soc.*, **145** (2017), 3795-3808.

[7] B. C. Berndt, Y. Lee, and J. Sohn, Koshliakov's formula and Guinand's formula in Ramanujan's lost notebook, *Surveys in number theory, Dev. Math.*, **17** (2012), 21-42.

[8] Yu. A. Brychkov, O. I. Marichev, N. V. Savischenko, Handbook of Mellin Transforms, CRC Press, 2019.

[9] K. Chandrasekharan and R. Narasimhan, Hecke's functional equation and arithmetical identities, *Ann. of Math.*, **74** (1961), 1-23.

[10] A. Dixit and R. Gupta, Koshliakov zeta functions I. Modular relations, *Adv. Math.* **393** (2021), Paper No. 108093.

[11] A. Dixit, Analogues of the general theta transformation formula, *Proc. Roy. Soc. Edinburgh*, Sect. A, **143** (2013), 371–399.

[12] A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, Higher Transcendental Functions, Vols. **1**, **2** and **3**, McGraw-Hill, New York, London and Toronto, 1953.

[13] A. P. Guinand, Some rapidly convergent series for the Riemann ζ-function, *Quart J. Math.* (Oxford) **6** (1955), 156—160.

[14] N. S. Koshliakov, (under the name N. S. Sergeev), Issledovanie odnogo klassa transtsendentnykh funktsii, opredelyaemykh obobshcennym yravneniem Rimana (A study of a class of transcendental functions defined by the generalized Riemann equation) (in Russian), Trudy Mat. Inst. Steklov, Moscow, 1949, available online at https://dds .crl .edu /crldelivery /14052.

[15] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, C. W. Clark (Eds.), NIST Handbook of Mathematical Functions, Cambridge University Press, Cambridge, 2010.

[16] A. I. Popov, On some summation formulas (in Russian), Bull. Acad. Sci. URSS, **6** (1934), 801–802 (in Russian). This paper can be found here.

[17] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, Integrals and Series: Vol. **1**: Elementary Functions, Gordon and Breach, New York, 1986; Vol. **2**: Special Functions, Gordon and Breach, New York, 1986; Vol. **3**: More Special Functions, Gordon and Breach, New York, 1989.

[18] S. Ramanujan, The Lost Notebook and Other Unpublished Papers, Narosa, New Delhi, 1988.

[19] P. Ribeiro, S. Yakubovich, Certain extensions of results of Siegel, Wilton and Hardy, *Adv. Appl. Math.*, in press.

[20] P. Ribeiro, S. Yakubovich, Generalizations (in the spirit of Koshliakov) of some formulas from Ramanujan's lost notebook, to appear in the special volume *Coimbra Math. Texts* dedicated to the Memory of Professor José Carlos Petronilho

[21] P. Ribeiro, On a class of summation formulas associated with products of Bessel functions, in preparation.

[22] N. M. Temme, Special Functions: An Introduction to the Classical Functions of Mathematical Physics, Wiley-Interscience, New York, 1996.

[23] E. C. Titchmarsh, The Theory of Functions, 2nd ed., Oxford University Press, London, 1939.

[24] G. N. Watson, Asymptotic expansions of hypergeometric functions, *Trans. Cambridge Philos. Soc.* **22** (1918) 277–308.

[25] G. N. Watson, Some self-reciprocal functions (1), *Quart. J. of Math.* (Oxford), **2** (1931), 298–309.

[26] S. B. Yakubovich, Index Transforms, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 1996.