ESSENTIALLY QUASI-DUO RINGS

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Abstract. A ring with identity is called essentially right quasi-duo if every essential maximal right ideal of it is a two-sided ideal. Essentially right quasi-duo rings generalize essentially right duo rings, a notion that arose in the study of hypercyclic rings, and right quasi-duo rings, as introduced by S.H. Brown. We prove that a ring R is essentially right quasi-duo if and only if R is semisimple or $R/\mathrm{soc}(R_R)$ is right quasi-duo. Although it is still unknown, whether a right quasi-duo ring is left quasi-duo, we provide an example of an essentially right quasi-duo ring that is not essentially left quasi-duo. Furthermore, while exchange right quasi-duo rings are known to be clean, there exist exchange essentially right quasi-duo rings that are not clean. A thorough study of essentially right quasi-duo rings is carried out and their relationship to skew power series rings, trivial extensions and formal triangular matrix rings is explored.

1. Introduction

Throughout, rings are associative with identity and modules are unitary. It is well-known that the following three conditions on a ring R are equivalent to R being semisimple Artinian: Every right ideal of R is a direct summand; every essential right ideal of R is a direct summand; every maximal right ideal of R is a direct summand. Moreover, because an essential maximal right ideal of a non-trivial ring can't be a direct summand, every essential maximal right ideal of R being a direct summand implies that R is a trivial ring. Here is another set of conditions on a ring R:

(1) Every right ideal of R is an ideal;
(2) Every essential right ideal of R is an ideal;
(3) Every maximal right ideal of R is an ideal;
(4) Every essential maximal right ideal of R is an ideal.

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In the literature, rings with (1) and rings with (3) are called right duo rings and, respectively, right quasi-duo rings. The definitions are due to Feller [1] and, respectively, Brown [2, Remark 3.21]. Right duo rings and right quasi-duo rings have been extensively studied (see, e.g. [7, 13, 15, 16, 20, 24]). Rings with (2) are termed essentially right duo rings and were motivated by the study of hypercyclic rings in [18]. In this paper, we initiate a study of rings satisfying (4), and these rings are called essentially right quasi-duo rings. They include right quasi-duo rings, essentially right duo rings, semisimple Artinian rings and some other classes of rings. The essentially right quasi-duo property of various ring extensions is considered in Section 2 and, as consequences, many examples of rings with this property are presented. Essentially right quasi-duo rings are characterized as semisimple Artinian rings or those rings R such that $R/\mathrm{soc}(R_R)$ is right quasi-duo (Theorem 2.11). Moreover, essentially right quasi-duo rings that are regular and right CS are completely determined (Corollary 2.15). The question as to whether a ring whose injective hull as a right module is cyclic must be right self-injective is answered positively under the assumption of the ring being essentially right quasi-duo (see Theorem 4.3). Section 3 presents a characterization of a formal triangular matrix ring that is essentially right quasi-duo (Theorem 3.1). This result is used to construct an essentially right quasi-duo ring that is not essentially left quasi-duo (Example 3.3), and to establish the structure of an essentially right quasi-duo semilocal ring (Theorem 3.7). In Section 4, we revisit the open question whether, for a ring R, the injective hull $E(R_R)$ being cyclic implies the (right) injectivity of R. It is known that the answer is "yes" if R is right quasi-duo. Here this is extended to R being essentially right quasi-duo in Theorem 4.3, as a consequence of the result that, for a finitely generated projective module M, every essential maximal submodule of M is fully invariant if and only if M does not contain proper submodules X, Y such that $M = X + Y$ and $M/X \cong M/Y$ is singular (Corollary 4.2).

For a ring R, the Jacobson radical, the right socle and the left socle of R are denoted by $J(R)$, $\mathrm{soc}(R_R)$ and $\mathrm{soc}({}_RR)$, respectively. The $n \times n$ matrix ring (resp. upper triangular matrix ring, or lower triangular matrix ring) over R is denoted by $\mathbb{M}_n(R)$ (resp. $\mathbb{UT}_n(R)$

or $\mathbb{LT}_n(R)$). The injective hull of the regular module R_R is denoted by $E(R_R)$. We write homomorphisms of right modules on the left of their arguments.

2. Essentially quasi-duo rings

Definition 2.1. A ring R is called essentially right quasi-duo if every essential maximal right ideal of R is an ideal.

Essentially right quasi-duo rings include right quasi-duo rings and essentially right duo rings.

Example 2.2. *There exist essentially right quasi-duo rings that are neither essentially right duo nor right quasi-duo.*

Proof. Let D be a division ring and R be an integral domain that is not a field. Then $\mathbb{M}_2(D)$ is an essentially quasi-duo ring that is not right quasi-duo and $\mathbb{LT}_2(R)$ is a quasi-duo ring that is not essentially right duo (see [18]). Hence $\mathbb{M}_2(D) \oplus \mathbb{LT}_2(R)$ is an essentially quasi-duo ring that is neither essentially right duo nor right quasi-duo. □

Indeed, there exist indecomposable essentially right quasi-duo rings that are neither essentially right duo nor right quasi-duo (see Example 3.3).

Next we study the essentially quasi-duo property of some standard ring extensions.

Proposition 2.3. (1) *If R is essentially right quasi-duo (resp. essentially right duo), then so is R/I for any proper ideal I of R.*

(2) *If $R = \prod_{i=1}^{n} R_i$ is a direct product of rings, then R is essentially right quasi-duo (resp. essentially right duo) if and only if so is each R_i.*

(3) *If $n \geq 2$, then $\mathbb{M}_n(R)$ is essentially right quasi-duo (resp. essentially right duo) if and only if R is a semisimple Artinian ring.*

Proof. (1) If T is an essential (maximal) right ideal of R/I, then $T = K/I$ where K is an essential (maximal) right ideal of R. So K is an ideal of R and hence T is an ideal of R/I.

(2) The necessity is by (1). For the sufficiency, let T be an essential (maximal) right ideal of R. Then $T = \prod_{i=1}^{n} TR_i$ where, for each i, TR_i is an essential (maximal) right ideal of R_i or $TR_i = R_i$. So each TR_i is an ideal of R_i. Hence T is an ideal of R.

(3) It is known that a ring is semisimple Artinian if and only if each of its maximal right ideals is a direct summand. Assume that R is not semisimple Artinian. Then R has a maximal right ideal I that is not a direct summand of R_R. Thus I is an essential maximal right ideal. So $\begin{pmatrix} I & I & \cdots & I \\ R & R & \cdots & R \\ \vdots & \vdots & \cdots & \vdots \\ R & R & \cdots & R \end{pmatrix}$ is an essential maximal right ideal, but not an ideal, of $\mathbb{M}_n(R)$. Hence $\mathbb{M}_n(R)$ is not essentially right quasi-duo.

If R is semisimple Artinian, then $\mathbb{M}_n(R)$ is semisimple Artinian, and hence has itself as the only essential right ideal. So $\mathbb{M}_n(R)$ is essentially right duo. □

For an endomorphism σ of a ring R, let $R[[t;\sigma]]$ (resp. $R[t;\sigma]$) be the ring of left skew power series (resp. left polynomials) over R with multiplication subject to the relation $ta = \sigma(a)t$ for all $a \in R$.

Proposition 2.4. *Let R be a ring, $\sigma : R \to R$ be an endomorphism of rings and $n \geq 1$. The following are equivalent:*

(1) *$R[[t;\sigma]]$ is essentially right quasi-duo (resp. right quasi-duo).*
(2) *$R[t;\sigma]/(t^{n+1})$ is essentially right quasi-duo (resp. right quasi-duo).*
(3) *R is right quasi-duo.*

Proof. (1) ⇒ (2). Since $R[t;\sigma]/(t^{n+1})$ is a homomorphic image of $R[[t;\sigma]]$, the implication follows from Proposition 2.3(1).

(2) ⇒ (3). Let I be a maximal right ideal of R. Then $T := I + Rt + \ldots + Rt^n$ is an essential maximal right ideal of $R[t;\sigma]/(t^{n+1})$, so it is an ideal. It follows that I is an ideal of R. So R is right quasi-duo.

(3) ⇒ (1). Let T be an (essential) maximal right ideal of $R[[t;\sigma]]$. Then $T = I + R[[t;\sigma]]t$ where I is a maximal right ideal of R. So I is an ideal of R, and hence T is an ideal of $R[[t;\sigma]]$. □

It is known that a left duo ring need not be right duo (see [3]). Below we show that, for an endomorphism $\sigma : R \to R$, if $R[t;\sigma]/(t^{n+1})$ is essentially right duo, then R is right duo; but the converse is not true as seen below. Below it is also seen that an essentially left duo ring need not be essentially right duo.

Examples 2.5. (1) *Let V_D be a vector space over a division ring D and $R = \mathrm{End}(V_D)$. Then R is essentially right quasi-duo (resp. essentially right duo) if and only if $\dim(V_D) < \infty$.*

(2) *If R is a simple essentially right quasi-duo ring, then R is a simple artinian ring.*

(3) *If R is an essentially right quasi-duo ring with $R/J(R)$ simple and idempotents lift modulo $J(R)$, then R is local or simple Artinian.*

(4) *An infinite direct product of essentially right quasi-duo (resp. essentially right duo) rings need not be essentially right quasi-duo (resp. essentially right duo).*

(5) *An essentially left duo ring need not be essentially right duo.*

(6) *Let F be a field with an endomorphism $\sigma : F \to F$ such that $0 \neq \sigma(F) \subsetneqq F$. Then $F[t;\sigma]/(t^{n+1})$ is essentially right duo if and only if $n = 1$.*

Proof. (1) The sufficiency is clear. If $\dim(V_D) = \infty$, then $V_D \cong V_D^2$, so $R = \mathrm{End}(V_D) \cong \mathrm{End}(V_D^2) \cong \mathbb{M}_2(R)$. Since R is not semisimple Artinian, $\mathbb{M}_2(R)$ is not essentially right quasi-duo by Proposition 2.3(3), so R is not essentially right quasi-duo.

(2) It suffices to show that R is semisimple Artinian. Assume that R is not semisimple Artinian. Then R has an essential maximal right ideal I, so I is a proper ideal of R. Hence $I = 0$ because R is simple, contradicting that I is essential in R_R. So R is semisimple Artinian.

(3) Since $R/J(R)$ is a simple essentially right quasi-duo ring, $R/J(R)$ is simple Artinian by (2), so $R/J(R) \cong \mathbb{M}_n(D)$ where $n \geq 1$ and D is a division ring. Since idempotents lift modulo $J(R)$, $R \cong \mathbb{M}_n(S)$ where $S/J(S) \cong D$. If $n = 1$, then $R \cong S$ is local. If $n \geq 2$, then $J(S) = 0$ by Proposition 2.3(3), so R is simple Artinian.

(4) Let $R = \prod_{i=1}^{\infty} R_i$ where $R_i = \mathbb{M}_2(F_i)$ and F_i is a field. Then each R_i is essentially right duo, while $R \cong \mathbb{M}_2(\prod_{i=1}^{\infty} F_i)$ is not essentially right quasi-duo, since $\prod_{i=1}^{\infty} F_i$ is not semisimple Artinian.

(5) By [12, Theorem 1], the skew power series ring $R := D[[x; \sigma]]$, where D is a commutative principal ideal domain and σ is a monomorphism from the quotient field of D to D, is left duo. If σ is not onto, then R is not right duo by [3, Examples, p.157]. Indeed, R is not essentially right duo, because $I := \sigma(D)x + Dx^2 + Dx^3 + \dots$ is an essential right ideal of R that is not an ideal.

(6) Let $R = F[t; \sigma]/(t^{n+1})$. If $n \geq 2$, then $\sigma(F)t + Ft^2 + \dots + Ft^n$ is an essential right ideal of R, which is not a left ideal. So R is not essentially right duo.

If $n = 1$, let I be an essential right ideal of R. We show that I is an ideal. We can assume that $I \neq R$. Then $I \subseteq Ft$. For any $0 \neq a \in F$, $0 \neq at \in R$, so there exists $u \in F$ such that $a\sigma(u)t \in I$. Then $at = a\sigma(u)\sigma(u^{-1})t = (a\sigma(u)t)u^{-1} \in I$. This shows that $I = Ft$, which is an ideal of R. □

Proposition 2.6. *Let R be a ring and $0 \neq e^2 = e \in R$.*

(1) *If R is essentially right duo, then so is eRe.*
(2) *If R is essentially right quasi-duo, then so is eRe.*

Proof. (1) Let I be an essential right ideal of eRe. Then IR is a submodule of $(eR)_R$. There exists a submodule Y of $(eR)_R$ such that $IR \oplus Y$ is essential in $(eR)_R$. Then $K := IR \oplus Y \oplus (1-e)R$ is an essential right ideal of R. Since R is essentially right duo, K is an ideal of R, so eKe is an ideal of eRe. Since $I \cap Ye \subseteq IR \cap Y = 0$, $I \cap Ye = 0$. So $Ye = 0$, because Ye is a right ideal of eRe and I is an essential right ideal of eRe. Hence $eKe = IRe + Ye = IRe = IeRe = I$, so I is an ideal of eRe.

(2) Let I be an essential maximal right ideal of eRe. Noting that $IR = eR$ would imply $e \in eRe = eIRe = I(eRe) = I$, we have $IR \subsetneq eR$. Let X be a maximal submodule of $(eR)_R$ such that $IR \subseteq X$. Then $K := X + (1-e)R$ is a maximal right ideal of R. We next verify that $I = Xe$ and X is an essential submodule of $(eR)_R$. Firstly, $I \subseteq Xe \subseteq eRe$. Secondly, $Xe \neq eRe$: If $Xe = eRe$, then $e \in Xe \subseteq X$, showing $X = eR$,

a contradiction. Since I is a maximal right ideal of eRe, it must be that $I = Xe$. Assume that X is not essential in $(eR)_R$. Then $X \oplus Y = eR$ where Y is a simple submodule of $(eR)_R$, and so $Xe \oplus Ye = eRe$. It follows that $Ye = 0$, so $Xe = eRe$, a contradiction. So X is essential in $(eR)_R$. Hence, K is an essential right ideal of R. Since R is essentially quasi-duo, K is an ideal of R. Hence $I = Xe = eXe = eKe$ is an ideal of eRe. □

Let C be a (unital) subring of a ring D with $1_C = 1_D$. The set

$$\mathcal{R}[D, C] := \{(d_1, \ldots, d_n, c, c, \ldots) : c \in C, n \geq 1, d_i \in D \text{ for } i = 1, \ldots, n\}$$

is a ring where addition and multiplication are defined component-wise.

Proposition 2.7. *If the ring $\mathcal{R}[D, C]$ is essentially right quasi-duo (resp. essentially right duo), then C and D are essentially right quasi-duo (resp. essentially right duo). The converse is false.*

Proof. Since D and C are homomorphic images of $\mathcal{R}[D, C]$, the necessity follows from Proposition 2.3(1).

If $D = C = \mathbb{M}_2(F)$ where F is a field, then D and C are essentially right duo. But, since $\mathcal{R}[F, F]$ is not semisimple Artinian, $\mathcal{R}[D, C] \cong \mathbb{M}_2(\mathcal{R}[F, F])$ is not essentially right quasi-duo. □

If M is a left module over a ring R, let $\mathbf{l}_R(M) = \{a \in R : aM = 0\}$.

Proposition 2.8. *Let R be a ring and M be an (R, R)-bimodule. Then the trivial extension $R \propto M$ is essentially right quasi-duo if and only if every maximal right ideal I of R such that $I \cap aR \neq 0$ for any $0 \neq a \in \mathbf{l}_R(M)$ is an ideal.*

Proof. ($\Rightarrow$). If I is a maximal right ideal of R such that $I \cap aR \neq 0$ for any $0 \neq a \in \mathbf{l}_R(M)$, then $I \propto M$ is an essential maximal right ideal of $R \propto M$, so it is an ideal. Hence I is an ideal of R.

($\Leftarrow$). Let T be an essential maximal right ideal of $R \propto M$. Then $T = I \propto M$ with I a maximal right ideal of R. Moreover, T essential in $R \propto M$ implies that $I \cap aR \neq 0$ for any $0 \neq a \in \mathbf{l}_R(M)$. So I is an ideal of R by our assumption, and hence T is an ideal of $R \propto M$. □

It is well-known that a ring R is right quasi-duo if and only if $R/J(R)$ is right quasi-duo. If R is essentially right quasi-duo, then so is $R/J(R)$ by Proposition 2.3(1); but the converse is not true.

Example 2.9. *Let R be a semilocal ring with $J(R) \neq 0$. Then, for $T = \mathbb{M}_n(R)$ where $n \geq 2$, $T/J(T) \cong \mathbb{M}_n(R/J(R))$ is essentially right duo, but T is not essentially right quasi-duo.*

In [25], a submodule N of a module M is called δ-small in M if $M \neq X + N$ for any proper submodule X with M/X singular. The sum of all δ-small submodules of a module M is denoted by $\delta(M)$. As shown in [25], $\delta(R_R)$ is an ideal of R which is the intersection of all essential maximal right ideals of R, so $\mathrm{soc}(R_R) \subseteq \delta(R_R)$ and $J(R) \subseteq \delta(R_R)$; indeed, $J(R/\mathrm{soc}(R_R)) = \delta(R_R)/\mathrm{soc}(R_R)$.

Below is an example of a ring R such that $R/\mathrm{soc}(R_R) = R/J(R) = R/\delta(R_R)$ is essentially right quasi-duo but R is not essentially right quasi-duo.

Example 2.10. *Let $R = \mathbb{M}_2(F) \propto \mathbb{M}_2(F)$ be the trivial extension where F is a field. Then $\mathrm{soc}(R_R) = 0 \propto \mathbb{M}_2(F) = J(R)$, which is an essential right ideal of R. Since $R/\mathrm{soc}(R_R) \cong \mathbb{M}_2(F)$, $J(R/\mathrm{soc}(R_R)) = 0$, so $\delta(R_R) = \mathrm{soc}(R_R)$. Hence $R/\delta(R_R) = R/\mathrm{soc}(R_R) = R/J(R)$ is an essentially right quasi-duo ring. Let $I = \{(a, x) \in R : a \in \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}, x \in \mathbb{M}_2(F)\}$. Then I is an essential maximal right ideal of R, but not an ideal. So R is not essentially right quasi-duo.*

Essentially right quasi-duo rings are characterized below.

Theorem 2.11. *The following are equivalent for a ring R:*

(1) *R is essentially right quasi-duo.*
(2) *R is semisimple Artinian or $R/\delta(R_R)$ is right quasi-duo.*
(3) *R is semisimple Artinian or $R/\mathrm{soc}(R_R)$ is right quasi-duo.*

Proof. (1) $\Rightarrow$ (3). Assume that R is not semisimple Artinian. Then $\mathrm{soc}(R_R) \neq R$. Let $K/\mathrm{soc}(R_R)$ be a maximal right ideal of $R/\mathrm{soc}(R_R)$. Then K is an essential maximal right ideal of R. So K is an ideal of R, and hence $K/\mathrm{soc}(R_R)$ is an ideal of $R/\mathrm{soc}(R_R)$.

(3) ⇒ (2). By [25], $\delta(R_R)/\mathrm{soc}(R_R) = J(R/\mathrm{soc}(R_R))$, so the implication follows from this and the fact that factor rings of a right quasi-duo are right quasi-duo.

(2) ⇒ (1). We can assume that R is not semisimple Artinian. So $R/\delta(R_R)$ is right quasi-duo. If K is an essential maximal right ideal of R, then $\delta(R_R) \subseteq K$, so $K/\delta(R_R)$ is a maximal right ideal of $R/\delta(R_R)$. Hence $K/\delta(R_R)$ is an ideal of $R/\delta(R_R)$, and it follows that K is an ideal of R. □

Note that Theorem 2.11 provides an alternative proof of Proposition 2.6(2), because if $e \in R$ is an idempotent of an essentially right quasi-duo ring R, then in case R is semisimple, also eRe is semisimple. In case $R/\mathrm{soc}(R_R)$ is right quasi-duo, then by Proposition 2.6(1) also $\overline{e}R/\mathrm{soc}(R_R)\overline{e}$ is right quasi-duo, where $\overline{e} = e + \mathrm{soc}(R_R)$. Since $\overline{e}\,(R/\mathrm{soc}(R_R))\,\overline{e} \simeq eRe/\mathrm{soc}(eRe_{eRe})$, we obtain that eRe is essentially right quasi-duo.

It follows from Theorem 2.11 that, for a ring R with $\mathrm{soc}(R_R) = 0$, R is essentially right quasi-duo if and only if it is right quasi-duo. A ring R is reduced if it contains no nonzero nilpotent elements.

Corollary 2.12. *If R is an essentially right quasi-duo ring that is not semisimple Artinian, then $R/\delta(R_R)$ is a reduced ring.*

Proof. By Theorem 2.11, $R/\delta(R_R)$ a right quasi-duo ring. Since $J(R/\delta(R_R)) = 0$, it follows from [24, Corollary 2.4] that $R/\delta(R_R)$ is reduced. □

Corollary 2.13. *A semilocal ring R is essentially right quasi-duo if and only if R is semisimple Artinian or $R/\delta(R_R)$ is a finite direct product of division rings.*

Proof. The sufficiency follows from Theorem 2.11. For the necessity, assume that R is not semisimple Artinian. Then $J(R) \subseteq \delta(R_R) \subsetneq R$ by [25]. So $R/\delta(R_R)$ is semisimple Artinian, and right quasi-duo (by Theorem 2.11). By Corollary 2.12, $R/\delta(R_R)$ is reduced, so it must be a finite direct product of division rings. □

A (von Neumann) regular ring is right quasi-duo if and only if it is strongly regular (see [24]).

Corollary 2.14. *A regular ring R is essentially right quasi-duo if and only if either R is semisimple Artinian, or $R/\mathrm{soc}(R_R)$ is strongly regular.*

Proof. The sufficiency follows from Theorem 2.11. For the necessity, assume that R is not semisimple Artinian. Then $R/\mathrm{soc}(R_R)$ is right quasi-duo by Theorem 2.11. Since R is regular, $R/\mathrm{soc}(R_R)$ is regular. So $R/\mathrm{soc}(R_R)$ is strongly regular by [24, Theorem 2.7]. □

The quasi-duo property of skew polynomial rings is extensively discussed in [15].

Corollary 2.15. *Let $\sigma : R \to R$ be an endomorphism of ring R. Then $R[t;\sigma]$ is essentially right quasi-duo if and only if $R[t;\sigma]$ is right quasi-duo.*

Proof. We only need to verify the necessity. For any nonzero right ideal T of $R[t;\sigma]$, Tt is a nonzero right ideal properly contained in T, so the right socle of $R[t;\sigma]$ is zero. Hence the necessity follows from Theorem 2.11. □

The ring in Example 2.5(1) is right self-injective regular. We next characterize right self-injective regular rings that are essentially right quasi-duo. Some related notions need be recalled. A ring R is directly finite if $ab = 1$ in R always implies $ba = 1$. An idempotent e in a regular ring R is called an Abelian idempotent if eRe is an Abelian ring (i.e. all idempotents of eRe are central), and is called a directly finite idempotent if eRe is a directly finite ring. An idempotent e in a right self-injective regular ring is called a faithful idempotent if 0 is the only central idempotent orthogonal to e.

Theorem 2.16. *Let R be a right self-injective regular ring. Then R is essentially right quasi-duo* (*resp. essentially right duo*) *if and only if R is a direct product of a semisimple Artinian ring and a strongly regular ring.*

Proof. Since semisimple Artinian rings and strongly regular rings are all essentially right duo, the sufficiency follows by Proposition 2.3(2).

To show the necessity, let R be essentially right quasi-duo. By [5, Theorem 10.22], there is a decomposition of rings

$$R = A \times B \times C,$$

where A is of type I_f (i.e. A is a directly finite ring containing a faithful Abelian idempotent), B is of type II_f (i.e. B is a directly finite ring that contains a faithful directly finite idempotent but contains no nonzero Abelian idempotents), and C is purely infinite (i.e. C contains no nonzero directly finite central idempotents). By [5, Theorem 10.24], $A \cong \prod_{i=1}^{\infty} \mathbb{M}_i(S_i)$ where each S_i is a strongly regular ring. Let $S = \prod_{i=2}^{\infty} \mathbb{M}_i(S_i)$. Then S has a corner ring isomorphic to $\prod_{i=2}^{\infty} \mathbb{M}_2(S_i) \cong \mathbb{M}_2(\prod_{i=2}^{\infty} S_i)$. By Proposition 2.6, S is essentially right quasi-duo, so $\prod_{i=2}^{\infty} S_i$ is semisimple Artinian by Proposition 2.3(3). Hence there exists $n_0 > 2$ such that $S_i = 0$ for all $i > n_0$. It follows that $S = \prod_{i=2}^{n_0} \mathbb{M}_i(S_i)$ is semisimple Artinian. So $A \cong S_1 \times S$ is a direct product of a strongly regular ring and a semisimple Artinian ring.

Since B is of type II_f, by [5, Proposition 10.28] there exists $e^2 = e \in B$ such that $B_B \cong (eB \oplus eB)_B$, so $B \cong \mathrm{End}((eB)^2_B) \cong \mathbb{M}_2(eBe)$, which is essentially right quasi-duo. So eBe is semisimple Artinian, and hence B is semisimple Artinian. Since B is of type II_f, It must be that $B = 0$.

Since C is purely infinite, by [5, Theorem 10.16] $C_C \cong (C \oplus C)_C$, so $C \cong \mathbb{M}_2(C)$, which is essentially right quasi-duo. So C is a semisimple Artinian ring. It must be that $C = 0$.

Therefore, $R \cong A \cong S_1 \times S$ is a direct product of a strongly regular ring and a semisimple Artinian ring. □

A ring R is called right CS if every submodule of R_R is essential in a direct summand of R_R, and a right CS ring R is called right continuous if the module R_R satisfies $C2$, i.e. every submodule of R_R isomorphic to a direct summand is itself a direct summand.

Corollary 2.17. *A regular right CS ring R is essentially right quasi-duo (resp. essentially right duo) if and only if it is the direct product of a semisimple Artinian ring and a strongly regular ring.*

Proof. The sufficiency is clear. For the necessity, since R is regular, the module R_R satisfies $C2$, so R is right continuous (and regular). By [19, Theorem 3.1, Corollary 3.13], $R \cong A \times B$ where A is right self-injective regular and B is strongly regular. Since

A is essentially right quasi-duo, by Theorem 2.16 A is a direct product of a semisimple Artinian ring and a strongly regular ring. Hence R is a direct product of a semisimple Artinian ring and a strongly regular ring. □

3. Formal triangular matrix rings and applications

While Corollary 2.13 gives a characterization of essentially right quasi-duo semilocal rings, here we will attain the structure of essentially right quasi-duo semilocal rings. To do so, we need to characterize when a formal triangular matrix ring is essentially right quasi-duo.

Theorem 3.1. *Let A, B be rings, M be an (A, B)-bimodule. The following are equivalent:*

(1) *The formal triangular matrix ring $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ is essentially right quasi-duo.*
(2) *Every maximal right ideal I of A such that $I \cap aA \neq 0$ for any $0 \neq a \in \mathbf{l}_A(M)$ is an ideal, and B is essentially right quasi-duo.*

Proof. (1) ⇒ (2). Let $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$. Being an image of R, B is essentially right quasi-duo. If I is a maximal right ideal of A such that $I \cap aA \neq 0$ for any $0 \neq a \in \mathbf{l}_A(M)$, then $\begin{pmatrix} I & M \\ 0 & B \end{pmatrix}$ is an essential maximal right ideal of R, so it is an ideal. Hence I is an ideal of A.

(2) ⇒ (1). Let T be an essential maximal right ideal of R. Then, by [9, Proposition 2.1], $T = \begin{pmatrix} I & M \\ 0 & B \end{pmatrix}$ with I a maximal right ideal of A or $T = \begin{pmatrix} A & M \\ 0 & K \end{pmatrix}$ with K a maximal right ideal of B. If $T = \begin{pmatrix} A & M \\ 0 & K \end{pmatrix}$, then T essential in R_R implies that K is essential in B_B. So K is an ideal of B, and hence T is an ideal of R. If $T = \begin{pmatrix} I & M \\ 0 & B \end{pmatrix}$, then T essential in R_R implies that $I \cap aA \neq 0$ for any $0 \neq a \in \mathbf{l}_A(M)$. So I is an ideal of A by (2), and hence T is an ideal of R. □

Corollary 3.2. *Let R be a ring and $n \geq 2$. Then $\mathbb{UT}_n(R)$ is essentially right quasi-duo if and only if R is right quasi-duo*

Proof. Write $\mathbb{UT}_n(R) = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, where $A = R, B = \mathbb{UT}_{n-1}(R)$ and $M = R^{n-1}$. Then $\mathbf{l}_R(M) = 0$. So the necessity follows from Theorem 3.1. For the sufficiency, B is essentially right quasi-duo by induction. Hence $\mathbb{UT}_n(R)$ is essentially right quasi-duo by Theorem 3.1. □

As mentioned earlier, an essentially right duo (resp. right duo) ring need not be essentially left duo (resp. left duo). It is still an open question whether a right quasi-duo ring is left quasi-duo. Here we show that an essentially right quasi-duo ring need not be essentially left quasi-duo.

Example 3.3. *Let* $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ *be the subring* $\begin{pmatrix} F & F & F \\ 0 & F & F \\ 0 & F & F \end{pmatrix}$ *of* $\mathbb{M}_3(F)$*, where* F *is a field with* $A = F$ *and* $B = \mathbb{M}_2(F)$*. Since* A *is right quasi-duo and* B *is essentially right quasi-duo,* R *is essentially right quasi-duo by Theorem 3.1. Let* $I = \begin{pmatrix} F & F & F \\ 0 & 0 & F \\ 0 & 0 & F \end{pmatrix}$*. One can check that* I *is an essential maximal left ideal of* R *that is not an ideal. So* R *is not essentially left quasi-duo. Indeed,* $\operatorname{soc}(R_R) = \begin{pmatrix} 0 & F & F \\ 0 & F & F \\ 0 & F & F \end{pmatrix}$ *and* $\operatorname{soc}({}_RR) = \begin{pmatrix} F & F & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$*. So* $R/\operatorname{soc}(R_R) \cong F$ *is right quasi-duo, but* $R/\operatorname{soc}({}_RR) \cong \mathbb{M}_2(F)$ *is not left quasi-duo.*

Note that the ring R in Example 3.3 is Artinian and can even be finite (when F is a finite field). We know that $\operatorname{soc}(R_R) = \operatorname{soc}({}_RR)$ for any regular ring (indeed for any semiprime ring) R. Thus, by Corollary 2.14, a regular ring is essentially right quasi-duo if and only if it is essentially left quasi-duo. This result has an extension. Recall that a ring R is an exchange ring if for any $a \in R$, there exists $e^2 = e$ such that $e \in aR$ and $1 - e \in (1 - a)R$.

Proposition 3.4. *Let* R *be a ring with* $\delta(R_R) = \delta({}_RR)$ *such that* $R/\delta(R_R)$ *is an exchange ring. Then* R *is essentially right quasi-duo if and only if it is essentially left quasi-duo.*

Proof. Let R be essentially right quasi-duo. It suffices to show that R is essentially left quasi-duo. We can assume that R is not semisimple Artinian. So, by Theorem 2.11,

$R/\mathrm{soc}(R_R)$ is right quasi-duo. Moreover, $\frac{R/\mathrm{soc}(R_R)}{J(R/\mathrm{soc}(R_R))} \cong R/\delta(R_R)$ is an exchange ring. By [13, Theorem 4.6], we infer that $R/\mathrm{soc}(R_R)$ is left quasi-duo. So, being an image of $R/\mathrm{soc}(R_R)$, $R/\delta({}_RR) = R/\delta(R_R)$ is left quasi-duo. Hence R is essentially left quasi-duo by Theorem 2.11. □

We remark that, for a ring R with $S := \mathrm{soc}(R_R) = \mathrm{soc}({}_RR)$, $\delta(R_R)/\mathrm{soc}(R_R) = J(R/S) = \delta({}_RR)/\mathrm{soc}({}_RR)$, so $\delta(R_R) = \delta({}_RR)$.

Corollary 3.5. *Let R be a ring with $\mathrm{soc}(R_R) = \mathrm{soc}({}_RR)$ such that $R/J(R)$ is an exchange ring. Then R is essentially right quasi-duo if and only if it is essentially left quasi-duo.*

For an essentially left and right quasi-duo ring R, $\delta(R_R)$ need not coincide with $\delta({}_RR)$: Let $R = \mathbb{UT}_2(F)$ where F is a field. Then R is essentially left and right quasi-duo ring, but $\delta(R_R) = \mathrm{soc}(R_R) = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$ and $\delta({}_RR) = \mathrm{soc}({}_RR) = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$.

Next we prove a structure theorem of essentially right quasi-duo semilocal rings. The proof of the following lemma is routine.

Lemma 3.6. *Let M be a semisimple module. If every maximal submodule of M is fully invariant, then M is a direct sum of non-isomorphic simple modules.*

Because homomorphisms of right modules are written on the left of their arguments, for two right modules M and N, $\mathrm{Hom}(M,N)$ is an $\big(\mathrm{End}(N), \mathrm{End}(M)\big)$-bimodule, $\mathrm{Hom}(N,M)$ is an $\big(\mathrm{End}(M), \mathrm{End}(N)\big)$-bimodule, and

$$\mathrm{End}(M \oplus N) \cong \begin{pmatrix} \mathrm{End}(M) & \mathrm{Hom}(N,M) \\ \mathrm{Hom}(M,N) & \mathrm{End}(N) \end{pmatrix}.$$

Theorem 3.7. *Let R be a semilocal ring. Then R is essentially right quasi-duo if and only if $R \cong \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, where $A/J(A)$ is a finite direct product of division rings, B is a semisimple Artinian ring, and M is an (A,B)-bimodule.*

Proof. Let R be a semilocal ring with Jacobson radical J. By [17, Theorem 3.5(c)], $R_R = P \oplus Q$, where Q is semisimple projective and P contains J as an essential submodule.

In particular, the radical of P is $\mathrm{Rad}(P) = J$ and $\mathrm{Hom}(P, Q) = \mathrm{Hom}(P/J, Q) = 0$, as Q is semisimple projective and P/J is semisimple singular. Hence we have

$$R \cong \mathrm{End}(R_R) \cong \begin{pmatrix} \mathrm{End}(P) & \mathrm{Hom}(Q, P) \\ 0 & \mathrm{End}(Q) \end{pmatrix}.$$

Then $B := \mathrm{End}(Q)$ is semisimple Artinian and $A := \mathrm{End}(P)$ is semilocal.

Assume R is essentially right quasi-duo, then A is essentially right quasi-duo by Proposition 2.6. Let M/J be a maximal submodule M/J of P/J and let $f \in \mathrm{End}(P/J)$. Because M contains J and J is essential in P, $M \oplus Q$ is an essential maximal right ideal of R. By hypothesis $M \oplus Q$ is fully invariant in R, which shows that M is fully invariant in P. Let $\pi : P \to P/J$ be the canonical epimorphism. Since P is projective, there exists $g \in \mathrm{End}(P)$ such that $\pi h = f\pi$. Hence $h(M) \subseteq M$, and so $f(M/J) = f\pi(M) = \pi h(M) \subseteq \pi(M) = M/J$. So M/J is fully invariant in P/J. By Lemma 3.6, P/J is a direct sum of non-isomorphic (singular) simple modules and therefore, by [23, 22.2], $A/J(A) \cong \mathrm{End}(P/J)$ is a (finite) direct product of division rings.

($\Leftarrow$). Let $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, where $A/J(A)$ is a finite direct product of division rings, B is a semisimple Artinian ring and M is an (A, B)-bimodule. Then A is right quasi-duo, and B is essentially right quasi-duo. So R is essentially right quasi-duo by Theorem 3.1. □

The opposite of a ring R is denoted by R^o. For a bimoddule ${}_AM_B$, let $M^o = \{x^o : x \in M\}$. Then M^o is a (B^o, A^o)-bimodule where, for $a^o \in A^o, b^o \in B^o$ and $x^0, y^o \in M^o$, $x^o + y^o = (x + y)^o$, $x^o a^o = (ax)^o$ and $b^o x^o = (xb)^o$. Moreover, if $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, then $R^o \cong \begin{pmatrix} A^o & 0 \\ M^o & B^o \end{pmatrix}$ via $\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}^o \mapsto \begin{pmatrix} a^o & 0 \\ x^o & b^o \end{pmatrix}$.

A ring R is called left pseudo-Frobenius if R is left self-injective, semilocal and the socle $\mathrm{soc}({}_RR)$ is an essential submodule of ${}_RR$.

Corollary 3.8. *A left pseudo-Frobenius ring R is essentially right quasi-duo if and only if $R \cong A \times B$ where B is semisimple Artinian and $A/J(A)$ is a finite direct product of division rings.*

Proof. The sufficiency is by Proposition 2.3(2). For the necessity, by Theorem 3.7 we have $R \cong \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, where $A/J(A)$ is a finite direct product of division rings, B is semisimple Artinian and M is an (A, B)-bimodule. Then $R^o \cong \begin{pmatrix} A^o & 0 \\ M^o & B^o \end{pmatrix}$ is right pseudo-Frobenius, and so $M^o = 0$ by [9, Theorem 5.6]. So $M = 0$ and hence $R \cong A \times B$. □

4. A question on rings with cyclic injective hull

It is an open question whether the injective hull $E(R_R)$ of R_R cyclic implies that R is right self-injective (see [11]). Some partial answers are presented in [11], one of which states that a right quasi-duo ring R with $E(R_R)$ cyclic is right self-injective. Here we extend this result to an essentially right quasi-duo ring.

A submodule N of a module M is said to be fully-invariant if $f(N) \subseteq N$ for all $f \in \text{End}(M)$. Following [21], a module P is called pseudo-projective if for any module X and any epimorphisms $f, g \in \text{Hom}(P, X)$, there exists $h \in \text{End}(P)$ such that $f = gh$. Note that a module P is projective if, for any modules M, X and any epimorphism $f \in \text{Hom}(P, X)$ and any homomorphism $g \in \text{Hom}(M, X)$, there exists $h \in \text{Hom}(P, M)$ such that $f = gh$ (see [22]). Quasi-projective modules are pseudo-projective, but the converse is not true (see [4]). The next result is motivated by the work in [10], and condition (1) can be compared with the notion of dual-square-free modules: A module M is called dual-square-free if M has no proper submodules A and B with $M = A + B$ and $M/A \cong M/B$ (see [8]).

Theorem 4.1. *The following are equivalent for a finitely generated pseudo-projective module M:*

1. *M does not contain proper submodules X and Y such that $M = X + Y$, and $M/X \cong M/Y$ is singular.*
2. *Every maximal submodule X of M with M/X singular is fully-invariant.*

Proof. (1) ⇒ (2). Assume that a maximal submodule X of M with M/X singular is not fully-invariant. Then $f(X) \not\subseteq X$ for some $f \in \text{End}(M)$. Let $\pi : M \to M/X$ be the canonical map. Then $\pi f : M \to M/X$ is an epimorphism. So $\frac{M}{X} \cong \frac{M}{\ker(\pi f)} = \frac{M}{f^{-1}(X)}$, which is singular and simple. So $f^{-1}(X)$ is also a maximal submodule of M. Since $f(X) \nsubseteq X$, $X \nsubseteq f^{-1}(X)$. It follows that $M = X + f^{-1}(X)$, so (1) fails.

(2) ⇒ (1). Assume that (1) fails. Then $M = A+B$ where A, B are proper submodules with $M/A \overset{\sigma}{\cong} M/B$ singular. Let Y be a maximal submodule of M with $B \subseteq Y$ (since M is finitely generated, such a Y exists), and let $\phi : M/B \to M/Y$ be the canonical map. Then $\phi\sigma : M/A \to M/Y$ is an epimorphism. Write $\ker(\phi\sigma) = X/A$. Then $M = X + Y$, and $M/Y \cong M/X$ which is singular and simple. Hence we can assume that both A and B are maximal submodules. Let $\pi_A : M \to M/A$ and $\pi_B : M \to M/B$ be the canonical maps. The pseudo-projectivity of M ensures that there exists a homomorphism $\lambda : M \to M$ such that $\sigma\pi_A\lambda = \pi_B$. Thus, $\frac{M}{B} = \frac{A+B}{B} = \pi_B(A) = \sigma\pi_A\lambda(A) = \sigma(\frac{\lambda(A)+A}{A})$. Since σ is an isomorphism, it follows that $\frac{M}{A} = \frac{\lambda(A)+A}{A}$, i.e. $M = \lambda(A)+A$. So $\lambda(A) \nsubseteq A$. Hence A is not fully-invariant, a contradiction to (2). □

The implication "(1) ⇒ (2)" in Theorem 4.1 does not need any hypothesis. Note that, for a maximal submodule X of a projective module P, P/X is singular if and only if X is essential in M. Theorem 4.1 has a quick consequence.

Corollary 4.2. *Let M be a finitely generated projective module. The following are equivalent:*

(1) *M does not contain proper submodules X and Y such that $M = X + Y$, and $M/X \cong M/Y$ is singular.*
(2) *Every essential maximal submodule of M is fully-invariant.*

The next theorem extends a result in [11] that a right quasi-duo ring R with $E(R_R)$ cyclic is right self-injective.

Theorem 4.3. *Let $E(R_R)$ be cyclic. If R is essentially right quasi-duo, then R_R is injective.*

Proof. Assume on the contrary that R_R is not injective. Write $E := E(R_R) = xR$, and let $i : R \to E$ be the inclusion, $\beta : R \to E$, $r \mapsto xr$, be the natural epimorphism.

Since E is injective, there exists a homomorphism $f : E \to E$ such that $\beta = fi$. Thus, $f(E) = \beta(R) = (fi)(R) = f(R)$, so $E = R + \ker(f)$. Let $\pi : E \to E/R$ be the natural homomorphism. Since $\ker(f) \subseteq \ker(\pi f)$, we have $E = R + \ker(\pi f)$, with $E/R \cong E/\ker(\pi f)$ being singular. Then $R = \beta^{-1}(R) + \beta^{-1}(\ker(\pi f))$ is a sum of two proper submodules. Moreover, $R/\beta^{-1}(R) \cong E/R$ and $R/\beta^{-1}(\ker(\pi f)) \cong E/\ker(\pi f)$ are all singular. So R is not essentially right quasi-duo by Corollary 4.2. □

There exists a regular ring R such that $R/\mathrm{soc}(R_R)$ is a field, but R is not clean (see [14, Example 1]). This shows that an exchange essentially right quasi-duo ring need not be clean, in contrast to a known result that an exchange right quasi-duo ring is always clean (see [10]).

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