On the number of zeros of $L-$functions attached to cusp forms of half-integral weight

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Abstract

Meher et al. [Proc. Amer. Math. Soc. 147 (2019)] have recently established that $L-$functions attached to certain cusp forms of half-integral weight have infinitely many zeros on the critical line. Kim [J. Numb. Th. 253 (2023)] obtained analogous results for $L-$functions attached to cusp forms twisted by an additive character $e\left(\frac{p}{q}n\right)$, $\frac{p}{q}\in\mathbb{Q}$. We extend the results of these authors by giving a lower bound for the number of such zeros.

We start by developing a variant of a method of de la Valée Poussin which seems to have interest as it avoids the evaluation of exponential sums. We finish the paper with an improvement of our first estimate by using Lekkerkerker's variant of the Hardy-Littlewood method.

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* *Keywords* : Zeros of Dirichlet series, Hardy's theorem, Half-integral weight cusp forms

2020 Mathematics Subject Classification : Primary: 11F37, 11F66, 11M41.

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1 Introduction and Main Results

For $k, N \in \mathbb{N}$, let us denote by $S_{k+\frac{1}{2}}(\Gamma_0(4N))$ the space of cusp forms of weight $k+\frac{1}{2}$ on the congruence subgroup $\Gamma_0(4N)$. This means that any element $f(z) \in S_{k+\frac{1}{2}}(\Gamma_0(4N))$ satisfies[1]

$$f\left(\frac{az+b}{cz+d}\right) = \left(\frac{c}{d}\right)^{2k+1} \epsilon_d^{-2k-1} (cz+d)^{k+\frac{1}{2}} f(z), \tag{1.1}$$

whenever $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N)$. Here, $\left(\frac{c}{d}\right)$ is Shimura's extension of the Jacobi symbol [[25], p.442] and ϵ_d is defined by

$$\epsilon_d = \begin{cases} 1 & d \equiv 1 \mod 4 \\ i & d \equiv 3 \mod 4 \end{cases}. \tag{1.2}$$

The theory of modular forms of half-integral weight was extensively developed by Shimura and we refer to the seminal paper [25] for some classical facts about these modular forms. Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(4N)$, $f(z)$ has a Fourier expansion of the form

$$f(z) = \sum_{n=1}^{\infty} a_f(n)\, e^{2\pi i n z}, \tag{1.3}$$

where $z \in \mathbb{H} = \{w \in \mathbb{C} : \text{Im}(w) > 0\}$. We can attach a Dirichlet series $L(s, f)$ to $f(z)$,

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s}, \tag{1.4}$$

and show that, for sufficiently large $\text{Re}(s)$, this series converges absolutely. The usual argument invoked to study the analytic continuation of $L-$functions of cusp forms with integral weight can also be applied to the half-integral case. Thus, $L(s, f)$ can be analytically continued to the whole complex plane $\mathbb{C}$ as an entire function of s. Moreover, it will satisfy Hecke's functional equation.

In order to state its functional equation, let W_{4N} denote the Fricke involution acting on $S_{k+\frac{1}{2}}(\Gamma_0(4N))$,

$$(f|W_{4N})(z) = i^{k+\frac{1}{2}} (4N)^{-\frac{k}{2}-\frac{1}{4}} z^{-k-\frac{1}{2}} f\left(-\frac{1}{4Nz}\right). \tag{1.5}$$

Then the function $L(s, f)$ satisfies the functional equation

$$\left(\frac{2\pi}{\sqrt{4N}}\right)^{-s} \Gamma(s)\, L(s, f) = \left(\frac{2\pi}{\sqrt{4N}}\right)^{-\left(k+\frac{1}{2}-s\right)} \Gamma\left(k+\frac{1}{2}-s\right) L\left(k+\frac{1}{2}-s, f|W_{4N}\right). \tag{1.6}$$

[1]Throughout this paper we define $\sqrt{z} = z^{1/2}$ so that $-\frac{\pi}{2} < \arg(z^{1/2}) \leq \frac{\pi}{2}$.

Adopting the notation given in [21], let us note that (1.6) can be written in the symmetric form

$$\Lambda\left(s,f\right)=\Lambda\left(k+\frac{1}{2}-s,f|W_{4N}\right), \tag{1.7}$$

where $\Lambda(s,f)$ represents the completed $L-$function

$$\Lambda\left(s,f\right):=\left(\frac{2\pi}{\sqrt{4N}}\right)^{-s}\Gamma\left(s\right)\,L\left(s,f\right). \tag{1.8}$$

Despite the formal similarities with the $L-$functions attached to a Hecke eigenform with integral weight, $L-$functions associated with cusp forms belonging to $S_{k+\frac{1}{2}}\left(\Gamma_0(4N)\right)$ do not possess an Euler product. This imposes some restrictions and adds complications in studying the location of zeros of such $L-$functions. We refer to the papers [21] and [22] for nice overviews of the non-vanishing results for $L(s,f)$ available in the literature.

Our focus in this paper will be the study of the zeros of $L(s,f)$, $f\in S_{k+\frac{1}{2}}(\Gamma_0(4N))$, on the critical line. According to [[21], p.132], Yoshida [30] was the first mathematician to study the zeros of such $L-$functions. To illustrate one of Yoshida's examples, let

$$\theta(z)=\sum_{n\in\mathbb{Z}}e^{2\pi in^2z},\quad \eta(z)=e^{\frac{\pi iz}{12}}\prod_{n=1}^{\infty}\left(1-e^{2\pi inz}\right),\quad z\in\mathbb{H}.$$

Then one can construct [[25], p.477] a cusp form $g(z)\in S_{\frac{9}{2}}\left(\Gamma_0(4)\right)$ by taking

$$g(z):=\theta(z)^{-3}\eta(2z)^{12}. \tag{1.9}$$

Yoshida considered the $L-$function attached to g, $L(s,g)$, and from the calculation of some of its zeros [[30], p.675], he showed that the analogue of the Riemann hypothesis for $L(s,g)$ is false.

Nevertheless, since there are Dirichlet series (such as the Epstein zeta function attached to a binary quadratic form) which do not satisfy the Riemann hypothesis but still possess a positive proportion of zeros on the critical line, one may study, in spite of Yoshida's counterexample, analogues of Hardy's Theorem for this class of $L-$functions.

As far as we know, the work of J. Meher, S. Pujahari and K. Srinivas [22] contains the first result of Hardy-type for $L-$functions associated with half-integral weight cusp forms. They established the following theorem.

Theorem A [22] *Suppose that $f(z)=\sum_{n=1}^{\infty}a_f(n)\,e^{2\pi inz}\in S_{k+\frac{1}{2}}\left(\Gamma_0(4)\right)$ is an eigenform for all Hecke operators T_{n^2} and for the operator W_4 (1.5). Assume also that all the Fourier coefficients of $f(z)$, $a_f(n)$, are either real or purely imaginary numbers. Then the $L-$function (1.4) attached to f has infinitely many zeros on the critical line $Re(s)=\frac{k}{2}+\frac{1}{4}$.*

We know that $\dim S_{\frac{9}{2}}\left(\Gamma_0(4)\right)=1$ [[25], p.477], so the cusp form $g(z)$ given by (1.9) is an eigenform for all the Hecke operators and satisfies $\left(g|W_4\right)(z)=g(z)$. Thus, despite violating the Riemann hypothesis, $L(s,g)$ has infinitely many zeros at the critical line $\mathrm{Re}(s)=\frac{9}{4}$.

Recently, Meher, Pujahari and Shankhadhar [21] improved Theorem A by extending it to forms of higher level.

Theorem B [21] *Let N be a perfect square and $f(z)=\sum_{n=1}^{\infty}a_f(n)\,e^{2\pi inz}\in S_{k+\frac{1}{2}}\left(\Gamma_0(4N)\right)$. Assume also that all the Fourier coefficients of $f(z)$, $a_f(n)$, are either real or purely imaginary numbers. Then the function*

$$L\left(s,f\right)\pm L\left(s,f|W_{4N}\right)$$

has infinitely many zeros of odd order on the critical line $Re(s)=\frac{k}{2}+\frac{1}{4}$.

The proofs of Theorems A and B are different in nature. The proof of Theorem A follows a classical argument first given by Landau [19] and its main idea is to contrast the behavior of the integrals $\left|\int_T^{2T}\Lambda\left(\frac{k}{2}+\frac{1}{4}+it,f\right)\,dt\right|$ and $\int_T^{2T}\left|\Lambda\left(\frac{k}{2}+\frac{1}{4}+it,f\right)\right|\,dt$ as $T\rightarrow\infty$. Due to the oscillations coming from the $\Gamma-$factor, the first integral $\int_T^{2T}\Lambda\left(\frac{k}{2}+\frac{1}{4}+it,f\right)\,dt$ has substantial cancellation and an upper bound for it can be obtained by estimating an exponential sum [[22], p.930, Lemma 3.2]. On the other hand, the proof of Theorem B uses Wilton's variant of Hardy's theorem [28] and it is much closer in spirit to Hardy's original proof [12].

Kim studied the zeros of this class $L-$functions in Wilton's setting. Using a very elegant argument involving distributions, he directed his study towards the zeros of the additively twisted $L-$function,

$$L_{p/q}(s,f)=\sum_{n=1}^{\infty}\frac{a_f(n)\,e^{\frac{2\pi ip}{q}n}}{n^s},\quad\frac{p}{q}\in\mathbb{Q},\quad f\in S_{k+\frac{1}{2}}\left(\Gamma_0(4N)\right). \tag{1.10}$$

Just like the Dirichlet series (1.4), $L_{p/q}(s,f)$ converges absolutely when $\text{Re}(s)$ is sufficiently large. Moreover, in analogy to the integral case [[29], Theorem 2.1 (a)], $L_{p/q}(s,f)$ satisfies Hecke's functional equation when $\frac{p}{q}$, $q>0$, is a rational number which is $\Gamma_0(4N)-$equivalent to $i\infty$. This functional equation can be explicitly written in the form [[18], p.174, Lemma 4.19]

$$\left(\frac{2\pi}{q}\right)^{-s}\Gamma(s)\,L_{p/q}(s,f)=i^{k+\frac{1}{2}}\left(\frac{-q}{p}\right)^{-2k-1}\epsilon_p^{2k+1}\left(\frac{2\pi}{q}\right)^{-\left(k+\frac{1}{2}-s\right)}\Gamma\left(k+\frac{1}{2}-s\right)\,L_{-\overline{p}/q}\left(k+\frac{1}{2}-s,f\right), \tag{1.11}$$

where $\overline{p}$ is such that $p\,\overline{p}\equiv 1\mod q$. Also, $\left(\frac{-q}{p}\right)$ and ϵ_p are given in (1.1). Using (1.11) and appealing to the notion of a distribution vanishing to infinite order, Kim established the following theorem [[18], p.160, Theorem 1.17].

Theorem C [18] *Let N be a positive integer and $\frac{p}{q}$, $q>0$, a rational number which is $\Gamma_0(4N)-$equivalent to $i\infty$ and such that $p^2\equiv 1\mod q$. Moreover, let $f(z)=\sum_{n=1}^{\infty}a_f(n)\,e^{2\pi inz}\in S_{k+\frac{1}{2}}\left(\Gamma_0(4N)\right)$, with $a_f(n)$ being either real or purely imaginary numbers.*

Then the additively twisted $L-$function $L_{p/q}(s,f)$ (1.10) has infinitely many zeros at the critical line $Re(s)=\frac{k}{2}+\frac{1}{4}$.

The main goal of our paper is to improve on Theorem B and Theorem C above by giving explicit quantitative estimates for the number of zeros of $L\left(\frac{k}{2}+\frac{1}{4}+it,f\right)$ and $L_{p/q}\left(\frac{k}{2}+\frac{1}{4}+it,f\right)$. Like Theorem B above, our first result uses the main setting of Wilton's variant of Hardy's theorem [28].

Theorem 1.1. *Let N be a perfect square and $f(z) = \sum_{n=1}^{\infty} a_f(n)\, e^{2\pi inz} \in S_{k+\frac{1}{2}}(\Gamma_0(4N))$. Assume also that the Fourier coefficients of $f(z)$, $a_f(n)$, are either real or purely imaginary numbers.*

Let $N_0^{\pm}(T)$ represent the number of zeros of odd order of the function $L(s,f) \pm L(s, f|W_{4N})$ written in the form $s = \frac{k}{2} + \frac{1}{4} + it$, $0 \le t \le T$. Then we have that

$$N_0^{\pm}(T) = \Omega\left(T^{\frac{1}{2}}\right), \tag{1.12}$$

or, by other words, there exists some $d > 0$ such that

$$\limsup_{T\to\infty} \frac{N_0^{\pm}(T)}{T^{1/2}} > d. \tag{1.13}$$

As far as we know, our Theorem 1.1 establishes for the first time a quantitative lower bound on the number of critical zeros of $L-$functions attached to half-integral weight cusp forms. As an immediate corollary, one obtains.

Corollary 1.1. *Let N be a perfect square and $f(z) = \sum_{n=1}^{\infty} a_f(n)\, e^{2\pi inz} \in S_{k+\frac{1}{2}}(\Gamma_0(4N))$ be such that $f|W_{4N} = f$ or $f|W_{4N} = -f$. Assume also that the Fourier coefficients of $f(z)$, $a_f(n)$, are either real or purely imaginary numbers.*

Let $N_0(T)$ denote the number of zeros of odd order of $L(s,f)$ of the form $s = \frac{k}{2} + \frac{1}{4} + it$, $0 \le t \le T$. Then we have that

$$N_0(T) = \Omega\left(T^{\frac{1}{2}}\right) \quad \text{or, equivalently,} \quad \limsup_{T\to\infty} \frac{N_0(T)}{T^{1/2}} > d, \tag{1.14}$$

for some $d > 0$.

We remark that it is possible to get an explicit numerical value for d in (1.13) and (1.14). In fact, d only depends on the Fourier expansion of a function arising from the slash operator (see (4.16) and the proof of Lemma 3.2 for details).

Our proof of Theorem 1.1 is inspired by a beautiful argument due to de la Vallée Poussin [5]. The belgian mathematician Charles-Jean de la Valeé Poussin is mostly famous for his proof of the prime number theorem. However, his contributions to the study of the zeros of $\zeta(s)$ on the line $\mathrm{Re}(s) = \frac{1}{2}$ are not so well-known. Shortly after the appearance of the papers by Hardy and Landau [12,19] on this topic, de la Vallée Poussin published two short notes. In one of them, he proved that the number of zeros of $\zeta(s)$ of the form $s = \frac{1}{2} + it$, $0 < t < T$, satisfies the estimate $N_0(T) = \Omega\left(T^{1/2}\right)$ as $T \to \infty$. However, as he remarks [[5], p.421] *"Cette méthode ne s'applique pas à toutes les fonctions qui interviennent dans l'étude de la progression arithmétique".* [2] In order to circumvent this difficulty, de la Vallée Poussin develops in a later note an argument providing estimates for the difference $N_0(T+H) - N_0(T)$, $T^{\frac{3}{4}+\epsilon} \le H(T) \le T$. His second note was the first improvement of the results of Landau about the zeros of Dirichlet $L-$functions.

Despite the beauty of the arguments presented in these short papers, the results and methods of de la Vallée Poussin quickly faded into oblivion [3] due to the much sharper estimates (proved around

[2] or by other words, his method does not work for certain Dirichlet $L-$functions.

the same time) by Hardy and Littlewood [13, 14][3] and, decades later, by Selberg and Levinson. Even Titchmarsh's text on the zeta function, known for its very precise historical background, omits de la Vallée Poussin's contributions in this regard, notwithstanding the fact that it exposes (just for the sake of historical interest) five different proofs of Hardy's theorem [[27], Chapter X]. [4]

With the proof of Theorem 1.1, we hope to bring de la Vallée's Poussin elegant arguments to a modern mathematical audience, as we believe that the method followed in the proof of the result (1.12) has a theoretical interest in itself. In a historical sense, it is closely connected to Hardy's original note [12], as well as to Wilton's variant [28], which, as mentioned above, was used in the proof of Theorem B. Thus, in a sense, our Theorem 1.1 can be seen as a complement to Theorem B due to Meher, Pujahari and Shankhadhar. It has also the advantage of giving a quantitative estimate for the number of zeros of a Dirichlet series without depending on any evaluation of exponential sums, which is rather unusual.

Similarly to de la Vallée Poussin's case, the result (1.12) in itself can be quickly surpassed by no more than the same ideas that Hardy and Littlewood used to prove that the function $\zeta\left(\frac{1}{2}+it\right)$, $0\leq t\leq T$, has $\gg T$ zeros [14]. Our next Theorem establishes an analogue of Hardy-Littlewood's famous result.

Theorem 1.2. *Let N be a perfect square and $f(z)=\sum_{n=1}^{\infty}a_f(n)\,e^{2\pi inz}\in S_{k+\frac{1}{2}}\left(\Gamma_0(4N)\right)$. Assume also that the Fourier coefficients of $f(z)$, $a_f(n)$, are either real or purely imaginary numbers.*

Let $N_0^{\pm}(T)$ represent the number of zeros of odd order of the function $L(s,f)\pm L(s,f|W_{4N})$ written in the form $s=\frac{k}{2}+\frac{1}{4}+it$, $0\leq t\leq T$.

Then we have that

$$N_0^{\pm}(T)\gg T \quad \textit{or, equivalently,} \quad \liminf_{T\to\infty}\frac{N_0^{\pm}(T)}{T}>d, \tag{1.15}$$

for some $d>0$.

By following the proof of Theorem 1.2, one sees that there are no extra complications in adapting the method to the case of additively twisted $L-$functions (1.10) (see Remark 6.1 below). Thus, the following extension of Kim's theorem holds.

Theorem 1.3. *Let N be a positive integer and $\frac{p}{q}$, $q>0$, a rational number which is $\Gamma_0(4N)-$equivalent to $i\infty$ and such that $p^2\equiv 1 \mod q$. Moreover, let $f(z)=\sum_{n=1}^{\infty}a_f(n)\,e^{2\pi inz}\in S_{k+\frac{1}{2}}\left(\Gamma_0(4N)\right)$, with $a_f(n)$ being either real or purely imaginary numbers.*

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[3]The first improvements on the number of critical zeros of $\zeta(s)$ by Hardy and Littlewood were in fact independent of those of de la Vallée Poussin. An additional note to the paper [[13], p.196] was written by Hardy and Littlewood just to acknowledge de la Vallée Poussin's contributions.

[4]It should be mentioned, however, that de la Vallée Poussin's papers [5, 6] appear as references in Titchmarsh's book but they are not mentioned in the main text.

If $N_{0,p/q}(T)$ denotes the number of zeros of $L_{p/q}(s,f)$ written in the form $s=\frac{k}{2}+\frac{1}{4}+it$, $0\leq t\leq T$, then we have that

$$N_{0,p/q}(T)\gg T \quad \textit{or, equivalently,} \quad \liminf_{T\to\infty}\frac{N_{0,p/q}(T)}{T}>d, \tag{1.16}$$

for some $d>0$.

Our proofs of Theorems 1.2 and 1.3 will rely on a variant of the Hardy-Littlewood method developed by Lekkerkerker [20]. In his doctoral dissertation, Lekkerkerker proved general theorems about the distribution of zeros of Dirichlet series satisfying Hecke's functional equation. The last chapter of his thesis is devoted to prove a result of Hardy-Littlewood type for entire Dirichlet series.

Lekkerkerker's results were mostly generalized by Berndt [1, 2], where the assumption of Hecke's functional equation was replaced by an equation with more $\Gamma-$factors. Epstein, Hafner and Sarnak [8,9], adapted Lekkerkerker's ideas to prove that the number of critical zeros of $L-$functions attached to Maass cusp forms also satisfy (1.15). [5]

It should be remarked that condition 3 on the statement of Theorem 13 of Lekkerkerker's dissertation is incomplete, because it is motivated by the incorrect lemma 7 on Chapter III of [20]. In order to avoid some of the incorrect steps motivated by this imprecision, we adapt to our case some of Lekkerkerker's arguments in a correct form and we present them in a way that avoids the use of the incorrect lemma 7. An alternative approach of our Lemma 5.3, based on Fourier analysis, is also sketched at Remark 5.4 below and it has the same spirit as the approach followed in [8].

Our paper is organized as follows. In the next section, we present some basic facts that will be important throughout our exposition. Next, we present the proof of Theorem 1.1 by modifying the argument of de la Vallée Poussin. Section 5 is devoted to a careful adaptation of Lekkerkerker's arguments to our case. We finish the paper with an exposition of the classical Hardy-Littlewood argument to prove Theorem 1.2 and with a remark on how the same ideas can be used to prove Theorem 1.3.

2 Preliminary results

Like cusp forms with integral weight, cusp forms of half-integral weight also satisfy uniform bounds on the upper half-plane [[15], Chapt.5, p.70]. We start by quoting a lemma in Shimura's textbook [[24], p.31, Lemma 6.2.].

If $f\in S_{k+\frac{1}{2}}(\Gamma_0(4N))$, with N being a perfect square, then $f\left(\frac{z}{\sqrt{4N}}\right)=\sum a_f(n)\,e^{2\pi inz/\ell}$ with $\ell\in\mathbb{N}$. Then, by [[24], p.31, Lemma 6.2.], there exists a positive constant $\mathcal{A}$ such that, for any $z\in\mathbb{H}$, the uniform

[5] Hafner [11] later generalized this result by proving the analogue of Selberg's celebrated theorem, showing that a positive proportion of the zeros of this $L-$functions lie on the critical line $\text{Re}(s)=\frac{1}{2}$.

bound takes place[6]

$$\left| f\left(\frac{z}{\sqrt{4N}}\right)\right| \leq \frac{\mathcal{A}}{\mathrm{Im}(z)^{\frac{k}{2}+\frac{1}{4}}}. \tag{2.1}$$

Connected with (2.1) is the mean value estimate for $a_f(m)$,

$$\sum_{m=1}^{M} |a_f(m)|^2 \ll M^{k+\frac{1}{2}}, \tag{2.2}$$

where, just like $\mathcal{A}$ in (2.1), the implied constant depends on f. The usual Hecke argument can be developed by taking (2.1) as a starting point. This classical procedure yields the bound

$$a_f(n) \ll n^{\frac{k}{2}+\frac{1}{4}}, \tag{2.3}$$

where, again, the implied constant depends only on f. Of course, much better bounds than (2.3) are currently available [4]. The uniform bound (2.1) will play a crucial role in some estimates necessary for the proof of Theorem 1.2.

Like in the integral case, one may define slash operators acting on the space $S_{k+\frac{1}{2}}\left(\Gamma_0(4N)\right)$. These are defined as follows [[25], p.447]. Let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}^{+}(2,\mathbb{R})$ and attach to it an analytic function on $\mathbb{H}$, $\phi(z)$, such that

$$\phi(z)^2 = \tau \, \det(\alpha)^{-\frac{1}{2}}(cz+d), \tag{2.4}$$

where τ is a complex number such that $|\tau| = 1$. If we let $\mathcal{G}$ denote the set of all pairs (α, ϕ), then $(\mathcal{G}, \star)$ forms a group, where the operation $\star$ is defined by

$$(\alpha, \phi) \star (\beta, \psi) = \left(\alpha\beta, \phi(\beta z)\, \psi(z)\right).$$

Under this setting, if $\xi = (\alpha, \phi(z)) \in \mathcal{G}$ and $f \in S_{k+\frac{1}{2}}\left(\Gamma_0(4N)\right)$, the slash operator is defined as

$$\left(f|\xi\right)(z) = f\left(\alpha z\right)\, \phi(z)^{-2k-1}. \tag{2.5}$$

As the functional equations (1.6) and (1.11) already clarify, in several occasions throughout this paper we shall need to estimate the asymptotic order of certain integrals involving the Dirichlet series $L(s,f)$. To justify most of the steps, we will often invoke the following version of Stirling's formula

$$\Gamma(\sigma+it) = (2\pi)^{\frac{1}{2}}\, t^{\sigma+it-\frac{1}{2}}\, e^{-\frac{\pi t}{2}-it+\frac{i\pi}{2}(\sigma-\frac{1}{2})}\left(1+\frac{1}{12(\sigma+it)}+O\left(\frac{1}{t^2}\right)\right), \tag{2.6}$$

as $t \to \infty$, uniformly for $-\infty < \sigma_1 \leq \sigma \leq \sigma_2 < \infty$. A similar formula can be written for $t<0$ as t tends to $-\infty$ by using the fact that $\Gamma(\overline{s}) = \overline{\Gamma(s)}$. Of course, a direct consequence of this exact version is

$$|\Gamma(\sigma+it)| = (2\pi)^{\frac{1}{2}}\, |t|^{\sigma-\frac{1}{2}}\, e^{-\frac{\pi}{2}|t|}\left(1+O\left(\frac{1}{|t|}\right)\right), \quad |t| \to \infty. \tag{2.7}$$

[6] Shimura's argument simply transforms the statement on half-integral weight cusp forms into the same statement about cusp forms with integral weight, which is well-known.

Before proceeding further, let us briefly mention the usual reasoning to get convex estimates for $L(\sigma + it, f)$. The estimate that we will be using in this paper can be obtained through the familiar argument invoking the classical Phragmén-Lindelöf theorem given in [[26], p.180, 5.65]. By Hecke's bound (2.3), we know that when $\sigma > \frac{k}{2} + \frac{5}{4}$, $L(\sigma + it, f|W_{4N}) = O(1)$. Hence, from the functional equation (1.6) and Stirling's formula (2.6) we have, whenever $\sigma < \frac{k}{2} - \frac{3}{4}$,

$$|L(\sigma + it, f)| \ll \left| \frac{\Gamma\left(k + \frac{1}{2} - s\right)}{\Gamma(s)} L\left(k + \frac{1}{2} - s, f|W_{4N}\right) \right| \ll |t|^{k+\frac{1}{2}-2\sigma}.$$

Thus, by the Phragmén-Lindelöf principle, we have the convex estimate for $L(s, f)$,

$$L(\sigma + it, f) \ll_{\epsilon} |t|^{\frac{k}{2}+\frac{5}{4}-\sigma+\epsilon}, \qquad \frac{k}{2} - \frac{3}{4} - \epsilon < \sigma < \frac{k}{2} + \frac{5}{4} + \epsilon. \tag{2.8}$$

3 Lemmas for the proof of Theorem 1.1

Since our first theorem concerns zeros of a combination of the form $L(s, f) \pm L(s, f|W_{4N})$, we will define two important functions related to this expression as

$$R_f(t) := \frac{\Lambda\left(\frac{k}{2} + \frac{1}{4} + it, f\right) + \Lambda\left(\frac{k}{2} + \frac{1}{4} + it, f|W_{4N}\right)}{2}, \tag{3.1}$$

and

$$I_f(t) := \frac{\Lambda\left(\frac{k}{2} + \frac{1}{4} + it, f\right) - \Lambda\left(\frac{k}{2} + \frac{1}{4} + it, f|W_{4N}\right)}{2i}. \tag{3.2}$$

It follows from the functional equation for $\Lambda(f, s)$ (1.7) that $R_f(t)$ can be written as

$$R_f(t) = \frac{\Lambda\left(\frac{k}{2} + \frac{1}{4} + it, f\right) + \Lambda\left(\frac{k}{2} + \frac{1}{4} - it, f\right)}{2}, \tag{3.3}$$

which means that $R_f(-t) = R_f(t)$. In particular, if the coefficients of $f(z)$, $a_f(n)$, are real numbers, then

$$R_f(t) = \frac{\Lambda\left(\frac{k}{2} + \frac{1}{4} + it, f\right) + \overline{\Lambda\left(\frac{k}{2} + \frac{1}{4} + it, f\right)}}{2} = \mathrm{Re}\left(\Lambda\left(\frac{k}{2} + \frac{1}{4} + it, f\right)\right). \tag{3.4}$$

Hence, when we assume that $a_f(n)$ are real numbers, $R_f(t)$ represents a real-valued and even function of t. Analogously, the use of the functional equation for $L(s, f)$ allows to write the representation for $I_f(t)$

$$I_f(t) := \frac{\Lambda\left(\frac{k}{2} + \frac{1}{4} + it, f\right) - \Lambda\left(\frac{k}{2} + \frac{1}{4} - it, f\right)}{2i}, \tag{3.5}$$

which means that $I_f(-t) = I_f(t)$. In particular, if the coefficients of $f(z)$, $a_f(n)$, are real numbers, then

$$I_f(t) = \frac{\Lambda\left(\frac{k}{2} + \frac{1}{4} + it, f\right) - \overline{\Lambda\left(\frac{k}{2} + \frac{1}{4} + it, f\right)}}{2i} = \mathrm{Im}\left(\Lambda\left(\frac{k}{2} + \frac{1}{4} + it, f\right)\right). \tag{3.6}$$

At last, we note that if the Fourier coefficients of $f(z)$, $a_f(n)$, are, instead, purely imaginary numbers, then the roles of $R_f(t)$ and $I_f(t)$ are somewhat reversed, this is

$$R_f(t)=\frac{\Lambda\left(\frac{k}{2}+\frac{1}{4}+it,f\right)+\Lambda\left(\frac{k}{2}+\frac{1}{4}-it,f\right)}{2}=\frac{\Lambda\left(\frac{k}{2}+\frac{1}{4}+it,f\right)-\overline{\Lambda\left(\frac{k}{2}+\frac{1}{4}+it,f\right)}}{2}$$
$$=i\,\mathrm{Im}\left(\Lambda\left(\frac{k}{2}+\frac{1}{4}+it,f\right)\right),$$

while

$$I_f(t)=\frac{\Lambda\left(\frac{k}{2}+\frac{1}{4}+it,f\right)-\Lambda\left(\frac{k}{2}+\frac{1}{4}-it,f\right)}{2i}=\frac{\Lambda\left(\frac{k}{2}+\frac{1}{4}+it,f\right)+\overline{\Lambda\left(\frac{k}{2}+\frac{1}{4}+it,f\right)}}{2i}$$
$$=-i\,\mathrm{Re}\left(\Lambda\left(\frac{k}{2}+\frac{1}{4}+it,f\right)\right).$$

Therefore, the difficulty in dealing with the case where $a_f(n)\in i\mathbb{R}$ is exactly the same as dealing with the assumption $a_f(n)\in\mathbb{R}$ and so, throughout this section and the next, we will only give the details for the case $a_f(n)\in\mathbb{R}$.

Lemma 3.1. *Let N be a perfect square and $f(z)=\sum_{n=1}^{\infty}a_f(n)\,e^{2\pi inz}\in S_{k+\frac{1}{2}}\left(\Gamma_0(4N)\right)$. If the Fourier coefficients of $f(z)$, $a_f(n)$, are real numbers, then the following integral representations take place*

$$\int_0^{\infty}R_f(t)\,\cosh\left(\left(\frac{\pi}{2}-u\right)t\right)\,dt=\frac{\pi}{2}i^{\frac{k}{2}+\frac{1}{4}}e^{-i\left(\frac{k}{2}+\frac{1}{4}\right)u}f\left(-\frac{e^{-iu}}{\sqrt{4N}}\right)+\frac{\pi}{2}i^{-\frac{k}{2}-\frac{1}{4}}e^{i\left(\frac{k}{2}+\frac{1}{4}\right)u}\,f\left(\frac{e^{iu}}{\sqrt{4N}}\right),\quad(3.7)$$

$$i\int_0^{\infty}I_f(t)\,\sinh\left(\left(\frac{\pi}{2}-u\right)t\right)\,dt=\frac{\pi}{2}i^{\frac{k}{2}+\frac{1}{4}}\,e^{-i\left(\frac{k}{2}+\frac{1}{4}\right)u}f\left(-\frac{e^{-iu}}{\sqrt{4N}}\right)-\frac{\pi}{2}i^{-\frac{k}{2}-\frac{1}{4}}e^{i\left(\frac{k}{2}+\frac{1}{4}\right)u}\,f\left(\frac{e^{iu}}{\sqrt{4N}}\right).\quad(3.8)$$

Proof. For any $-\frac{\pi}{2}<\omega<\frac{\pi}{2}$, we evaluate the integral

$$\mathcal{J}(\omega):=\int_{-\infty}^{\infty}R_f(t)\,e^{\omega t}dt,\quad(3.9)$$

by using the Cahen-Mellin formula. Since $|\omega|<\frac{\pi}{2}$, it follows from Stirling's formula (2.7) and the Phragmén-Lindelöf principle (2.8) that $|R_f(t)|e^{\omega t}\in L_1(\mathbb{R})$. By (3.3), we may write $\mathcal{J}(\omega)$ as an integral over the vertical line $\mathrm{Re}(z)=\frac{k}{2}+\frac{1}{4}$, i.e.,

$$\mathcal{J}(\omega)=\int_{-\infty}^{\infty}\frac{\Lambda\left(\frac{k}{2}+\frac{1}{4}+it,f\right)+\Lambda\left(\frac{k}{2}+\frac{1}{4}-it,f\right)}{2}\,e^{\omega t}dt$$
$$=\frac{1}{2}\int_{-\infty}^{\infty}\Lambda\left(\frac{k}{2}+\frac{1}{4}+it,f\right)e^{\omega t}\,dt+\frac{1}{2}\int_{-\infty}^{\infty}\Lambda\left(\frac{k}{2}+\frac{1}{4}+it,f\right)\,e^{-\omega t}dt$$
$$=\frac{e^{i\omega\left(\frac{k}{2}+\frac{1}{4}\right)}}{2i}\int_{\frac{k}{2}+\frac{1}{4}-i\infty}^{\frac{k}{2}+\frac{1}{4}+i\infty}\Lambda\left(z,f\right)\,e^{-i\omega z}\,dz+\frac{e^{-i\omega\left(\frac{k}{2}+\frac{1}{4}\right)}}{2i}\int_{\frac{k}{2}+\frac{1}{4}-i\infty}^{\frac{k}{2}+\frac{1}{4}+i\infty}\Lambda\left(z,f\right)\,e^{i\omega z}\,dz.\quad(3.10)$$

We will now evaluate the first integral, as the second can be analogously computed by replacing ω by $-\omega$. We shift the line of integration to $\mathrm{Re}(z)=\frac{k+3}{2}\pm iT$: to do this we just need to integrate integrating along a positively oriented rectangular contour $\mathcal{R}(T)$ with vertices $\frac{k}{2}+\frac{1}{4}\pm iT$ and $\frac{k+3}{2}\pm iT$, $T>0$. Since $\Gamma(z)$ and $L(z,f)$ are analytic inside $\mathcal{R}(T)$, an application of Cauchy's theorem gives

$$\left\{\int_{\frac{k}{2}+\frac{1}{4}-iT}^{\frac{k}{2}+\frac{1}{4}+iT}+\int_{\frac{k}{2}+\frac{1}{4}+iT}^{\frac{k+3}{2}+iT}+\int_{\frac{k+3}{2}-iT}^{\frac{k}{2}+\frac{1}{4}-iT}+\int_{\frac{k+3}{2}+iT}^{\frac{k+3}{2}-iT}\right\}\Lambda\left(z,f\right)\,e^{-i\omega z}\,dz=0. \tag{3.11}$$

Since $|\omega|<\frac{\pi}{2}$, using Stirling's formula and the convex estimates (2.8), we see that the integrals along the horizontal segments $\left[\frac{k}{2}+\frac{1}{4}\pm iT,\frac{k+3}{2}\pm iT\right]$ can be bounded as follows

$$\left|\int_{\frac{k}{2}+\frac{1}{4}+iT}^{\frac{k+3}{2}+iT}\Lambda\left(z,f\right)\,e^{-i\omega z}\,dz\right|\ll_{\epsilon}T^{\frac{k}{2}+\frac{3}{4}+\epsilon}e^{-\left(\frac{\pi}{2}-\omega\right)T}$$

and so they tend to zero as $T\to\infty$. Taking $T\to\infty$ in (3.11) and using the fact that the Dirichlet series (1.4) converges absolutely when $\mathrm{Re}(s)=\frac{k+3}{2}$, we deduce that

$$\int_{\frac{k}{2}+\frac{1}{4}-i\infty}^{\frac{k}{2}+\frac{1}{4}+i\infty}\Lambda\left(z,f\right)\,e^{-i\omega z}\,dz=\int_{\frac{k+3}{2}-i\infty}^{\frac{k+3}{2}+i\infty}\left(\frac{2\pi}{\sqrt{4N}}\right)^{-z}\Gamma(z)\,L(z,f)\,e^{-i\omega z}dz=2\pi i\sum_{n=1}^{\infty}\frac{a_f(n)}{2\pi i}\int_{\frac{k+3}{2}-i\infty}^{\frac{k+3}{2}+i\infty}\Gamma(z)\,\left(\frac{2\pi n e^{i\omega}}{\sqrt{4N}}\right)^{-z}dz$$

$$=2\pi i\sum_{n=1}^{\infty}a_f(n)\,\exp\left(-\frac{2\pi n e^{i\omega}}{\sqrt{4N}}\right)=2\pi i\,f\left(\frac{ie^{i\omega}}{\sqrt{4N}}\right).$$

Note that, on the third equality, we have invoked the Cahen-Mellin integral,

$$e^{-z}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\,z^{-s}ds,\quad c>0,\;\;\mathrm{Re}(z)>0, \tag{3.12}$$

which can be applied because $\mathrm{Re}(e^{i\omega})=\cos(\omega)>0$ when $|\omega|<\frac{\pi}{2}$. Thus, returning to (3.10) we find that

$$\mathcal{J}(\omega)=\pi e^{i\omega\left(\frac{k}{2}+\frac{1}{4}\right)}\,f\left(\frac{ie^{i\omega}}{\sqrt{4N}}\right)+\pi e^{-i\omega\left(\frac{k}{2}+\frac{1}{4}\right)}\,f\left(\frac{ie^{-i\omega}}{\sqrt{4N}}\right). \tag{3.13}$$

This implies immediately (3.7), after we substitute ω by $\frac{\pi}{2}-u$ and use the fact that $R_f(t)=R_f(-t)$. The proof of the second formula (3.8) uses the same computations but uses instead the fact that $I_f(t)=-I_f(-t)$. □

Remark 3.1. We can adapt the computations given in the evaluation of (3.9) to give an even more general expression. In fact, for any $z\in\mathbb{H}$, the following integral representation is valid

$$\int_{-\infty}^{\infty}R_f(t)\,(-iz)^{-it}dt=\pi i^{-\frac{k}{2}-\frac{1}{4}}z^{\frac{k}{2}+\frac{1}{4}}\,\left\{f\left(\frac{z}{\sqrt{4N}}\right)+(f|W_{4N})\left(\frac{z}{\sqrt{4N}}\right)\right\}. \tag{3.14}$$

In the previous lemma, we have connected the functions $R_f(t)$ and $I_f(t)$ (whose zeros we pretend to study) with an evaluation of the cusp form $f\left(-e^{-iu}/\sqrt{4N}\right)$, for $0 < u < \frac{\pi}{2}$. The next lemma uses the slash operator (2.5) in order to establish a uniform bound for any derivative of $f\left(-e^{-iu}/\sqrt{4N}\right)$ with respect to u.

Lemma 3.2. *There exist four positive constants A, B, C and D (only depending on the weight and level of f) such that, for any $0 < u < \frac{\pi}{2}$ and $p \in \mathbb{N}_0$,*

$$\left|\frac{d^p}{du^p} f\left(-\frac{e^{-iu}}{\sqrt{4N}}\right)\right| < C\, \frac{2^p\, p!}{u^{p+k+\frac{1}{2}}} e^{-\frac{A}{u}} \tag{3.15}$$

and

$$\left|\frac{d^p}{du^p} f\left(\frac{e^{iu}}{\sqrt{4N}}\right)\right| < D\, \frac{2^p\, p!}{u^{p+k+\frac{1}{2}}} e^{-\frac{B}{u}}. \tag{3.16}$$

Proof. Let $0 < u_0 < \frac{\pi}{2}$. The purpose of our proof is to bound the absolute value of $\left[\frac{d^p}{du^p} f\left(-\frac{e^{-iu}}{\sqrt{4N}}\right)\right]_{u=u_0}$. The derivative of the function $f\left(-\frac{e^{-iu}}{\sqrt{4N}}\right)$ at the point u_0 will be given by integrating along a positively oriented circle with center u_0 and having radius λu_0, $0 < \lambda < 1$. Let us denote this circle by $C_{\lambda u_0}(u_0)$: then for any $w \in D_{\lambda u_0}(u_0) := \text{int}(C_{\lambda u_0}(u_0))$, one can easily check that $\text{Im}\left(-\frac{e^{-iw}}{\sqrt{4N}}\right) > 0$. Hence, $f\left(-\frac{e^{-iw}}{\sqrt{4N}}\right)$ must be analytic inside the circle $C_{\lambda u_0}(u_0)$ and so, by Cauchy's formula,

$$\left[\frac{d^p}{du^p} f\left(-\frac{e^{-iu}}{\sqrt{4N}}\right)\right]_{u=u_0} = \frac{p!}{2\pi i} \int_{C_{\lambda u_0}(u_0)} \frac{f\left(-\frac{e^{-iw}}{\sqrt{4N}}\right)}{(w-u_0)^{p+1}}\, dw. \tag{3.17}$$

Our next task will be to bound the integral on the right-hand side of (3.17). This will be done by considering the matrix

$$\gamma = \begin{pmatrix} -1 & 0 \\ \sqrt{4N} & -1 \end{pmatrix} \in \text{SL}(2,\mathbb{Z}), \tag{3.18}$$

and using it to construct the slash operator (2.5)

$$g(z) := (f|\gamma)(z) = \tau\left(\sqrt{4N}z - 1\right)^{-k-\frac{1}{2}} f\left(\frac{z}{1-\sqrt{4N}z}\right), \tag{3.19}$$

where τ is a complex number such that $|\tau| = 1$. If we apply the construction (3.19) with

$$z := -\frac{1}{\sqrt{4N}\,(e^{iw}-1)} \in \mathbb{H},$$

then we see that

$$f\left(-\frac{e^{-iw}}{\sqrt{4N}}\right) = \frac{1}{\tau}\left(-\frac{e^{iw}}{e^{iw}-1}\right)^{k+\frac{1}{2}} g\left(-\frac{1}{\sqrt{4N}\,(e^{iw}-1)}\right). \tag{3.20}$$

Since $f(z)$ is a cusp form and, by hypothesis, N is a perfect square, $g(z)$ admits the Fourier expansion

$$g(z) := \sum_{n=1}^{\infty} b(n)\, e^{\frac{2\pi i n}{r}z}, \quad \text{for some } r \in \mathbb{N}, \tag{3.21}$$

with $b(n) = O\left(n^{\alpha}\right)$ for some large $\alpha > 0$. Using (3.20) and (3.21) we obtain

$$\left|f\left(-\frac{e^{-iw}}{\sqrt{4N}}\right)\right| \leq \left(\frac{|e^{iw}|}{|e^{iw}-1|}\right)^{k+\frac{1}{2}} \sum_{n=1}^{\infty} |b(n)| \exp\left(\frac{\pi n}{r\sqrt{N}} \operatorname{Im}\left(\frac{1}{e^{iw}-1}\right)\right)$$
$$= \left(\frac{e^{-\operatorname{Im}(w)/2}}{2\sqrt{\sin^2\left(\frac{\operatorname{Re}(w)}{2}\right)+\sinh^2\left(\frac{\operatorname{Im}(w)}{2}\right)}}\right)^{k+\frac{1}{2}} \sum_{n=1}^{\infty} |b(n)| \exp\left(-\frac{\pi n}{2r\sqrt{N}} \operatorname{Re}\left(\frac{1}{\tan\left(\frac{w}{2}\right)}\right)\right)$$
$$\leq \left(\frac{e^{\frac{\lambda u_0}{2}}}{2\sin\left(\frac{\operatorname{Re}(w)}{2}\right)}\right)^{k+\frac{1}{2}} \sum_{n=1}^{\infty} |b(n)| \exp\left(-\frac{\pi n}{2r\sqrt{N}} \frac{\sin\left(\operatorname{Re}(w)\right)}{\cosh\left(\operatorname{Im}(w)\right)-\cos\left(\operatorname{Re}(w)\right)}\right), \tag{3.22}$$

where in the last step we just have used the fact that, for any $w \in C_{\lambda u_0}(u_0)$, $|\operatorname{Im}(w)| \leq \lambda u_0$. Using now the elementary Jordan inequality $\sin(x) > \frac{2x}{\pi}$, $0 < x < \frac{\pi}{2}$, and the fact that $\frac{1}{2}(1-\lambda)u_0 \leq \frac{1}{2}\operatorname{Re}(w) \leq \frac{1}{2}(1+\lambda)u_0 < \frac{\pi}{2}$, we obtain

$$\left|f\left(-\frac{e^{-iw}}{\sqrt{4N}}\right)\right| < \left(\frac{\pi e^{\frac{\lambda u_0}{2}}}{2\operatorname{Re}(w)}\right)^{k+\frac{1}{2}} \sum_{n=1}^{\infty} |b(n)| \exp\left(-\frac{\pi n}{2r\sqrt{N}} \frac{\sin\left(\operatorname{Re}(w)\right)}{\cosh\left(\operatorname{Im}(w)\right)-\cos\left(\operatorname{Re}(w)\right)}\right)$$
$$\leq \left(\frac{\pi e^{\frac{\lambda u_0}{2}}}{2\left(1-\lambda\right)u_0}\right)^{k+\frac{1}{2}} \sum_{n=1}^{\infty} |b(n)| \exp\left(-\frac{\pi n}{2r\sqrt{N}} \frac{\sin\left(\operatorname{Re}(w)\right)}{\cosh\left(\operatorname{Im}(w)\right)-\cos\left(\operatorname{Re}(w)\right)}\right). \tag{3.23}$$

Our proof shall be concluded once we bound the terms of the exponential function in (3.23). First, let us note that

$$\cosh(\operatorname{Im}(w)) - \cos(\operatorname{Re}(w)) = \int_0^{\operatorname{Im}(w)} \sinh(t)\,dt + \int_0^{\operatorname{Re}(w)} \sin(t)\,dt \leq \int_0^{\operatorname{Im}(w)} t\,e^{\frac{t^2}{6}}\,dt + \int_0^{\operatorname{Re}(w)} t\,dt$$
$$\leq \frac{\operatorname{Im}(w)^2}{2}\,e^{\frac{\operatorname{Im}(w)^2}{6}} + \frac{\operatorname{Re}(w)^2}{2} < e^{\frac{\lambda^2}{6}u_0^2}\left(1+\lambda\right)^2 u_0^2, \tag{3.24}$$

where we have used the fact that $(1-\lambda)u_0 \leq \operatorname{Re}(w) \leq (1+\lambda)u_0$, $-\lambda u_0 \leq \operatorname{Im}(w) \leq \lambda u_0$ and the well-known inequality $\sinh(x) \leq x\,e^{x^2/6}$, $x > 0$. But since $0 < (1-\lambda)u_0 \leq \operatorname{Re}(w) \leq (1+\lambda)u_0 < \pi$, another application of Jordan's inequality gives

$$\sin(\operatorname{Re}(w)) > \begin{cases} \frac{2}{\pi}\operatorname{Re}(w), & 0 < \operatorname{Re}(w) < \frac{\pi}{2} \\ 2 - \frac{2}{\pi}\operatorname{Re}(w), & \frac{\pi}{2} \leq \operatorname{Re}(w) < \pi \end{cases} \geq \frac{2}{\pi}\left(1-\lambda\right)u_0. \tag{3.25}$$

Hence, the combination of (3.24) and (3.25) yields the lower bound

$$\frac{\sin\left(\operatorname{Re}(w)\right)}{\cosh\left(\operatorname{Im}(w)\right)-\cos\left(\operatorname{Re}(w)\right)} > \frac{2(1-\lambda)}{\pi e^{\frac{\lambda^2}{6}u_0^2}\left(1+\lambda\right)^2 u_0} > \frac{2(1-\lambda)e^{-\frac{\pi^2\lambda^2}{24}}}{\pi(1+\lambda)^2 u_0}. \tag{3.26}$$

Therefore, returning to (3.23) and denoting by c the smallest integer such that $b(c) \neq 0$,

$$\begin{aligned}\left|f\left(-\frac{e^{-iw}}{\sqrt{4N}}\right)\right| &\leq \left(\frac{\pi e^{\frac{\lambda u_0}{2}}}{2\left(1-\lambda\right)u_0}\right)^{k+\frac{1}{2}} \sum_{n=1}^{\infty} |b(n)| \exp\left(-\frac{\pi n}{2r\sqrt{N}} \frac{\sin\left(\mathrm{Re}(w)\right)}{\cosh\left(\mathrm{Im}(w)\right)-\cos\left(\mathrm{Re}(w)\right)}\right) \\ &< \left(\frac{\pi e^{\frac{\lambda u_0}{2}}}{2\left(1-\lambda\right)u_0}\right)^{k+\frac{1}{2}} \sum_{n=1}^{\infty} |b(n)| \exp\left(-\frac{n(1-\lambda)e^{-\frac{\pi^2\lambda^2}{24}}}{r\sqrt{N}\left(1+\lambda\right)^2 u_0}\right) \\ &= \left(\frac{\pi e^{\frac{\lambda u_0}{2}}}{2\left(1-\lambda\right)u_0}\right)^{k+\frac{1}{2}} |b(c)| \exp\left(-\frac{\left(1-\lambda\right)c\, e^{-\frac{\pi^2\lambda^2}{24}}}{r\sqrt{N}\,\left(1+\lambda\right)^2 u_0}\right) \sum_{n\geq c}\left|\frac{b(n)}{b(c)}\right| \exp\left(-\frac{\left(1-\lambda\right)e^{-\frac{\pi^2\lambda^2}{24}}\left(n-c\right)}{r\sqrt{N}\left(1+\lambda\right)^2 u_0}\right).\end{aligned} \tag{3.27}$$

Finally, since $0 < u_0 < \frac{\pi}{2}$ and $0 < \lambda < 1$ is arbitrary, we may now take $\lambda = \frac{1}{2}$ and get the bound

$$\left|f\left(-\frac{e^{-iw}}{\sqrt{4N}}\right)\right| \leq |b(c)| \left(\frac{\pi e^{\frac{\pi}{8}}}{2u_0}\right)^{k+\frac{1}{2}} \exp\left(-\frac{2ce^{-\frac{\pi^2}{96}}}{9r\sqrt{N}}\frac{1}{u_0}\right) \sum_{n\geq c}\left|\frac{b(n)}{b(c)}\right| \exp\left(-\frac{4e^{-\frac{\pi^2}{96}}}{9\pi r\sqrt{N}}(n-c)\right) \tag{3.28}$$

$$\leq \frac{C'}{u_0^{k+\frac{1}{2}}} \exp\left(-\frac{A}{u_0}\right), \tag{3.29}$$

where C' and A only depend on the weight of the cusp form $f(z)$ and on the level $4N$. An explicit expression for A is, therefore,

$$A = \frac{2ce^{-\frac{\pi^2}{96}}}{9r\sqrt{N}}, \tag{3.30}$$

where, as already described, r and c only depend on the cusp form $g(z)$ given by (3.19).[7] Note that C contains the numerical value of the infinite series on (3.28), which is clearly convergent because $b(n) = O(n^{\alpha})$. This proves (3.15) for $p = 0$. For every $p \geq 1$, let us return to Cauchy's integral formula (3.17) with $\lambda = \frac{1}{2}$ and use (3.29) to get

$$\left|\left[\frac{d^p}{du^p} f\left(-\frac{e^{-iu}}{\sqrt{4N}}\right)\right]_{u=u_0}\right| \leq \frac{2^p p!}{\pi u_0^{p+1}} \int_{C_{\frac{u_0}{2}}(u_0)} \left|f\left(-\frac{e^{-iw}}{\sqrt{4N}}\right)\right| |dw| < C\,\frac{2^p p!}{u_0^{p+k+\frac{1}{2}}} e^{-\frac{A}{u_0}}, \tag{3.31}$$

which completes the proof of (3.15). The proof of the uniform bound (3.16) is the same, the only difference being in taking the slash operator. Instead of the matrix γ given in (3.18), we consider

$$\gamma^{\star} = \begin{pmatrix} 1 & 0 \\ \sqrt{4N} & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}), \qquad h(z) := \left(f|\gamma^{\star}\right)(z) = \tau\left(\sqrt{4N}z+1\right)^{-k-\frac{1}{2}} f\left(\frac{z}{1+\sqrt{4N}z}\right), \tag{3.32}$$

where, as in (3.19), τ is a complex number such that $|\tau| = 1$. Applying (3.32) with

$$z := \frac{1}{\sqrt{4N}\left(e^{-iw}-1\right)} \in \mathbb{H},$$

[7] Of course, there are other possible expressions for A with a different choice of λ.

we obtain, in analogy to (3.20),

$$f\left(\frac{e^{iw}}{\sqrt{4N}}\right)=\frac{1}{\tau}\left(\frac{e^{-iw}}{e^{-iw}-1}\right)^{k+\frac{1}{2}}h\left(\frac{1}{\sqrt{4N}\left(e^{-iw}-1\right)}\right).$$

Since f is a cusp form and N is a perfect square, we know that $h(z)$ also admits the Fourier expansion

$$h(z):=\sum_{n=1}^{\infty}b^{\star}(n)\,e^{\frac{2\pi in}{r'}z},\quad\text{for some } r'\in\mathbb{N}. \tag{3.33}$$

From this point on, the proof of (3.16) is exactly the same as before and, just like (3.30), we can find an explicit expression for the constant B in (3.16) in the form

$$B=\frac{2c'e^{-\frac{\pi^2}{96}}}{9r'\sqrt{N}}, \tag{3.34}$$

where c' is the smallest integer such that $b^{\star}(c')\neq 0$ and r' is the integer appearing in the Fourier expansion (3.33).

□

4 Proof of Theorem 1.1

Throughout our proof we shall assume that $a_f(n)$ are real numbers. The case where $a_f(n)$ are purely imaginary is analogous and the necessary changes in the argument are already outlined in [21]. First we show that $N_0^{+}(T)=\Omega\left(T^{1/2}\right)$, which is the first part of (1.12). The proof that $N_0^{-}(T)=\Omega\left(T^{1/2}\right)$, as we shall see, presents no extra difficulties.

Let $(\rho_n)_{n\in\mathbb{N}}$ be the sequence of zeros of odd order of $L\left(s,f\right)+L\left(s,f|W_{4N}\right)$ such that $\text{Re}(\rho_n)=\frac{k}{2}+\frac{1}{4}$. Then we can write $\rho_n:=\frac{k}{2}+\frac{1}{4}+i\tau_n$, with $\tau_n>0$ being an increasing sequence[8]. If we show that there is some $h>0$ such that, for infinitely many values of n, $\tau_n<h\,n^2$, we are done. This is the case because, if we choose the sequence $T_n:=hn^2$, then we find that $N_0^{+}\left(T_n\right)\geq N_0^{+}(\tau_n)=n=\sqrt{\frac{T_n}{h}}$. Thus, we have built in this way a sequence $(T_n)_{n\in\mathbb{N}}$ such that $N_0^{+}\left(T_n\right)>\sqrt{\frac{T_n}{h}}$, which ultimately establishes

$$\limsup_{T\to\infty}\frac{N_0^{+}(T)}{\sqrt{T}}>\frac{1}{\sqrt{h}},\quad\text{or, equivalently,}\quad N_0^{+}(T)=\Omega\left(T^{\frac{1}{2}}\right). \tag{4.1}$$

Hence, for the sake of contradiction, let us assume that there is some N_0 such that, for every $n\geq N_0$ and any $h>0$, $\tau_n\geq h\,n^2$. We will now show that there exists some (large enough) h for which this assumption is contradicted. Indeed, if we construct the entire function[9]

$$\varphi\left(y\right)=\prod_{j=1}^{\infty}\left(1-\frac{y^2}{\tau_j^2}\right)=\sum_{j=0}^{\infty}(-1)^j a_{2j}\,y^{2j}, \tag{4.2}$$

[8] Note that if $L\left(\frac{k}{2}+\frac{1}{4},f\right)+L\left(\frac{k}{2}+\frac{1}{4},f|W_{4N}\right)=0$, we are excluding this real zero from the sequence $(\tau_n)_{n\in\mathbb{N}}$.

[9] The infinite product can be written because, due to the result in [21] (Theorem B above), we already know that $L\left(s,f\right)+L\left(s,f|W_{4N}\right)$ has infinitely many zeros of the form $\frac{k}{2}+\frac{1}{4}+i\tau_n$.

we see that $a_0 = 1$ and, for $j \geq 1$,

$$a_{2j} = \sum_{r_1 \geq 1} \sum_{r_2 > r_1} \dots \sum_{r_j > r_{j-1}} \frac{1}{\tau_{r_1}^2 \cdot \dots \cdot \tau_{r_j}^2} = \sum_{1 \leq r_1 < r_2 < \dots < r_j} \frac{1}{\tau_{r_1}^2 \cdot \dots \cdot \tau_{r_j}^2}, \tag{4.3}$$

where we are summing over $(r_1, ..., r_j) \in \mathbb{N}^j$ such that $r_1 < r_2 < ... < r_j$. Note that, in the k^{th} nested series in (4.3), the index r_k always satisfies $r_k \geq k$, due to the condition $r_k > r_{k-1} > ... > r_1 \geq 1$.

From this point on, we just need to find a suitable bound for a_{2j}. By considering one of the nested infinite series above, we have two possibilities: if, for some $1 \leq k \leq j$, $r_k \geq N_0$, we know by the contradiction hypothesis that $\tau_{r_k}^{-2} \leq \frac{r_k^{-4}}{h^2}$. On the other hand, if $1 \leq r_k \leq N_0 - 1$, then $\tau_{r_k}^{-2} \leq \frac{r_k^{-4}}{h^{\star 2}}$, where $h^\star := \min_{1 \leq n \leq N_0 - 1} \left\{ \frac{\tau_n}{n^2} \right\}$. Let us take a generic series in (4.3): by the above reasoning, if $r_{k-1} \geq N_0 - 1$, then the contradiction hypothesis gives

$$\sum_{r_k > r_{k-1}} \frac{1}{\tau_{r_k}^2} = \sum_{r_k > r_{k-1} \geq N_0 - 1} \frac{1}{\tau_{r_k}^2} \leq \frac{1}{h^2} \sum_{r_k > r_{k-1}} \frac{1}{r_k^4}. \tag{4.4}$$

On the other hand, if $1 \leq r_{k-1} < N_0 - 1$,

$$\begin{aligned}
\sum_{r_k > r_{k-1}} \frac{1}{\tau_{r_k}^2} &= \sum_{r_k = r_{k-1}+1}^{N_0 - 1} \frac{1}{\tau_{r_k}^2} + \sum_{r_k = N_0}^{\infty} \frac{1}{\tau_{r_k}^2} \leq \frac{1}{h^{\star 2}} \sum_{r_k = r_{k-1}+1}^{N_0 - 1} \frac{1}{r_k^4} + \frac{1}{h^2} \sum_{r_k = N_0}^{\infty} \frac{1}{r_k^4} \\
&= \frac{1}{h^2} \sum_{r_k > r_{k-1}} \frac{1}{r_k^4} + \frac{1}{h^2} \left(\frac{h^2}{h^{\star 2}} - 1 \right) \sum_{r_k = r_{k-1}+1}^{N_0 - 1} \frac{1}{r_k^4} \leq \frac{1}{h^2} \sum_{r_k > r_{k-1}} \frac{1}{r_k^4} + \frac{1}{h^2} \left| \frac{h^2}{h^{\star 2}} - 1 \right| \sum_{j=1}^{N_0 - 1} \frac{1}{j^4} \\
&\leq \frac{\mathscr{A}}{h^2} \sum_{r_k > r_{k-1}} \frac{1}{r_k^4},
\end{aligned} \tag{4.5}$$

for some constant $\mathscr{A}$ depending on h and N_0 but not on k. We can explicitly choose this constant by taking the observation that there exists some A such that

$$\left| \frac{h^2}{h^{\star 2}} - 1 \right| \sum_{j=1}^{N_0 - 1} \frac{1}{j^4} \leq A \sum_{j \geq N_0} \frac{1}{j^4} \leq A \sum_{r_k > r_{k-1}} \frac{1}{r_k^4},$$

where the last inequality is due to the fact that $1 \leq r_{k-1} < N_0 - 1$. Picking $\mathscr{A} := 1 + A \geq 1$ gives (4.5). Since $r_\ell \geq \ell$ always, a sufficient condition for $r_{k-1} \geq N_0 - 1$ is that $k \geq N_0$. Thus, we have at most $N_0 - 1$ infinite series in (4.3) where we need to apply (4.5). Hence, the sequence a_{2j} can be bounded in a simple form

$$\begin{aligned}
a_{2j} &= \sum_{r_1 \geq 1} \sum_{r_2 > r_1} \dots \sum_{r_j > r_{j-1}} \frac{1}{\tau_{r_1}^2 \cdot \dots \cdot \tau_{r_j}^2} = \sum_{r_1 \geq 1} \sum_{r_2 > r_1} \dots \sum_{r_{N_0} > r_{N_0 - 1} \geq N_0 - 1} \dots \sum_{r_j > r_{j-1}} \frac{1}{\tau_{r_1}^2 \cdot \dots \cdot \tau_{r_j}^2} \\
&\leq \frac{\mathscr{A}^{N_0 - 1}}{h^{2j}} \sum_{r_1 \geq 1} \sum_{r_2 > r_1} \dots \sum_{r_j > r_{j-1}} \frac{1}{r_1^4 \cdot \dots \cdot r_j^4} \leq \frac{\mathcal{K}}{h^{2j}} \sum_{r_1 \geq 1} \sum_{r_2 > r_1} \dots \sum_{r_j > r_{j-1}} \frac{1}{r_1^4 \cdot \dots \cdot r_j^4} = \frac{\mathcal{K}}{h^{2j}} \sum_{1 \leq r_1 < r_2 < \dots < r_j} \frac{1}{r_1^4 \cdot \dots \cdot r_j^4},
\end{aligned} \tag{4.6}$$

for some $\mathcal{K}$ only depending on h and N_0.

We now see that a bound for a_{2j} will depend on finding a bound for the new coefficient

$$b_{2j} := \sum_{r_1 \geq 1} \sum_{r_2 > r_1} \cdots \sum_{r_j > r_{j-1}} \frac{1}{r_1^4 \cdot \ldots \cdot r_j^4} = \sum_{1 \leq r_1 < \ldots < r_j} \frac{1}{r_1^4 \cdot \ldots \cdot r_j^4}. \tag{4.7}$$

Whoever is familiar with one of Euler's proofs of the formula for $\zeta(2n)$ recognizes this sequence of numbers as coming from the Weierstrass factorization of the sine function, which takes the form

$$\frac{\sinh(\pi\sqrt{y})\sin(\pi\sqrt{y})}{\pi^2 y} = \prod_{j=1}^{\infty}\left(1+\frac{y}{j^2}\right)\prod_{j=1}^{\infty}\left(1-\frac{y}{j^2}\right) = \prod_{j=1}^{\infty}\left(1-\frac{y^2}{j^4}\right) := 1+\sum_{j=1}^{\infty}(-1)^j b_{2j} y^{2j}. \tag{4.8}$$

Thus, we can get precise information about b_{2j} (and, consequently, about a_{2j}) by interpreting them as the Taylor coefficients of the function on the left-hand side of (4.8).[10] Indeed, it is quite simple[11] to see that, for any $y \in \mathbb{R}$,

$$\sinh(\pi\sqrt{y})\,\sin(\pi\sqrt{y}) = -\frac{i}{2}\left[\cos\left((1-i)\pi\sqrt{y}\right) - \cos\left((1+i)\pi\sqrt{y}\right)\right] = -\frac{i}{2}\sum_{k=0}^{\infty}\frac{(-1)^k\pi^{2k}}{(2k)!}y^k\left\{(1-i)^{2k}-(1+i)^{2k}\right\}$$

$$= -\sum_{k=0}^{\infty}\frac{(-1)^k\pi^{2k}}{(2k)!}(2y)^k\sin\left(\frac{\pi}{2}k\right) = \sum_{j=0}^{\infty}\frac{(-1)^j 2^{2j+1}\pi^{4j+2}}{(4j+2)!}y^{2j+1}. \tag{4.9}$$

Thus, comparing (4.8) with (4.9) and returning to the inequality (4.6), we find that

$$b_{2j} := \sum_{1 \leq r_1 < \ldots < r_j} \frac{1}{r_1^4 \cdot \ldots \cdot r_j^4} = \frac{2^{2j+1}\pi^{4j}}{(4j+2)!} \implies a_{2j} \leq \mathcal{K}\,\frac{2^{2j+1}\pi^{4j}}{h^{2j}(4j+2)!}. \tag{4.10}$$

Our proof will be concluded by seeing that (4.10) contradicts (3.7) and the bounds for the derivatives of f found in Lemma 3.2. By construction of $\varphi(t)$, we know that $R_f(t)\varphi(t)$ must have constant sign for any $t \in \mathbb{R}$. If $R_f(t)\varphi(t) \geq 0$ for any $t \in \mathbb{R}$, the continuous function $Q_f : \left(0, \frac{\pi}{2}\right) \longmapsto \mathbb{R}$ defined by the integral

$$Q_f(u) := \int_0^{\infty} R_f(t)\varphi(t)\cosh\left(\left(\frac{\pi}{2}-u\right)t\right)dt, \quad 0 < u < \frac{\pi}{2},$$

is positive decreasing. Analogously, if $R_f(t)\varphi(t) \leq 0$, $Q_f(u)$ will be negative increasing. In both cases, we have that the continuous function

$$|Q_f(u)| := \left|\int_0^{\infty} R_f(t)\varphi(t)\cosh\left(\left(\frac{\pi}{2}-u\right)t\right)dt\right|, \quad 0 < u < \frac{\pi}{2},$$

will be positive decreasing. On the other hand, if we use the power series (4.2) for $\varphi(t)$, we see that $Q_f(u)$ can be written as an infinite series of the form

$$Q_f(u) = \int_0^{\infty} R_f(t)\,\varphi(t)\cosh\left(\left(\frac{\pi}{2}-u\right)t\right)dt = \sum_{j=0}^{\infty}(-1)^j a_{2j}\int_0^{\infty} R_f(t)\,t^{2j}\cosh\left(\left(\frac{\pi}{2}-u\right)t\right)dt. \tag{4.11}$$

[10] It is somewhat poetic that the same idea that led Euler to have the first grasp on the nature of $\zeta(2n)$ can be used to understand the zeros of the same function.

[11] Although these computations are standard, we did not find any reference containing this Taylor expansion. In order to be self-contained, we have decided to briefly present it.

Note that the interchange of the orders of summation and integration in (4.11) is justified by Fubini's theorem and the bounds for a_{2j} given in (4.10): indeed,

$$\begin{aligned}
\int_0^\infty \sum_{j=0}^\infty |a_{2j}| t^{2j} \, |R_f(t)| \cosh\left(\left(\frac{\pi}{2}-u\right)t\right) dt &\le \mathcal{K} \int_0^\infty \sum_{j=0}^\infty \frac{2^{2j+1}\pi^{4j}}{h^{2j}(4j+2)!} t^{2j} \, |R_f(t)| \cosh\left(\left(\frac{\pi}{2}-u\right)t\right) dt \\
&\le 2\mathcal{K} \int_0^\infty \sum_{j=0}^\infty \frac{1}{(4j)!} \left(\frac{2\pi^2 t}{h}\right)^{2j} |R_f(t)| \cosh\left(\left(\frac{\pi}{2}-u\right)t\right) dt \\
&< 2\mathcal{K} \int_0^\infty \sum_{j=0}^\infty \frac{1}{j!} \left(\pi\sqrt{\frac{2t}{h}}\right)^{j} |R_f(t)| \cosh\left(\left(\frac{\pi}{2}-u\right)t\right) dt \\
&< 2\mathcal{K} \int_0^\infty |R_f(t)| \exp\left(\pi\sqrt{\frac{2t}{h}} + \left(\frac{\pi}{2}-u\right)t\right) dt \\
&\le 2\mathcal{K}\, C \int_0^\infty t^A \exp\left(-ut + \pi\sqrt{\frac{2t}{h}}\right) dt < \infty, \qquad (4.12)
\end{aligned}$$

because $u > 0$ by hypothesis. In the last step we have used Stirling's formula for $\Gamma(s)$ (2.7) as well as the convex estimate for $L(s,f)$ (2.8), which show that $|R_f(t)| \le C\,|t|^A e^{-\frac{\pi}{2}|t|}$ for sufficiently large C and A. Having assured that we can perform the operation (4.11), it now follows from (4.12) that the Leibniz rule holds for the integral on the right-hand side of (4.11) and, after using (3.7), we find the expression

$$\begin{aligned}
Q_f(u) &= \sum_{j=0}^\infty (-1)^j a_{2j} \frac{d^{2j}}{du^{2j}} \left\{ \int_0^\infty R_f(t) \cosh\left(\left(\frac{\pi}{2}-u\right)t\right) dt \right\} \\
&= \frac{\pi}{2} \sum_{j=0}^\infty (-1)^j\, a_{2j} \frac{d^{2j}}{du^{2j}} \left[e^{\frac{i\pi}{2}\left(\frac{k}{2}+\frac{1}{4}\right)} e^{-i\left(\frac{k}{2}+\frac{1}{4}\right)u} f\left(-\frac{e^{-iu}}{\sqrt{4N}}\right) + e^{-i\frac{\pi}{2}\left(\frac{k}{2}+\frac{1}{4}\right)} e^{i\left(\frac{k}{2}+\frac{1}{4}\right)u} f\left(\frac{e^{iu}}{\sqrt{4N}}\right) \right]. \qquad (4.13)
\end{aligned}$$

Using Lemma 3.2 and our estimates for a_{2j} (4.10), we can bound uniformly the previous series with respect to u. Firstly, according to Lemma 3.2, we have

$$\begin{aligned}
&\left| \frac{d^{2j}}{du^{2j}} \left[e^{-i\left(\frac{k}{2}+\frac{1}{4}\right)u} f\left(-\frac{e^{-iu}}{\sqrt{4N}}\right) \right] \right| \le \sum_{\ell=0}^{2j} \binom{2j}{\ell} \left(\frac{k}{2}+\frac{1}{4}\right)^{2j-\ell} \left| \frac{d^\ell}{du^\ell} f\left(-\frac{e^{-iu}}{\sqrt{4N}}\right) \right| \\
&< C \frac{e^{-\frac{A}{u}}}{u^{k+\frac{1}{2}}} \sum_{\ell=0}^{2j} \frac{(2j)!}{(2j-\ell)!} \left(\frac{k}{2}+\frac{1}{4}\right)^{2j-\ell} \frac{2^\ell}{u^\ell} \le C \frac{e^{-\frac{A}{u}}(2j)!}{u^{k+\frac{1}{2}}} \left(\frac{k}{2}+\frac{1}{4}\right)^{2j} \sum_{\ell=0}^{2j} \left(\frac{8}{(2k+1)u}\right)^\ell \\
&= C \frac{e^{-\frac{A}{u}}(2j)!}{u^{k+2j+\frac{1}{2}} 2^{4j}} \frac{8^{2j+1} - ((2k+1)u)^{2j+1}}{8-(2k+1)u}.
\end{aligned}$$

If we now pick $0 < u < \frac{4}{2k+1} < \frac{\pi}{2}$, we see immediately that

$$\left| \frac{d^{2j}}{du^{2j}} \left[e^{-i\left(\frac{k}{2}+\frac{1}{4}\right)u} f\left(-\frac{e^{-iu}}{\sqrt{4N}}\right) \right] \right| < C' \frac{2^{2j}(2j)! e^{-A/u}}{u^{2j+k+\frac{1}{2}}},$$

while

$$\left| \frac{d^{2j}}{du^{2j}} \left[e^{i\left(\frac{k}{2}+\frac{1}{4}\right)u} f\left(\frac{e^{iu}}{\sqrt{4N}}\right) \right] \right| < D' \frac{2^{2j}(2j)! e^{-B/u}}{u^{2j+k+\frac{1}{2}}}.$$

Thus

$$|Q_f(u)| \leq \frac{\pi}{2}\sum_{j=0}^{\infty}|a_{2j}|\frac{2^{2j}(2j)!}{u^{2j+k+\frac{1}{2}}}\left\{C' e^{-A/u} + D'\, e^{-B/u}\right\} \leq \frac{\pi\beta}{u^{k+\frac{1}{2}}}\, e^{-\alpha/u}\sum_{j=0}^{\infty}|a_{2j}|\frac{2^{2j}(2j)!}{u^{2j}}, \tag{4.14}$$

where $\beta = \max\{C', D'\}$ and $\alpha = \min\{A, B\}$. To finish, we must estimate the infinite series on the right-hand side of (4.14): we do this by employing (4.10), which yields

$$\begin{aligned}\sum_{j=0}^{\infty}|a_{2j}|\frac{2^{2j}(2j)!}{u^{2j}} &\leq \mathcal{K}\sum_{j=0}^{\infty}\frac{2^{4j+1}\pi^{4j}(2j)!}{h^{2j}(4j+2)!}\cdot\frac{1}{u^{2j}} = \frac{\sqrt{\pi}\,\mathcal{K}}{2}\sum_{j=0}^{\infty}\frac{\pi^{4j}}{\Gamma\left(2j+\frac{3}{2}\right)(2j+1)(hu)^{2j}} \\ &< \sqrt{\pi}\,\mathcal{K}\sum_{j=0}^{\infty}\frac{\pi^{4j}}{(2j+1)!(hu)^{2j}} < \sqrt{\pi}\,\mathcal{K}\cosh\left(\frac{\pi^2}{h\,u}\right) \leq \sqrt{\pi}\,\mathcal{K}\exp\left(\frac{\pi^2}{h\,u}\right).\end{aligned} \tag{4.15}$$

Combining (4.14) and (4.15) gives

$$|Q_f(u)| < \frac{\pi^{\frac{3}{2}}\beta\,\mathcal{K}}{u^{k+\frac{1}{2}}}\exp\left(-\left(\alpha - \frac{\pi^2}{h}\right)\frac{1}{u}\right).$$

Hence, if we pick h such that $h > \frac{\pi^2}{\alpha}$, then

$$\lim_{u\to 0^+}|Q_f(u)| < \pi^{\frac{3}{2}}\beta\,\mathcal{K}\lim_{u\to 0^+}u^{-k-\frac{1}{2}}\exp\left(-\left(\alpha-\frac{\pi^2}{h}\right)\frac{1}{u}\right) = 0,$$

which contradicts the fact that $|Q_f(u)|$ is positive decreasing. Consequently, we have found $h > \frac{\pi^2}{\alpha}$ such that $\tau_n < h\,n^2$ for infinitely many values of n. This shows (1.13) and so we complete the proof of our Theorem to $N_0^+(T)$.

Finally, we note that we can replace the value of d in (1.13) by an explicit constant depending on the properties of the cusp form f. Indeed, since $\alpha = \min\{A, B\}$, we may assume without any loss of generality that $\alpha = A$ and so α is explicitly given by (3.30). Using the inequality in (4.1) and taking $h = \frac{4\pi^2}{\alpha}$, we find that[12]

$$\limsup_{T\to\infty}\frac{N_0^+(T)}{\sqrt{T}} > \frac{e^{-\frac{\pi^2}{192}}}{6\pi}\sqrt{\frac{2c}{r\sqrt{N}}}, \tag{4.16}$$

where c and r are defined with respect to the Fourier expansion (3.21) of $g(z) := (f|\gamma)(z)$, defined by the slash operator (3.19). This completes the proof.

In order to study the number $N_0^-(T)$, the argument is the same with a small modification. Replacing $R_f(t)$ by $I_f(t)$, instead of considering $Q_f(u)$ we study the similar function

$$P_f(u) = \int_0^{\infty} tI_f(t)\,\varphi(t)\,\cosh\left(\left(\frac{\pi}{2}-u\right)t\right)\,dt,$$

where $\varphi(t)$ is given by (4.2). Invoking (3.8) and using the bounds for the coefficients a_{2j} (4.10), we can see that $|P_f(u)| \to 0$ as $u \to 0^+$, which again contradicts the first hypothesis over the zeros of $N_0^-(T)$. ■

[12] It is possible to improve the value of the explicit constant on (4.16) by taking a more careful choice of λ in (3.27). Recall that the constant (3.30) was obtained by the choice $\lambda = \frac{1}{2}$.

5 Lemmas for the proof of Theorem 1.2

Throughout this section, we will assume that $f(z)$ satisfies the conditions of Theorem 1.2. Thus, N will always be a perfect square and the Fourier coefficients, $a_f(n)$, will always be real or purely imaginary numbers.

We start with a lemma which was proved by Wilton for Ramanujan's Dirichlet series (see Lemma 3 of [28]). Wilton used it to give a Voronoï-type formula for the Riesz sum

$$\sum_{n\leq x}{}^{'}\tau(n)\,(x-n)^{\rho}\,e^{\frac{2\pi ip}{q}n},\quad 0<p<q,\;\;\rho>0,\;\;x>0.$$

To bring Wilton's result into our study, we shall adapt a lemma in Lekkerkerker's thesis [[20], Chpt. 3, Lemma 3.8], which acts as a generalization of Wilton's lemma in the setting of Dirichlet series. Our proof will take into consideration the uniform bound (2.1) for a generic cusp form f.

Lemma 5.1. *Let M be an integer ≥ 2 and define $\delta:=\delta_{z,M}$ as follows*

$$\delta=\begin{cases}0 & 0<Im(z)\leq\frac{\sqrt{N}}{\pi M},\\ 1 & Im(z)>\frac{\sqrt{N}}{\pi M}.\end{cases}\tag{5.1}$$

Then we have that

$$\sum_{m=1}^{M}a_f(m)\,e^{\frac{2\pi im}{\sqrt{4N}}z}-\delta\,f\left(\frac{z}{\sqrt{4N}}\right)=O\left(e^{-\frac{2\pi Im(z)}{\sqrt{4N}}(M+1)}\,M^{\frac{k}{2}+\frac{1}{4}}\log(M)\right)\tag{5.2}$$

uniformly in z.

Proof. Note that, if we show (5.2) for $0\leq\mathrm{Re}(z)<\sqrt{4N}$ we are done due to the periodicity of the expansion on the left-hand side of (5.2). Therefore, throughout our argument, we shall assume that $0\leq\mathrm{Re}(z)<\sqrt{4N}$ and we will let M be an integer greater than 2. Consider the power series

$$F(\zeta):=\sum_{n=1}^{\infty}a_f(n)\,\zeta^n.$$

This infinite series converges in the circle $|\zeta|=1$ because $f(z)=F\left(e^{2\pi iz}\right)$. Following Wilton and Lekkerkerker [20, 28], we define a parameter β in the following form

$$\beta:=\begin{cases}\frac{1}{2} & \text{if }\mathrm{Im}(z)>\frac{\sqrt{N}}{\pi M}\\ \frac{3}{2} & \text{if }\mathrm{Im}(z)\leq\frac{\sqrt{N}}{\pi M}\end{cases},\tag{5.3}$$

and, for each value of β, we consider a sequence of integrals, $\left(I_{\beta,M}\right)_{M\in\mathbb{N}}$, defined as follows

$$I_{\beta,M}=\frac{e^{\frac{2\pi iz}{\sqrt{4N}}(M+1)}}{2\pi i}\int\limits_{|\zeta|=e^{-\beta/M}}\frac{F(\zeta)}{\zeta^{M+1}\left(\zeta-e^{2\pi iz/\sqrt{4N}}\right)}\,d\zeta.\tag{5.4}$$

The value of the integral will be dependent on the choice of the parameter β in (5.3). We are left with two possibilities:

1. First, let $\beta = \frac{3}{2}$, so that $\mathrm{Im}(z) \le \frac{\sqrt{N}}{\pi M}$. Then $\left|e^{2\pi i z/\sqrt{4N}}\right| = e^{-\frac{\pi \mathrm{Im}(z)}{\sqrt{N}}} \ge e^{-\frac{1}{M}} > e^{-\frac{3}{2M}}$. Hence, the only pole that the above integrand has inside the circle $|\zeta| = e^{-\frac{3}{2M}}$ is located at $\zeta = 0$ and its order is $M+1$. A standard computation of the residue at this point gives

$$I_{\frac{3}{2},M} = -\sum_{m=1}^{M} a_f(m)\, e^{\frac{2\pi i z}{\sqrt{4N}} m}. \tag{5.5}$$

2. Second, let $\beta = \frac{1}{2}$, so that $\mathrm{Im}(z) > \frac{\sqrt{N}}{\pi M}$. Therefore, $\left|e^{2\pi i z/\sqrt{4N}}\right| = e^{-\frac{\pi \mathrm{Im}(z)}{\sqrt{N}}} < e^{-\frac{1}{M}} < e^{-\frac{1}{2M}}$, which shows that the point $e^{2\pi i z/\sqrt{4N}}$ is located inside the circle $|\zeta| = e^{-\frac{1}{2M}}$. Thus, besides the pole at $\zeta = 0$, the integrand in (5.4) has an additional simple pole, $\zeta = e^{2\pi i z/\sqrt{4N}}$. Once more, a standard application of the residue theorem yields

$$I_{\frac{1}{2},M} = F\left(e^{\frac{2\pi i z}{\sqrt{4N}}}\right) - \sum_{m=1}^{M} a_f(m)\, e^{\frac{2\pi i z}{\sqrt{4N}} m} = f\left(\frac{z}{\sqrt{4N}}\right) - \sum_{m=1}^{M} a_f(m)\, e^{\frac{2\pi i z}{\sqrt{4N}} m}. \tag{5.6}$$

Note that, in the cases (5.5) and (5.6), the symmetric value of the left-hand side of (5.2) is displayed. Hence, if we show that $I_{\beta,M}$ satisfies the bound at the right-hand side of (5.2), we finish our argument. Indeed,

$$\begin{aligned} |I_{\beta,M}| &\le \frac{e^{-\frac{2\pi \mathrm{Im}(z)}{\sqrt{4N}}(M+1)} e^{\frac{3}{2}\left(1+\frac{1}{M}\right)}}{2\pi} \int_0^{2\pi} \frac{\left|f\left(\frac{\theta}{2\pi} + \frac{i\beta}{2\pi M}\right)\right|}{\left|1 - e^{\frac{\beta}{M} + \frac{2\pi i z}{\sqrt{4N}} - i\theta}\right|}\, d\theta \\ &\le \frac{K\, e^{-\frac{2\pi \mathrm{Im}(z)}{\sqrt{4N}}(M+1)} e^{\frac{3}{2}\left(1+\frac{1}{M}\right)}}{2\pi} \left(\frac{\beta}{2\pi M}\right)^{-\frac{k}{2}-\frac{1}{4}} \int_0^{2\pi} \frac{d\theta}{\left|1 - e^{\frac{\beta}{M} - \frac{2\pi \mathrm{Im}(z)}{\sqrt{4N}} + i\left(\frac{2\pi \mathrm{Re}(z)}{\sqrt{4N}} - \theta\right)}\right|}, \end{aligned} \tag{5.7}$$

where in the last step we have used the fact that $|f(x+iy)| = O\left(y^{-\frac{k}{2}-\frac{1}{4}}\right)$ uniformly in x (2.1). Comparing (5.7) with our aim (5.2), in order to finish our proof we just need to show that

$$\int_0^{2\pi} \frac{d\theta}{\left|1 - e^{\frac{\beta}{M} - \frac{2\pi \mathrm{Im}(z)}{\sqrt{4N}} + i\left(\frac{2\pi \mathrm{Re}(z)}{\sqrt{4N}} - \theta\right)}\right|} = O\left(\log(M)\right). \tag{5.8}$$

To do so, let us take a new variable $\eta := \frac{\beta}{M} - \frac{2\pi \mathrm{Im}(z)}{\sqrt{4N}}$. Once more, we have two possibilities on the range of η: if $\beta = \frac{1}{2}$, then $|\eta| > \frac{1}{2M}$. On the other hand, if $\beta = \frac{3}{2}$, then $|\eta| \ge \frac{1}{2M}$. Therefore, we shall estimate (5.8) only assuming that $\beta = \frac{3}{2}$, as for the case $\beta = \frac{1}{2}$, the same computations are allowed.

If we set $\varphi := \frac{2\pi \mathrm{Re}(z)}{\sqrt{4N}}$, then by the hypothesis $0 \le \mathrm{Re}(z) < \sqrt{4N}$, the range of φ will be $0 \le \varphi < 2\pi$. Taking this change of variables, we just need to estimate the integral

$$J(\eta) := \int_{\varphi - 2\pi}^{\varphi} \frac{d\phi}{|1 - e^{\eta + i\phi}|},$$

subject to the condition that $|\eta| > \frac{1}{2M}$. First, if we assume that $|\eta| \geq 1$, then $J(\eta)$ can be simply estimated as follows

$$J(\eta) = \int_{\varphi-2\pi}^{\varphi} \frac{d\phi}{|1-e^{\eta+i\phi}|} \leq \frac{2\pi}{|1-e^{\eta}|} \leq \frac{2\pi}{1-e^{-1}},$$

which shows that our integral is uniformly bounded in this range. Thus, from now on we may suppose that $\frac{1}{2M} \leq |\eta| < 1$. Since

$$\left|1-e^{\eta+i\phi}\right| = \sqrt{(1-e^{\eta})^2 + 4e^{\eta}\sin^2\left(\frac{\phi}{2}\right)} \geq \sqrt{\left(1-e^{-\frac{1}{2M}}\right)^2 + 4e^{\eta}\sin^2\left(\frac{\phi}{2}\right)},$$

we have that

$$\int_{\varphi-2\pi}^{\varphi} \frac{d\phi}{|1-e^{\eta+i\phi}|} \leq 2\int_{\frac{\varphi}{2}-\pi}^{\frac{\varphi}{2}} \frac{d\phi}{\sqrt{\left(1-e^{-\frac{1}{2M}}\right)^2 + 4e^{\eta}\sin^2(\psi)}} \leq 2\int_{-\pi}^{\pi} \frac{d\psi}{\sqrt{\left(1-e^{-\frac{1}{2M}}\right)^2 + 4e^{\eta}\sin^2(\psi)}}$$

$$\leq 8\int_{0}^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{\left(\frac{1}{2M}-\frac{1}{8M^2}\right)^2 + \frac{4}{e}\sin^2(\psi)}} < 8\int_{0}^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{\frac{1}{16M^2} + \frac{4}{e}\sin^2(\psi)}}, \tag{5.9}$$

where we have used the fact that $M \geq 2$ and $1-e^{-x} \geq x - \frac{x^2}{2}$, as well as the hypothesis $\frac{1}{2M} \leq |\eta| < 1$. Now, by Jordan's inequality, $\sin(\psi) > \frac{2}{\pi}\psi$, $0 < \psi < \frac{\pi}{2}$, we finally reach the bound

$$\int_{0}^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{\frac{1}{16M^2} + \frac{4}{e}\sin^2(\psi)}} < \int_{0}^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{\frac{1}{16M^2} + \frac{16}{\pi^2 e}\psi^2}} < 4\int_{0}^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{\frac{1}{M^2} + \psi^2}}$$

$$= 4\int_{0}^{1/M} \frac{d\psi}{\sqrt{\frac{1}{M^2} + \psi^2}} + 4\int_{1/M}^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{\frac{1}{M^2} + \psi^2}} < 4 + 4\int_{1/M}^{\frac{\pi}{2}} \frac{d\psi}{\psi} = 4\log\left(\frac{\pi e}{2}M\right) = O\left(\log(M)\right), \tag{5.10}$$

which finishes the proof of (5.8) and, consequently, the proof of the whole lemma. □

Remark 5.1. Consider the Fourier expansion of the Fricke involution of f,

$$(f|W_{4N})\left(\frac{z}{\sqrt{4N}}\right) = \sum_{n=1}^{\infty} a_{f|W_{4N}}(n)\, e^{\frac{2\pi i z}{\sqrt{4N}}}.$$

Then the previous lemma gives

$$\sum_{m=1}^{M} a_{f|W_{4N}}(m)\, e^{\frac{2\pi i m}{\sqrt{4N}}z} - \delta\,(f|W_{4N})\left(\frac{z}{\sqrt{4N}}\right) = O\left(e^{-\frac{2\pi \mathrm{Im}(z)}{\sqrt{4N}}(M+1)}\, M^{\frac{k}{2}+\frac{1}{4}}\log(M)\right)$$

uniformly in z.

Since our proof of Theorem 1.2 will use Lekkerkerker's variant of the Hardy-Littlewood method, we shall reach a point in the argument below where we will have to find a lower bound for the function

$$J(t) := \int_{t}^{t+H} |R_f(u)|\, e^{\left(\frac{\pi}{2}-\epsilon\right)u} du,$$

where $R_f(u)$ is given by (3.3). The following simple lemma will be crucial to achieve this lower bound. We omit its proof, as it is standard.

Lemma 5.2. *Let c be the smallest positive integer for which $a_f(c) \neq 0$. Consider the Dirichlet series*

$$L^{\star}(s,f) := c^s L(s,f) - a_f(c). \tag{5.11}$$

Also, let $L(s,f_1)$ be defined as

$$L(s,f_1) := \sum_{n=c+1}^{\infty} \frac{a_f(n)}{\log(n/c)\, n^s}, \quad Re(s) > \frac{k}{2} + \frac{5}{4}. \tag{5.12}$$

Then $L(s,f_1)$ can be analytically continued as an entire function of s and, for any $s \in \mathbb{C}$, the following relation holds

$$c^{-s} \int_s^{\infty} L^{\star}(z,f)\, dz = L(s,f_1). \tag{5.13}$$

Remark 5.2. The previous lemma also holds when we replace $f(z)$ by $(f|W_{4N})(z)$. Indeed, if d is the smallest positive integer such that $a_{f|W_{4N}}(d) \neq 0$, then, considering the Dirichlet series

$$L^{\star}(s, f|W_{4N}) := d^s L(s, f|W_{4N}) - a_{f|W_{4N}}(d), \tag{5.14}$$

as well as

$$L(s,(f|W_{4N})_1) := \sum_{n=d+1}^{\infty} \frac{a_{f|W_{4N}}(n)}{\log(n/d)\, n^s}, \quad \mathrm{Re}(s) > \frac{k}{2} + \frac{5}{4}, \tag{5.15}$$

one may see that $L(s,(f|W_{4N})_1)$ can be analytically continued as an entire function of s. The relation analogous to (5.13) in this case is

$$d^{-s} \int_s^{\infty} L^{\star}(z, f|W_{4N})\, dz = L(s,(f|W_{4N})_1). \tag{5.16}$$

The previous lemma defined a Dirichlet series, $L(s,f_1)$, which possesses certain characteristics that will be of interest in the forthcoming argument. To get some of these properties, we need to attach it to a series resembling a cusp form,

$$f_1\left(\frac{z}{\sqrt{4N}}\right) := \sum_{n=c+1}^{\infty} \frac{a_f(n)}{\log(n/c)}\, e^{\frac{2\pi i n}{\sqrt{4N}} z}.$$

Our next lemma will provide bounds for $f_1\left(z/\sqrt{4N}\right)$ in two ranges of $\mathrm{Im}(z)$. The estimates given below will be of crucial importance for an application of Parseval's formula in the proof of Claim 6.1 at the end of this section. The proof of the next lemma is essentially the same as Lekkerkerker's [[20], Chapt. 3, Lemma 10 and Theorem 12], but we present it in a simpler way, avoiding the erroneous lemma 7 of his thesis. We also remark that there is an alternative argument using Fourier analysis (see Remark 5.4 below).

Lemma 5.3. *Let $z \in \mathbb{H}$ and consider*

$$f_1\left(\frac{z}{\sqrt{4N}}\right) := \sum_{n=c+1}^{\infty} \frac{a_f(n)}{\log(n/c)}\, e^{\frac{2\pi i n}{\sqrt{4N}} z}, \tag{5.17}$$

where c is the smallest integer for which $a_f(c) \neq 0$. Then, for all $Im(z) \geq Im(z_0) > 0$, $f_1\left(z/\sqrt{4N}\right)$ satisfies the estimate

$$f_1\left(\frac{z}{\sqrt{4N}}\right) \ll e^{-\frac{2\pi}{\sqrt{4N}} Im(z)}. \tag{5.18}$$

Moreover, for small $Im(z)$, $f_1\left(z/\sqrt{4N}\right)$ satisfies

$$f_1\left(\frac{z}{\sqrt{4N}}\right) \ll \frac{1}{Im(z)^{\frac{k}{2}+\frac{1}{4}}} \left(\log\left(\frac{1}{Im(z)}\right)\right)^{-1}. \tag{5.19}$$

Proof. Let us define the integer $\ell := 1 + \left[\frac{\sqrt{N}}{\pi \mathrm{Im}(z)}\right]$. Moreover, for $M \in \mathbb{N}_0$, consider $P_M\left(z/\sqrt{4N}\right)$ and $Q_M\left(z/\sqrt{4N}\right)$ as the functions defined by

$$P_0(0) := 0, \qquad P_M\left(\frac{z}{\sqrt{4N}}\right) = \sum_{m=1}^{M} a_f(m)\, e^{\frac{2\pi i m}{\sqrt{4N}} z}, \tag{5.20}$$

$$Q_M(z) = P_M(z) - f\left(\frac{z}{\sqrt{4N}}\right). \tag{5.21}$$

Note that $P_M(z) = 0$ if $M \leq c-1$. By estimate (5.2) given at lemma 5.1, we have

$$P_M\left(\frac{z}{\sqrt{4N}}\right) = O\left(e^{-\frac{2\pi \mathrm{Im}(z)}{\sqrt{4N}}(M+1)}\, M^{\frac{k}{2}+\frac{1}{4}} \log(M)\right), \text{ for } M = 2, 3, ..., \ell - 1, \tag{5.22}$$

$$Q_M\left(\frac{z}{\sqrt{4N}}\right) = O\left(e^{-\frac{2\pi \mathrm{Im}(z)}{\sqrt{4N}}(M+1)}\, M^{\frac{k}{2}+\frac{1}{4}} \log(M)\right), \text{ for } M = \ell, \ell+1, ..., \tag{5.23}$$

uniformly in z. Estimate (5.22) holds because if $2 \leq M \leq \ell - 1 \leq \frac{\sqrt{N}}{\pi \mathrm{Im}(z)}$, then $\mathrm{Im}(z) \leq \frac{\sqrt{N}}{\pi M}$, and so we can apply (5.2) with $\delta = 0$. Analogously, the estimate (5.23) follows from (5.2) with $\delta = 1$.

We shall use these bounds to estimate the modified "polynomial"

$$P_{1,M}\left(\frac{z}{\sqrt{4N}}\right) := \sum_{m=c+1}^{M} \frac{a_f(m)}{\log(m/c)} e^{\frac{2\pi i m}{\sqrt{4N}} z}. \tag{5.24}$$

Note that, as $M \to \infty$, $P_{1,M}\left(z/\sqrt{4N}\right) \to f_1\left(z/\sqrt{4N}\right)$. Due to the choice of ℓ, we have two possibilities: $\ell \geq c+1$ or $1 \leq \ell \leq c$. By definition of $\ell := 1 + \left[\frac{\sqrt{N}}{\pi \mathrm{Im}(z)}\right]$, $\mathrm{Im}(z) \leq \frac{\sqrt{N}}{\pi c}$ in the first case and, in the second,

$\mathrm{Im}(z) \geq \frac{\sqrt{N}}{\pi c}$. Using partial summation and assuming first that $\ell \geq c+1$, we find that

$$P_{1,M}\left(\frac{z}{\sqrt{4N}}\right) = \sum_{m=c+1}^{M} \frac{a_f(m)}{\log(m/c)} e^{\frac{2\pi i m}{\sqrt{4N}} z} = \sum_{m=c+1}^{\ell} \frac{a_f(m)}{\log(m/c)} e^{\frac{2\pi i m}{\sqrt{4N}} z} + \sum_{m=\ell+1}^{M} \frac{a_f(m)}{\log(m/c)} e^{\frac{2\pi i m}{\sqrt{4N}} z}$$

$$= \sum_{m=c+1}^{\ell} \left\{P_m\left(\frac{z}{\sqrt{4N}}\right) - P_{m-1}\left(\frac{z}{\sqrt{4N}}\right)\right\} \frac{1}{\log(m/c)} + \sum_{m=\ell+1}^{M} \left\{Q_m\left(\frac{z}{\sqrt{4N}}\right) - Q_{m-1}\left(\frac{z}{\sqrt{4N}}\right)\right\} \frac{1}{\log(m/c)}$$

$$= \sum_{m=c+1}^{\ell-1} P_m\left(\frac{z}{\sqrt{4N}}\right) \left\{\frac{1}{\log(m/c)} - \frac{1}{\log((m+1)/c)}\right\} + P_\ell\left(\frac{z}{\sqrt{4N}}\right) \frac{1}{\log(\ell/c)} - P_c\left(\frac{z}{\sqrt{4N}}\right) \frac{1}{\log((c+1)/c)}$$

$$+ \sum_{m=\ell}^{M-1} Q_m\left(\frac{z}{\sqrt{4N}}\right) \left\{\frac{1}{\log(m/c)} - \frac{1}{\log((m+1)/c)}\right\} + Q_M\left(\frac{z}{\sqrt{4N}}\right) \frac{1}{\log(M/c)} - Q_\ell\left(\frac{z}{\sqrt{4N}}\right) \frac{1}{\log((\ell+1)/c)}. \tag{5.25}$$

Recalling that $P_\ell(z/\sqrt{4N}) - Q_\ell(z/\sqrt{4N}) = f\left(z/\sqrt{4N}\right)$, we deduce

$$P_{1,M}\left(\frac{z}{\sqrt{4N}}\right) = \sum_{m=c+1}^{\ell-1} P_m\left(\frac{z}{\sqrt{4N}}\right) \left\{\frac{1}{\log(m/c)} - \frac{1}{\log((m+1)/c)}\right\} + \frac{f\left(z/\sqrt{4N}\right)}{\log(\ell/c)} + \frac{Q_M\left(z/\sqrt{4N}\right)}{\log(M/c)}$$

$$+Q_\ell\left(\frac{z}{\sqrt{4N}}\right) \left\{\frac{1}{\log(\ell/c)} - \frac{1}{\log((\ell+1)/c)}\right\} + \sum_{m=\ell}^{M-1} Q_m\left(\frac{z}{\sqrt{4N}}\right) \left\{\frac{1}{\log(m/c)} - \frac{1}{\log((m+1)/c)}\right\} - \frac{P_c\left(z/\sqrt{4N}\right)}{\log((c+1)/c)}. \tag{5.26}$$

Analogously, if $1 \leq \ell \leq c$, we have

$$P_{1,M}\left(\frac{z}{\sqrt{4N}}\right) = \sum_{m=c+1}^{M} \frac{a_f(m)}{\log(m/c)} e^{\frac{2\pi i m}{\sqrt{4N}} z} = \sum_{m=\ell+1}^{M} \left\{Q_m\left(\frac{z}{\sqrt{4N}}\right) - Q_{m-1}\left(\frac{z}{\sqrt{4N}}\right)\right\} \frac{1}{\log(m/c)}$$

$$= \sum_{m=c+1}^{M-1} Q_m\left(\frac{z}{\sqrt{4N}}\right) \left\{\frac{1}{\log(m/c)} - \frac{1}{\log((m+1)/c)}\right\} + Q_M\left(\frac{z}{\sqrt{4N}}\right) \frac{1}{\log(M/c)} - Q_c\left(\frac{z}{\sqrt{4N}}\right) \frac{1}{\log((c+1)/c)}. \tag{5.27}$$

Taking $M \to \infty$ in both expressions (5.26) and (5.27) and also invoking (5.23), we see that

$$f_1\left(\frac{z}{\sqrt{4N}}\right) = \sum_{m=c+1}^{\ell-1} P_m\left(\frac{z}{\sqrt{4N}}\right) \left\{\frac{1}{\log(m/c)} - \frac{1}{\log((m+1)/c)}\right\} + \frac{f\left(z/\sqrt{4N}\right)}{\log(\ell/c)} - P_c\left(\frac{z}{\sqrt{4N}}\right) \frac{1}{\log((c+1)/c)}$$

$$+ \sum_{m=\ell}^{\infty} Q_m\left(\frac{z}{\sqrt{4N}}\right) \left\{\frac{1}{\log(m/c)} - \frac{1}{\log((m+1)/c)}\right\} + Q_\ell\left(\frac{z}{\sqrt{4N}}\right) \left\{\frac{1}{\log(\ell/c)} - \frac{1}{\log((\ell+1)/c)}\right\}, \tag{5.28}$$

whenever $\ell \geq c+1$, while

$$f_1\left(\frac{z}{\sqrt{4N}}\right) = \sum_{m=c+1}^{\infty} Q_m\left(\frac{z}{\sqrt{4N}}\right) \left\{\frac{1}{\log(m/c)} - \frac{1}{\log((m+1)/c)}\right\} - Q_c\left(\frac{z}{\sqrt{4N}}\right) \frac{1}{\log((c+1)/c)}, \tag{5.29}$$

if $1 \leq \ell \leq c$.

The rest of the proof will consist in estimating the terms on the right-hand sides of (5.28) and (5.29). We start by dealing with (5.28) and so we are under the hypothesis that $\ell \geq c+1$. Using the mean value theorem for the function $F(x) = 1/\log(x/c)$, as well as the inequalities (5.22) and (5.23), we see that there exists some constant $C_1 > 0$ such that

$$\left| f_1\left(\frac{z}{\sqrt{4N}}\right)\right| < C_1 \left\{ \sum_{m=2}^{\infty} e^{-\frac{2\pi \mathrm{Im}(z)}{\sqrt{4N}}(m+1)} \frac{m^{\frac{k}{2}-\frac{3}{4}}}{\log(m)} + \frac{\left|f\left(z/\sqrt{4N}\right)\right|}{\log(\ell/c)} + \frac{\left|Q_\ell\left(z/\sqrt{4N}\right)\right|}{\ell \log^2(\ell/c)} + \frac{\left|P_c\left(z/\sqrt{4N}\right)\right|}{\log((c+1)/c)} \right\}. \tag{5.30}$$

If $\frac{\sqrt{N}}{\pi c} \geq \mathrm{Im}(z) \geq \mathrm{Im}(z_0) > 0$, then $P_c(z/\sqrt{4N}) \ll e^{-2\pi \mathrm{Im}(z)/\sqrt{4N}}$. Thus, (5.30) gives

$$\begin{aligned} \left| f_1\left(\frac{z}{\sqrt{4N}}\right)\right| &< C \left\{ \sum_{m=2}^{\infty} e^{-\frac{2\pi \mathrm{Im}(z)}{\sqrt{4N}}(m+1)} \frac{m^{\frac{k}{2}-\frac{3}{4}}}{\log(m)} + \frac{\left|f\left(z/\sqrt{4N}\right)\right|}{\log(\ell/c)} + \frac{\left|Q_\ell\left(z/\sqrt{4N}\right)\right|}{\ell \log^2(\ell/c)} \right\} \\ &\leq C \left\{ e^{-\frac{2\pi \mathrm{Im}(z)}{\sqrt{4N}}} \sum_{m=2}^{\infty} e^{-\frac{2\pi \mathrm{Im}(z)}{\sqrt{4N}}m} m^{\frac{k}{2}-\frac{3}{4}} + \frac{\left|f\left(z/\sqrt{4N}\right)\right|}{\log(\ell/c)} + \frac{\left|Q_\ell\left(z/\sqrt{4N}\right)\right|}{\ell \log^2(\ell/c)} \right\} \\ &< C' \left\{ \frac{e^{-\frac{2\pi \mathrm{Im}(z)}{\sqrt{4N}}}}{\mathrm{Im}(z)^{\frac{k}{2}+\frac{1}{4}}} + \left(\log\left(\frac{\sqrt{N}}{\pi \mathrm{Im}(z)}\right)\right)^{-1} \left|f\left(\frac{z}{\sqrt{4N}}\right)\right| + \frac{\mathrm{Im}(z)\left|Q_\ell\left(\frac{z}{\sqrt{4N}}\right)\right|}{\sqrt{N}} \left(\log\left(\frac{\sqrt{N}}{\pi \mathrm{Im}(z)}\right)\right)^{-2} \right\} \\ &< D\, e^{-\frac{2\pi \mathrm{Im}(z)}{\sqrt{4N}}}, \end{aligned} \tag{5.31}$$

for some positive D. At the final step we have used the fact that $\ell > \frac{\sqrt{N}}{\pi \mathrm{Im}(z)}$, as well as the elementary estimate

$$\sum_{m=2}^{\infty} m^{h-1} e^{-m\alpha} = O\left(\alpha^{-h}\right), \quad h, \alpha > 0. \tag{5.32}$$

Moreover, the bounds for $\left|f\left(z/\sqrt{4N}\right)\right|$ and $\left|Q_\ell(z/\sqrt{4N})\right|$ are obtained using the elementary observation that, for $\frac{\sqrt{N}}{\pi c} \geq \mathrm{Im}(z) \geq \mathrm{Im}(z_0) > 0$,

$$\begin{aligned} \left| f\left(\frac{z}{\sqrt{4N}}\right)\right| &\leq e^{-\frac{2\pi \mathrm{Im}(z)}{\sqrt{4N}}} \sum_{n=1}^{\infty} |a_f(n)|\, e^{-\frac{2\pi \mathrm{Im}(z)}{\sqrt{4N}}(n-1)} \\ &\leq e^{-\frac{2\pi \mathrm{Im}(z)}{\sqrt{4N}}} \sum_{n=1}^{\infty} |a_f(n)|\, e^{-\frac{2\pi \mathrm{Im}(z_0)}{\sqrt{4N}}(n-1)} = O\left(e^{-\frac{2\pi \mathrm{Im}(z)}{\sqrt{4N}}}\right). \end{aligned} \tag{5.33}$$

Analogously, for the case where $1 \leq \ell \leq c$ we already know that $\mathrm{Im}(z) \geq \frac{\sqrt{N}}{\pi c}$, so we can bound (5.29) using (5.33) with $\mathrm{Im}(z_0)$ replaced by $\frac{\sqrt{N}}{\pi c}$. Therefore, (5.31) holds also in the second case $1 \leq \ell \leq c$. This gives our first bound (5.18).

We will now get (5.19). From now on we shall assume that $0 < \mathrm{Im}(z) < \min\left\{\frac{\sqrt{N}}{3\pi}, \frac{\sqrt{N}}{\pi c}\right\}$ and our goal will be to prove that the estimate

$$f_1\left(\frac{z}{\sqrt{4N}}\right) \ll \frac{1}{\mathrm{Im}(z)^{\frac{k}{2}+\frac{1}{4}}} \left(\log\left(\frac{1}{\mathrm{Im}(z)}\right)\right)^{-1} \tag{5.34}$$

holds in this range. By the hypothesis $0 < \mathrm{Im}(z) < \min\left\{\frac{\sqrt{N}}{3\pi}, \frac{\sqrt{N}}{\pi c}\right\}$, we know that $\ell \geq \max\{4, c+1\}$ by definition of ℓ. Thus, in order to prove (5.34), we will use the expression for $f_1(z/\sqrt{4N})$ holding for $\ell \geq c+1$, which is (5.28). Recalling the starting point (5.30) and some of the steps given in (5.31), we know that $f_1(z/\sqrt{4N})$ can be bounded in the form

$$\left|f_1\left(\frac{z}{\sqrt{4N}}\right)\right| \ll e^{-\frac{2\pi \mathrm{Im}(z)}{\sqrt{4N}}} \sum_{m=2}^{\infty} e^{-\frac{2\pi \mathrm{Im}(z)}{\sqrt{4N}}m} m^{\frac{k}{2}-\frac{3}{4}} + \left(\log\left(\frac{\sqrt{N}}{\pi \mathrm{Im}(z)}\right)\right)^{-1} \left|f\left(\frac{z}{\sqrt{4N}}\right)\right|$$
$$+\frac{\mathrm{Im}(z)}{\sqrt{N}}\left(\log\left(\frac{\sqrt{N}}{\pi \mathrm{Im}(z)}\right)\right)^{-2}\left|Q_\ell(z/\sqrt{4N})\right| + \frac{\left|P_c\left(z/\sqrt{4N}\right)\right|}{\log\left((c+1)/c\right)}. \tag{5.35}$$

From now on, we shall proceed by estimating each term individually. First, by (5.20), $P_c\left(z/\sqrt{4N}\right) = O(1)$ in the range $0 < \mathrm{Im}(z) < \min\left\{\frac{\sqrt{N}}{3\pi}, \frac{\sqrt{N}}{\pi c}\right\}$. Furthermore, using the uniform bound (2.1) for $f\left(z/\sqrt{4N}\right)$, we know that

$$\left(\log\left(\frac{\sqrt{N}}{\pi \mathrm{Im}(z)}\right)\right)^{-1}\left|f\left(\frac{z}{\sqrt{4N}}\right)\right| \ll \frac{1}{\mathrm{Im}(z)^{\frac{k}{2}+\frac{1}{4}}}\left(\log\left(\frac{1}{\mathrm{Im}(z)}\right)\right)^{-1}. \tag{5.36}$$

In the meantime, in order to treat $|Q_\ell\left(z/\sqrt{4N}\right)|$, we use (5.21) and (2.1) to obtain

$$\left|Q_\ell\left(\frac{z}{\sqrt{4N}}\right)\right| \leq \left|f\left(\frac{z}{\sqrt{4N}}\right)\right| + \left|P_\ell\left(\frac{z}{\sqrt{4N}}\right)\right| \leq \frac{\mathcal{A}}{\mathrm{Im}(z)^{\frac{k}{2}+\frac{1}{4}}} + \mathcal{B}_1 \ell^{\frac{k}{2}+\frac{1}{4}} \log(\ell)\, e^{-\frac{2\pi \mathrm{Im}(z)}{\sqrt{4N}}(\ell+1)}$$
$$\leq \frac{\mathcal{A}}{\mathrm{Im}(z)^{\frac{k}{2}+\frac{1}{4}}} + \frac{\mathcal{B}_2}{\mathrm{Im}(z)^{\frac{k}{2}+\frac{1}{4}}}\log\left(\frac{1}{\mathrm{Im}(z)}\right),$$

which, in its turn, provides the estimate

$$\frac{\mathrm{Im}(z)}{\sqrt{N}}\left(\log\left(\frac{\sqrt{N}}{\pi \mathrm{Im}(z)}\right)\right)^{-2}\left|Q_\ell\left(\frac{z}{\sqrt{4N}}\right)\right| \ll \left(\log\left(\frac{1}{\mathrm{Im}(z)}\right)\right)^{-2}\left\{\frac{1}{\mathrm{Im}(z)^{\frac{k}{2}-\frac{3}{4}}} + \frac{1}{\mathrm{Im}(z)^{\frac{k}{2}-\frac{3}{4}}}\log\left(\frac{1}{\mathrm{Im}(z)}\right)\right\}$$
$$\ll \frac{1}{\mathrm{Im}(z)^{\frac{k}{2}+\frac{1}{4}}}\left(\log\left(\frac{1}{\mathrm{Im}(z)}\right)\right)^{-1}. \tag{5.37}$$

If we now return to (5.35) and combine (5.36) with (5.37), we deduce that

$$f_1\left(\frac{z}{\sqrt{4N}}\right) = O\left(\sum_{m=2}^{\infty} e^{-\frac{2\pi \mathrm{Im}(z)}{\sqrt{4N}}m}\frac{m^{\frac{k}{2}-\frac{3}{4}}}{\log(m)}\right) + O\left(\frac{1}{\mathrm{Im}(z)^{\frac{k}{2}+\frac{1}{4}}}\left(\log\left(\frac{1}{\mathrm{Im}(z)}\right)\right)^{-1}\right) + O(1). \tag{5.38}$$

The second O term already contains the desired bound (5.34). We shall now estimate the first O term containing the infinite series by using (5.32), from which we deduce

$$\sum_{m=2}^{\infty} e^{-\frac{2\pi \mathrm{Im}(z)}{\sqrt{4N}}m} \frac{m^{\frac{k}{2}-\frac{3}{4}}}{\log(m)} = \sum_{m=2}^{\ell} e^{-\frac{2\pi \mathrm{Im}(z)}{\sqrt{4N}}m} \frac{m^{\frac{k}{2}-\frac{3}{4}}}{\log(m)} + \sum_{m=\ell+1}^{\infty} e^{-\frac{2\pi \mathrm{Im}(z)}{\sqrt{4N}}m} \frac{m^{\frac{k}{2}-\frac{3}{4}}}{\log(m)}$$
$$< \sum_{m=2}^{\ell} e^{-\frac{2\pi \mathrm{Im}(z)}{\sqrt{4N}}m} \frac{m^{\frac{k}{2}-\frac{3}{4}}}{\log(m)} + \frac{1}{\log(\ell)} \sum_{m=\ell+1}^{\infty} m^{\frac{k}{2}-\frac{3}{4}} e^{-\frac{2\pi \mathrm{Im}(z)}{\sqrt{4N}}m}$$
$$< \sum_{m=2}^{\ell} e^{-\frac{2\pi \mathrm{Im}(z)}{\sqrt{4N}}m} \frac{m^{\frac{k}{2}-\frac{3}{4}}}{\log(m)} + O\left\{\left(\log\left(\frac{\sqrt{N}}{\pi \mathrm{Im}(z)}\right)\right)^{-1} \mathrm{Im}(z)^{-\frac{k}{2}-\frac{1}{4}}\right\}$$
$$= \sum_{m=2}^{\ell} e^{-\frac{2\pi \mathrm{Im}(z)}{\sqrt{4N}}m} \frac{m^{\frac{k}{2}-\frac{3}{4}}}{\log(m)} + O\left\{\left(\log\left(\frac{1}{\mathrm{Im}(z)}\right)\right)^{-1} \mathrm{Im}(z)^{-\frac{k}{2}-\frac{1}{4}}\right\}. \tag{5.39}$$

The second term of (5.39) already contains what we want, this is, (5.34). Therefore, if we show that a similar estimate takes place for the finite sum in (5.39), we are done. This final estimate will depend on the range of k, for which we have two possibilities: $k = 1$ or $k \geq 2$.

1. If $k = 1$, we need to sum over the expression $\frac{1}{m^{\frac{1}{4}} \log(m)}$. Since the real function $f(x) = \frac{1}{x^{1/4} \log(x)}$ is steadily decreasing for $x \geq 2$, we have

$$\sum_{m=2}^{\ell} e^{-\frac{2\pi \mathrm{Im}(z)}{\sqrt{4N}}m} \frac{m^{-\frac{1}{4}}}{\log(m)} < \sum_{m=2}^{\ell} \frac{m^{-\frac{1}{4}}}{\log(m)} < \frac{2^{-\frac{1}{4}}}{\log(2)} + \int_2^{\ell} \frac{x^{-1/4}}{\log(x)}\, dx$$
$$= \frac{2^{-\frac{1}{4}}}{\log(2)} + \int_2^{\sqrt{\ell}} \frac{x^{-1/4}}{\log(x)}\, dx + \int_{\sqrt{\ell}}^{\ell} \frac{x^{-1/4}}{\log(x)}\, dx$$
$$< \frac{2^{-\frac{1}{4}}}{\log(2)} + \frac{4\ell^{\frac{3}{8}}}{3\log(2)} + \frac{8\ell^{\frac{3}{4}}}{3\log(\ell)} < \frac{4\ell^{\frac{3}{8}}}{3\log(2)} + \frac{16\ell^{\frac{3}{4}}}{3\log(\ell)}. \tag{5.40}$$

The real function $h(x) := x^{-\frac{3}{8}} \log(x)$ has a maximum equal to $\frac{8}{3e}$ which is attained at the point $x = e^{\frac{8}{3}}$. Thus, for every $\ell \geq 4$,

$$\ell^{-\frac{3}{8}} < \frac{16}{3e \log(\ell)}. \tag{5.41}$$

Using (5.41) in (5.40), one finds that

$$\sum_{m=2}^{\ell} e^{-\frac{2\pi \mathrm{Im}(z)}{\sqrt{4N}}m} \frac{m^{-\frac{1}{4}}}{\log(m)} < \frac{4\ell^{\frac{3}{8}}}{3\log(2)} + \frac{16\ell^{\frac{3}{4}}}{3\log(\ell)} < \left(1 + \frac{4}{3\log(2)}\right) \frac{16\,\ell^{\frac{3}{4}}}{3\log(\ell)} := \frac{A_1}{\log(\ell)} \ell^{\frac{3}{4}}. \tag{5.42}$$

Recalling once more that $\ell := 1 + \left[\frac{\sqrt{N}}{\pi \mathrm{Im}(z)}\right]$, (5.42) yields

$$\sum_{m=2}^{\ell} e^{-\frac{2\pi \mathrm{Im}(z)}{\sqrt{4N}}m} \frac{m^{-\frac{1}{4}}}{\log(m)} < \frac{A_1}{\log(\ell)} \ell^{\frac{3}{4}} = O\left(\frac{1}{\mathrm{Im}(z)^{\frac{3}{4}}} \left\{\log\left(\frac{1}{\mathrm{Im}(z)}\right)\right\}^{-1}\right), \tag{5.43}$$

which proves (5.34) for $k = 1$.

2. For $k \geq 2$, it is even easier to get an estimate of the form (5.43). Indeed,

$$\sum_{m=2}^{\ell} e^{-\frac{2\pi \mathrm{Im}(z)}{\sqrt{4N}}m} \frac{m^{\frac{k}{2}-\frac{3}{4}}}{\log(m)} < \sum_{m=2}^{\ell} \frac{m^{\frac{k}{2}-\frac{3}{4}}}{\log(m)} = \sum_{m=2}^{\sqrt{\ell}} \frac{m^{\frac{k}{2}-\frac{3}{4}}}{\log(m)} + \sum_{m=\sqrt{\ell}+1}^{\ell} \frac{m^{\frac{k}{2}-\frac{3}{4}}}{\log(m)}$$
$$< \frac{\ell^{\frac{k}{4}+\frac{1}{8}}}{\log(2)} + 2\frac{\ell^{\frac{k}{2}+\frac{1}{4}}}{\log(\ell)}.$$

As in (5.41), we know that $h(x) := x^{-\frac{k}{4}-\frac{1}{8}}\log(x)$ has a maximum equal to $\frac{8}{(2k+1)e}$, which is attained at the point $x = e^{\frac{8}{2k+1}}$. Therefore, for every $\ell \geq 4$, the inequality takes place

$$\ell^{-\frac{k}{4}-\frac{1}{8}} < \frac{16}{(2k+1)e\,\log(\ell)},$$

and so, since $\ell = 1 + \left[\frac{\sqrt{N}}{\pi \mathrm{Im}(z)}\right]$,

$$\sum_{m=2}^{\ell} e^{-\frac{2\pi \mathrm{Im}(z)}{\sqrt{4N}}m} \frac{m^{\frac{k}{2}-\frac{3}{4}}}{\log(m)} < \frac{\ell^{\frac{k}{4}+\frac{1}{8}}}{\log(2)} + 2\frac{\ell^{\frac{k}{2}+\frac{1}{4}}}{\log(\ell)} < \left(1 + \frac{8}{(2k+1)e\log(2)}\right)\frac{2\ell^{\frac{k}{2}+\frac{1}{4}}}{\log(\ell)}$$
$$= O\left(\frac{1}{\mathrm{Im}(z)^{\frac{k}{2}+\frac{1}{4}}}\left\{\log\left(\frac{1}{\mathrm{Im}(z)}\right)\right\}^{-1}\right),$$

which finally proves (5.34).

□

Remark 5.3. In the same lines as those of Remark 5.2, it is clear from the properties of the Fricke involution that, if we consider

$$(f|W_{4N})_1\left(\frac{z}{\sqrt{4N}}\right) := \sum_{n=d+1}^{\infty} \frac{a_{f|W_{4N}}(n)}{\log(n/d)}\, e^{\frac{2\pi i n}{\sqrt{4N}}z},$$

then, for small $\mathrm{Im}(z)$, $(f|W_{4N})_1(z)$ obeys to the estimate

$$(f|W_{4N})_1\left(\frac{z}{\sqrt{4N}}\right) \ll \frac{1}{\mathrm{Im}(z)^{\frac{k}{2}+\frac{1}{4}}}\left(\log\left(\frac{1}{\mathrm{Im}(z)}\right)\right)^{-1}.$$

Remark 5.4. An alternative proof of the above lemma, which is short but invokes Fourier analysis, is given for $L-$functions associated with Maass cusp forms in [[8], pp.126-127]. Finding a bound for (5.17) is equivalent to finding a bound for

$$f_1(z) := \sum_{n=c+1}^{\infty} \frac{a_f(n)}{\log(n/c)}\, e^{2\pi i n z}.$$

It is clear that $f_1(x+iy)$ can be written in terms of the convolution

$$f_1(x+iy) = \int_0^1 f\,(t+iy)\,\mathcal{H}(x-t)\,dt,$$

where

$$\mathcal{H}(x) := \sum_{n=c+1}^{\infty} \frac{\cos(2\pi n x)}{\log(n/c)}. \tag{5.44}$$

The convexity of the sequence $(b_n)_{n\geq c+1} := \frac{1}{\log(n/c)}$ assures that $\mathcal{H} \in L^1([0,2\pi))$ [[17], p.23]. Hence, the proof of lemma 5.3 follows from an estimate of the L^1 modulus of continuity of $\mathcal{H}(x)$ combined with an estimate of the form (5.2). See [[8], p.127] for details.

6 Proof of Theorem 1.2

As is customary, any variant of the Hardy-Littlewood method starts by taking a positive number H (to be chosen at a later stage of the proof), some $0 < \epsilon < \frac{\pi}{2}$ (to be very small later), and by considering the integrals (with $t \in \mathbb{R}$)

$$I(t) := \int_t^{t+H} R_f(u) e^{\left(\frac{\pi}{2}-\epsilon\right)u}\, du \tag{6.1}$$

and

$$J(t) := \int_t^{t+H} |R_f(u)|\, e^{\left(\frac{\pi}{2}-\epsilon\right)u} du. \tag{6.2}$$

The idea of the proof is to contrast the behavior of the integrals $I(t)$ and $J(t)$ as $t \to \infty$.

We start the argument by estimating $I(t)$.

6.1 Estimating $I(t)$

The estimate of $I(t)$ is done via the representation (3.14), in which we consider ξ real, pick $0 < \epsilon < \frac{\pi}{2}$ and take $z = -\exp\{\xi - i\epsilon\}$ (which clearly satisfies the condition $\mathrm{Im}(z) > 0$). We can therefore rewrite (3.14) as the Fourier transform

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} R_f(t) e^{\left(\frac{\pi}{2}-\epsilon\right)t}\, e^{-i\xi t}\, dt = \sqrt{\frac{\pi}{2}} \exp\left\{\left(\frac{k}{2}+\frac{1}{4}\right)\xi + i\left(\frac{k}{2}+\frac{1}{4}\right)\left(\frac{\pi}{2}-\epsilon\right)\right\}$$
$$\times \left[f\left(-\frac{\exp\{\xi - i\epsilon\}}{\sqrt{4N}}\right) + (f|W_{4N})\left(-\frac{\exp\{\xi - i\epsilon\}}{\sqrt{4N}}\right)\right]. \tag{6.3}$$

From the Phragmén-Lindelöf principle (2.8) and Stirling's formula (2.7), we know that, as $|t| \to \infty$,

$$R_f(t) e^{\left(\frac{\pi}{2}-\epsilon\right)t} = O\left(|t|^A \exp\left\{-\frac{\pi}{2}\left(|t|-t\right) - \epsilon t\right\}\right), \tag{6.4}$$

for some $A > 0$. This estimate proves that $|R_f(t)|\, e^{\left(\frac{\pi}{2}-\epsilon\right)t}$ and $|R_f(t)|^2\, e^{(\pi-2\epsilon)t}$ are integrable over $\mathbb{R}$ and the same must be true for the function $I(t)$ defined by (6.1), as well as to $|I(t)|$ and $|I(t)|^2$. An integration by parts gives the Fourier transform

$$\hat{I}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} I(t)\, e^{-i\xi t} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi t} \int_{t}^{t+H} R_f(u)\, e^{\left(\frac{\pi}{2}-\epsilon\right)u}\, du\, dt$$

$$= \frac{1}{i\xi\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi t} \left\{ R_f(t+H)\, e^{\left(\frac{\pi}{2}-\epsilon\right)(t+H)} - R_f(t)\, e^{\left(\frac{\pi}{2}-\epsilon\right)t} \right\} dt$$

$$= \sqrt{\frac{\pi}{2}}\, \frac{e^{i\xi H}-1}{i\xi}\, \exp\left\{ \left(\frac{k}{2}+\frac{1}{4}\right)\xi + i\left(\frac{k}{2}+\frac{1}{4}\right)\left(\frac{\pi}{2}-\epsilon\right) \right\} \left[f\left(-\frac{\exp\{\xi - i\epsilon\}}{\sqrt{4N}}\right) + (f|W_{4N})\left(-\frac{\exp\{\xi - i\epsilon\}}{\sqrt{4N}}\right) \right], \tag{6.5}$$

and so, by Parseval's formula,

$$\int_{-\infty}^{\infty} |I(t)|^2 dt = 2\pi \int_{-\infty}^{\infty} \frac{\sin^2\left(\frac{\xi H}{2}\right)}{\xi^2} e^{\left(k+\frac{1}{2}\right)\xi} \left| f\left(-\frac{\exp\{\xi - i\epsilon\}}{\sqrt{4N}}\right) + (f|W_{4N})\left(-\frac{\exp\{\xi - i\epsilon\}}{\sqrt{4N}}\right) \right|^2 d\xi. \tag{6.6}$$

We now give an estimate for the latter integral: invoking the uniform bound for cusp forms (2.1), we know that there exist two positive constants A_1 and A_2 such that

$$\left| f\left(-\frac{\exp\{\xi - i\epsilon\}}{\sqrt{4N}}\right) \right| \leq \frac{A_1}{\sin^{\frac{k}{2}+\frac{1}{4}}(\epsilon)} e^{-\left(\frac{k}{2}+\frac{1}{4}\right)\xi} \tag{6.7}$$

and

$$\left| (f|W_{4N})\left(-\frac{\exp\{\xi - i\epsilon\}}{\sqrt{4N}}\right) \right| \leq \frac{A_2}{\sin^{\frac{k}{2}+\frac{1}{4}}(\epsilon)} e^{-\left(\frac{k}{2}+\frac{1}{4}\right)\xi}. \tag{6.8}$$

Supposing, without any loss of generality, that $A_1 \geq A_2$ and returning to (6.7), we find the mean inequality for $I(t)$,

$$\int_{-\infty}^{\infty} |I(t)|^2 dt < \frac{8\pi A_1^2}{\sin^{k+\frac{1}{2}}(\epsilon)} \int_{-\infty}^{\infty} \frac{\sin^2\left(\frac{\xi H}{2}\right)}{\xi^2} d\xi < K_1 \epsilon^{-k-\frac{1}{2}} H, \tag{6.9}$$

where we have used the fact that $0 < \epsilon < \frac{\pi}{2}$ and invoked Jordan's inequality, $\sin(\epsilon) > \frac{2}{\pi}\epsilon$. The constant K_1 is, of course, absolute, not depending on H or ϵ.

6.2 Estimating $J(t)$

Now, let us define the Dirichlet series associated with $R_f(t)$ (3.1),

$$\varphi_f(s) := \frac{1}{2}\left(L(s,f) + L(s, f|W_{4N})\right) = \frac{1}{2}\sum_{n=1}^{\infty} \frac{a_f(n) + a_{f|W_{4N}}(n)}{n^s} := \sum_{n=1}^{\infty} \frac{\alpha_f(n)}{n^s},$$

and assume that r is the smallest positive integer for which $\alpha_f(r) \neq 0$. We will apply the reasoning of Lemma 5.2 to the Dirichlet series

$$\varphi_f^{\star}(s) := r^s \varphi_f(s) - \alpha_f(r).$$

From Stirling's formula, we know that there exist $T_1, K_2 > 0$ such that

$$|R_f(t)| > K_2\,|t|^{\frac{k}{2}-\frac{1}{4}}\,e^{-\frac{\pi}{2}|t|}\left|\varphi_f\left(\frac{k}{2}+\frac{1}{4}+it\right)\right|,\quad |t|\geq T_1.$$

With the construction of this number T_1, let

$$T > T_0 := \max\{1, 6H, T_1\}, \tag{6.10}$$

and assume that $t \in [T, 2T]$ (recall that this is the argument of $I(t)$ and $J(t)$). If we now take the choice $\epsilon = T^{-1}$ in our integrals (6.1) and (6.2), the condition $t \leq u \leq t + H$ implies that $\epsilon u < \epsilon(2T + H) < 3$ and $u^{\frac{k}{2}-\frac{1}{4}} > T^{\frac{k}{2}-\frac{1}{4}}$. Consequently, we find the inequality

$$\begin{aligned}
J(t) &= \int_t^{t+H} |R_f(u)|\,e^{\left(\frac{\pi}{2}-\epsilon\right)u}du > K'\,T^{\frac{k}{2}-\frac{1}{4}}\int_t^{t+H}\left|\varphi_f\left(\frac{k}{2}+\frac{1}{4}+iu\right)\right|du \\
&= K'\,T^{\frac{k}{2}-\frac{1}{4}}\int_t^{t+H}\left|\alpha_f(r)\,r^{-\frac{k}{2}-\frac{1}{4}-iu}+r^{-\frac{k}{2}-\frac{1}{4}-iu}\varphi_f^{\star}\left(\frac{k}{2}+\frac{1}{4}+iu\right)\right|du \\
&= K'|\alpha_f(r)|\,r^{-\frac{k}{2}-\frac{1}{4}}\,T^{\frac{k}{2}-\frac{1}{4}}\int_t^{t+H}\left|1+\frac{1}{\alpha_f(r)}\varphi_f^{\star}\left(\frac{k}{2}+\frac{1}{4}+iu\right)\right|du \\
&\geq K'|\alpha_f(r)|\,r^{-\frac{k}{2}-\frac{1}{4}}\,T^{\frac{k}{2}-\frac{1}{4}}\left\{H+\operatorname{Re}\int_t^{t+H}\frac{1}{\alpha_f(r)}\varphi_f^{\star}\left(\frac{k}{2}+\frac{1}{4}+iu\right)du\right\} \\
&= K_3\,T^{\frac{k}{2}-\frac{1}{4}}\left\{H+\operatorname{Re}\int_t^{t+H}\frac{1}{\alpha_f(r)}\varphi_f^{\star}\left(\frac{k}{2}+\frac{1}{4}+iu\right)du\right\}.
\end{aligned} \tag{6.11}$$

Analogously to $L(s, f_1)$ in (5.12), we can construct the Dirichlet series

$$\psi_f(s) := \sum_{n=r+1}^{\infty}\frac{\alpha_f(n)}{\log(n/r)\,n^s},\quad \operatorname{Re}(s) > \frac{k}{2}+\frac{5}{4}, \tag{6.12}$$

and, due to Lemma 5.2, say that $\psi_f(s)$ can be continued to the whole complex plane as an entire function which satisfies

$$r^{-s}\int_s^{\infty}\varphi_f^{\star}(z)\,dz = \psi_f(s). \tag{6.13}$$

Employing (6.13), we find the representation

$$\frac{1}{\alpha_f(r)}\int_t^{t+H}\varphi_f^{\star}\left(\frac{k}{2}+\frac{1}{4}+iu\right)du = -\frac{ir^{\frac{k}{2}+\frac{1}{4}}}{\alpha_f(r)}\left\{r^{it}\psi_f\left(\frac{k}{2}+\frac{1}{4}+it\right)-r^{i(t+H)}\psi_f\left(\frac{k}{2}+\frac{1}{4}+i(t+H)\right)\right\}. \tag{6.14}$$

Let $\Psi(t)$ denote the right-hand side of (6.14): then (6.11) says that

$$J(t) \geq K_3\,T^{\frac{k}{2}-\frac{1}{4}}\left\{H+\operatorname{Re}\Psi(t)\right\}, \tag{6.15}$$

with K_3 not depending on any of the parameters H, t or T. If, at last, we deduce a suitable estimate for $\int_T^{2T} |\Psi(t)|^2 dt$, we can finish the application of the Hardy-Littlewood method. The next claim contains the necessary fact to conclude the proof of our result.

Claim 6.1. *For large T, there exists some absolute constant $K_4 > 0$ such that*

$$\int_T^{2T} |\Psi(t)|^2 dt \leq K_4 T. \tag{6.16}$$

Before giving a proof of this claim, we show how this assertion can be used to complete the proof of Theorem 1.2. From now on, the reasoning is standard and the corresponding known argument for the zeta case can be found in Titchmarsh's text [[27], pp.267-268]. However, for completeness, we shall also present it. Let S be the subset of $(T, 2T)$ where $|I(t)| = J(t)$. Note that this happens if and only if $R_f(u)$ does not have a zero of odd order on the interval $(t, t+H)$. We shall estimate the measure of S, $m(S)$, by comparing (6.11) and (6.15). Indeed, using (6.15) and claim 6.1, we get

$$\begin{aligned}\int_S J(t)\,dt &\geq K_3 T^{\frac{k}{2}-\frac{1}{4}} \left\{ H m(S) + \int_S \operatorname{Re}\Psi(t)\,dt \right\} \geq K_3 H T^{\frac{k}{2}-\frac{1}{4}}\, m(S) - K_3 T^{\frac{k}{2}+\frac{1}{4}} \left\{ \int_T^{2T} |\Psi(t)|^2 dt \right\}^{1/2} \\ &\geq K_3 H T^{\frac{k}{2}-\frac{1}{4}}\, m(S) - K_3 K_4^{\frac{1}{2}}\, T^{\frac{k}{2}+\frac{3}{4}}. \end{aligned} \tag{6.17}$$

On the other hand, we can bound the previous integral by above if we use (6.9) with $\epsilon = T^{-1}$, which gives

$$\int_S J(t)\,dt = \int_S |I(t)|\,dt \leq \int_T^{2T} |I(t)|\,dt \leq T^{1/2} \left\{ \int_T^{2T} |I(t)|^2 dt \right\}^{1/2} \leq K_1^{\frac{1}{2}} H^{\frac{1}{2}} T^{\frac{k}{2}+\frac{3}{4}}. \tag{6.18}$$

Combining (6.17) and (6.18), we find an upper bound for $m(S)$ of the form

$$K_3 H\, m(S) < K_1^{\frac{1}{2}} H^{\frac{1}{2}} T + K_3 K_4^{\frac{1}{2}}\, T.$$

Now, we can choose H so large that

$$K_4^{\frac{1}{2}} H^{-1} + K_1^{\frac{1}{2}} K_3^{-1} H^{-\frac{1}{2}} < \frac{1}{12},$$

and, with this choice of H, conclude that

$$m(S) < \frac{T}{12}. \tag{6.19}$$

The argument is completed if we subdivide $(T, 2T)$ into $[T/2H]$ pairs of abutting subintervals in the form

$$(T, 2T) = \bigcup_{j=1}^{[T/2H]} \left(I_1^{(j)} \cup I_2^{(j)} \right),$$

with $|I_1^{(j)}| = H$, $j = 1, ..., \left[\frac{T}{2H}\right]$ and $|I_2^{(j)}| = H$, $j = 1, ..., \left[\frac{T}{2H}\right] - 1$. Suppose that, for some $\nu \in \mathbb{N}$, the intervals $I_1^{(j_1)}$,..., $I_1^{(j_\nu)}$ consist entirely of points of S. Since each of these intervals has length H and there are ν such intervals, then $\nu \le m(S)/H$. Of the remaining $\left[\frac{T}{2H}\right] - \nu$ pairs of subintervals of $(T, 2T)$, either $I_1^{(j)}$ or the corresponding $I_2^{(j)}$ contains a zero of odd order of $R_f(t)$. By our choice of T (6.10) and the measure of S (6.19), we find that

$$\left[\frac{T}{2H}\right] - \nu > \frac{T}{3H} - \frac{m(S)}{H} > \frac{T}{3H} - \frac{T}{12H} = \frac{T}{4H}.$$

Hence, the interval $(T, 2T)$ has at least $T/4H$ zeros of odd order of $R_f(t)$. The same is evidently true for $L\left(\frac{k}{4} + \frac{1}{2} + it, f\right) + L\left(\frac{k}{4} + \frac{1}{2} + it, f|W_{4N}\right)$ and, with the necessary changes, the same can be concluded for $L\left(\frac{k}{4} + \frac{1}{2} + it, f\right) - L\left(\frac{k}{4} + \frac{1}{2} + it, f|W_{4N}\right)$. This completes the proof. ■

6.2.1 Proof of Claim 6.1

Since, by definition,

$$\Psi(t) := -\frac{ir^{\frac{k}{2}+\frac{1}{4}}}{\alpha_f(r)} \left\{ r^{it}\psi_f\left(\frac{k}{2} + \frac{1}{4} + it\right) - r^{i(t+H)}\psi_f\left(\frac{k}{2} + \frac{1}{4} + i(t+H)\right)\right\},$$

in order to prove (6.16) it is enough to show that

$$\int_T^{2T} \left|\psi_f\left(\frac{k}{2} + \frac{1}{4} + it\right)\right|^2 dt \ll T, \tag{6.20}$$

where

$$\psi_f(s) := \sum_{n=r+1}^{\infty} \frac{\alpha_f(n)}{\log(n/r)\, n^s}, \quad \mathrm{Re}(s) > \frac{k}{2} + \frac{5}{4}.$$

Just like in the case of (5.17), we can construct a series resembling a cusp form attached to $\psi_f(s)$ of the form

$$\sum_{n=r+1}^{\infty} \frac{\alpha_f(n)}{\log(n/r)} e^{\frac{2\pi i n z}{\sqrt{4N}}} := g_1\left(\frac{z}{\sqrt{4N}}\right). \tag{6.21}$$

By adapting the proof of lemma 5.3, one can see that $g_1\left(z/\sqrt{4N}\right)$ must satisfy the bounds given in (5.18) and (5.19), which are dependent on the range of $\mathrm{Im}(z)$. We will find the estimate (6.20) as a consequence of a L^2 estimate of the function

$$\Phi_f(t) := \left(\frac{2\pi}{\sqrt{4N}}\right)^{-\frac{k}{2}-\frac{1}{4}-it} \Gamma\left(\frac{k}{2} + \frac{1}{4} + it\right) \psi_f\left(\frac{k}{2} + \frac{1}{4} + it\right), \tag{6.22}$$

which, in its turn, will be found by using the estimates for $g_1(z/\sqrt{4N})$.

Just like (3.14), the inversion formula for the Mellin transform gives

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_f(t)\, (-iz)^{-\frac{k}{2}-\frac{1}{4}-it} dt = g_1\left(\frac{z}{\sqrt{4N}}\right), \quad \forall z \in \mathbb{H}. \tag{6.23}$$

Let us now take (as above) $\xi \in \mathbb{R}$, $0 < \epsilon < \frac{\pi}{2}$ and write $z = -\exp\{\xi - i\epsilon\}$: then (6.23) can be viewed in terms of the Fourier transform

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi_f(t) e^{\left(\frac{\pi}{2}-\epsilon\right)t} e^{-i\xi t}\, dt = \sqrt{2\pi}\, \exp\left\{\left(\frac{k}{2}+\frac{1}{4}\right)\xi + i\left(\frac{k}{2}+\frac{1}{4}\right)\left(\frac{\pi}{2}-\epsilon\right)\right\} g_1\left(-\frac{\exp\{\xi - i\epsilon\}}{\sqrt{4N}}\right)$$

and so, by Parseval's formula,

$$\int_{-\infty}^{\infty} |\Phi_f(t)|^2 e^{(\pi-2\epsilon)t}\, dt = 2\pi \int_{-\infty}^{\infty} \exp\left\{\left(k+\frac{1}{2}\right)\xi\right\} \left|g_1\left(-\frac{\exp\{\xi-i\epsilon\}}{\sqrt{4N}}\right)\right|^2 d\xi = 2\pi \int_0^{\infty} y^{k-\frac{1}{2}} \left|g_1\left(-\frac{ye^{-i\epsilon}}{\sqrt{4N}}\right)\right|^2 dy$$

By lemma 5.3 (adapted to $g_1(z/\sqrt{4N})$), we know that there exists some $\eta \in (0,1)$ such that

$$g_1\left(-\frac{ye^{-i\epsilon}}{\sqrt{4N}}\right) \ll \frac{\left\{\log\left(\frac{1}{y\sin(\epsilon)}\right)\right\}^{-1}}{(y\sin(\epsilon))^{\frac{k}{2}+\frac{1}{4}}}, \quad 0 < y\sin(\epsilon) < \eta < 1, \tag{6.24}$$

while

$$g_1\left(-\frac{ye^{-i\epsilon}}{\sqrt{4N}}\right) \ll e^{-\frac{2\pi\sin(\epsilon)}{\sqrt{4N}}y}, \quad y\sin(\epsilon) \geq \eta. \tag{6.25}$$

Thus, invoking (6.24) and (6.25) in the respective ranges of y, we deduce that

$$\int_{-\infty}^{\infty} |\Phi_f(t)|^2 e^{(\pi-2\epsilon)t}\, dt = 2\pi \int_0^{\infty} y^{k-\frac{1}{2}} \left|g_1\left(-\frac{ye^{-i\epsilon}}{\sqrt{4N}}\right)\right|^2 dy$$

$$< C\left\{\int_0^{\eta/\sin(\epsilon)} \frac{\left\{\log\left(\frac{1}{y\sin(\epsilon)}\right)\right\}^{-2}}{\sin^{k+\frac{1}{2}}(\epsilon)} \frac{dy}{y} + \int_{\eta/\sin(\epsilon)}^{\infty} y^{k-\frac{1}{2}} e^{-\frac{4\pi\sin(\epsilon)}{\sqrt{4N}}y}\, dy\right\}$$

$$< \frac{D}{\sin^{k+\frac{1}{2}}(\epsilon)} \left\{\int_0^{\eta} \frac{dv}{v\log^2(v)} + \int_{\eta}^{\infty} v^{k-\frac{1}{2}} e^{-\frac{4\pi v}{\sqrt{4N}}}\, dv\right\} < D'\epsilon^{-k-\frac{1}{2}}, \tag{6.26}$$

for some absolute constant $D' > 0$.

Using (6.26) and the relation between $\Phi_f(t)$ and $\psi_f\left(\frac{k}{4}+\frac{1}{2}+it\right)$, we are ready to establish (6.20). By Stirling's formula, there exists some $T_0 > 0$ for which the following inequality holds

$$\left|\Gamma\left(\frac{k}{2}+\frac{1}{4}+it\right)\right| \geq \sqrt{\frac{\pi}{2}}\, |t|^{\frac{k}{2}-\frac{1}{4}} e^{-\frac{\pi}{2}|t|}, \quad |t| \geq T_0. \tag{6.27}$$

Thus, if we pick $T > \max\left\{T_0, \frac{2}{\pi}\right\}$ and put $\epsilon = T^{-1}$, we have from (6.22) and (6.22),

$$\int_T^{2T} |\Phi_f(t)|^2 e^{\left(\pi-\frac{2}{T}\right)t}\, dt > A \int_T^{2T} t^{k-\frac{1}{2}} \left|\psi_f\left(\frac{k}{2}+\frac{1}{4}+it\right)\right|^2 dt > A\, T^{k-\frac{1}{2}} \int_T^{2T} \left|\psi_f\left(\frac{k}{2}+\frac{1}{4}+it\right)\right|^2 dt,$$

for some positive constant A. Combining this inequality and (6.26) with $\epsilon = T^{-1}$, we finally deduce

$$\int_T^{2T} \left|\psi_f\left(\frac{k}{2}+\frac{1}{4}+it\right)\right|^2 dt < \frac{1}{AT^{k-\frac{1}{2}}} \int_T^{2T} |\Phi_f(t)|^2 e^{\left(\pi-\frac{2}{T}\right)t}\, dt < \frac{1}{AT^{k-\frac{1}{2}}} \int_{-\infty}^{\infty} |\Phi_f(t)|^2 e^{\left(\pi-\frac{2}{T}\right)t}\, dt < \frac{D'}{A} T,$$

which proves (6.20) and establishes our claim.

Remark 6.1. The proof of the result stated as Theorem 1.3 follows the same ideas. Let

$$\eta_{p/q}\,(s,f) := \left(\frac{2\pi}{q}\right)^{-s}\Gamma(s)\,L_{p/q}(s,f).$$

We start by recalling the functional equation for $L_{p/q}(s,f)$ (1.11),

$$\left(\frac{2\pi}{q}\right)^{-s}\Gamma(s)\,L_{p/q}(s,f) = i^{k+\frac{1}{2}}\left(\frac{-q}{p}\right)^{-2k-1}\epsilon_p^{2k+1}\left(\frac{2\pi}{q}\right)^{-\left(k+\frac{1}{2}-s\right)}\Gamma\left(k+\frac{1}{2}-s\right)L_{-\overline{p}/q}\left(k+\frac{1}{2}-s,f\right),$$

where $p\,\overline{p}\equiv 1 \mod q$ and the symbols $\left(\frac{-q}{p}\right)$ and ϵ_p are defined by (1.1). One can see that, if $p^2\equiv 1 \mod q$ and the the Fourier coefficients of $f(z)$, $a_f(n)$, are real numbers (resp. purely imaginary numbers) then the function

$$Z_{f,p/q}\,(t) := i^{-\frac{k}{2}-\frac{1}{4}}\left(\frac{-q}{p}\right)^{k+\frac{1}{2}}\epsilon_p^{k+\frac{1}{2}}\,\eta_{p/q}\left(\frac{k}{2}+\frac{1}{4}+it,f\right)$$

is always real for any $t\in\mathbb{R}$ (resp. purely imaginary for any $t\in\mathbb{R}$). Analogously to Lemma 3.1, it is then straightforward to deduce the integral representation

$$\int_{-\infty}^{\infty} Z_{f,p/q}\,(t)\,(-iz)^{-it}dt = 2\pi\,i^{k+\frac{1}{2}}\left(\frac{-q}{p}\right)^{k+\frac{1}{2}}\epsilon_p^{k+\frac{1}{2}}z^{\frac{k}{2}+\frac{1}{4}}f\left(\frac{z+p}{q}\right),\quad z\in\mathbb{H}. \tag{6.28}$$

If we consider $\xi\in\mathbb{R}$ and let $0<\epsilon<\frac{\pi}{2}$, then substituting $z=-\exp\{\xi-i\epsilon\}$ in (6.28) yields the Fourier transform

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} Z_{f,p/q}\,(t)\,e^{\left(\frac{\pi}{2}-\epsilon\right)t}e^{-i\xi t}dt = \sqrt{2\pi}\left(\frac{-q}{p}\right)^{k+\frac{1}{2}}\epsilon_p^{k+\frac{1}{2}}\exp\left\{\left(\frac{k}{2}+\frac{1}{4}\right)\xi - i\left(\frac{k}{2}+\frac{1}{4}\right)\epsilon\right\}f\left(-\frac{\exp\{\xi-i\epsilon\}}{q}+\frac{p}{q}\right),$$

which is analogous to (6.3). Since Lemmas 5.1-5.3 also apply to $L_{p/q}(s,f)$, the bounds (6.9) and (6.16) are also valid in this case and the conclusion of Theorem 1.3 must follow from them.

7 Concluding Remarks

Similar results to the ones described here can be established for $L-$functions attached to cusp forms with integral weight and level N. In this case, let us recall the definition of the Fricke involution as

$$(f|W_N)\,(z) = i^k N^{-k/2}z^{-k}f\left(-\frac{1}{Nz}\right).$$

If we adapt the proof of our Theorem 1.2, we can deduce the following integral analogue.

Theorem 7.1. *Let N be a perfect square and $f(z)=\sum_{n=1}^{\infty}a_f(n)\,e^{2\pi i n z}\in S_k\left(\Gamma_0(N)\right)$, with $a_f(n)$ being either real or purely imaginary numbers.*

If $\mathcal{N}_0^{\pm}(T)$ represents the number of zeros of odd order of $L(s,f)\pm L(s,f|W_N)$ written in the form $s=\frac{k}{2}+it$, $0\leq t\leq T$, then there exists some $d>0$ such that

$$\liminf_{T\to\infty}\frac{\mathcal{N}_0^{\pm}(T)}{T} > d. \tag{7.1}$$

Taking into consideration the simple modifications outlined in remark 6.1, a similar extension holds for the twisted $L-$functions studied by Kim.

Theorem 7.2. *Let N be a positive integer and $\frac{p}{q}$ a rational number which is $\Gamma_0(N)-$equivalent to $i\infty$ and such that $p^2 \equiv 1 \mod q$. Moreover, let $f(z) = \sum_{n=1}^{\infty} a_f(n)\, e^{2\pi i n z} \in S_k\left(\Gamma_0(N)\right)$, with $a_f(n)$ being either real or purely imaginary numbers.*

If $\mathcal{N}_{0,p/q}(T)$ denotes the number of zeros of $L_{p/q}(s,f)$ written in the form $s = \frac{k}{2} + it$, $0 \leq t \leq T$, then there exists some $d > 0$ such that

$$\liminf_{T\to\infty} \frac{\mathcal{N}_{0,p/q}(T)}{T} > d. \tag{7.2}$$

Of course, when f is a cusp form of weight k with respect to the full modular group, then (7.1) and (7.2) are given in Lekkerkerker's thesis [[20], Chapter IV, Theorems 15 and 16]. It is also well-known that Lekkerkerker's result was superseded by Hafner [10], who proved that $L-$functions attached to holomorphic cusp forms of integral weight have a positive proportion of their zeros at the critical line $\text{Re}(s) = \frac{k}{2}$.

When $f \in S_{k+\frac{1}{2}}\left(\Gamma_0(4N)\right)$, one might study the distribution of the zeros of $L(s,f)$ at the line $\text{Re}(s) = \frac{k}{2} + \frac{1}{4}$ by taking another point of view. For instance, one may consider the problem of finding suitable bounds for the gaps between consecutive zeros of $L\left(\frac{k}{2} + \frac{1}{4} + it, f\right)$. When $f(z)$ is a cusp form of integral weight for the full modular group with real or purely imaginary coefficients, it was found by Jutila [[16], p. 140, Theorem 5] that, for any fixed $\epsilon > 0$ and $T \geq T_0(\epsilon)$, there is always a zero $\frac{k}{2} + i\tau$ of $L\left(s,f\right)$ such that $\tau \in [T, T + T^{\frac{1}{3}+\epsilon}]$. The proof of Jutila is particularly beautiful because it involves the transformation of some exponential sums via Voronoï's summation formula. Yogananda [[29], p. 21, Theorem 4.1] extended Jutila's result to certain $L-$functions attached to cusp forms belonging to $S_k\left(\Gamma_0(N)\right)$.

An examination of Yogananda's argument (which is motivated by a remark at the end of Jutila's paper [[16], p. 156]) shows that only Rankin's mean value estimate for the coefficients of f (and not something stronger like Deligne's bound) is needed to establish this gap between zeros of $L(s,f)$. Motivated by this, it might be plausible (provided that all transformation formulas of Voronoï type are valid in this case) to conjecture an analogous result for $L-$functions in the half-integral case.

Conjecture 7.1. *Let $N \in \mathbb{N}$ and $f(z) = \sum_{n=1}^{\infty} a_f(n)\, e^{2\pi i n z} \in S_{k+\frac{1}{2}}\left(\Gamma_0(4N)\right)$ be such that $f|W_{4N} = f$ or $f|W_{4N} = -f$. Assume also that $a_f(n)$ are either real or purely imaginary numbers. Then, for any $\epsilon > 0$, there exists $T_0(\epsilon)$ such that, for all $T \geq T_0(\epsilon)$, $L(s,f)$ has a zero of the form $s = \frac{k}{2} + \frac{1}{4} + i\tau$ with $\tau \in [T, T + T^{\frac{1}{3}+\epsilon}]$.*

There are yet other directions of research that can be taken. Together with Yakubovich [23], the author of this paper derived the following result.

Theorem D [23]: *Let $f(z)$ be a cusp form of weight k for the full modular group with real Fourier coefficients $a_f(n)$. Also, let $L(s,f)$ represent the corresponding Dirichlet series.*

If $(c_j)_{j\in\mathbb{N}}$ is a sequence of non-zero real numbers such that $\sum_{j=1}^{\infty}|c_j| < \infty$, $(\lambda_j)_{j\in\mathbb{N}}$ is a bounded sequence of distinct real numbers[13] *and z is a complex number satisfying the condition*

$$z \in \mathscr{D} := \left\{z \in \mathbb{C}\,:\,|Re(z)| < 2\sqrt{\pi},\;|Im(z)| < 2\sqrt{\pi}\right\}, \tag{7.3}$$

then the function

$$\sum_{j=1}^{\infty} c_j\,(2\pi)^{-s-i\lambda_j}\,\Gamma(s+i\lambda_j)\,L\left(s+i\lambda_j,f\right)\left\{{}_1F_1\left(k-s-i\lambda_j;\,k;\,\frac{z^2}{4}\right)+{}_1F_1\left(k-\overline{s}+i\lambda_j;\,k;\,\frac{\overline{z}^2}{4}\right)\right\} \tag{7.4}$$

has infinitely many zeros on the critical line $Re(s)=\frac{k}{2}$.

In the statement above, ${}_1F_1(a;c;w)$ represents the confluent hypergeometric function. The previous result is, in fact, a generalization of a Theorem of Dixit, Kumar, Maji and Zaharescu [7]. Like the proof presented in [7], deriving it requires the use of a generalization of the theta transformation formula. We conclude this paper by mentioning that, using our variant of de la Vallée Poussin's method given in the proof of Theorem 1.1, it is possible to obtain an estimate for the number of critical zeros of the function (7.4). The details of this study and related topics will be presented elsewhere.

Acknowledgements: This work was partially supported by CMUP, member of LASI, which is financed by national funds through FCT - Fundação para a Ciência e a Tecnologia, I.P., under the projects with reference UIDB/00144/2020 and UIDP/00144/2020. We also acknowledge the support from FCT (Portugal) through the PhD scholarship 2020.07359.BD. The author would like to thank to Semyon Yakubovich for unwavering support and guidance throughout the writing of this paper.

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