

# FOURIER–MUKAI TRANSFORMS AND NORMALIZATION OF NODAL CURVES

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ABSTRACT. In this article we study the relation between Arinkin’s Poincaré sheaf on the compactified Jacobian over an integral nodal curve and the classical Poincaré bundle on the Jacobian of the normalization. This is applied to derive a formula for the Fourier-Mukai transform of complexes on the compactified Jacobian that come from pushforward along the normalization map. This is achieved via a third intermediate geometry: the parabolic modules introduced by Bhosle and Cook, which encode fiber jumps at the nodes.

## CONTENTS

|  |    |
|--|----|
| 1. Introduction  | 2  |
| 1.1. Summary of the paper                                    | 2  |
| 1.2. Higgs bundles and mirror symmetry                       | 3  |
| 1.3. Structure   | 4  |
| 1.4. Acknowledgements  | 4  |
| 2. Preliminaries   | 4  |
| 2.1. Fourier–Mukai transforms on fine compactified Jacobians | 4  |
| 2.2. The case of nodal curves                                | 7  |
| 2.3. Parabolic modules                                       | 11 |
| 3. Parabolic modules and the curvilinear Hilbert scheme      | 15 |
| 4. Fourier–Mukai and normalization                           | 20 |
| 4.1. Relation between Poincaré sheaves                       | 21 |
| 4.2. Relation between Fourier–Mukai functors                 | 25 |
| References   | 27 |

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## 1. INTRODUCTION

In the seminal work of Mukai [Muk], the Fourier–Mukai transform provides a derived equivalence between an abelian variety  $A$  and the dual  $A^\vee := \text{Pic}^0(A)$ . The equivalence is defined using  $\mathcal{P}_A$ , the Poincaré bundle on  $A \times A^\vee$ , which is used to twist a composition of derived push-pull functors along the projections to  $A$  and  $A^\vee$ . Given a smooth curve  $\Sigma$ , Mukai’s transform applies to the Jacobian variety  $A = \text{Jac}_\Sigma^0 \cong (\text{Jac}_\Sigma^0)^\vee$  to give a derived auto-equivalence  $D^b(\text{Jac}_\Sigma^0) \cong D^b(\text{Jac}_\Sigma^0)$ , such that skyscraper sheaves are sent to line bundles. The twisting in this case is achieved by  $\mathcal{P}_\Sigma = \mathcal{P}_{\text{Jac}_\Sigma^0}$ , the universal family for degree zero line bundles on  $\text{Jac}_\Sigma^0$ .

Since [Muk], new Fourier–Mukai transform techniques have provided derived equivalences between geometric spaces in many contexts. Given a singular curve  $C$ , a natural case to consider is the compactified Jacobian  $\overline{\text{Jac}}_C^0$  parameterizing torsion-free sheaves on  $C$  of degree zero and rank one, the geometry of which has been studied by numerous authors [AK, Cap, D’So, Est, EGK, EK, MRV1, Sim]. With respect to a natural compactification of the duality functor:  $(\cdot)^\vee = \overline{\text{Pic}}^0(\cdot)$ , one has an extension of the autoduality property:  $\overline{\text{Jac}}_C^0 \cong (\overline{\text{Jac}}_C^0)^\vee$ , and correspondingly an auto-equivalence  $D^b(\overline{\text{Jac}}_C^0) \cong D^b((\overline{\text{Jac}}_C^0))$ . This equivalence was established for integral planar curves via a Fourier–Mukai transform of Arinkin [Ari2], extended to reduced planar curves by Melo–Rapagnetta–Viviani [MRV2, MRV3]. The definition of these transforms involves the construction of a Poincaré sheaf  $\mathcal{P}_C$  on  $\overline{\text{Jac}}_C^0 \times (\overline{\text{Jac}}_C^0)^\vee$ , universal for the family of degree zero and rank one torsion-free sheaves on  $\overline{\text{Jac}}_C^0$ .

**1.1. Summary of the paper.** This paper addresses the situation where  $C$  is an integral nodal curve with singularity divisor of length  $k$ . We equip  $C$  with the normalisation  $\nu : \Sigma \rightarrow C$ , along which pushforward defines an embedding  $\check{\nu} := \nu_* : \text{Jac}_\Sigma^{-k} \rightarrow \overline{\text{Jac}}_C^0$ . With respect to this morphism, we construct a relation between the two Fourier–Mukai transforms at hand: Arinkin’s transform over  $\overline{\text{Jac}}_C^0$  and Mukai’s transform over  $\text{Jac}_\Sigma^{-k}$ . Our calculations make essential use of a third intermediate geometry: the parabolic modules introduced by Bhosle and Cook [Bho, Cool], which consist of pairs  $(M, V)$  where  $M \in \text{Jac}_\Sigma^0$  and  $V$  is a vector subspace of pre-specified fibers of  $M$ . By specifying the fibers to be taken at  $\nu^{-1}(\text{Sing}(C))$ , the moduli space of such objects  $\text{PMod}_C^0$  defines a resolution of singularities  $\rho : \text{PMod}_C^0 \rightarrow \overline{\text{Jac}}_C^0$ , an essential morphism for studying the geometry at hand. Furthermore, the natural projection

$$\begin{aligned} \check{\nu} : \text{PMod}_C^0 &\longrightarrow \text{Jac}_\Sigma^0 \\ (M, V) &\longmapsto M \end{aligned} ,$$

makes  $\text{PMod}_C^0$  into a fiber bundle over  $\text{Jac}_\Sigma^0$ . By fixing  $y_0 \in \Sigma$  such that  $\nu(y_0)$  lies in the smooth locus of  $C$ , we define a translation

$$\begin{aligned} \tau_{-k, y_0} : \text{Jac}_\Sigma^{-k} &\longrightarrow \text{Jac}_\Sigma^0 \\ M &\longmapsto M \otimes \mathcal{O}(ky_0) \end{aligned} .$$

Our first main theorem establishes a relation between the aforementioned geometry and the Poincaré sheaves associated to the two curves.

**Theorem A** (Theorem 4.4). *Let  $C$  denote an integral nodal curve with singularity divisor of length  $k$  and normalisation  $\nu : \Sigma \rightarrow C$ . Then, the Poincaré bundle  $\mathcal{P}_\Sigma$  on  $\text{Jac}_\Sigma^{-k} \times \text{Jac}_\Sigma^{-k}$  and*

the Poincaré sheaf  $\mathcal{P}_C$  on  $\overline{\text{Jac}}_C^0 \times \overline{\text{Jac}}_C^0$  are related by the isomorphism

$$(\text{id} \times \check{\nu})^* \mathcal{P}_C \cong (\rho \times \text{id})_* \left( \mathfrak{q}_2^* \omega_{\text{PMod}}^{1/2} \otimes (\check{\nu} \times \text{id})^* (\text{id} \times \tau_{-k, y_0})^* \mathcal{P}_\Sigma \right),$$

where  $\mathfrak{q}_2 : \text{PMod}_C^0 \times \text{Jac}_\Sigma^{-k} \rightarrow \text{PMod}_C^0$  is the natural projection and  $\omega_{\text{PMod}}^{1/2}$  is a (preferred choice of) square root of the canonical bundle of  $\text{PMod}_C^0$ .

Taking  $\mathcal{P}_C$  and  $\mathcal{P}_\Sigma$  as integral kernels defines the respective Fourier–Mukai transforms

$$\Phi^{\mathcal{P}_C} : D^b(\overline{\text{Jac}}_C^0) \rightarrow D^b(\overline{\text{Jac}}_C^0), \quad \Phi^{\mathcal{P}_\Sigma} : D^b(\text{Jac}_\Sigma^{-k}) \rightarrow D^b(\text{Jac}_\Sigma^{-k}),$$

and a natural consequence of Theorem A is the following relation between the two transforms.

**Theorem B** (Theorem 4.6). *Adopt the same hypothesis as Theorem A. Then, for every object  $\mathcal{F}^\bullet \in D^b(\text{Jac}_\Sigma^{-k})$ , one has the following isomorphisms in  $D^b(\overline{\text{Jac}}_C^0)$ :*

$$\Phi^{\mathcal{P}_C} (R\check{\nu}_* \mathcal{F}^\bullet) \cong R\rho_* \left( \omega_{\text{PMod}}^{1/2} \otimes \check{\nu}^* \Phi^{\mathcal{P}_\Sigma} (\tau_{-k, y_0, *} \mathcal{F}^\bullet) \right).$$

This formula can be compared to the abelian case, where given an embedding between abelian varieties  $f : A \rightarrow B$ , the dual functor  $f^\vee : B^\vee \rightarrow A^\vee$  and the corresponding Fourier–Mukai transforms  $\Phi^{A^\vee} : D^b(A^\vee) \rightarrow D^b(A)$ ,  $\Phi^{B^\vee} : D^b(B^\vee) \rightarrow D^b(B)$ , satisfy the natural relation

$$(1.1) \quad \Phi^{A^\vee} \circ R(f^\vee)_* \simeq Rf^* \circ \Phi^{B^\vee},$$

as established for isogenies by Mukai [Muk, (3.4)] and for embeddings by Chen–Jiang [CJ, Proposition 2.3]. Theorem B can be seen as the extension of (1.1) that takes the geometry of the compactification  $\overline{\text{Jac}}_C^0$  into account.

**1.2. Higgs bundles and mirror symmetry.** Our work is motivated by the study of mirror symmetry on  $\mathcal{M}_{\text{Higgs}}(X)$ , the hyperkähler moduli space of  $\text{GL}_n$ -Higgs bundles over a smooth projective curve  $X$ . A Lagrangian torus fibration is provided by the *Hitchin fibration*  $h : \mathcal{M}_{\text{Higgs}}(X) \rightarrow \mathcal{B}$ , an algebraic integrable system over an affine space  $\mathcal{B}$  that parameterises *spectral curves*  $C \subset T^*X$  [Hit1]. The *Hitchin fiber* at  $C \in \mathcal{B}$  is a torsor over  $\overline{\text{Jac}}_C$ , so the SYZ mirror symmetry self-duality of  $\mathcal{M}_{\text{Higgs}}(X)$  corresponds to the self-duality of the compactified Jacobian. The translation to homological formalisms of mirror symmetry involves mirror dual branes related by Fourier–Mukai transforms along the Hitchin fibers - derived autodualities  $D^b(\overline{\text{Jac}}_C) \cong D^b(\overline{\text{Jac}}_C)$ .

By varying the spectral curves  $C$  in the locus  $\mathcal{B}^k \subset \mathcal{B}$  consisting of spectral curves with  $k$  nodal singularities, Hitchin [Hit3] uses the normalization maps  $\nu : \Sigma \rightarrow C$  to define subintegrable systems  $(\mathcal{I}^k \rightarrow \mathcal{B}^k) \subset (\mathcal{M}_{\text{Higgs}}(X) \rightarrow \mathcal{B})$ , with fibers given by torsors over  $\nu_* : \text{Jac}_\Sigma^{-k} \hookrightarrow \overline{\text{Jac}}_C$ . Hitchin’s paper is one of several recent advancements that shine light on singular Hitchin fibers [GO, HP, Hor], whilst the normalization of spectral curves has appeared in other studies of the Hitchin fibration [Ngo2, FGOP, FP, MS].

In a sequel paper [FHHO], the authors study the mirror symmetry of Hitchin’s subintegrable systems. This required a detailed analysis of the relation between nodal curves, normalisation and the Fourier–Mukai transform of Arinkin, which led us to the findings outlined in the present paper.

In [FHHO], the mirror branes to the subintegrable systems are holomorphic Lagrangian branes that consist of Hecke cycles. The brane structure consists of a spin bundle, or Chan-Paton bundle, which is baked into Theorems A and B above via the appearance of  $\omega_{\text{PMod}}^{1/2}$ . Recent developments in holomorphic Floer theory demonstrate that the spin bundle plays a key role in the categorisation of such branes [BBDJS, GS2, Pri], for given a holomorphic Lagrangians  $\mathcal{L}$ ,  $\mathcal{N}$ , choices of square root  $\omega_{\mathcal{L}}^{1/2}$  and  $\omega_{\mathcal{N}}^{1/2}$  are used to orient Lagrangian intersections  $[\mathcal{L} \cap \mathcal{N}]$ . We hope for Fourier-Mukai transform techniques similar to those treated in this paper to provide new tools for studying mirror symmetry in this context.

To implement this at scale requires extensions of our results, and to this end we suggest two possible generalisations. The first is on the compactified Jacobian of a singular curve beyond the nodal case, towards the generality of any locally planar singularity. This necessarily involves other techniques than those treated in this paper, for in the case of a nodal curve  $C$ , we make essential use of: (a) an established theory of parabolic modules, and (b) the fact that the Abel-Jacobi map  $\alpha_C : \text{Hilb}_{n,C}^{\text{cur}} \rightarrow \overline{\text{Jac}}_C^0$  is surjective. The second generalisation we suggest replaces a compactified Jacobian with a compactified Prym variety, adapting our results to the Fourier–Mukai transforms constructed in [FHR, GS1]. Whilst this step is straightforward for nodal curves, we expect it to be non-trivial for other singularities, for instance in the endoscopic locii of the  $\text{SL}_n$ -Hitchin system. The authors wish to pursue such ideas in future work.

**1.3. Structure.** This paper starts with Section 2.1, dedicated to the geometry of fine compactified Jacobian varieties and the corresponding Fourier–Mukai transform technology. Section 2.2 studies a local model of compactified Jacobians, and Section 2.3 recall the theory of parabolic modules and their moduli spaces.

Section 3 concerns the interplay between curvilinear Hilbert schemes and the moduli spaces of parabolic modules, aimed at subsequent applications to our Fourier–Mukai transform techniques.

Section 4.1 establishes the relation between the Poincaré sheaf for the compactified Jacobian over a nodal curve and the Poincaré bundle for the usual Jacobian over the normalized curve. In Section 4.2, this naturally leads to the relation between the corresponding Fourier–Mukai transforms.

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## 2. PRELIMINARIES

**2.1. Fourier–Mukai transforms on fine compactified Jacobians.** Let  $C$  be an integral projective curve with arithmetic genus  $g$  and, at most, locally planar singularities. The main points of this section treat the case where  $C$  singular, but everything also applies if  $C$  is smooth.

Whenever  $C$  is not smooth, the corresponding Jacobian  $\text{Jac}_C$  is not projective in general, although one has at hand a broad literature on its compactification [AK, Est, Sim]. Given a sheaf  $\mathcal{F}$  on  $C$ , we denote its degree by  $\text{deg}(\mathcal{F}) = \chi(\mathcal{F}) - \chi(\mathcal{O}_C)$ . As  $C$  is reduced and irreducible, every rank 1 pure dimension sheaf on it has no subsheaves of dimension 1, so it is automatically

stable. In that case, the compactified Jacobian  $\overline{\text{Jac}}_C^d$  is a *fine* moduli space for the classification of rank 1 and degree  $d$  pure dimension 1 sheaves over  $C$ , as there exists a universal sheaf  $\mathcal{U}_{d,C}$ . For smooth  $C$ , a pure sheaf is automatically locally free and hence  $\overline{\text{Jac}}_C^d = \text{Jac}_C^d$ . In the general case, it is well known that [MRV2],

- $\overline{\text{Jac}}_C^d$  is connected reduced irreducible projective scheme of dimension  $g = g(C)$  with locally complete intersection singularities;
- $\overline{\text{Jac}}_C^d$  has trivial dualizing sheaf;
- the smooth locus of  $\overline{\text{Jac}}_C^d$  is the open subset  $\text{Jac}_C^d$  parametrizing line bundles.

The compactified Jacobian is equipped with the corresponding *Abel–Jacobi map* associated to the smooth point  $x_0 \in C$ ,

$$(2.1) \quad \begin{array}{ccc} \alpha_C : \text{Hilb}_{n,C} & \longrightarrow & \overline{\text{Jac}}_C^0 \\ \mathcal{I} & \longmapsto & \mathcal{I}^\vee(-nx_0), \end{array}$$

which is smooth and surjective for sufficiently large  $n$ ; cf. [AK]. Since both  $\text{Hilb}_{n,C}$  and  $\overline{\text{Jac}}_C^0$  are projective schemes (see [Sim] for instance), it follows that the Abel–Jacobi map in (2.1) is projective. In fact, we know again by [AK] that, for large enough  $n$ , the map  $\alpha_C$  is a projective bundle over  $\overline{\text{Jac}}_C^0$  with fiber over  $\mathcal{F}$  equal to  $\mathbb{P}(H^0(C, \mathcal{F}(nx_0)))$ .

In his seminal work [Muk], Mukai constructed an equivalence between the derived category of an abelian variety, and the derived category of its dual abelian variety, the so-called Fourier–Mukai transform. Since the Jacobian of a smooth curve is autodual, the Fourier–Mukai transform provides a non-trivial autoequivalence of the derived category of the Jacobian of a smooth curve. Arinkin [Ari2] extended the aforementioned derived autoequivalence to the case of compactified Jacobians of integral planar curves (which are fine), and Melo–Rapagnetta–Viviani [MRV2, MRV3] to fine compactified Jacobians of reduced curves. As in Mukai’s original work, these equivalences are provided by an integral functor having the so-called Poincaré sheaf as a kernel. The later is a maximal Cohen–Macaulay sheaf, flat over each factor [Ari2], which is obtained from the descent of a certain maximal Cohen–Macaulay sheaf via the Abel–Jacobi map. This construction is based on the choice of an embedding of our curve into a smooth surface, although it is proven to be independent from such choice.

Consider the Hilbert scheme  $\text{Hilb}_{n,S}$  consisting of finite subschemes of length  $n$  in a smooth projective surface  $S$ . Let  $\mathcal{D}_S \subset \text{Hilb}_{n,S} \times S$  denote the universal family of length  $n$  subschemes of  $S$  and let  $h_S : \mathcal{D}_S \rightarrow \text{Hilb}_{n,S}$  be the restriction to  $\mathcal{D}_S$  of the projection map. This is a finite flat morphism of degree  $n$ . Then

$$\mathcal{A}_{S,n} := h_{S,*} \mathcal{O}_{\mathcal{D}_S}$$

is a locally free coherent sheaf on  $\text{Hilb}_{n,S}$ .

Let us also consider the flag scheme,  $\text{Flag}_{n,S}$ , parameterizing filtrations

$$\emptyset = D_0 \subset \cdots \subset D_k \subset \cdots \subset D_n$$

of finite subschemes of length  $n$  in  $S$ . Denote by  $\text{Hilb}_{n,S}^{cur}$  and  $\text{Flag}_{n,S}^{cur}$  the so-called *curvilinear* locii of subschemes of  $S$  that can be embedded into  $\mathbb{A}^1$ , or any other smooth curve for that

matter. These are open subschemes of  $\mathrm{Hilb}_{n,S}$  and  $\mathrm{Flag}_{n,S}$  respectively, as described by Arinkin [Ari2, Section 3]. The morphism

$$\psi_S^{cur} : \mathrm{Flag}_{n,S}^{cur} \longrightarrow \mathrm{Hilb}_{n,S}^{cur}, \quad D_0 \subset \cdots \subset D_n \mapsto D_n,$$

is a degree  $n!$  finite flat morphism [Ari2, Proposition 3.5]. Moreover,

$$\sigma_S^{cur} : \mathrm{Flag}_{n,S}^{cur} \longrightarrow S^n, \quad D_0 \subset \cdots \subset D_n \mapsto \mathrm{supp}(\ker(\mathcal{O}_{D_i} \longrightarrow \mathcal{O}_{D_{i-1}})_{i=1}^n),$$

is a  $\mathfrak{S}_n$ -equivariant morphism, with respect to the permutation action on  $S^n$  and the induced action of  $\mathfrak{S}_n$  on  $\mathrm{Flag}_{n,S}^{cur}$  [Ari2, Proposition 3.5]. Thanks to the work of Haiman [Hai] (see also [Ari2, Proposition 3.7]) there exists a scheme  $\widetilde{\mathrm{Hilb}}_{n,S}$ , *the isospectral Hilbert scheme*, making the right square of

$$\begin{array}{ccccc} S^n & \xleftarrow{\sigma_S^{cur}} & \mathrm{Flag}_{n,S}^{cur} & \xrightarrow{\psi_S^{cur}} & \mathrm{Hilb}_{n,S}^{cur} \\ \parallel & & \downarrow & & \downarrow \\ S^n & \xleftarrow{\sigma_S} & \widetilde{\mathrm{Hilb}}_{n,S} & \xrightarrow{\psi_S} & \mathrm{Hilb}_{n,S} \end{array},$$

Cartesian. The bottom line of the above diagram is crucial in Arinkin's construction [Ari2] of the Poincaré sheaf.

Let us go back to our irreducible projective curve  $C$  with  $x_0 \in C$  being a smooth point. Denote the universal sheaf normalized at  $\{x_0\} \times \overline{\mathrm{Jac}}_C^d$  by

$$\mathcal{U}_{d,C} \longrightarrow C \times \overline{\mathrm{Jac}}_C^d.$$

Consider a closed embedding  $\iota_C : C \hookrightarrow S$  of an integral and nodal curve  $C$  into our smooth surface. Set also  $\iota_C^n : C^n \hookrightarrow S^n$  as the induced embedding. With the diagram

$$\begin{array}{ccccccc} \mathrm{Hilb}_{n,S} \times \overline{\mathrm{Jac}}_C^0 & \xleftarrow{\psi_S \times \mathrm{id}_{\overline{\mathrm{Jac}}}} & \widetilde{\mathrm{Hilb}}_{n,S} \times \overline{\mathrm{Jac}}_C^0 & \xrightarrow{\sigma_S \times \mathrm{id}_{\overline{\mathrm{Jac}}}} & S^n \times \overline{\mathrm{Jac}}_C^0 & \xleftarrow{\iota_C^n \times \mathrm{id}_{\overline{\mathrm{Jac}}}} & C^n \times \overline{\mathrm{Jac}}_C^0 \\ \downarrow p_1 & & & & & & \\ \mathrm{Hilb}_{n,S} & & & & & & \end{array}$$

at hand, we now construct the sheaf over  $\mathrm{Hilb}_{n,S} \times \overline{\mathrm{Jac}}_C^0$ ,

$$\mathcal{G}_{0,S} := \left( (\psi_S \times \mathrm{id}_{\overline{\mathrm{Jac}}})_* (\sigma_S \times \mathrm{id}_{\overline{\mathrm{Jac}}})^* (\iota_C^n \times \mathrm{id}_{\overline{\mathrm{Jac}}})_* \mathcal{U}_{0,C}^{\boxtimes n} \right)^{sign} \otimes p_1^* \det(\mathcal{A}_{S,n})^{-1}.$$

The following statement is crucial in the construction.

**Proposition 2.1** (Proposition 4.1 of [Ari2]). *Consider  $\iota_C : C \rightarrow S$ ,  $\psi_S$  and  $\sigma_S$  as above. Given any sheaf  $M \in \overline{\mathrm{Jac}}_C^0$ , one has that  $(\psi_{S,*} \sigma_S^* \iota_C^n M^{\boxtimes n})^{sign}$  is a Cohen–Macaulay sheaf supported on  $\mathrm{Hilb}_{n,C} \subset \mathrm{Hilb}_{n,S}$ .*

It follows from Proposition 2.1 and the properties of Cohen–Macaulay sheaves, that  $\mathcal{G}_{0,S}|_{\mathrm{Hilb}_{n,C} \times \{M\}}$  is Cohen–Macaulay and supported on  $\mathrm{Hilb}_{n,C} \times \overline{\mathrm{Jac}}_C^0$ . Let us denote by  $\mathcal{G}_{0,C}$  the restriction there of  $\mathcal{G}_{0,S}$ . The last step in Arinkin's construction consists on showing that  $\mathcal{G}_{0,C}$  descends along the Abel–Jacobi map.

**Theorem 2.2** (Theorem A of [Ari2]). *There exists a maximal Cohen–Macaulay sheaf  $\mathcal{P}_C$  over  $\overline{\text{Jac}}_C^0 \times \overline{\text{Jac}}_C^0$ , called the Poincaré sheaf, which is flat over both factors, symmetric under permutation of these factors, and such that*

$$(2.2) \quad \mathcal{G}_{0,C} \cong (\alpha_C \times \text{id}_{\overline{\text{Jac}}})^* \mathcal{P}_C.$$

Moreover  $\mathcal{P}_C$  is the universal family for the moduli problem consisting of rank one torsion-free sheaves on  $\overline{\text{Jac}}_C^0$

With  $\mathcal{P}_C$  as the kernel we can write down the Fourier–Mukai transform. Define the projections  $p_1, p_2$  onto the first and second factors

$$(2.3) \quad \begin{array}{ccc} & \overline{\text{Jac}}_C^0 \times \overline{\text{Jac}}_C^0 & \\ p_1 \swarrow & & \searrow p_2 \\ \overline{\text{Jac}}_C^0 & & \overline{\text{Jac}}_C^0. \end{array}$$

and consider the integral functor

$$(2.4) \quad \begin{array}{ccc} \Phi_{1 \rightarrow 2}^{\mathcal{P}_C} : D^b(\overline{\text{Jac}}_C^0) & \longrightarrow & D^b(\overline{\text{Jac}}_C^0) \\ \mathcal{F}^\bullet & \longmapsto & R p_{2,*} (p_1^* \mathcal{F}^\bullet \otimes \mathcal{P}_C). \end{array}$$

Denote by  $\mathcal{P}_C^\vee$  the dual sheaf of  $\mathcal{P}_C$ , which is flat over each factor as  $\mathcal{P}_C$  is. Associated to  $\mathcal{P}_C^\vee$ , we consider the integral functor

$$(2.5) \quad \begin{array}{ccc} \Phi_{2 \rightarrow 1}^{\mathcal{P}_C^\vee} : D^b(\overline{\text{Jac}}_C^0) & \longrightarrow & D^b(\overline{\text{Jac}}_C^0) \\ \mathcal{F}^\bullet & \longmapsto & R p_{1,*} (p_2^* \mathcal{F}^\bullet \otimes \mathcal{P}_C^\vee). \end{array}$$

Extending the work of Mukai to the singular case, Arinkin proves the following.

**Theorem 2.3** (Theorem C [Ari2]). *Let  $C$  be an integral projective curve with locally planar singularities and arithmetic genus  $g$ . Then,  $\Phi_{1 \rightarrow 2}^{\mathcal{P}_C}$  is a derived equivalence with quasi-inverse  $\Phi_{2 \rightarrow 1}^{\mathcal{P}_C^\vee}[g]$ .*

**2.2. The case of nodal curves.** From now on and until the end of the article,  $C$  shall denote a projective curve with at most nodal singularities. Suppose that the singularity divisor  $\text{Sing}(C)$  is given by  $k$  reduced points  $b_1, \dots, b_k$ . In this context, Casalania–Martin–Kass–Viviani in [CKV] provide a local description of  $\overline{\text{Jac}}_C^d$ . Although they work with arbitrary nodal curves, possibly reducible, here, we provide a simplified version of their main result restricted to the case of nodal and integral curves.

**Theorem 2.4** ([CKV]). *For an integral projective curve  $C$  with simple nodal singularities and arithmetic genus  $g = g(C)$ , the local ring of its compactified Jacobian  $\overline{\text{Jac}}_C^d$  at the point  $\mathcal{F} \in \overline{\text{Jac}}_C^d$  is*

$$\mathcal{O}_{\overline{\text{Jac}}_C, \mathcal{F}} \cong k[z_1^+, z_1^-, \dots, z_\ell^+, z_\ell^-, z_{\ell+1}, \dots, z_g] / \langle z_1^+ z_1^-, \dots, z_\ell^+ z_\ell^- \rangle,$$

where  $\ell$  is the number of points of  $\text{Sing}(C)$  where  $\mathcal{F}$  is not locally free.

Consider a closed embedding  $\iota_C : C \hookrightarrow S$  of an integral and nodal curve  $C$  into our smooth surface. Set the curvilinear Hilbert scheme of  $C$  to be

$$(2.6) \quad \text{Hilb}_{n,C}^{cur} = \text{Hilb}_{n,S}^{cur} \cap \text{Hilb}_{n,C},$$

where one can see  $\text{Hilb}_{n,C}$  as a subscheme of  $\text{Hilb}_{n,S}$  by means of  $\iota_C$ . As in the case of the smooth surface  $S$ , it classifies subschemes embedded into  $\mathbb{A}^1$ , or in any other smooth curve. Arinkin showed that for nodal curves, one can have a surjective map even restricting to the curvilinear Hilbert scheme.

**Proposition 2.5** (Proposition 4.5 of [Ari2]). *There exists  $n$  sufficiently large such that the restriction of the associated Abel–Jacobi map,*

$$\alpha_C : \text{Hilb}_{n,C}^{cur} \longrightarrow \overline{\text{Jac}}_C^0$$

*is surjective.*

We will always assume that such  $n$  has been chosen.

Thanks to Proposition 2.5 the construction of the Poincaré sheaf is more natural in the case of nodal curves, as one just need to perform Arinkin’s construction over the curvilinear Hilbert scheme, as

$$\mathcal{G}_{n,C}^{cur} := \mathcal{G}_{n,C} |_{\text{Hilb}_{n,C}^{cur} \times \overline{\text{Jac}}}$$

descends via the Abel–Jacobi map.

The closed embedding of  $C$  into the smooth surface  $S$  is crucial in the proof of Proposition 2.1, but once we know that our curve is planar, we can express  $\mathcal{G}_{d,C}^{cur}$  without mentioning the embedding. Given

$$(2.7) \quad \begin{array}{ccc} \text{Hilb}_{n,C}^{cur} \times \overline{\text{Jac}}_C^d & \xleftarrow{\psi_C \times \text{id}_{\overline{\text{Jac}}}} & \text{Flag}_{n,C}^{cur} \times \overline{\text{Jac}}_C^d \xrightarrow{\sigma_C \times \text{id}_{\overline{\text{Jac}}}} C^n \times \overline{\text{Jac}}_C^d \\ \downarrow p_1 & & \\ \text{Hilb}_{n,C}^{cur} & & \end{array}$$

one has

$$(2.8) \quad \mathcal{G}_{d,C}^{cur} := \left( (\psi_C \times \text{id}_{\overline{\text{Jac}}})_* (\sigma_C \times \text{id}_{\overline{\text{Jac}}})^* \mathcal{U}_{d,C}^{\boxtimes n} \right)^{\text{sign}} \otimes p_1^* \det(\mathcal{A}_{C,n})^{-1},$$

which is a maximal Cohen-Macaulay sheaf as we shall see below. Observe that  $\mathcal{A}_{C,n}$ , being the restriction of  $\mathcal{A}_S$  to  $\text{Hilb}_{n,C}^{cur} \subset \text{Hilb}_{n,S}$ , can also be expressed independently of the embedding  $\iota_C$  as

$$(2.9) \quad \mathcal{A}_{C,n} := h_{C,*} \mathcal{O}_{\mathcal{D}_C},$$

where  $\mathcal{D}_C \subset C \times \text{Hilb}_{n,C}^{cur}$  is the universal subscheme, and  $h_C$  the obvious flat projection onto  $\text{Hilb}_{n,C}^{cur}$ .

In the next paragraphs, we will study the relation between the sheaves  $\mathcal{G}_{0,C}$  and  $\mathcal{G}_{d,C}$  which will be crucial in Section 4.



Recall that  $x_0$  a fixed smooth point of  $C$ . Tensorization by powers of  $\mathcal{O}_C(x_0)$  produces a map

$$\tau_{d,x_0} : \overline{\text{Jac}}_C^{d'+d} \longrightarrow \overline{\text{Jac}}_C^{d'},$$

which is an isomorphism with inverse  $\tau_{-d,x_0} : \overline{\text{Jac}}_C^{d'} \longrightarrow \overline{\text{Jac}}_C^{d'+d}$ , as  $C$  is taken to be irreducible.

The universal sheaves  $\mathcal{U}_{0,C} \longrightarrow C \times \overline{\text{Jac}}_C^0$  and  $\mathcal{U}_{d,C} \longrightarrow C \times \overline{\text{Jac}}_C^d$  are related via pullback under this map, so

$$(2.10) \quad \mathcal{U}_{d,C} \cong (\text{id}_C \times \tau_{d,x_0})^* (\mathcal{U}_{0,C} \otimes p_C^* \mathcal{O}_C(dx_0)),$$

with  $p_C : C \times \overline{\text{Jac}}_C^0 \longrightarrow C$  the projection.

Given a point  $x \in C$ , the line bundle  $\mathcal{O}_C(x) \boxtimes \cdots \boxtimes \mathcal{O}_C(x)$  over  $C^n$  is  $\mathfrak{S}_n$ -equivariant under permutation in  $C^n$ , and, since the tensorization is invariant under the permutation, it follows that that  $\mathfrak{S}_n$  acts trivially on its fibres over the  $\mathfrak{S}_n$ -fixed points. By Kempf's criterium (see Theorem 2.3 of Drézet–Narasimhan [DN]), it descends to a line bundle  $\tilde{\mathcal{J}}_{C,x}$  on the symmetric product  $\text{Sym}_C^n$ , *i.e.*

$$(2.11) \quad \mathcal{O}_C(x) \boxtimes \cdots \boxtimes \mathcal{O}_C(x) \cong \pi_C^* \tilde{\mathcal{J}}_{C,x},$$

with  $\pi_C : C^n \longrightarrow \text{Sym}_C^n$  the quotient map. Since  $\mathcal{O}_C(x)$  comes naturally equipped with a section vanishing at  $x$ , it follows that  $\tilde{\mathcal{J}}_{C,x}$  can be endowed with a section vanishing at any tuple containing  $x$ , *i.e.* vanishing over the subvariety

$$\tilde{\Xi}_{C,x} := \text{Image}(f_{C,x}),$$

of  $\text{Sym}_C^n$ , where

$$f_{C,x} : \begin{array}{ccc} \text{Sym}_C^{n-1} & \longrightarrow & \text{Sym}_C^n \\ [(x_1, \dots, x_{n-1})]_{\mathfrak{S}_{n-1}} & \longmapsto & [(x, x_1, \dots, x_{n-1})]_{\mathfrak{S}_n} \end{array}$$

Denoting by  $\text{chow}_C : \text{Hilb}_{n,C}^{\text{cur}} \longrightarrow \text{Sym}_C^n$  the Chow morphism, let us consider the pullback

$$(2.12) \quad \mathcal{J}_{C,x} := \text{chow}_C^* \tilde{\mathcal{J}}_{C,x},$$

which is associated to the divisor

$$(2.13) \quad \Xi_{C,x} := \text{chow}_C^{-1}(\tilde{\Xi}_{C,x})$$

in  $\text{Hilb}_{n,C}^{\text{cur}}$ .

*Remark 2.6.* By the work of Schwarzenberger [Sch], for  $n$  is large enough and  $C$  a smooth curve, the isomorphism class of  $\mathcal{J}_{C,x}$  does not depend on the choice of  $x \in C$  and we simply denote it  $\mathcal{J}_C$  in that case.

We now have the following relation between the sheaves  $\mathcal{G}_{d,C}$  and  $\mathcal{G}_{0,C}$  over  $\text{Hilb}_{n,C}^{\text{cur}} \times \overline{\text{Jac}}_C^d$  and  $\text{Hilb}_{n,C}^{\text{cur}} \times \overline{\text{Jac}}_C^0$  respectively; recall (2.8). Let  $\mathcal{J}_{C,x_0}^d$  be the line bundle over  $\text{Hilb}_{n,C}^{\text{cur}}$  obtained by taking  $d$ -times the tensor product of (2.12) for the smooth point  $x_0 \in C$ . If  $p_1$  is the projection in (2.7) (for  $d = 0$ ), then  $p_1^* \mathcal{J}_{C,x_0}^d$  is a line bundle over  $\text{Hilb}_{n,C}^{\text{cur}} \times \overline{\text{Jac}}_C^0$ .

**Lemma 2.7.** *In the above notation,*

$$\mathcal{G}_{d,C}^{cur} \cong (\mathrm{id}_{\mathrm{Hilb}} \times \tau_{d,x_0})^* \left( \mathcal{G}_{0,C}^{cur} \otimes p_1^* \mathcal{J}_{C,x_0}^d \right),$$

with  $\tau_{d,x_0} : \overline{\mathrm{Jac}}_C^d \rightarrow \overline{\mathrm{Jac}}_C^0$  the tensorization morphism. It follows that  $\mathcal{G}_{d,C}^{cur}$  is a maximal Cohen–Macaulay sheaf.

We shall use this lemma later to induce compatible Poincaré sheaves over a nodal curve and its normalisation.

*Proof.* Plugging (2.10) into (2.8), and, applying functoriality with respect to the right hand side, and proper base change with respect to the left hand side of the commutative diagram

$$\begin{array}{ccccc} \mathrm{Hilb}_{n,C}^{cur} \times \overline{\mathrm{Jac}}_C^d & \xleftarrow{\psi_C \times \mathrm{id}_{\overline{\mathrm{Jac}}}} & \mathrm{Flag}_{n,C}^{cur} \times \overline{\mathrm{Jac}}_C^d & \xrightarrow{\sigma_C \times \mathrm{id}_{\overline{\mathrm{Jac}}}} & C^n \times \overline{\mathrm{Jac}}_C^d \\ \mathrm{id}_{\mathrm{Hilb}} \times \tau_{d,x_0} \downarrow & & \downarrow \mathrm{id}_{\mathrm{Flag}} \times \tau_{d,x_0} & & \downarrow \mathrm{id}_{C^n} \times \tau_{d,x_0} \\ \mathrm{Hilb}_{n,C}^{cur} \times \overline{\mathrm{Jac}}_C^0 & \xleftarrow{\psi_C \times \mathrm{id}_{\overline{\mathrm{Jac}}}} & \mathrm{Flag}_{n,C}^{cur} \times \overline{\mathrm{Jac}}_C^0 & \xrightarrow{\sigma_C \times \mathrm{id}_{\overline{\mathrm{Jac}}}} & C^n \times \overline{\mathrm{Jac}}_C^0, \end{array}$$

one gets the identification

$$\begin{aligned} \mathcal{G}_{d,C}^{cur} \cong (\mathrm{id}_{\mathrm{Hilb}} \times \tau_{d,x_0})^* & \left( (\psi_C \times \mathrm{id}_{\overline{\mathrm{Jac}}})_* (\sigma_C \times \mathrm{id}_{\overline{\mathrm{Jac}}})^* \left( \mathcal{U}_{0,C} \otimes p_C^* \mathcal{O}(x_0)^d \right)^{\boxtimes C^n} \right)^{sign} \\ & \otimes p_1^* \det(\mathcal{A}_{C,n})^{-1}, \end{aligned}$$

where we made use of the fact that pullback under  $(\mathrm{id}_{\mathrm{Hilb}} \times \tau_{d,x_0})$  is invariant under the symmetric group action. Considering the commutative diagram

$$\begin{array}{ccc} C^n \times \overline{\mathrm{Jac}}_C^0 & \xrightarrow{q_i \times \mathrm{id}_{\overline{\mathrm{Jac}}}} & C \times \overline{\mathrm{Jac}}_C^0 \\ q_{C^n} \downarrow & & \downarrow p_C \\ C^n & \xrightarrow{q_i} & C, \end{array}$$

and recalling from (2.11) that  $\mathcal{O}(dx_0)^{\boxtimes n}$  is isomorphic to  $\pi_C^* \tilde{\mathcal{J}}_{C,x_0}^d$ , we obtain

$$\mathcal{G}_{d,C}^{cur} \cong (\mathrm{id}_{\mathrm{Hilb}} \times \tau_{d,x_0})^* \left( (\psi_C \times \mathrm{id}_{\overline{\mathrm{Jac}}})_* (\sigma_C \times \mathrm{id}_{\overline{\mathrm{Jac}}})^* \left( \mathcal{U}_{0,C}^{\boxtimes C^n} \otimes q_{C^n}^* \pi_C^* \tilde{\mathcal{J}}_{C,x_0}^d \right) \right)^{sign} \otimes p_1^* \det(\mathcal{A}_{C,n})^{-1}.$$

Next, we apply functoriality with respect to

$$\begin{array}{ccc} \mathrm{Flag}_{n,C}^{cur} \times \overline{\mathrm{Jac}}_C^0 & \xrightarrow{\sigma_C \times \mathrm{id}_{\overline{\mathrm{Jac}}}} & C^n \times \overline{\mathrm{Jac}}_C^0 \\ q_{\mathrm{Flag}} \downarrow & & \downarrow q_{C^n} \\ \mathrm{Flag}_{n,C}^{cur} & \xrightarrow{\sigma_C} & C^n, \\ \\ \mathrm{Flag}_{n,C}^{cur} & \xrightarrow{\sigma_C} & C^n \\ \psi_C \downarrow & & \downarrow \pi_C \\ \mathrm{Hilb}_{n,C}^{cur} & \xrightarrow{\mathrm{chow}_C} & \mathrm{Sym}_C^n, \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Hilb}_{n,C}^{\mathrm{cur}} \times \overline{\mathrm{Jac}}_C^0 & \xleftarrow{\psi_C \times \mathrm{id}_{\overline{\mathrm{Jac}}}} & \mathrm{Flag}_{n,C}^{\mathrm{cur}} \times \overline{\mathrm{Jac}}_C^0 \\ \mathrm{p}_1 \downarrow & & \downarrow \mathrm{q}_{\mathrm{Flag}} \\ \mathrm{Hilb}_{n,C}^{\mathrm{cur}} & \xleftarrow{\psi_C} & \mathrm{Flag}_{n,C}^{\mathrm{cur}}, \end{array}$$

to observe that

$$\mathcal{G}_{d,C}^{\mathrm{cur}} \cong (\mathrm{id}_{\mathrm{Hilb}} \times \tau_{d,x_0})^* \left( (\psi_C \times \mathrm{id}_{\overline{\mathrm{Jac}}})_* \left( (\sigma_C \times \mathrm{id}_{\overline{\mathrm{Jac}}})^* \mathcal{U}_{0,C}^{\boxtimes n} \otimes (\psi_C \times \mathrm{id}_{\overline{\mathrm{Jac}}})^* p_1^* \mathcal{J}_{C,x_0}^d \right) \right)^{\mathrm{sign}} \otimes p_1^* \det(\mathcal{A}_{C,n})^{-1}.$$

The first statement follows after applying projection formula with respect to  $(\psi_C \times \mathrm{id}_{\overline{\mathrm{Jac}}})$  followed by the formula (2.8) for  $d = 0$ .

The second statement is a consequence of the first and the fact that  $\mathcal{G}_{0,C}^{\mathrm{cur}}$  is a maximal Cohen–Macaulay sheaf.  $\square$

**2.3. Parabolic modules.** A key ingredient in our work will be the use of *parabolic modules* [Reg, Bho, Cool, Cool2, GO, FGOP]. The existing literature concerning these objects is still limited, having been studied over an integral curve with only simple singularities (*i.e.* of type ADE). In this paper we further restrict to the case where  $C$  is a projective, integral curve with only *simple nodal* (thus of type  $A_1$ ) singularities.

Denote by  $\nu : \Sigma \rightarrow C$  the normalization of  $C$  and the induced pullback morphism,

$$\begin{array}{ccc} \hat{\nu} : \mathrm{Jac}_C^d & \longrightarrow & \mathrm{Jac}_\Sigma^d \\ L & \longmapsto & \nu^* L. \end{array}$$

This does not extend to the compactification  $\overline{\mathrm{Jac}}_C^d$  and an important motivation to introduce parabolic modules is that their moduli space is a compactification of  $\mathrm{Jac}_C^d$  (different to  $\overline{\mathrm{Jac}}_C^d$ ) where  $\hat{\nu}$  naturally extends.

Let  $\mathrm{Sing}(C) = \{b_1, \dots, b_k\}$  be the set of simple nodes of  $C$ . So each  $b_i$  is a reduced point of  $C$ . Define the effective degree  $2k$  divisor

$$(2.14) \quad B = \nu^{-1}(\mathrm{Sing}(C)),$$

on the smooth curve  $\Sigma$ , and decompose it as

$$B = B_1 + \dots + B_k$$

so that  $B_i = \nu^{-1}(b_i) = b_i^+ + b_i^-$ . We can now define a parabolic module on  $\Sigma$  with respect to this data; see [Bho, Cool, Cool2] for more details.

**Definition 2.8.** A rank 1 and degree  $d$  *parabolic module* over  $\Sigma$ , associated to the divisor  $B$  in (2.14), is a pair  $(M, V)$  where  $M \in \mathrm{Jac}_\Sigma^d$  and  $V$  is a subsheaf of  $M \otimes \mathcal{O}_B$  such that

$$V = V_1 \oplus \dots \oplus V_k$$

with  $V_i$  a 1-dimensional vector subspace of  $M \otimes \mathcal{O}_{B_i} \cong M_{b_i^+} \oplus M_{b_i^-}$  and  $M_{b_i^\pm} = M \otimes \mathcal{O}_{b_i^\pm}$ .

*Remark 2.9.*

- (1) The notion of parabolic module, associated to other kinds of singularities, defined by Cook in [Coo1] is more general. For example, the subspaces  $V_i$  may be higher dimensional and not just 1-dimensional. In this situation, one needs to impose that  $V_i$  is an  $\mathcal{O}_{b_i}$ -submodule of  $M \otimes \mathcal{O}_{B_i}$  via pushforward under  $\nu$  *i.e.* via the inclusion  $\mathcal{O}_C \hookrightarrow \nu_* \mathcal{O}_\Sigma$ .
- (2) A parabolic module is a special case of a generalized parabolic bundle as introduced by Bhosle in [Bho] (in the case of only one simple node).

Denote by  $\text{PMod}_C^d = \text{PMod}_C^d(\Sigma, B)$  the moduli space of rank 1 and degree  $d$  parabolic modules over  $\Sigma$  associated to  $B$ . We refer again to [Coo1, Coo2, Bho] for its construction.

By [Coo2, Theorem 1] or [Bho, Theorem 3], there is a finite morphism

$$(2.15) \quad \begin{aligned} \rho : \text{PMod}_C^d &\longrightarrow \overline{\text{Jac}}_C^d \\ (M, V) &\longmapsto \mathcal{F} \end{aligned}$$

where  $\mathcal{F}$  is defined by the short exact sequence

$$(2.16) \quad 0 \longrightarrow \mathcal{F} \longrightarrow \nu_* M \longrightarrow \nu_*(M \otimes \mathcal{O}_B/V) \longrightarrow 0,$$

and the second map is the composition  $\nu_* M \longrightarrow \nu_*(M \otimes \mathcal{O}_B) \longrightarrow \nu_*(M \otimes \mathcal{O}_B/V)$ . Note that  $\deg(\nu_*(M \otimes \mathcal{O}_{B_i})/V_i) = 1$ , hence

$$(2.17) \quad \nu_*(M \otimes \mathcal{O}_B/V) \cong \mathcal{O}_{\text{Sing}(C)}.$$

Since the quotient  $\nu_*(M \otimes \mathcal{O}_B/V)$  is an  $\mathcal{O}_C$ -module,  $\mathcal{F}$  inherits an  $\mathcal{O}_C$ -module structure too. In addition,  $\deg(\nu_* M) = \deg(\nu_*(M \otimes \mathcal{O}_B/V)) = k$ , thus indeed  $\deg(\mathcal{F}) = 0$ .

Let  $\rho_0$  denote the restriction of  $\rho$  to  $\rho^{-1}(\text{Jac}_C^d)$ . From [Coo2, Theorem 1] or [Bho, Theorem 3], we know it is an isomorphism

$$(2.18) \quad \rho_0 : \rho^{-1}(\text{Jac}_C^d) \xrightarrow{\cong} \text{Jac}_C^d,$$

so  $\text{Jac}_C^d$  can be seen as a dense open subspace of  $\text{PMod}_C^d$  via  $\rho_0^{-1}$ . In other words,  $\text{PMod}_C^d$  is a compactification of  $\text{Jac}_C^d$ . It is however different from  $\overline{\text{Jac}}_C^d$  since we have the natural projection

$$(2.19) \quad \begin{aligned} \dot{\nu} : \text{PMod}_C^d &\longrightarrow \text{Jac}_\Sigma^d \\ (M, V) &\longmapsto M \end{aligned}$$

extending  $\hat{\nu}$ .

The description of  $\text{PMod}_C^d$  and  $\rho$  is particularly simple in our case and appears in [Bho]. We provide a proof of the following result for completeness. Recall that

$$B = \nu^{-1}(\text{Sing}(C)) = b_1^+ + b_1^- + \cdots + b_k^+ + b_k^-,$$

with  $\nu^{-1}(b_i) = \{b_i^+, b_i^-\}$ .

**Lemma 2.10.** *The moduli space  $\text{PMod}_C^d$  is a  $(\mathbb{P}^1)^k$ -bundle over  $\text{Jac}_\Sigma^d$  via  $\dot{\nu}$  in (2.19). In particular,  $\text{PMod}_C^d$  is smooth. Moreover, the morphism  $\rho : \text{PMod}_C^d \longrightarrow \overline{\text{Jac}}_C^d$  in (2.15) is a resolution of singularities. Two points of  $\text{PMod}_C^d$  have the same image*

$$\rho((M, V)) = \rho((M', V')),$$

if and only if there exists a line bundle

$$J = \mathcal{O}_\Sigma(b_1^- - b_1^+)^{a_1} \otimes \cdots \otimes \mathcal{O}_\Sigma(b_k^- - b_k^+)^{a_k},$$

with  $a_i \in \{-1, 0, 1\}$ , such that

$$(2.20) \quad M' = M \otimes J$$

and, for those  $i$  with  $a_i = 1$ , those  $j$  with  $a_j = -1$  and those  $k$  with  $a_k = 0$ , one has

$$(2.21) \quad V_i = M \otimes \mathcal{O}_{b_i^+} \quad \text{and} \quad V_i' = M' \otimes \mathcal{O}_{b_i^-},$$

$$(2.22) \quad V_j = M \otimes \mathcal{O}_{b_j^-} \quad \text{and} \quad V_j' = M' \otimes \mathcal{O}_{b_j^+}$$

and

$$(2.23) \quad V_k = V_k'.$$

Finally, if  $\mathcal{F} \in \overline{\text{Jac}}_C^d$  is not locally free at  $\ell$  points of  $\text{Sing}(C)$ , the fibre of  $\rho$  over  $\mathcal{F}$  can be identified with  $\mathbb{Z}_2^\ell$ .

*Proof.* The bundle description of  $\text{PMod}_C^d$  via  $\nu$  is clear from the definition of a parabolic module. Surjectivity of  $\rho$  is proven in [FGOP, Lemma 5.9]. Given any two different points  $(M, V)$  and  $(M', V')$  with the same image under  $\rho$ , it follows that both lie outside  $\text{Jac}_C$ , the smooth locus of  $\overline{\text{Jac}}_C$ , as  $\rho$  is an isomorphism restricted there. Hence,  $\rho$  is a resolution of singularities.

The stalk of  $\mathcal{F} = \rho(M, V)$  at  $b \in \text{Sing}(C)$  is the kernel of a map  $\mathcal{O}_{\Sigma, b^+} \oplus \mathcal{O}_{\Sigma, b^-} \longrightarrow \mathcal{O}_b$ . If  $\mathcal{F}_b$  is not locally-free at  $b$ , then

$$(2.24) \quad \mathcal{F}_b \cong \mathcal{O}_{\Sigma, b^+} \oplus \mathfrak{m}_{b^-},$$

if  $V$  is locally  $\mathcal{O}_{b^+}$ , or

$$(2.25) \quad \mathcal{F}_b \cong \mathfrak{m}_{b^+} \oplus \mathcal{O}_{\Sigma, b^-},$$

when  $V$  is given by  $\mathcal{O}_{b^-}$ . The previous distinction is redundant since one can build the isomorphism of  $\mathcal{O}_{C, b}$ -modules between (2.24) and (2.25), by means of a meromorphic section of  $\mathcal{O}_\Sigma(b^- - b^+)$ .

It then follows that, given  $(M, V)$ ,  $(M', V')$  and  $J$  satisfying (2.20), (2.21), (2.22) and (2.23) one obtains an isomorphism between  $\mathcal{F} := \rho(M, V) = \ker(\nu_* M \longrightarrow \nu_*(M \otimes \mathcal{O}_B/V))$  and  $\mathcal{F}' := \rho(M', V') = \ker(\nu_* M' \longrightarrow \nu_*(M' \otimes \mathcal{O}_B/V'))$  by means of a meromorphic section of  $J$  vanishing on the  $b_i^-$  and on the  $b_j^+$ , and having poles on the  $b_i^+$  and on the  $b_j^-$ .

Assuming that  $\mathcal{F} \cong \mathcal{F}'$ , one has for every point  $b_i \in \text{Sing}(C)$  that the stalks  $\mathcal{F}_{b_i}$  and  $\mathcal{F}'_{b_i}$  may be locally-free, or if not, be described as (2.24) or as (2.25). If both have the same local description we set  $a_i = 0$  and, else, we set  $a_i = 1$  if we need an isomorphism given by a meromorphic section vanishing on  $b_i^+$  or  $a_i = -1$  in the remaining case. Repeating this construction for every point of  $\text{Sing}(C)$ , we obtain  $J$  satisfying (2.20) and the conditions (2.21), (2.22) and (2.23) hold.

For a sheaf  $\mathcal{F} \in \overline{\text{Jac}}_C^d$  which is not locally free at  $\{b_{j_1}, \dots, b_{j_\ell}\} \subset \text{Sing}(C)$ , one immediately sees that its preimages under  $\rho$  are of the form  $(M, V)$  with  $V_{j_i}$  of the form  $M \otimes \mathcal{O}_{b_{j_i}^+}$  or  $M \otimes \mathcal{O}_{b_{j_i}^-}$ . The last statement follows easily from this observation.  $\square$

The image under  $\rho$  of each fibre of  $\hat{\nu}$  is a collection of Hecke cycles,

$$\text{Hecke}(M) := \rho(\hat{\nu}^{-1}(M)),$$

as is described below.

**Lemma 2.11** (Lemma 5.10 of [FGOP]). *For every  $M \in \text{Jac}_\Sigma^d$ ,  $\text{Hecke}(M) \cong (\mathbb{P}^1)^k$  and it is the closure of  $\hat{\nu}^{-1}(M)$  inside  $\overline{\text{Jac}}_C^d$  and classifies Hecke transforms of the pushforward sheaf  $\nu_* M$  along  $\text{Sing}(C) \subset C$ .*

*Remark 2.12.* Let  $\mathcal{F} \in \overline{\text{Jac}}_C^d$  so that it is not locally free at  $\{b_{j_1}, \dots, b_{j_\ell}\} \subset \text{Sing}(C)$ . Then it represents a singular point of  $\overline{\text{Jac}}_C^d$ . The local description of  $\overline{\text{Jac}}_C^d$  over the singular point  $\mathcal{F}$  given by Theorem 2.4 can be understood as the union of irreducible components, labelled by  $a = (a_1, \dots, a_\ell) \in \mathbb{Z}_2^\ell$ , each of them being the image under  $\rho$  of the local neighbourhood of  $\text{PMod}_C^d$  at  $\tilde{\mathcal{F}}_a = (M_a, V_a) \in \text{PMod}_C^d$ . Hence, the projection  $\rho : \text{PMod}_C^d \rightarrow \overline{\text{Jac}}_C^d$  induces the following injection of local rings:

$$(2.26) \quad \mathcal{O}_{\overline{\text{Jac}}, \mathcal{F}} \hookrightarrow \bigoplus_{a \in \mathbb{Z}_2^\ell} \mathcal{O}_{\text{PMod}, \tilde{\mathcal{F}}_a},$$

with  $\text{PMod} = \text{PMod}_C^d$ .

Let

$$\mathcal{U}_{d, \Sigma} \longrightarrow \Sigma \times \text{Jac}_\Sigma^d$$

be the universal line bundle for the classification of line bundles of degree  $d$  over the smooth curve  $\Sigma$ . Associated to each of the singular points  $\{b_1, \dots, b_k\} = \text{Sing}(C)$  and to the morphism (2.19), we consider the tautological line bundle over  $\text{PMod}_C^d$ ,

$$(2.27) \quad \mathcal{V}_i \subset \hat{\nu}^*(\mathcal{U}_{d, \Sigma}|_{\{b_i^-\} \times \text{Jac}}) \oplus \hat{\nu}^*(\mathcal{U}_{d, \Sigma}|_{\{b_i^+\} \times \text{Jac}}),$$

with fibre  $V_i$  over the point  $(M, V_1 \oplus \dots \oplus V_k)$  of  $\text{PMod}_C^d$ , where we recall that  $V_i$  is a 1-dimensional subspace of  $(M \otimes \mathcal{O}_{b_i^-}) \oplus (M \otimes \mathcal{O}_{b_i^+})$ . In particular, the restriction is  $\mathcal{V}_i|_{\hat{\nu}^{-1}(M)} \cong \mathcal{O}(-1)$ .

Having set up this notation, we end the section describing the canonical bundle  $\omega_{\text{PMod}}$  of  $\text{PMod}_C^d$ .

**Lemma 2.13.** *Let  $C$  be an irreducible nodal curve with  $k$  singular points and normalization  $\nu : \Sigma \rightarrow C$ . Then,*

$$\omega_{\text{PMod}} \cong \mathcal{V}_1^2 \otimes \dots \otimes \mathcal{V}_k^2.$$

*Proof.* The statement is a straightforward consequence of the fact that  $\text{PMod}_C^d$  is a  $(\mathbb{P}^1)^k$ -bundle over  $\text{Jac}_\Sigma^d$  and the triviality of the cotangent bundle of  $\text{Jac}_\Sigma^d$ .  $\square$

## 3. PARABOLIC MODULES AND THE CURVILINEAR HILBERT SCHEME

In this section we present some technical results that will be used in Section 4.1 concerning the interplay between the curvilinear Hilbert scheme of  $C$  and the moduli space of parabolic modules. As before, we take  $C$  to be an irreducible nodal projective curve, with simple nodes at  $b_1, \dots, b_k \in C$ , and denote by  $\nu : \Sigma \rightarrow C$  its normalization. Recall that we have chosen a smooth point  $x_0 \in C$ . Let  $y_0 = \nu^{-1}(x_0) \in \Sigma$ .

Consider the following Cartesian product,

$$(3.1) \quad \begin{array}{ccc} \mathrm{Hilb}_{n,C}^{\mathrm{cur}} \times_{\mathrm{Sym}_C^n} \mathrm{Sym}_\Sigma^n & \xrightarrow{q} & \mathrm{Sym}_\Sigma^n \\ \beta \downarrow & & \downarrow \nu^{(n)} \\ \mathrm{Hilb}_{n,C}^{\mathrm{cur}} & \xrightarrow{\mathrm{chow}_C} & \mathrm{Sym}_C^n, \end{array}$$

where  $\nu^{(n)}$  is naturally induced from the normalization. Choose  $n$  sufficiently large so that the restriction of the Abel–Jacobi map  $\alpha_C : \mathrm{Hilb}_{n,C}^{\mathrm{cur}} \rightarrow \overline{\mathrm{Jac}}_C^0$  is surjective (cf. Proposition 2.5), and consider another Cartesian diagram

$$(3.2) \quad \begin{array}{ccc} \mathrm{Hilb}_{n,C}^{\mathrm{cur}} \times_{\overline{\mathrm{Jac}}_C^0} \mathrm{PMod}_C^0 & \xrightarrow{\tilde{\alpha}} & \mathrm{PMod}_C^0 \\ \tilde{\rho} \downarrow & & \downarrow \rho \\ \mathrm{Hilb}_{n,C}^{\mathrm{cur}} & \xrightarrow{\alpha_C} & \overline{\mathrm{Jac}}_C^0. \end{array}$$

Observe that  $\mathrm{Hilb}_{n,C}^{\mathrm{cur}} \times_{\overline{\mathrm{Jac}}_C^0} \mathrm{PMod}_C^0$  equipped with  $\tilde{\alpha}$  is an open subset of a projective bundle over  $\mathrm{PMod}_C^0$ , as so is  $\alpha_C$  over  $\overline{\mathrm{Jac}}_C^0$ . It is therefore smooth, because  $\mathrm{PMod}_C^0$  is so.

The points of  $\mathrm{Hilb}_{n,C}^{\mathrm{cur}} \times_{\overline{\mathrm{Jac}}_C^0} \mathrm{PMod}_C^0$  are given by pairs  $\mathcal{I}_D \in \mathrm{Hilb}_{n,C}^{\mathrm{cur}}$  and  $(M, V) \in \mathrm{PMod}_C^0$  such that

$$\alpha_C(\mathcal{I}_D) = \mathcal{F} = \rho(M, V).$$

By the definition of  $\alpha_C$ , the sub-scheme  $D$  naturally yields a section  $s_D : \mathcal{O}_C \rightarrow \mathcal{F}(nx_0)$ . On the other hand,  $(M, V)$  corresponds to an embedding  $j : \mathcal{F} \hookrightarrow \nu_* M$ . Hence, the composition  $j \circ s_D$  is a section of  $(\nu_* M)(nx_0) \cong \nu_*(M(ny_0))$ . By adjointness this gives a section of the line bundle  $M(ny_0)$  whose vanishing locus determines, in turn, a degree  $n$  effective divisor  $E \in \alpha_\Sigma^{-1}(M) \subset \mathrm{Sym}_\Sigma^n$ , where  $\alpha_\Sigma : \mathrm{Hilb}_{n,\Sigma} \rightarrow \mathrm{Jac}_\Sigma^0$  is defined by  $\alpha_\Sigma(\mathcal{I}_E) = \mathcal{I}_E^\vee(-ny_0) = \mathcal{O}_\Sigma(E - ny_0)$ , and where we are using that  $\mathrm{chow}_\Sigma : \mathrm{Hilb}_{n,\Sigma}^{\mathrm{cur}} \rightarrow \mathrm{Sym}_\Sigma^n$  is an isomorphism due to the smoothness of  $\Sigma$ . In view of this, by abuse of notation, we use the same notation for the symmetric class in  $\mathrm{Sym}_\Sigma^n$  and its associated divisor. Note that the support of the vanishing locus of  $s_D$  and of  $j \circ s_D$  is  $D$  in both cases. Then, by push-pull adjointness, we have  $\nu(E) \subset D$ , so  $\mathrm{chow}_C(\mathcal{I}_D) = \nu^{(n)}(E)$  as both have the same length. This gives rise to the morphism

$$(3.3) \quad j : \mathrm{Hilb}_{n,C}^{\mathrm{cur}} \times_{\overline{\mathrm{Jac}}_C^0} \mathrm{PMod}_C^0 \hookrightarrow \mathrm{Hilb}_{n,C}^{\mathrm{cur}} \times_{\mathrm{Sym}_C^n} \mathrm{Sym}_\Sigma^n, \quad j(\mathcal{I}_D, (M, V)) = (\mathcal{I}_D, E).$$

In order to describe  $j$ , we first describe the points of  $\mathrm{Hilb}_{n,C}^{\mathrm{cur}}$ .

**Lemma 3.1.** *Let  $C$  be an irreducible nodal projective curve with  $k$  nodes at  $b_1, \dots, b_k \in C$ . Denote by  $\nu : \Sigma \rightarrow C$  the normalization of  $C$  and by  $\nu^{-1}(b_i) = \{b_i^+, b_i^-\}$  the preimages of  $b_i$ .*

Observe that one can decompose the subscheme  $D \in \text{Hilb}_{n,C}^{\text{cur}}$  as

$$(3.4) \quad D = D_0 + D_1 + \cdots + D_k,$$

with  $D_0$  being supported on the smooth part of  $C$  and  $D_i$  supported on the nodal point  $b_i$ .

Then, either

- (1)  $D_i$  is 0 and  $\mathcal{I}_{D_i} \cong \mathcal{O}_C$  is principal;
- (2)  $D_i$  is supported on one of the branches of  $C$  at  $b_i$ , in which case  $\mathcal{I}_{D_i}$  is not principal;
- (3)  $D_i$  is  $D_{b_i, a_i}$  for  $a_i \in \mathbb{C}^*$ , where

$$D_{b_i, a_i} := \text{Spec}(\mathbb{C}[y_i^+, y_i^-] / \langle y_i^+ y_i^-, y_i^+ - a_i y_i^- \rangle),$$

with  $y_i^+$  and  $y_i^-$  being local coordinates of  $C$  at the node  $b_i$ . In that case,  $\text{length}(D_i) = 2$  and  $\mathcal{I}_{D_i}$  is principal.

*Proof.* Knowing that  $C$  is locally isomorphic to  $\text{Spec}(\mathbb{C}[y_i^+, y_i^-] / \langle y_i^+ y_i^- \rangle)$ , the proof becomes a simple exercise. If  $D_i$  is not 0 nor contained in  $\text{Spec}(\mathbb{C}[y_i^\pm])$  it must have length at least 2. As, by hypothesis, it is contained in some  $\mathbb{A}^1$ , forcibly  $D_i \subset D_{b_i, a_i}$  for some  $a_i \in \mathbb{C}^*$ . Observe that the right hand side is a length 2 subscheme, hence the later is identified with  $D_i$  itself. It follows from this identification that  $\mathcal{I}_{D_i}$  is principal.  $\square$

Our next goal is to describe the points  $(\mathcal{I}_D, E)$  of  $\text{Image}(j)$ . From the description prior to (3.3) we know that the points of  $\text{Image}(j)$  satisfy  $\nu(E) \subset D$ , and this is the condition that we shall combine with the description provided in Lemma 3.1. Recall that we use the same notation for the symmetric class  $E \in \text{Sym}_{\Sigma}^n$  and its associated divisor.

**Corollary 3.2.** *Let  $C$  be as above. Consider  $(\mathcal{I}_D, E) \in \text{Hilb}_{n,C}^{\text{cur}} \times_{\text{Sym}_C^n} \text{Sym}_{\Sigma}^n$  satisfying  $\nu(E) \subset D$ . Considering the decomposition (3.4) of  $D$ , one has that  $E$  decomposes as*

$$(3.5) \quad E = E_0 + E_1^+ + E_1^- + \cdots + E_k^+ + E_k^-,$$

with  $E_0$  being supported on  $\Sigma - \{b_1^+, b_1^-, \dots, b_k^+, b_k^-\}$ ,  $E_i^+$  supported at  $b_i^+$  and  $E_i^-$  at  $b_i^-$ .

Then  $E_0 = \nu(D_0)$ , and

- (1) if  $D_i$  is 0 then  $E_i^+ = 0$  and  $E_i^- = 0$ ;
- (2) if  $D_i$  is supported on one of the branches of  $C$  at  $b_i$ , then either  $D_i = \nu(E_i^+)$  and  $E_i^- = 0$ , or  $D_i = \nu(E_i^-)$  and  $E_i^+ = 0$ ;
- (3) if  $D_i$  is  $D_{b_i, a_i}$  for  $a_i \in \mathbb{C}^*$ , then  $E_i^+ = b_i^+$  and  $E_i^- = b_i^-$ .

We are now in position of completing our description of  $j$  as a closed immersion.

**Lemma 3.3.** *In the conditions stated above, the morphism*

$$j : \text{Hilb}_{n,C}^{\text{cur}} \times_{\text{Jac}_C^0} \text{PMod}_C^0 \hookrightarrow \text{Hilb}_{n,C}^{\text{cur}} \times_{\text{Sym}_C^n} \text{Sym}_{\Sigma}^n$$

is a closed immersion.



*Proof.* Let  $Z \subset \mathbf{Hilb}_{n,C}^{cur} \times_{\mathbf{Sym}_C^n} \mathbf{Sym}_\Sigma^n$  be the closed subset given by those pairs  $(\mathcal{I}_D, E)$  such that  $\nu(E) \subset D$ . The strategy of our proof will be to construct, on  $Z$ , an explicit inverse map to  $j$ . To this end, let  $(\mathcal{I}_D, E) \in Z$  and decompose  $D$  and  $E$  as in (3.4) and (3.5). The geometry we require is an understanding of  $\nu^*\mathcal{O}_D$ . After Corollary 3.2, one has  $\mathcal{O}_{E_0} = \nu^*\mathcal{O}_{D_0}$ , and the following case by case analysis of the decomposition:

- (1) either  $D_i$  and  $E_i$  are empty (so  $\nu^*\mathcal{O}_{D_i} = \mathcal{O}_\Sigma = \mathcal{O}_{E_i}$ );
- (2) or  $D_i$  is supported on one of the branches of  $C$  at  $b_i$ ;
- (3) or  $D_i$  is  $D_{b_i, a_i}$  for  $a_i \in \mathbb{C}^*$  (so  $\nu^*\mathcal{O}_{D_i} \cong \mathcal{O}_{b_i^+ \cup b_i^-} = \mathcal{O}_{E_i}$ ).

In all cases there exists a divisor  $F$  on  $\Sigma$ , supported on a subset of  $\nu^{-1}(\mathbf{Sing}(C))$ , such that  $\nu^*\mathcal{O}_D = \mathcal{O}_{E \cup F}$  and  $F \cap E = \emptyset$ .

The following homological analysis determines the pullback  $\nu^*\mathcal{I}^\vee$ . We have the exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{I}_D^\vee \longrightarrow \mathcal{O}_D \longrightarrow 0,$$

and pullback along  $\nu : \Sigma \longrightarrow C$  yields a 5-term exact sequence

$$0 \longrightarrow \mathrm{Tor}_1(\mathcal{I}_D^\vee, \mathcal{O}_\Sigma) \longrightarrow \mathrm{Tor}_1(\mathcal{O}_D, \mathcal{O}_\Sigma) \xrightarrow{(*)} \mathcal{O}_\Sigma \longrightarrow \nu^*\mathcal{I}_D^\vee \longrightarrow \nu^*\mathcal{O}_D = \mathcal{O}_{E+F} \longrightarrow 0$$

Similarly, the pullback of the defining sequence

$$0 \longrightarrow \mathcal{I}_D \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_D \longrightarrow 0$$

shows that  $\mathrm{Tor}_1(\mathcal{O}_D, \mathcal{O}_\Sigma)$  is a torsion sheaf. Hence, the map  $(*)$  is 0 and we obtain the short exact sequence

$$(3.6) \quad 0 \longrightarrow \mathcal{O}_\Sigma \longrightarrow \nu^*\mathcal{I}_D^\vee \longrightarrow \nu^*\mathcal{O}_D = \mathcal{O}_{E+F} \longrightarrow 0.$$

The coherent sheaf  $\nu^*\mathcal{I}_D^\vee$  on the smooth curve  $\Sigma$  decomposes into a locally free sheaf and a torsion sheaf

$$\nu^*\mathcal{I}_D^\vee = \nu^{\mathrm{tf}}\mathcal{I}_D^\vee \oplus \mathcal{T}_D,$$

where we refer to the locally-free part as the torsion-free pullback  $\nu^{\mathrm{tf}}\mathcal{I}_D^\vee$ . Let  $\mathbf{Sing}(C) = S_{1,3} \cup S_2$ , where  $S_{1,3}$  contains all nodes, where  $D$  is principal, i.e. described by case (1) or (3) above and  $S_2$  are the nodes, where it is not principal and we are in case (2). By a local computation it is easy to see that  $\mathcal{T}_D = \mathcal{O}_{\nu^{-1}S_2} = \mathcal{O}_{F+\sigma F}$ , where  $\sigma : \nu^{-1}\mathbf{Sing}(C) \longrightarrow \nu^{-1}\mathbf{Sing}(C)$  is the involution that pairwise interchanges the preimages  $b_i^+, b_i^-$ . Hence, equation (3.6) reduces to

$$0 \longrightarrow \mathcal{O}_\Sigma \longrightarrow \nu^{\mathrm{tf}}\mathcal{I}_D^\vee \longrightarrow \nu^*\mathcal{O}_D = \mathcal{O}_{E-\sigma F} \longrightarrow 0.$$

Therefore, we conclude

$$(3.7) \quad \nu^{\mathrm{tf}}\mathcal{I}_D^\vee = \mathcal{O}_\Sigma(E - \sigma F).$$

Using the previous calculation we can now write down an exact sequence of the form (2.16), therefore defining the parabolic module corresponding to  $\mathcal{I}_D^\vee(-nx_0)$ . Consider the partial normalisation  $\nu_2 : C' \longrightarrow C$  at the nodes in  $S_2$ . Then the normalisation factors as

$$\nu : \Sigma \xrightarrow{\nu_1} C' \xrightarrow{\nu_2} C.$$

By Cook [Coo2, Lemma 1], there is a push-pull isomorphism

$$\mathcal{I}_D^\vee \cong \nu_{2*} \nu_2^{\text{tf}} \mathcal{I}_D^\vee,$$

where  $\nu_2^{\text{tf}}$  is the torsion-free pullback. Now,  $\nu_2^{\text{tf}} \mathcal{I}_D^\vee$  is a locally free sheaf on  $C'$ . Hence, there exists an exact sequence

$$0 \longrightarrow \nu_2^{\text{tf}} \mathcal{I}_D^\vee \longrightarrow \nu_{1*} \nu_1^* \nu_2^{\text{tf}} \mathcal{I}_D^\vee \longrightarrow \mathcal{O}_{\text{Sing}(C')} \longrightarrow 0.$$

Note that  $\text{Sing}(C') = \nu_2^{-1}(S_{1,3})$  and  $\nu_1^* \nu_2^{\text{tf}} \mathcal{I}_D^\vee = \nu^{\text{tf}} \mathcal{I}_D^\vee$ . Pushing forward this exact sequence to  $C$ , we obtain

$$(3.8) \quad 0 \longrightarrow \mathcal{I}_D^\vee \xrightarrow{f_1} \nu_* \nu^{\text{tf}} \mathcal{I}_D^\vee \longrightarrow \mathcal{O}_{S_{1,3}} \longrightarrow 0.$$

From equation (3.7) we have the exact sequence

$$0 \longrightarrow \nu^{\text{tf}} \mathcal{I}_D^\vee(-ny_0) = \mathcal{O}_\Sigma(E - \sigma F - ny_0) \xrightarrow{f_2} \mathcal{O}_\Sigma(E - ny_0) \longrightarrow \mathcal{O}_{\sigma F} \longrightarrow 0.$$

Composing its pushforward with  $f_1$  of (3.8) results in the desired sequence

$$(3.9) \quad 0 \longrightarrow \mathcal{I}_D^\vee(-nx_0) \xrightarrow{\nu_* f_2 \circ f_1} \nu_* \mathcal{O}_\Sigma(E - ny_0) \longrightarrow \mathcal{O}_{\text{Sing}(C)} \longrightarrow 0.$$

Then, since the map  $\rho : \text{PMod}_C^0 \longrightarrow \overline{\text{Jac}}_C^0$  is surjective,  $\mathcal{F} := \alpha_C(\mathcal{I}_D) = \mathcal{I}_D^\vee(-nx_0)$  and  $M := \alpha_\Sigma(\mathcal{I}_E) = \mathcal{O}(E - ny_0)$  determine a point  $(M, V)$  in  $\text{PMod}_C^0$ . Since we are specifying both  $\mathcal{F}$  and  $M$ , Lemma 2.10 says that the point  $(M, V)$  is uniquely determined by (3.9). This gives rise to a map

$$g : Z \longrightarrow \text{Hilb}_{n,C}^{\text{cur}} \times_{\overline{\text{Jac}}_C^0} \text{PMod}_C^0, \quad g(\mathcal{I}_D, E) = (\mathcal{O}_\Sigma(E - ny_0), V).$$

One can easily show that  $g$  is inverse to  $j$ , hence  $\text{Hilb}_{n,C}^{\text{cur}} \times_{\overline{\text{Jac}}_C^0} \text{PMod}_C^0$  is naturally isomorphic to  $Z$  closed in  $\text{Hilb}_{n,C}^{\text{cur}} \times_{\text{Sym}_C^n} \text{Sym}_\Sigma^n$ .  $\square$

The next technical result will be applied in Section 4.1.

**Lemma 3.4.** *The complement of  $\text{Image}(j)$  inside  $\text{Hilb}_{n,C}^{\text{cur}} \times_{\text{Sym}_C^n} \text{Sym}_\Sigma^n$  is supported in the big diagonal of  $\text{Sym}_C^n$ .*

*Proof.* Recalling the decomposition (3.4), if  $(\mathcal{I}_D, E)$  is a point of  $\text{Hilb}_{n,C}^{\text{cur}} \times_{\text{Sym}_C^n} \text{Sym}_\Sigma^n$  such that every  $D_i$  has length at most 1, after Corollary 3.2 one immediately has that the  $D_i$  are contained in the branches of  $C$ . In that case  $\nu(E)$  embeds into  $D$  so the pair lies on the image of  $j$ . Therefore, the points lying outside the big diagonal of  $\text{Sym}_C^n$  are contained in the image of  $j$ , and the last statement is an straightforward consequence of this fact.  $\square$

We now state the main result of this section. It describes the relation between the pullback under the map  $j$  in (3.3) of line bundles associated to  $\det \mathcal{A}_{C,n}$  and  $\det \mathcal{A}_{\Sigma,n}$  constructed in Section 2.1. The definition of  $\mathcal{A}_{\Sigma,n}$  is also the one in (2.9), applied to the case of smooth curves.

As  $\text{chow}_\Sigma : \text{Hilb}_{n,\Sigma} \longrightarrow \text{Sym}_\Sigma^n$  is an isomorphism, we abuse of notation to denote the pushforward  $\text{chow}_{\Sigma,*} \mathcal{A}_{\Sigma,n}$  simply by  $\mathcal{A}_{\Sigma,n}$ . Recall the line bundle  $\mathcal{J}_\Sigma$  constructed in (2.12) after specifying a point  $y_0 \in \Sigma$ . As stated in Remark 2.6, since  $\Sigma$  is smooth, the isomorphism class of this line bundle does not depend on the choice of the point, so we drop it from the notation. Also by

abuse of notation, we consider  $\mathcal{J}_\Sigma$  to be a line bundle over  $\text{Sym}_\Sigma^n$ . Recall diagrams (3.1) and (3.2).

**Proposition 3.5.** *Let  $C$  be an irreducible nodal projective curve with  $k$  simple nodes at  $b_1, \dots, b_k \in C$ . Then,*

$$(3.10) \quad j^* q^* \left( \det \mathcal{A}_{\Sigma, n} \otimes \mathcal{J}_\Sigma^{-k} \right) \otimes \tilde{\rho}^* \left( \det \mathcal{A}_{C, n} \right)^{-1} \cong \tilde{\alpha}^* \omega_{\text{PMod}}^{1/2}.$$

*Proof.* Decompose  $D$  and  $E$  as in (3.4) and (3.5). Observe that  $\mathcal{A}_{\Sigma, n}$  and  $\mathcal{A}_{C, n}$  are vector bundles over  $\text{Sym}_\Sigma^n$  and  $\text{Hilb}_{n, C}^{\text{cur}}$ , respectively, and whose fibres over  $E$  and  $\mathcal{I}_D$  are

$$\mathcal{A}_{\Sigma, n}|_E = H^0(\Sigma, \mathcal{O}_E) = H^0(\Sigma, \mathcal{O}_{E_0}) \oplus \bigoplus_{i=1}^k \left( H^0(\Sigma, \mathcal{O}_{E_i^+}) \oplus H^0(\Sigma, \mathcal{O}_{E_i^-}) \right)$$

and

$$\mathcal{A}_{C, n}|_{\mathcal{I}_D} = H^0(C, \mathcal{O}_D) = H^0(C, \mathcal{O}_{D_0}) \oplus \bigoplus_{i=1}^k H^0(C, \mathcal{O}_{D_i}).$$

Thanks to Lemma 3.1 and Corollary 3.2, one has the following natural isomorphisms. Firstly,

$$(3.11) \quad H^0(C, \mathcal{O}_{D_0}) \cong H^0(\Sigma, \mathcal{O}_{E_0}).$$

Secondly,

$$(3.12) \quad H^0(C, \mathcal{O}_{D_i}) \cong H^0(\Sigma, \mathcal{O}_{E_i^+}) \quad \text{and} \quad H^0(\Sigma, \mathcal{O}_{E_i^-}) = 0,$$

for those  $D_i$  supported on the local branch of  $C$  on  $b_i$  associated to  $b_i^+$ , and

$$(3.13) \quad H^0(C, \mathcal{O}_{D_i}) \cong H^0(\Sigma, \mathcal{O}_{E_i^-}) \quad \text{and} \quad H^0(\Sigma, \mathcal{O}_{E_i^+}) = 0,$$

for those  $D_i$  supported on the branch of  $C$  associated to  $b_i^-$ . Finally, if  $D_i$  is isomorphic to some  $D_{b_i, a_i}$ , then

$$(3.14) \quad H^0(C, \mathcal{O}_{D_i}) \cong H^0(C, \mathcal{O}_{b_i}) \oplus \left( H^0(\Sigma, \mathcal{O}_{b_i^+}) \oplus H^0(\Sigma, \mathcal{O}_{b_i^-}) \right) / W_{i, a_i}$$

$$H^0(\Sigma, \mathcal{O}_{E_i^+}) \oplus H^0(\Sigma, \mathcal{O}_{E_i^-}) \cong H^0(\Sigma, \mathcal{O}_{b_i^+}) \oplus H^0(\Sigma, \mathcal{O}_{b_i^-}),$$

where  $W_{i, a_i}$  is the subspace of  $H^0(\Sigma, \mathcal{O}_{b_i^+}) \oplus H^0(\Sigma, \mathcal{O}_{b_i^-})$  generated by  $\delta_i^+ + a_i \delta_i^-$ , with  $\delta_i^\pm$  being the section of  $H^0(\Sigma, \mathcal{O}_{b_i^\pm})$  satisfying  $\delta_i^\pm(b_i^\pm) = 1$  and  $\delta_i^\pm(b_i^\mp) = 0$ .

Recall from (2.27) the tautological bundle  $\mathcal{V}_i$  over  $\text{PMod}_C^0$ . Observe that its pullback  $\tilde{\alpha}^* \mathcal{V}_i$  to  $\text{Hilb}_{n, C}^{\text{cur}} \times_{\text{Jac}_C^0} \text{PMod}_C^0$  is a line bundle whose fibre over  $(\mathcal{I}_D, (M, V_1 \oplus \dots \oplus V_k))$  with  $D_i = D_{b_i, a_i}$ , is equal to

$$V_i \cong M \otimes W_{i, a_i}.$$

Hence,

$$(3.15) \quad \tilde{\alpha}^* \mathcal{V}_i \cong \mathcal{W}_i,$$

where  $\mathcal{W}_i$  is the corresponding the tautological bundle over  $\text{Hilb}_{n, C}^{\text{cur}} \times_{\text{Jac}_C^0} \text{PMod}_C^0$ ,

$$(3.16) \quad \mathcal{W}_i \subset \left( H^0(\Sigma, \mathcal{O}_{b_i^+}) \oplus H^0(\Sigma, \mathcal{O}_{b_i^-}) \right) \otimes \mathcal{O}_{\text{Hilb} \times \text{PMod}},$$

having fibre  $W_{i,a_i}$  over  $(\mathcal{I}_D, (M, V_1 \oplus \cdots \oplus V_k))$  with  $D_i = D_{b_i, a_i}$ . Pick the symmetric product  $\mathrm{Sym}_{\Sigma}^{n+2k}$  and consider the morphism

$$\begin{aligned} \theta : \quad \mathrm{Sym}_{\Sigma}^n &\longrightarrow \mathrm{Sym}_{\Sigma}^{n+2k} \\ [(y_1, \dots, y_n)]_{\mathfrak{S}_n} &\longmapsto [(b_1^+, b_1^-, \dots, b_k^+, b_k^-, y_1, \dots, y_n)]_{\mathfrak{S}_{n+2k}} \end{aligned}$$

Hence,  $\Theta := \mathrm{Image}(\theta)$  is the locus of  $\mathrm{Sym}_{\Sigma}^{n+2k}$  of the symmetric classes containing  $b_1^+, b_1^-, \dots, b_k^+, b_k^-$ . One then has, by construction, the following trivial subsheaf of the pullback of  $\mathcal{A}_{\Sigma, n+2k}$ ,

$$(3.17) \quad \left( H^0(\Sigma, \mathcal{O}_{b_1^+}) \oplus H^0(\Sigma, \mathcal{O}_{b_1^-}) \oplus \cdots \oplus H^0(\Sigma, \mathcal{O}_{b_k^+}) \oplus H^0(\Sigma, \mathcal{O}_{b_k^-}) \right) \otimes \mathcal{O}_{\Theta} \subset \theta^* \mathcal{A}_{\Sigma, n+2k},$$

and their quotient is isomorphic to  $\mathcal{A}_{\Sigma, n}$ . Thus,

$$(3.18) \quad \det(\mathcal{A}_{\Sigma, n}) \cong \det(\theta^* \mathcal{A}_{\Sigma, n+2k}).$$

It follows from (3.16) and (3.17) that  $\mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_k$  is, naturally, a subsheaf of  $j^* q^* \theta^* \mathcal{A}_{\Sigma, n+2k}$ . Let us denote their quotient by

$$\mathcal{F} := j^* q^* \theta^* \mathcal{A}_{\Sigma, n+2k} / (\mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_k),$$

Note by (3.11), (3.12), (3.13) and (3.14), that  $\tilde{\rho}^* \mathcal{A}_{C, n}$  is isomorphic to a subsheaf of  $\mathcal{F}$ . Furthermore, its quotient amounts to

$$\mathcal{F} / \tilde{\rho}^* \mathcal{A}_{C, n} \cong j^* q^* \left( \mathcal{O}_{\Xi_{\Sigma, b_1^-}} \oplus \cdots \oplus \mathcal{O}_{\Xi_{\Sigma, b_k^-}} \right).$$

Observe that such isomorphism depends on the choice of a point among  $\{b_i^+, b_i^-\}$  for each  $i = 1, \dots, k$  in order to pick the second or third terms in (3.14). Taking determinants,

$$\det \mathcal{F} \otimes \tilde{\rho}^* (\det \mathcal{A}_{C, n})^{-1} \cong j^* q^* \left( \mathcal{O}_{\mathrm{Sym}(\Xi_{\Sigma, b_1^-})} \otimes \cdots \otimes \mathcal{O}_{\mathrm{Sym}(\Xi_{\Sigma, b_k^-})} \right) \cong j^* q^* \mathcal{J}_{\Sigma}^k,$$

where we recall from Remark 2.6 that  $\mathcal{J}_{\Sigma}$  is associated to the divisor  $\Xi_{\Sigma, y}$  for any choice of point  $y \in \Sigma$ . Furthermore, by definition of  $\mathcal{F}$ ,

$$\det \mathcal{F} \cong j^* q^* \det(\theta^* \mathcal{A}_{\Sigma, n+2k}) \otimes (\mathcal{W}_1^{-1} \otimes \cdots \otimes \mathcal{W}_k^{-1}) \cong j^* q^* \det(\mathcal{A}_{\Sigma, n}) \otimes \tilde{\alpha}^* (\mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_k)^{-1},$$

after (3.15) and (3.18). At this point, the result follows from Lemma 2.13.  $\square$

*Remark 3.6.* Observe that

$$j^* q^* \det \left( \mathcal{O}_{\Xi_{\Sigma, b_i^+}} \right) \cong j^* q^* \mathcal{J}_{\Sigma} \cong j^* q^* \det \left( \mathcal{O}_{\Xi_{\Sigma, b_i^-}} \right),$$

so a different choice for the isomorphism between  $\tilde{\rho}^* \mathcal{A}_{C, n}$  with a subsheaf of  $\mathcal{F}$  will lead to the same conclusion.

#### 4. FOURIER–MUKAI AND NORMALIZATION

In this section we explore the relation between two Fourier–Mukai transforms: the one associated to a nodal curve and the one associated to its normalization. To achieve it, we first need a description of the restriction of the Poincaré sheaf to the locus of the compactified Jacobian described by pushforward under the normalization morphism, which we address in Section 4.1. Making use of this, we can provide the relation of the associated integral functors that we describe in Section 4.2.

**4.1. Relation between Poincaré sheaves.** For the rest of this section we work over a projective, integral and (simple) nodal curve  $C$  with  $k$  singular points and normalization  $\nu : \Sigma \rightarrow C$ . Recall that there is no need to specify a polarization on  $C$  as every rank 1 torsion-free sheaf is already stable over an irreducible curve. Let us consider the closed embedding

$$\begin{aligned} \check{\nu} : \text{Jac}_{\Sigma}^{-k} &\longrightarrow \overline{\text{Jac}}_C^0 \\ \mathcal{F} &\longmapsto \nu_*\mathcal{F}. \end{aligned}$$

Consider the smooth point  $x_0 \in C$  and the point  $y_0 \in \nu^{-1}(x_0) \subset \Sigma$ , and take the corresponding translation map

$$\tau_{-k, y_0} : \text{Jac}_{\Sigma}^{-k} \longrightarrow \text{Jac}_{\Sigma}^0.$$

Our first goal is study the restriction of the Poincaré sheaf  $\mathcal{P}_C$  (recall Theorem 2.2) and its dual  $\mathcal{P}_C^{\vee}$  to the image of the embedding

$$(\text{id} \times \check{\nu}) : \overline{\text{Jac}}_C^0 \times \text{Jac}_{\Sigma}^{-k} \hookrightarrow \overline{\text{Jac}}_C^0 \times \overline{\text{Jac}}_C^0.$$

This will be carried out by providing such description in terms of the line bundles  $(\text{id} \times \tau_{-k, y_0})^*\mathcal{P}_{\Sigma}$  and  $(\text{id} \times \tau_{-k, y_0})^*\mathcal{P}_{\Sigma}^{\vee}$ . Here

$$\mathcal{P}_{\Sigma} \longrightarrow \text{Jac}_{\Sigma}^0 \times \text{Jac}_{\Sigma}^0$$

is the Poincaré bundle of  $\Sigma$ , and

$$\mathcal{P}_{\Sigma}^{\vee} \longrightarrow \text{Jac}_{\Sigma}^0 \times \text{Jac}_{\Sigma}^0$$

its dual.

Given a sheaf on  $C$  representing a point  $\mathcal{F} \in \overline{\text{Jac}}_C^0$ , we use the standard notation

$$\mathcal{P}_{C, \mathcal{F}} = \mathcal{P}_C|_{\{\mathcal{F}\} \times \overline{\text{Jac}}_C^0}$$

and similarly for  $\mathcal{P}_{\Sigma}$  and for a line bundle on  $\Sigma$  representing a point on  $\text{Jac}_{\Sigma}^0$ .

With all these ingredients at hand, we now provide a fibrewise description of the Poincaré sheaf restricted to the image of  $\check{\nu}$ . Recall the maps  $\rho : \text{PMod}_C^0 \rightarrow \overline{\text{Jac}}_C^0$  from (2.15) and  $\check{\nu} : \text{PMod}_C^0 \rightarrow \text{Jac}_{\Sigma}^0$  from (2.19).

**Proposition 4.1.** *Let  $\mathcal{L} \in \text{Jac}_{\Sigma}^{-k}$ . Then,*

$$(4.1) \quad \mathcal{P}_{C, \nu_*\mathcal{L}} \cong \rho_* \left( \check{\nu}^* \mathcal{P}_{\Sigma, \mathcal{L}(y_0)} \otimes \omega_{\text{PMod}}^{1/2} \right),$$

and

$$(4.2) \quad \mathcal{P}_{C, \nu_*\mathcal{L}}^{\vee} \cong \rho_* \left( \check{\nu}^* \mathcal{P}_{\Sigma, \mathcal{L}(y_0)}^{\vee} \otimes \omega_{\text{PMod}}^{1/2} \right).$$

*Proof.* Since  $C$  is nodal and integral, by means of Proposition 2.5, one can choose  $n \gg 0$  such that the restriction of the Abel–Jacobi map  $\alpha_C$  to the curvilinear Hilbert scheme is surjective. One can assume, without loss of generality, that  $\alpha_{\Sigma}$  is surjective too.

As an immediate application of flat base change and projection formula, one has the identification  $(\nu_*\mathcal{L})^{\boxtimes n} \cong \nu_*^n(\mathcal{L}^{\boxtimes n})$ , where  $\nu^n : \Sigma^n \rightarrow C^n$  is induced by  $\nu$ . In light of this, and after (2.8), one has the following description for the restriction  $\mathcal{G}_{0, C, \nu_*\mathcal{L}} = \mathcal{G}_{0, C}|_{\text{Hilb} \times \{\nu_*\mathcal{L}\}}$  of  $\mathcal{G}_{0, C}$  to

the slice associated to  $\nu_*\mathcal{L}$ ,

$$\mathcal{G}_{0,C,\nu_*\mathcal{L}} \cong (\psi_{C,*}\sigma_C^*\nu_*^n\mathcal{L}^{\boxtimes n})^{sign} \otimes \det(\mathcal{A}_{C,n})^{-1}.$$

Consider the following Cartesian diagram

$$(4.3) \quad \begin{array}{ccc} \mathbf{Flag}_{n,C}^{cur} \times_{C^n} \Sigma^n & \xrightarrow{\tilde{\sigma}} & \Sigma^n \\ \tilde{\nu} \downarrow & & \downarrow \nu^n, \\ \mathbf{Flag}_{n,C}^{cur} & \xrightarrow{\sigma_C} & C^n \end{array}$$

and the commutative square

$$(4.4) \quad \begin{array}{ccc} \mathbf{Flag}_{n,C}^{cur} \times_{C^n} \Sigma^n & \xrightarrow{\tilde{\psi}} & \mathbf{Hilb}_{n,C}^{cur} \times_{\mathbf{Sym}_C^n} \mathbf{Sym}_\Sigma^n \\ \tilde{\nu} \downarrow & & \downarrow \beta \\ \mathbf{Flag}_{n,C}^{cur} & \xrightarrow{\psi_C} & \mathbf{Hilb}_{n,C}^{cur} \end{array},$$

where we made use of the Hilbert–Chow map when constructing the fibre product on the top right corner. Recalling that  $\nu^n$  is proper (because  $\nu$  is so), we use proper base change around (4.3), and functoriality around (4.4), to get

$$\mathcal{G}_{0,C,\nu_*\mathcal{L}} \cong (\beta_*\tilde{\psi}_*\tilde{\sigma}^*\mathcal{L}^{\boxtimes n})^{sign} \otimes \det(\mathcal{A}_{C,n})^{-1},$$

where the permutation action lifts to  $\mathbf{Flag}_{n,C}^{cur} \times_{C^n} \Sigma^n$  by pullback under  $\tilde{\nu}$ . By invariance of  $\beta$  with respect to this action we obtain,

$$\mathcal{G}_{0,C,\nu_*\mathcal{L}} \cong \beta_* \left( \tilde{\psi}_*\tilde{\sigma}^*\mathcal{L}^{\boxtimes n} \right)^{sign} \otimes \det(\mathcal{A}_{C,n})^{-1}.$$

We shall see next that the previous construction factors through the closed embedding  $j : \mathbf{Hilb}_{n,C}^{cur} \times_{\mathbf{Jac}_C^0} \mathbf{PMod}_C \hookrightarrow \mathbf{Hilb}_{n,C}^{cur} \times_{\mathbf{Sym}_C^n} \mathbf{Sym}_\Sigma^n$  of (3.3).

We claim that the support of  $(\tilde{\psi}_*\tilde{\sigma}^*\mathcal{L}^{\boxtimes n})^{sign}$  lies in the image of  $j$ . This will be proved in Lemma 4.3 below. Consequently, one has that

$$\mathcal{G}_{0,C,\nu_*\mathcal{L}} \cong \tilde{\rho}_* j^* \left( \tilde{\psi}_*\tilde{\sigma}^*\mathcal{L}^{\boxtimes n} \right)^{sign} \otimes \det(\mathcal{A}_{C,n})^{-1},$$

where we observe, as in (3.2), that  $\tilde{\rho} = \beta \circ j$ .

The obvious projection  $\mathbf{Flag}_{n,C}^{cur} \rightarrow \mathbf{Hilb}_{n,C}^{cur}$ , gives rise to  $\mathbf{Flag}_{n,C}^{cur} \times_{C^n} \Sigma^n \rightarrow \mathbf{Hilb}_{n,C}^{cur} \times_{\mathbf{Sym}_C^n} \Sigma^n$ . By definition, the elements of  $\mathbf{Flag}_{n,C}^{cur}$  and  $\mathbf{Hilb}_{n,C}^{cur}$  are locally contained in smooth curves. Hence, starting from  $D \in \mathbf{Hilb}_{n,C}^{cur}$ , one determines uniquely a filtration of  $D$  out of a filtration on  $\mathbf{chow}_C(D) \in \mathbf{Sym}_C^n$ . This naturally provides an inverse to the previous morphism, so

$$\mathbf{Flag}_{n,C}^{cur} \times_{C^n} \Sigma^n \cong \mathbf{Hilb}_{n,C}^{cur} \times_{\mathbf{Sym}_C^n} \Sigma^n.$$

The statement above, followed by the isomorphism

$$\mathbf{Hilb}_{n,C}^{cur} \times_{\mathbf{Sym}_C^n} \Sigma^n \cong (\mathbf{Hilb}_{n,C}^{cur} \times_{\mathbf{Sym}_C^n} \mathbf{Sym}_\Sigma^n) \times_{\mathbf{Sym}_\Sigma^n} \Sigma^n,$$

provides us with a Cartesian diagram,

$$\begin{array}{ccc} \mathrm{Flag}_{n,C}^{\mathrm{cur}} \times_{C^n} \Sigma^n & \xrightarrow{\tilde{\psi}} & \mathrm{Hilb}_{n,C}^{\mathrm{cur}} \times_{\mathrm{Sym}_C^n} \mathrm{Sym}_{\Sigma}^n \\ \tilde{\sigma} \downarrow & & \downarrow q \\ \Sigma^n & \xrightarrow{\pi_{\Sigma}} & \mathrm{Sym}_{\Sigma}^n \end{array},$$

where  $q$  denotes the obvious projection. Recall that  $\pi_{\Sigma}$  finite, hence proper. Then, proper base change with respect to it gives

$$\mathcal{G}_{0,C,\nu_*\mathcal{L}} \cong \tilde{\rho}_* j^* (q^* \pi_{\Sigma,*} \mathcal{L}^{\boxtimes n})^{\mathrm{sign}} \otimes \det(\mathcal{A}_{C,n})^{-1}.$$

and by equivariance with respect to the action of the symmetric group,

$$(4.5) \quad \mathcal{G}_{0,C,\nu_*\mathcal{L}} \cong \tilde{\rho}_*(q \circ j)^* (\pi_{\Sigma,*} \mathcal{L}^{\boxtimes n})^{\mathrm{sign}} \otimes \det(\mathcal{A}_{C,n})^{-1}.$$

We now turn our attention to the sheaf  $\mathcal{G}_{-k,\Sigma}$ . After (2.8), we have that the restriction of  $\mathcal{G}_{-k,\Sigma}$  to slice associated to  $\mathcal{L} \in \mathrm{Jac}_{\Sigma}^{-k}$  is

$$\mathcal{G}_{-k,\Sigma,\mathcal{L}} \cong (\psi_{\Sigma,*} \sigma_{\Sigma}^* (\mathcal{L}^{\boxtimes n}))^{\mathrm{sign}} \otimes \det(\mathcal{A}_{\Sigma,n})^{-1}.$$

Since  $\Sigma$  is smooth the morphism  $\sigma_{\Sigma} : \mathrm{Flag}_{n,\Sigma} \rightarrow \Sigma^n$  is an isomorphism and we may identify  $\mathrm{Hilb}_{n,\Sigma} = \mathrm{Sym}_{\Sigma}^n$ , so  $\pi_{\Sigma} = \psi_{\Sigma} \circ \sigma_{\Sigma}^{-1}$  and  $\psi_{\Sigma,*} \circ \sigma_{\Sigma}^* = \pi_{\Sigma,*}$  it follows that

$$\mathcal{G}_{-k,\Sigma,\mathcal{L}} \cong (\pi_{\Sigma,*} \mathcal{L}^{\boxtimes n})^{\mathrm{sign}} \otimes \det(\mathcal{A}_{\Sigma,n})^{-1}.$$

Recalling Lemma 2.7,

$$\mathcal{G}_{-k,\Sigma,\mathcal{L}} \cong \mathcal{G}_{0,\Sigma,\mathcal{L}(y_0)} \otimes \mathcal{J}_{\Sigma}^{-k}.$$

Substituting this into (4.5) gives

$$\mathcal{G}_{0,C,\nu_*\mathcal{L}} \cong \tilde{\rho}_* j^* q^* \left( \mathcal{G}_{0,\Sigma,\mathcal{L}(y_0)} \otimes \det(\mathcal{A}_{\Sigma,n}) \otimes \mathcal{J}_{\Sigma}^{-k} \right) \otimes \det(\mathcal{A}_{C,n})^{-1}.$$

Statement (2.2) and functoriality with respect the commutative square below

$$\begin{array}{ccc} \mathrm{Hilb}_{n,C}^{\mathrm{cur}} \times_{\mathrm{Jac}_C} \mathrm{PMod}_C^0 & \xrightarrow{\tilde{\alpha}} & \mathrm{PMod}_C^0 \\ r := q \circ j \downarrow & & \downarrow \dot{\nu} \\ \mathrm{Sym}_{\Sigma}^n & \xrightarrow{\alpha_{\Sigma}} & \mathrm{Jac}_{\Sigma}^0 \end{array},$$

provides us with the identification

$$\mathcal{G}_{0,C,\nu_*\mathcal{L}} \cong \tilde{\rho}_* \left( \tilde{\alpha}^* \dot{\nu}^* \mathcal{P}_{\Sigma,\mathcal{L}(y_0)} \otimes r^* \det(\mathcal{A}_{\Sigma,n}) \otimes r^* \mathcal{J}_{\Sigma}^{-k} \right) \otimes \det(\mathcal{A}_{C,n})^{-1}.$$

An application of the projection formula alongside Proposition 3.5 allows us to write

$$\mathcal{G}_{0,C,\nu_*\mathcal{L}} \cong \tilde{\rho}_* \left( \tilde{\alpha}^* \dot{\nu}^* \mathcal{P}_{\Sigma,\mathcal{L}(y_0)} \otimes \tilde{\alpha}^* \omega_{\mathrm{PMod}}^{1/2} \right).$$

Proper base change with respect to the Cartesian diagram (3.2), gives

$$\mathcal{G}_{0,C,\nu_*\mathcal{L}} \cong \alpha_C^* \rho_* \left( \dot{\nu}^* \mathcal{P}_{\Sigma,\mathcal{L}(y_0)} \otimes \omega_{\mathrm{PMod}}^{1/2} \right),$$

what implies our first statement (4.1).

We address the last statement making use of Grothendieck-Verdier duality, which can be expressed as a natural equivalence

$$R\rho_! \circ \mathbb{D}_{\mathrm{PMod}_C} \simeq \mathbb{D}_{\overline{\mathrm{Jac}}_C} \circ R\rho_*.$$

One can replace  $\rho_!$  with  $\rho_*$  by noting that  $\rho$  is proper.

**Lemma 4.2** (Lemma 2.1 of [Ari2]). *Let  $Y$  and  $Z$  be schemes of pure dimension and suppose that  $Z$  is Gorenstein. Consider a coherent sheaf  $\mathcal{G}$  on  $Y \times Z$  flat over  $Z$ , such that for every  $z \in Z$  the restriction  $\mathcal{G}|_{Y \times \{z\}}$  is maximal Cohen–Macaulay.*

*Then,  $\mathcal{G}$  is maximal Cohen–Macaulay and  $\mathbb{D}_{Y \times Z}(\mathcal{G})|_{Y \times \{z\}} = \mathbb{D}_{Y \times \{z\}}(\mathcal{G}|_{Y \times \{z\}})$ .*

Hence, (4.2) follows naturally from (4.1) and Lemma 4.2.  $\square$

**Lemma 4.3.** *The sheaf  $(\tilde{\psi}_* \tilde{\sigma}^* \mathcal{L}^{\boxtimes n})^{\mathrm{sign}}$  is supported on the image of  $\mathrm{Hilb}_{n,C}^{\mathrm{cur}} \times_{\overline{\mathrm{Jac}}_C} \mathrm{PMod}_C$  under  $j$ .*

*Proof.* Recall that the action of  $\mathfrak{S}_n$  on  $\mathrm{Flag}_{n,C}^{\mathrm{cur}}$  is induced by the action of this group on  $C^n$ . Hence, it follows that the fixed point set  $(\mathrm{Flag}_{n,C}^{\mathrm{cur}})^\gamma$  coincides with the preimage of  $(C^n)^\gamma$  inside  $\mathrm{Flag}_{n,C}^{\mathrm{cur}}$ . Furthermore, since we consider the action of  $\mathfrak{S}_n$  on  $\mathrm{Flag}_{n,C}^{\mathrm{cur}} \times_{C^n} \Sigma^n$  to be the one obtained by pullback under  $\tilde{\nu}$  of the permutation action on  $\mathrm{Flag}_{n,C}^{\mathrm{cur}}$ , one has that the fixed point locus  $(\mathrm{Flag}_{n,C}^{\mathrm{cur}} \times_{C^n} \Sigma^n)^\gamma$  amounts to  $(\mathrm{Flag}_{n,C}^{\mathrm{cur}})^\gamma \times_{(C^n)^\gamma} (\nu^n)^{-1}(C^n)^\gamma$ .

Since  $\mathrm{Hilb}_{n,C}^{\mathrm{cur}} \times_{\overline{\mathrm{Jac}}_C} \mathrm{PMod}_C$  is closed in  $\mathrm{Hilb}_{n,C}^{\mathrm{cur}} \times_{\mathrm{Sym}_C^n} \mathrm{Sym}_\Sigma^n$ , one has that its complement is an open subset of a union of irreducible components of  $\mathrm{Hilb}_{n,C}^{\mathrm{cur}} \times_{\mathrm{Sym}_C^n} \mathrm{Sym}_\Sigma^n$ . By Lemma 3.4, these irreducible components lie over the big diagonal  $\Delta$ , hence contained in  $\mathrm{chow}^{-1}(\Delta) \times_\Delta \tilde{\nu}^{-1}(\Delta) \subset \mathrm{Hilb}_{n,C}^{\mathrm{cur}} \times_{C^n} \Sigma^n$ . Observe that the big diagonal  $\Delta$  is the image under  $\pi_C : C^n \rightarrow \mathrm{Sym}_C^n$  of  $(C^n)^\gamma$  for some  $\gamma \in \mathfrak{S}_n$ . It follows that the complement of  $\mathrm{Hilb}_{n,C}^{\mathrm{cur}} \times_{\overline{\mathrm{Jac}}_C} \mathrm{PMod}_C$  is contained in the union of  $\tilde{\psi}((\mathrm{Flag}_{n,C}^{\mathrm{cur}} \times_{C^n} \Sigma^n)^\gamma)$  for all  $\gamma \in \mathfrak{S}_n$ .

Consider  $\gamma \in \mathfrak{S}_n$  odd, since  $\gamma$  is the identity on  $(\mathrm{Flag}_{n,C}^{\mathrm{cur}} \times_{C^n} \Sigma^n)^\gamma$ , it follows that the restriction of  $(\tilde{\psi}_* \tilde{\sigma}^* \mathcal{L}^{\boxtimes n})^{\mathrm{sign}}$  to an irreducible component contained in  $\tilde{\psi}((\mathrm{Flag}_{n,C}^{\mathrm{cur}} \times_{C^n} \Sigma^n)^\gamma)$  vanishes as the sections must satisfy  $-s = \gamma^* s = s$ . Hence,  $(\tilde{\psi}_* \tilde{\sigma}^* \mathcal{L}^{\boxtimes n})^{\mathrm{sign}}$  vanishes on the complement of  $\mathrm{Hilb}_{n,C}^{\mathrm{cur}} \times_{\overline{\mathrm{Jac}}_C} \mathrm{PMod}_C$  and the proof follows.  $\square$

With the fibrewise description at hand, one can now address the proof of the description of the Poincaré sheaf and its dual to  $\overline{\mathrm{Jac}}_C^0 \times \check{\nu}(\mathrm{Jac}_\Sigma^{-s})$ .

**Theorem 4.4.** *Let  $C$  be an integral nodal curve with singularity divisor of length  $k$  and normalization  $\nu : \Sigma \rightarrow C$ . Pick  $y_0 \in \Sigma$  such that  $\nu(y_0)$  lies in the smooth locus of  $C$ . Then,*

$$(4.6) \quad (\mathrm{id} \times \check{\nu})^* \mathcal{P}_C \cong (\rho \times \mathrm{id})_* \left( \mathfrak{q}_2^* \omega_{\mathrm{PMod}}^{1/2} \otimes (\dot{\nu} \times \mathrm{id})^* (\mathrm{id} \times \tau_{-k, y_0})^* \mathcal{P}_\Sigma \right),$$

where  $\mathfrak{q}_2 : \mathrm{PMod}_C \times \mathrm{Jac}_\Sigma^{-k} \rightarrow \mathrm{PMod}_C$  is the obvious projection.

Also, the restriction of the dual of the Poincaré sheaf is

$$(4.7) \quad (\mathrm{id} \times \check{\nu})^* \mathcal{P}_C^\vee \cong (\rho \times \mathrm{id})_* \left( \mathfrak{q}_2^* \omega_{\mathrm{PMod}}^{1/2} \otimes (\dot{\nu} \times \mathrm{id})^* (\mathrm{id} \times \tau_{-k, y_0})^* \mathcal{P}_\Sigma^\vee \right).$$



We need to recall the *see-saw* principle before addressing the proof of Theorem 4.4. Here, we reproduce the statement as in [MRV3, Lemma 5.5], adapting the hypothesis to our case.

**Lemma 4.5** (See-saw principle). *Let  $Z$  and  $T$  be two reduced locally Noetherian schemes with  $Z$  proper and connected. Let  $\mathcal{E}$  and  $\mathcal{F}$  be two sheaves on  $Z \times T$ , flat over  $T$ , such that*

$$(i) \mathcal{F}|_{Z \times \{t\}} \cong \mathcal{E}|_{Z \times \{t\}}, \text{ for all } t \in T;$$

$$(ii) \mathcal{F}|_{Z \times \{t\}} \text{ is simple for every } t \in T;$$

$$(iii) \text{ there exists } z_0 \in Z \text{ and an isomorphism of line bundles } \mathcal{F}|_{\{z_0\} \times T} \cong \mathcal{E}|_{\{z_0\} \times T}.$$

Then  $\mathcal{E}$  and  $\mathcal{F}$  are isomorphic.

*Proof of Theorem 4.4.* The theorem would follow if the right and left hand sides of (4.6) and (4.7) satisfy all hypothesis of the see-saw principle.

Take the line bundle  $\mathcal{H} = \mathfrak{q}_2^* \omega_{\text{PMod}}^{1/2} \otimes (\check{\nu} \times \text{id})^*(\text{id} \times \tau_{-k, y_0})^* \mathcal{P}_\Sigma$  over  $\text{PMod}_C \times \text{Jac}_\Sigma^{-k}$ , which is flat over  $\text{Jac}_\Sigma^{-k}$ . After the description of  $\rho$  that we obtain from Remark 2.12, the stalk of  $(\rho \times \text{id})_* \mathcal{H}$  at a point of  $\overline{\text{Jac}}_C^0 \times \text{Jac}_\Sigma^{-k}$  is the direct sum of the stalks at their preimages in  $\text{PMod}_C^0 \times \text{Jac}_\Sigma^{-k}$ , hence flat over  $\text{Jac}_\Sigma^{-k}$ , so the right-hand side of (4.6) is a flat sheaf over the second factor. Similarly, starting with the line bundle  $\mathfrak{q}_1^* \omega_{\text{PMod}} \otimes \mathcal{H}^{-1}$  one can show that the right-hand side of (4.7) is a flat sheaf over the second factor.

One can also check flatness of the left-hand sides of (4.6) and (4.7). Since both  $\mathcal{P}_C$  and  $\mathcal{P}_C^\vee$  are flat over the second factor, it follows that  $(\text{id} \times \check{\nu})^* \mathcal{P}_C$  and  $(\text{id} \times \check{\nu})^* \mathcal{P}_C^\vee$  are flat over  $\text{Jac}_\Sigma^{-k}$ , as flatness is preserved under base change.

Proposition 4.1 implies that condition (i) holds for (4.6) and (4.7). Condition (ii) is automatic since  $\Phi^{\mathcal{P}_C}$  and  $\Phi^{\mathcal{P}_C^\vee}$  are derived equivalences. Since  $\rho$  is locally an isomorphism at  $\mathcal{O}_C \in \overline{\text{Jac}}_C^0$ , hypothesis (iii) holds as well as  $\mathcal{P}_C$  and  $\mathcal{P}_C^\vee$  are both normalized at  $\{\mathcal{O}_C\} \times \overline{\text{Jac}}_C^0$ , and so are  $\mathcal{P}_\Sigma$  and  $\mathcal{P}_\Sigma^\vee$  over  $\{\mathcal{O}_\Sigma\} \times \text{Jac}_\Sigma^0$ .  $\square$

**4.2. Relation between Fourier–Mukai functors.** Using the description of the Poincaré sheaves obtained in Section 4.1, we provide here a relation of the Fourier–Mukai transform associated to a singular curve and the Fourier–Mukai transform associated to its normalization.

**Theorem 4.6.** *Let  $C$  be an integral nodal curve with arithmetic genus  $g$  and singularity divisor of length  $k$  and normalization  $\nu : \Sigma \rightarrow C$ . Pick  $y_0 \in \Sigma$  such that  $\nu(y_0)$  lies in the smooth locus of  $C$ . Then, for every  $\mathcal{F}^\bullet \in D^b(\text{Jac}_\Sigma^{-k})$ , one has*

$$(4.8) \quad \Phi_{1 \rightarrow 2}^{\mathcal{P}_C} (R\check{\nu}_* \mathcal{F}^\bullet) \cong R\rho_* \left( \omega_{\text{PMod}}^{1/2} \otimes \check{\nu}^* \Phi_{1 \rightarrow 2}^{\mathcal{P}_\Sigma} (\tau_{-k, y_0, *} \mathcal{F}^\bullet) \right).$$

and

$$(4.9) \quad \Phi_{2 \rightarrow 1}^{\mathcal{P}_C^\vee} (R\check{\nu}_* \mathcal{F}^\bullet) \cong R\rho_* \left( \omega_{\text{PMod}}^{1/2} \otimes \check{\nu}^* \Phi_{2 \rightarrow 1}^{\mathcal{P}_\Sigma^\vee} (\tau_{-k, y_0, *} \mathcal{F}^\bullet) \right).$$

*Proof.* Recall the construction of  $\Phi_{1 \rightarrow 2}^{\mathcal{P}_C}$  described in (2.4). Following the Cartesian diagram

$$\begin{array}{ccc} \overline{\text{Jac}}_C^0 \times \text{Jac}_\Sigma^{-k} & \xrightarrow{t_1} & \text{Jac}_\Sigma^{-k} \\ (\text{id} \times \check{\nu}) \downarrow & & \downarrow \check{\nu} \\ \overline{\text{Jac}}_C^0 \times \overline{\text{Jac}}_C^0 & \xrightarrow{p_1} & \overline{\text{Jac}}_C^0, \end{array}$$

one has

$$(4.10) \quad \Phi_{1 \rightarrow 2}^{\mathcal{P}_C}(R\check{\nu}_* \mathcal{F}^\bullet) \cong R t_{2,*} (t_1^* \mathcal{F}^\bullet \otimes (\text{id} \times \check{\nu})^* \mathcal{P}_C),$$

where we have applied flat base change, projection formula and functoriality with respect to

$$\begin{array}{ccc} \overline{\text{Jac}}_C^0 \times \text{Jac}_\Sigma^{-k} & & \\ (\text{id} \times \check{\nu}) \downarrow & \searrow t_2 & \\ \overline{\text{Jac}}_C^0 \times \overline{\text{Jac}}_C^0 & \xrightarrow{p_2} & \overline{\text{Jac}}_C^0. \end{array}$$

Substituting (4.6) in (4.10), applying the projection formula with respect to  $t_2$  and  $(\rho \times \text{id})$ , functoriality with respect to

$$\begin{array}{ccc} \text{PMod}_C^0 \times \text{Jac}_\Sigma^{-k} & & \text{PMod}_C^0 \times \text{Jac}_\Sigma^{-k} \xrightarrow{q_2} \text{PMod}_C^0 \\ (\rho \times \text{id}) \downarrow & \searrow q_1 & \downarrow \rho \\ \overline{\text{Jac}}_C^0 \times \text{Jac}_\Sigma^{-k} \xrightarrow{t_1} \text{Jac}_\Sigma^{-k}, & & \overline{\text{Jac}}_C^0 \times \text{Jac}_\Sigma^{-k} \xrightarrow{t_2} \overline{\text{Jac}}_C^0, \end{array}$$

followed by the projection formula with respect to  $q_1$  (only in the case of  $\Phi_{2 \rightarrow 1}^{\mathcal{P}_C^\vee}$ ), we get

$$(4.11) \quad \Phi_{1 \rightarrow 2}^{\mathcal{P}_C}(\check{\nu}_* \mathcal{F}^\bullet) \cong R \rho_* \left( \omega_{\text{PMod}}^{1/2} \otimes R q_{2,*} (q_1^* \mathcal{F}^\bullet \otimes (\check{\nu} \times \text{id})^* ((\text{id} \times \tau_{y_0})^* \mathcal{P}_\Sigma)) \right),$$

Due to functoriality with respect to the diagram on the left, and flat base change for the diagram on the right,

$$\begin{array}{ccc} \text{PMod}_C^0 \times \text{Jac}_\Sigma^{-k} & & \text{PMod}_C^0 \times \text{Jac}_\Sigma^{-k} \xrightarrow{q_2} \text{PMod}_C^0 \\ (\check{\nu} \times \text{id}) \downarrow & \searrow q_1 & \downarrow \check{\nu} \\ \text{Jac}_\Sigma^0 \times \text{Jac}_\Sigma^{-k} \xrightarrow{r_1} \text{Jac}_\Sigma^{-k}, & & \text{Jac}_\Sigma^0 \times \text{Jac}_\Sigma^{-k} \xrightarrow{r_2} \text{Jac}_\Sigma^0, \end{array}$$

one observes that (4.11) becomes

$$\Phi_{1 \rightarrow 2}^{\mathcal{P}_C}(R\check{\nu}_* \mathcal{F}^\bullet) \cong R \rho_* \left( \omega_{\text{PMod}}^{1/2} \otimes \check{\nu}^* (R r_{2,*} (r_1^* \mathcal{F}^\bullet \otimes (\text{id} \times \tau_{-k, y_0})^* \mathcal{P}_\Sigma)) \right).$$

This proves the statement (4.8) after applying projection formula with respect to  $\text{id} \times \tau_{k, y_0}$ .

Since the definition of  $\Phi_{2 \rightarrow 1}^{\mathcal{P}_C^\vee}$  given in (2.5) is analogous to that of (2.4) and the relation between  $\mathcal{P}_C^\vee$  and  $\mathcal{P}_\Sigma^\vee$  of (4.7) is analogous to (4.6), the proof (4.9) follows mutatis mutandis the proof of (4.8).  $\square$

We conclude the section by studying the above Fourier–Mukai transform in the case of a topologically trivial line bundle  $\mathcal{M} \in \text{Pic}^0(\text{Jac}_\Sigma^{-k})$ . Using the translation and self-duality isomorphisms

$$\text{Pic}^0(\text{Jac}_\Sigma^{-k}) \xrightarrow{\cong} \text{Pic}^0(\text{Jac}_\Sigma^0) \xrightarrow{\cong} \text{Jac}_\Sigma^0,$$

$$\mathcal{M} \mapsto \tau_{-k, y_0, *} \mathcal{M} \mapsto \widetilde{\mathcal{M}},$$

one can relate  $\tau_{-k, y_0, *} \mathcal{M} \in D^b(\text{Jac}_\Sigma^0)$  and the skyscraper sheaf  $\mathcal{O}_{\widetilde{\mathcal{M}}} \in D^b(\text{Jac}_\Sigma^0)$  by the transform

$$\Phi_{1 \rightarrow 2}^{\mathcal{P}_\Sigma}(\tau_{-k, y_0, *} \mathcal{M}) \cong \mathcal{O}_{\widetilde{\mathcal{M}}} \in D^b(\text{Pic}^0(\text{Jac}_\Sigma^0)).$$

To describe Theorem 4.6 in the case  $\mathcal{F}^\bullet = \mathcal{M}$ , we take note of the term  $\check{\nu}^* \mathcal{O}_{\widetilde{\mathcal{M}}} \cong \mathcal{O}_{\check{\nu}^{-1}(\widetilde{\mathcal{M}})}$ , the structure sheaf of the fibre of  $\check{\nu} : \text{PMod}_C^0 \rightarrow \text{Jac}_\Sigma^0$ . The geometry of the fibers is the subject of Lemma 2.11, where it is described in terms of a locus of Hecke cycles:

$$\check{\nu}^{-1}(\widetilde{\mathcal{M}}) \cong \text{Hecke}(\widetilde{\mathcal{M}}) \cong \mathbb{P}^1 \times \cdots \times \mathbb{P}^1.$$

Now we make use of the fact that  $\rho$  is an isomorphism restricted to the fibres to write Theorem 4.6, evaluated at  $\mathcal{F}^\bullet = \mathcal{M} \in \text{Pic}^0(\text{Jac}_\Sigma^{-k}) \subset D^b(\text{Jac}_\Sigma^{-k})$ , as follows.

**Corollary 4.7.** *Given a topologically trivial line bundle  $\mathcal{M} \in \text{Pic}^0(\text{Jac}_\Sigma^{-k})$  and the corresponding  $\widetilde{\mathcal{M}} \in \text{Jac}_\Sigma^0$ , the transform of the pushforward along  $\check{\nu} : \text{Jac}_\Sigma^{-k} \hookrightarrow \overline{\text{Jac}}_C^0$  satisfies*

$$\begin{aligned} \Phi_{1 \rightarrow 2}^{\mathcal{P}_C}(R\check{\nu}_* \mathcal{M}) &\cong R\rho_* \left( \mathcal{O}_{\text{PMod}}(-1, \dots, -1) \otimes \mathcal{O}_{\check{\nu}^{-1}(\widetilde{\mathcal{M}})} \right) \\ &\cong \mathcal{O}_{\text{Hecke}(\widetilde{\mathcal{M}})}(-1) \\ &\cong \mathcal{O}_{\mathbb{P}^1}(-1) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathbb{P}^1}(-1). \end{aligned}$$

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