DECOMPOSITION OF RECTANGLES IN DOMINOS

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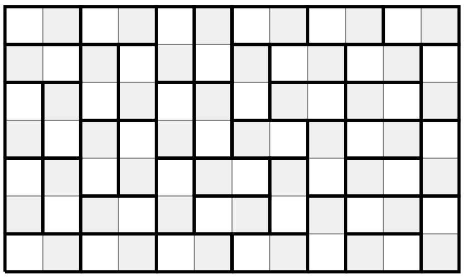
1. Introduction

Let us consider a $\ell \times k$ "chess board" B, with ℓ and k positive integers, and suppose we have dominos that cover exactly two adjacent squares of R. In how many ways can we cover the whole board with dominos without overlapping any two domino pieces? For example, below you find a way to cover a 12×7 chess board. How many different ways are there of doing this?



We know the exact number of *domino tilings* for every ℓ and k. In the case here depicted the exact number is $T(12,7) = 2\,188\,978\,117$.[1] Surprisingly, this number is given by the "very strange" formula (note that the factors are in most cases far from being integers)

$$T(12,7) = \prod_{p=1}^{12} \prod_{q=1}^{7} \sqrt[4]{4\cos^2\left(\frac{p\,\pi}{13}\right) + 4\cos^2\left(\frac{q\,\pi}{8}\right)}$$

or, in the general case, by

$$T(\ell,k) = \prod_{p=1}^{\ell} \prod_{q=1}^{k} \sqrt[4]{4\cos^2\left(\frac{p\,\pi}{\ell+1}\right) + 4\cos^2\left(\frac{q\,\pi}{k+1}\right)} . \tag{1}$$

In this paper we explain this formula, obtained by the physicist Pieter Kasteleyn in order to solve a concrete problem in Physics. In the first part, we prove that the number of tilings equals the absolute value of

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[1]The tiling presented above was *randomly generated* with J. Rangel-Mondragon's application for Mathematica "Random Domino Tilings", Wolfram Demonstrations Project, `http://demonstrations.wolfram.com/RandomDominoTilings/`.

a given determinant; in the second part, we show that Formula 1 gives the value of the determinant we are looking for.

For example, for $\ell = 4$ and $k = 3$, with black squares numbered with integers from 1 to 6, and white squares numbered with integers from 7 to 12, we obtain:

5	11	6	12
9	3	10	4
1	7	2	8

When ℓ and k are both odd, clearly $T(\ell,k) = 0$. From now on, we consider ℓk to be even, and consider the $(\ell k) \times (\ell k)$ matrix $M(\ell,k) = \big(a_{pq}\big)_{1\leqslant p,q\leqslant \ell\, k}$ given by

$$a_{pq} = \begin{cases} 0, & \text{if squares } p \text{ and } q \text{ are not adjacent;} \\ 1, & \text{if squares } p \text{ and } q \text{ are placed side by side;} \\ i = \sqrt{-1}, & \text{if squares } p \text{ and } q \text{ are placed one on top of the other.} \end{cases}$$

Hence, for example,

$$M(4,3) = \left(\begin{array}{cccccc|cccccc} 0&0&0&0&0&0&1&0&i&0&0&0 \\ 0&0&0&0&0&0&1&1&0&i&0&0 \\ 0&0&0&0&0&0&i&0&1&1&i&0 \\ 0&0&0&0&0&0&0&i&0&1&0&i \\ 0&0&0&0&0&0&0&0&i&0&1&0 \\ 0&0&0&0&0&0&0&0&0&i&1&1 \\ \hline 1&1&i&0&0&0&0&0&0&0&0&0 \\ 0&1&0&i&0&0&0&0&0&0&0&0 \\ i&0&1&0&i&0&0&0&0&0&0&0 \\ 0&i&1&1&0&i&0&0&0&0&0&0 \\ 0&0&i&0&1&1&0&0&0&0&0&0 \\ 0&0&0&i&0&1&0&0&0&0&0&0 \end{array}\right)$$

Then, $M(4,3) = \left(\begin{smallmatrix} 0 & m(4,3) \\ m(4,3)^T & 0 \end{smallmatrix}\right)$ for the matrix $m(4,3)$ that we shall consider in next section.

In fact, we shall see in Section 2 that in general the number of tilings is $T(\ell,k) = \big\| \det\big(m(\ell,k)\big) \big\|$ for the $(\ell k/2) \times (\ell k/2)$ matrix $m(\ell,k)$ such that

$$M(\ell,k) =: \begin{pmatrix} 0 & m(\ell,k) \\ m(\ell,k)^T & 0 \end{pmatrix}.$$

Note that

$$\det\big(M(\ell,k)\big) = \det\big(m(\ell,k)\big)\det\big(m(\ell,k)^T\big) = \det\big(m(\ell,k)\big)^2.$$

In Section 4 we shall see that

$$\det\big(M(\ell,k)\big) = \prod_{p=1}^{\ell}\prod_{q=1}^{k}\sqrt{\left(2\cos\left(\frac{p\,\pi}{\ell+1}\right)\right)^2 + \left(2\cos\left(\frac{q\,\pi}{k+1}\right)\right)^2},$$

with which we have fulfilled our purpose. Finally, Section 3 prepares Section 4.

2. From the tilings to the determinant

By definition, $m(4,3)$ is a 6×6 matrix, namely,

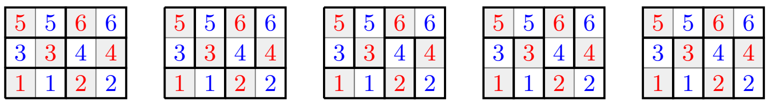
$$m(4,3) = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6 \end{array}\!\!\begin{array}{c} \begin{matrix} 1&2&3&4&5&6 \end{matrix} \\ \begin{pmatrix} 1&0&i&0&0&0\\ 1&1&0&i&0&0\\ i&0&1&1&i&0\\ 0&i&0&1&0&i\\ 0&0&i&0&1&0\\ 0&0&0&i&1&1 \end{pmatrix} \end{array}.$$

Accordingly, we relabel both black and white board squares with integers from 1 to $d = \ell\, k/2$, but, in order to avoid any confusion, *black* board squares are labeled in *red* and *white* board squares are labeled in *blue*. We may assign a *permutation* $\sigma \in \mathfrak{S}_d$ to every tiling in an obvious way. In the fourth tiling below, for example, (red) square 1 and (blue) square 3 are covered by the same domino, squares 2 and 2 are also covered by the same domino, and so on. Hence, we obtain permutation $\rho = \binom{1\ 2\ 3\ 4\ 5\ 6}{3\ 2\ 1\ 6\ 5\ 4}$.

Define, for a permutation $\tau \in \mathfrak{S}_6$,

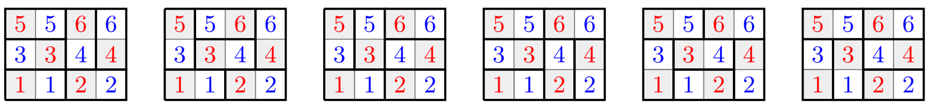
$$s(\tau) := a_{1\tau_1} a_{2\tau_2} a_{3\tau_3} a_{4\tau_4} a_{5\tau_5} a_{6\tau_6}\,.$$

Then $s(\tau) = 0$ if there is no tiling associated with τ. In fact, for any $\tau \in \mathfrak{S}_d$, when $s(\tau)$ is defined similarly, $s(\tau) = 0$ *if and only if*, for some $p \in \{1, 2, \dots, d\}$, p and τ_p are not adjacent. When $\ell = 4$ and $k = 3$, there are eleven permutations τ with $s(\tau) \neq 0$. Five of the eleven permutations are positive:



(1, 2, 3, 4, 5, 6) (1, 2, 5, 6, 3, 4) (1, 4, 5, 2, 3, 6) (3, 2, 1, 6, 5, 4) (3, 4, 1, 2, 5, 6)

and six are negative:



(1, 2, 3, 6, 5, 4) (1, 2, 4, 6, 3, 5) (1, 2, 5, 4, 3, 6) (1, 4, 3, 2, 5, 6) (3, 1, 4, 2, 5, 6) (3, 2, 1, 4, 5, 6)

Let $\mathcal{S}$ be the set formed by the permutations τ for which $s(\tau) \neq 0$, and note that, by definition,

$$\begin{aligned} \det\big(m(\ell,k)\big) &= \sum_{\tau \in \mathfrak{S}_n} \operatorname{sign}(\tau)\, s(\tau) \\ &= \sum_{\tau \in \mathcal{S}} \operatorname{sign}(\tau)\, s(\tau)\,. \end{aligned}$$

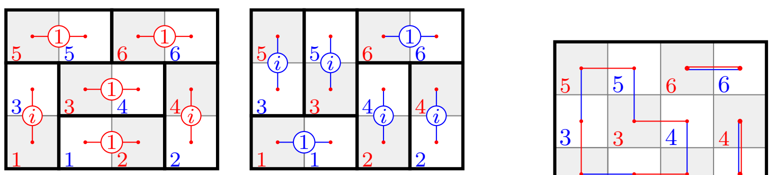
We shall see that $\operatorname{sign}(\sigma)\, s(\sigma) = \operatorname{sign}(\mu)\, s(\mu)$ for any two $\sigma, \mu \in \mathcal{S}$. Hence, since $\|\operatorname{sign}(\sigma)\, s(\sigma)\| = 1$,

$$\begin{aligned} \|\det\big(m(\ell,k)\big)\| &= \Big\| \sum_{\tau\in\mathcal{S}} \operatorname{sign}(\tau)\, s(\tau) \Big\| \\ &= |\mathcal{S}|\, \|\operatorname{sign}(\sigma)\, s(\sigma)\| \\ &= \text{number of domino tilings of } B\,. \end{aligned} \tag{2}$$

For example, the tilings below define the permutations σ and μ, where

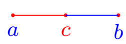
$$\operatorname{sign}(\sigma)\, c(\sigma) = (-1)\, i^2 = \operatorname{sign}(\mu)\, c(\mu) = (+1)\, i^4 = 1\,.$$

Let us see why this is so in general, based on this example. First, suppose that the coordinates of the centers of the squares form the set $\{1,\ldots,\ell\} \times \{1,\ldots,k\}$. We connect the centers covered with the same domino by a line, red in the first tiling and blue in the second tiling. Then, we put together red and blue lines connecting the squares. We obtain in this case three *integer lattice polygons*, that is, polygons with vertices of integer coordinates such that two consecutive vertices either differ by $(\pm 1, 0)$ or by $(0, \pm 1)$: one is an octagon and the remaining two are bigons. If we read the *blue* vertices of the polygons counter-clockwise, we obtain $1\,4\,5\,3$, 2 and 6, respectively.



$\sigma = \binom{1\ 2\ 3\ 4\ 5\ 6}{3\ 1\ 4\ 2\ 5\ 6}$ $\quad$ $\mu = \binom{1\ 2\ 3\ 4\ 5\ 6}{1\ 4\ 5\ 2\ 3\ 6}$

$c(\sigma) = i^2 = -1$ $\quad$ $c(\mu) = i^4 = 1$

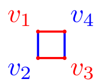
Now, note that $\frac{\operatorname{sign}(\sigma)}{\operatorname{sign}(\mu)} = \operatorname{sign}(\sigma\circ\mu^{-1})$, and that $\sigma\circ\mu^{-1} = \big((1\,4\,5\,3)(2)(6)\big)$ as a product of disjoints cycles. The similarity between this decomposition and the decomposition of blue vertices as consecutive vertices of the polygons that occurs is no coincidence, of course. In fact, $\sigma\circ\mu^{-1}(a) = b$ means that $\mu^{-1}(a) = \sigma^{-1}(b)$, that is, there exists c such that $a = \mu(c)$, and $b = \sigma(c)$. Graphically, we have then  .

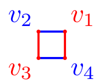
The sign of a cycle of length m is $(-1)^{m-1}$ and the sign of a product of cycles is the product of the signs of the cycles. If we prove that, for a polygon $\langle r_1\, b_1\, r_2\, b_2 \cdots r_m\, b_m\rangle$, it is also $\frac{m_{r_1 b_1} m_{r_2 b_2} \cdots m_{r_m b_m}}{m_{b_1 r_2} m_{b_2 r_3} \cdots m_{b_m r_1}} = (-1)^{m-1}$, then we have proved (2), and we may conclude this section. For this purpose,

• First, note that, in our situation, every vertex belongs to a unique polygon, which has an even number of sides, by definition. Hence, if there exist lattice points (that is, points of integer coordinates) inside a given lattice polygon, they must be in even number.

• Second, suppose that, in polygon $P = \langle r_1\, b_1\, r_2\, b_2 \cdots r_m\, b_m \rangle$, segment $r_1\, b_1$ is blue. Then, $\frac{m_{r_1 b_1} m_{r_2 b_2} \cdots m_{r_m b_m}}{m_{b_1 r_2} m_{b_2 r_3} \cdots m_{b_m r_1}} = i^{p-q}$, where p is the number of *blue* vertical segments of the polygon, whereas q is the number of *red* vertical segments. Let us prove, by induction on the area of a *general* length $2m$ integer lattice polygon P with t lattice points in the interior, that

$$i^{p-q} = (-1)^{m-t-1}\,.$$

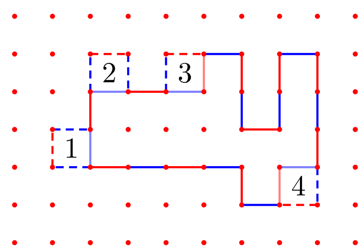
If the area of P is 1, we either have 

or 

.

In both cases, $m = 2$ and $t = 0$. In the first case, $p = 2$ and $q = 0$; in the second one, $p = 0$ and $q = 2$.

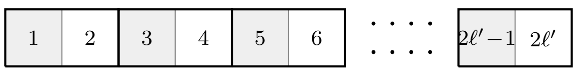
If the area of P is bigger than 1, we can always delete a square as indicated in the figure, reducing the area by 1. Let P' be the reduced polygon.



In the case where we delete a square in the position of 1 or 2, m decreases by 1 from P to P', and t remains constant, and either p increases and q decreases by 1, or q increases and p decreases by 1. When we delete a square in the position of 3 or 4, t decreases by 1, and p, q and m remain constant. In all cases, i^{p-q} is multiplied by -1, as well as $(-1)^{m-t-1}$.

3. Determinant associated with a linear board

Counting the number of domino tiling is easiest, of course, in the case where $k = 1$.[2] Let us renumber again the squares, so that to make calculations easier. Note that this does not change the absolute value of the determinant of the matrix. Then, there exists exactly *one* solution, if ℓ is even, and none, if ℓ is odd.



Consider the matrix below,

$$\begin{pmatrix} -x & 1 & \cdots & 0 & 0 & 0 \\ 1 & -x & \ddots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & -x & 1 & 0 \\ 0 & 0 & \cdots & 1 & -x & 1 \\ 0 & 0 & \cdots & 0 & 1 & -x \end{pmatrix}.$$

We define $N(\ell, 1)$ to be this matrix *for* $x = 0$, and $\chi_\ell(x)$ be the determinant of the same matrix for every $x \in \mathbb{C}$. Thus, $\chi_\ell(x)$ is the characteristic polynomial of $N(\ell, 1)$, and so, of course, $\chi_\ell(0) = \det\big(N(\ell, 1)\big)$. On

[2] The case where $k = 2$ is still very easy: we obtain the *Fibonacci numbers*. Why?

the other hand, the value of $\det\big(N(\ell,1)\big)$ is the product of the *eigenvalues* of $N(\ell,1)$, taking into account their multiplicities. Note that $N(ell,1)$ and $M(\ell,1)$ on differ on the labels of the squares we use. Then, since $\det\big(M(\ell,1)\big) = \det(m\big(\ell,1)\big)^2$ is non-negative, $\det\big(M(\ell,1)\big) = \big|\det\big(N(\ell,1)\big)\big|$.[3]

Although we already know $|\det\big(N(\ell,1)\big)|$, it will be important to find all roots of the characteristic polynomial, in here ℓ different values. Note that, by deleting the last row and the last column of the matrix associated with $\chi_\ell(x)$, we obtain the matrix associated with $\chi_{\ell-1}(x)$. Hence, by Laplace expansion along the last row, first, and then along the last column, we obtain, for every $n \geqslant 3$ and $x \in \mathbb{C}$,

$$\chi_\ell(x) = -x\,\chi_{\ell-1}(x) - \chi_{\ell-2}(x)\,,$$

and, in particular,

$$\chi_\ell(2\cos(t)) = -2\cos(t)\big(\chi_{\ell-1}(2\cos(t))\big) - \chi_{\ell-2}(2\cos(t))\,. \tag{3}$$

But, from the usual expansions of $\sin(\ell\,t+t)$ and $\sin(\ell\,t-t)$,[4]

$$\sin\big((\ell+1)t\big) = 2\cos(t)\sin(\ell\,t) - \sin\big((\ell-1)t\big)\,,$$

and hence

$$(-1)^\ell\frac{\sin\big((\ell+1)t\big)}{\sin(t)} = -2\cos(t)(-1)^{\ell-1}\frac{\sin(\ell t)}{\sin(t)} - (-1)^{\ell-2}\frac{\sin\big((\ell-1)t\big)}{\sin(t)}\,. \tag{4}$$

Comparing (3) with (4), we see that

$$\chi_\ell(2\cos(t)) = (-1)^\ell\frac{\sin\big((\ell+1)t\big)}{\sin(t)}\,, \tag{5}$$

if this expression holds true for $\ell = 1$ and $\ell = 2$, that is, if

$$\chi_1(2\cos(t)) = -\frac{\sin\big(2t\big)}{\sin(t)}\,,$$

and

$$\chi_2(2\cos(t)) = \frac{\sin\big(3t\big)}{\sin(t)}\,.$$

But

$$\chi_1(x) = \det\left(-x\right) = -x \quad\text{and}\quad \chi_2(x) = \det\begin{pmatrix}-x & 1\\ 1 & -x\end{pmatrix} = x^2-1\,,$$

whereas

$$\sin\big(2t\big) = 2\cos(t)\sin(t)\,,$$

and

$$\sin\big(3t\big) = 2\cos(t)\sin(2t) - \sin(t) = (4\cos^2(t)-1)\sin(t)\,.$$

[3]In fact, $\det\big(M(\ell,1)\big)$ is a positive integer, according to section 2.

[4]That is, $\sin(\ell\,t \pm t) = \sin(\ell\,t)\cos(t) \pm \cos(\ell\,t)\sin(t)$.

We can now see that the eigenvalues of $m(\ell,1)$ are, for $p=1,2,\dots,\ell$, $2\cos\left(\frac{p\pi}{\ell+1}\right)$ since, by (5),

$$\chi_\ell\Big(2\cos\Big(\frac{p\,\pi}{\ell+1}\Big)\Big)=(-1)^\ell\sin(p\,\pi)=0\,.$$

By the results of last section, hence, we have proved in particular that

$$\left|\prod_{p=1}^{\ell}2\cos\Big(\frac{p\,\pi}{\ell+1}\Big)\right|=\begin{cases}1, & \text{if } \ell \text{ is even;}\\ 0, & \text{if } \ell \text{ is odd.}^5\end{cases}$$

4. Determinant associated with a general board

In this section we consider the general case, with squares labeled as in Section 3. In our example, where $\ell=4$ and $k=3$,

1	2	3	4
5	6	7	8
9	10	11	12

Now, we consider three $(\ell\,k)\times(\ell\,k)$ matrices, $N(\ell,k)=\big(a_{pq}\big)_{1\leqslant p,q\leqslant \ell\,k}$, where

$$a_{pq}=\begin{cases}0, & \text{if squares } p \text{ and } q \text{ are not adjacent,}\\ 1, & \text{if squares } p \text{ and } q \text{ are placed side by side,}\\ i=\sqrt{-1}, & \text{if squares } p \text{ and } q \text{ are placed one on top of the other;}\end{cases}$$

and $N_h(\ell,k)=\big(b_{pq}\big)_{1\leqslant p,q\leqslant \ell\,k}$ and $N_v(\ell,k)=\big(c_{pq}\big)_{1\leqslant p,q\leqslant \ell\,k}$ given by

$$b_{pq}=\begin{cases}1, & \text{if squares } p \text{ and } q \text{ are placed side by side,}\\ 0, & \text{otherwise;}\end{cases}$$

$$c_{pq}=\begin{cases}i, & \text{if squares } p \text{ and } q \text{ are placed one on top of the other,}\\ 0, & \text{otherwise.}\end{cases}$$

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[5]In fact, by considering the $(2\ell+2)$th complex roots of 1, we may write

$$\begin{aligned}X^{2\ell+2}-1&=\prod_{p=1}^{2\ell+2}\Big(X-\Big(\cos\Big(\frac{p\,\pi}{\ell+1}\Big)+\sin\Big(\frac{p\,\pi}{\ell+1}\Big)\,i\Big)\Big)\\&=(X^2-1)\prod_{p=1}^{\ell}\Big(X-\cos\Big(\frac{p\,\pi}{\ell+1}\Big)-\sin\Big(\frac{p\,\pi}{\ell+1}\Big)\,i\Big)\Big(X-\cos\Big(\frac{p\,\pi}{\ell+1}\Big)+\sin\Big(\frac{p\,\pi}{\ell+1}\Big)\,i\Big)\\&=(X^2-1)\prod_{p=1}^{\ell}\Big(X^2-2\cos\Big(\frac{p\,\pi}{\ell+1}\Big)X+1\Big)\end{aligned}$$

and so

$$\prod_{p=1}^{\ell}\Big(X^2-2\cos\Big(\frac{p\,\pi}{\ell+1}\Big)X+1\Big)=1+X^2+\cdots+X^{2\ell}\,.$$

Taking $X=i$, this shows that

$$\prod_{p=1}^{\ell}2\cos\Big(\frac{p\,\pi}{\ell+1}\Big)=\begin{cases}(-1)^{\ell/2}, & \text{if } \ell \text{ is even;}\\ 0, & \text{if } \ell \text{ is odd.}\end{cases}$$

(Compare with the roots of the Chebyshev polynomials of the second kind U_n.)

Of course again $\det\big(M(\ell,k)\big) = \big|\det\big(N(\ell,k)\big)\big|$ and

$$N(\ell,k) = N_h(\ell,k) + N_v(\ell,k)\,.$$

Before finding the eigenvalues of $N(\ell,k)$, we consider the eigenvalues that are common to $N_h(\ell,k)$ and $N_v(\ell,k)$. Their number is again $\ell\,k$. We first note that, if $(a_1,a_2,\ldots,a_\ell)$ is an eigenvector of $N(\ell,1)$ corresponding to the eigenvalue λ, and $m_1,\ldots,m_k$ are nonzero real numbers, then

$$v = (\underbrace{m_1a_1, m_1a_2, \ldots, m_1a_\ell}, \underbrace{m_2a_1, m_2a_2, \ldots, m_2a_\ell}, \ldots, \underbrace{m_ka_1, m_ka_2, \ldots, m_ka_\ell})$$

is an eigenvalue of $N_h(\ell,k)$, since $N_h(\ell,k)\cdot v^T = \lambda\, v^T$.

A similar thing happens with the columns: If $(b_1,b_2,\ldots,b_k)$ is an eigenvector of $N(k,1)$, corresponding to the eigenvalue μ, and $n_1,\ldots,n_\ell$ are nonzero real numbers, then

$$w = (\underbrace{n_1b_1, n_2b_1, \ldots, n_\ell b_1}, \underbrace{n_1b_2, n_2b_2, \ldots, n_\ell b_2}, \ldots, \underbrace{n_1b_k, n_2b_k, \ldots, n_\ell b_k})$$

is also an eigenvalue of $N_v(\ell,k)$, since $N_v(\ell,k)\cdot w^T = (\mu\, i)w^T$.

Hence, by taking $n_p = a_p$ for $p = 1,\ldots,\ell$ and $m_q = b_q$ for $q = 1,\ldots,k$, $v = w$ is a common eigenvalue of $N_h(\ell,k)$ and $N_v(\ell,k)$; more importantly, $\lambda + \mu\, i$ *is an eigenvalue of* $N(\ell,k)$, since

$$N(\ell,k)\cdot v^T = N_h(\ell,k)\cdot v^T + N_v(\ell,k)\cdot v^T = \lambda\, v^T + (\mu\, i)v^T = (\lambda + \mu\, i)\, v^T.$$

We have seen that $\lambda = 2\cos\left(\frac{p\,\pi}{\ell+1}\right)$ for some $p \in \{1,\ldots,\ell\}$, whereas $\mu = 2\cos\left(\frac{q\,\pi}{k+1}\right)$ for some $q \in \{1,\ldots,k\}$, and so we have $\ell\,k$ different eigenvalues of $N(\ell,k)$, and $\det\big(N(\ell,k)\big)$ is their product, that is,

$$\det\big(N(\ell,k)\big) = \prod_{p=1}^{\ell}\prod_{q=1}^{k}\left(2\cos\left(\frac{p\,\pi}{\ell+1}\right) + 2\cos\left(\frac{q\,\pi}{k+1}\right) i\right),$$

and, finally,

$$\begin{aligned}\det\big(M(\ell,k)\big) &= |\det\big(N(\ell,k)\big)| \\ &= \prod_{p=1}^{\ell}\prod_{q=1}^{k}\left\|2\cos\left(\frac{p\,\pi}{\ell+1}\right) + 2\cos\left(\frac{q\,\pi}{k+1}\right) i\right\| \\ &= \prod_{p=1}^{\ell}\prod_{q=1}^{k}\sqrt{4\cos^2\left(\frac{p\,\pi}{\ell+1}\right) + 4\cos^2\left(\frac{q\,\pi}{k+1}\right)}\,.\end{aligned}$$

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