CLUSTER DISTRIBUTIONS FOR DYNAMICALLY DEFINED POINT PROCESSES

CORENTIN CORREIA, ANA CRISTINA MOREIRA FREITAS, AND JORGE MILHAZES FREITAS

Abstract. The emergence of clustering of rare events for chaotic dynamics was first observed as a consequence of periodicity, *i.e.*, by considering target sets that shrink to a periodic point, one was able to create fast returns to these target sets which were responsible for the appearance of a bulk of high observations of observable functions that were maximised at the periodic point. This meant that the Rare Events Point Processes, counting the number of entrances in these target sets, converge to a compound Poisson process, with a geometric multiplicity distribution ruling the cluster sizes. In [AFFR16], a new mechanism to create clustering of rare events was introduced by considering observable functions maximised at a finite number of points that were linked by belonging to the same orbit. We make a deep study of the potential of this mechanism to produce different multiplicity distributions. Namely, we show that with the right choice of a system and observable, one can obtain any given finitely supported cluster size distribution. We also study the impact of symmetry and other properties of the systems on the possible clustering size distributions, which are also classified for the case of periodic maximal orbits.

Contents

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1. Introduction

In the last decade, the study of extreme value theory for dynamical systems, which turned out to be intimately related with quantitative recurrence, has captured the attention of many dynamicists (see for example [LFF$^+$16, BLR19, AFF20, FFMa20, HV20, PS20, CHN21, FFT21] and references therein) and inspired applications to climate dynamics (see [FMACY17, MCB$^+$18, MCF17, CFM$^+$19, LFG$^+$22] and references therein).

Observations of extreme or rare events are detected by abnormal high values of a given observable function and correspond to the entrance of the orbits of the system into sensitive regions of the phase space, which are assumed to shrink, as time evolves, to a set, $\mathcal{M}$, of zero measure, where the observable is maximised. We will refer to this set as the maximal set.

In this context, Point Processes of Rare Events have revealed to be a very useful tool to keep track of the number of rare events on a normalised time frame. Initially, $\mathcal{M}$ was assumed to be a typical point and then one could prove that these point processes converged to a standard Poisson process (see [HSV99, FFT10]). However, when $\mathcal{M}$ is a periodic point, the extremal observations have a tendency to appear concentrated in the time line, forming clusters of rare events associated to the fast recurrence imposed by the periodicity, which is responsible for the appearance of compound Poisson process in the limit. Since, the systems considered have nice mixing properties, there is an expansion at these periodic points which explains the appearance of the geometric distribution for the size of the clusters, which is of course associated to the Pólya–Aeppli distribution of the total number of extreme events ([HV09, FFT13]).

In [AFFR16], the authors introduced a new mechanism to create clustering of rare events by considering that $\mathcal{M}$ is made out of multiple points which are dynamically linked, in the sense that they belong to the same orbit. This dynamical link between the points of $\mathcal{M}$ is responsible for the creation of what was called a fake periodic behaviour, which was able to generate cluster size distributions other than the geometric one. In fact, other cluster size distributions had already been observed, for example in [AFV15] but they were still associated to periodic points, where the dynamical system was discontinuous.

The main purpose of this paper is to study deeper the device introduced in [AFFR16] to create clustering and, in particular, its potential to generate different cluster size distributions. We will show that with an appropriate choice of the dynamics and observables we can generate any finitely supported given distribution. However, as we will also show, the choice of the dynamics plays a surprisingly important role since, for example, the existence of symmetry may preclude the possibility of certain distributions appearing. Moreover, we will provide a more dynamical heuristics for the interpretation of the cluster distribution formulae obtained in [FFT13]. A more probabilistic interpretation of these formulae already appeared in [AFF20, Section 2.1], but the one we provide here is particularly useful for describing for example, the class of cluster size distributions that can arise when $\mathcal{M}$ is made out of a periodic orbit.

The paper is organised as follows. In Section 2 we establish the setting, give the definitions and conditions in order to prove convergence of the Rare Events Point Processes. In Section 3, we consider the case where the observable functions are maximised on multiple points of a non-periodic orbit and study the possible cluster size distributions that can arise in this situation. In Section 4, we study clustering patterns and provide a more dynamical interpretation of the formulae that give the asymptotic cluster size distribution. In Section 5, we study the possible cluster size distributions that may appear for observable functions maximised at multiple points of the same periodic orbit. In Section 6, we provide the proofs of the main results stated in Section 3, as well as a small simulation study to illustrate the ability to generate any given finitely supported distribution for cluster size. In Section 7, we provide the proof of the main result stated in Section 5 and we also consider some case studies from [AFFR16] to illustrate the performance of the methods we introduce here to obtain the cluster size distribution.

2. Context and background

In order to study the extremal behaviour of the chaotic dynamical systems, we consider stochastic processes generated by such systems, simply by evaluating a given observable function along its orbits,

2.1. **The setting.** Let $(\mathcal{X}, \mathcal{B}, f, \mu)$ be an ergodic dynamical system where $\mathcal{X}$ is a a compact Riemannian manifold with a Riemannian metric that we denote by $dist$, $\mathcal{B}$ is the Borel σ-algebra, $f\colon \mathcal{X} \to \mathcal{X}$ is a measurable map and μ is an f-invariant probability measure on $(\mathcal{X}, \mathcal{B})$. In this paper, μ will always be absolutely continuous with respect to Leb, we will say that it is an **acip** (absolutely continuous invariant probability). Given an observable $\varphi\colon \mathcal{X} \to \mathbb{R} \cup \{\pm\infty\}$, we consider the stochastic process $(X_n)_{n\in\mathbb{N}_0}$ given by :

$$X_n = \varphi \circ f^n.$$

As mentioned earlier, a realisation of the process $(X_n)_{n\in\mathbb{N}_0}$ corresponds to obtaining a point $x \in \mathcal{X}$, chosen according to the acip μ, and then evaluate the observable along its orbit $(f^n(x))_{n\in\mathbb{N}_0}$.

As in [AFFR16], we assume that φ achieves N global maxima at the points $\xi_1, \ldots, \xi_N$. We denote by $u_F := \varphi(\xi_i)$ the maximum value (we allow $u_F = +\infty$) and for each ξ_i, φ is defined on a neighbourhood of ξ_i by

$$\varphi(x) = h_i(dist(x, \xi_i))$$

where $h_i\colon V_i \to h_i(V_i)$ is a decreasing bijection on a neighbourhood V_i of 0 in $\mathbb{R}_0^+$ and $h_i(0) = u_F$. These requirements imply that the following set

$$U(u) := \{X_0 > u\}$$

is the union of the balls $\mathrm{B}\left(\xi_i, h_i^{-1}(u)\right)$ when the level u is sufficiently large. We suppose that the maps h_i and the f-invariant measure μ are sufficiently regular in the following sense : the quantity $\mu(U(u))$, as a function of u, varies continiously on a neighbourhood of u_F[1].

[1]Example : when the maps h_i are continuous and μ has no atoms (absolutely continuous for instance).

The extremal behaviour of the systems can then be partly described by the distributional limit of the sequence of partial maxima defined for each $n \in \mathbb{N}$ as:

$$M_n = \max\{X_0, \dots, X_{n-1}\}.$$

Ergodicity implies that M_n converges a.s. to u_F and in order to obtain a non-degenerate weak limit for M_n, motivated by the i.i.d. setting we consider a sequence of levels $(u_n)_{n\in\mathbb{N}}$ such that:

$$n\mu\left(X_0 > u_n\right) \underset{n\to\infty}{\longrightarrow} \tau > 0. \tag{2.1}$$

We succeed in proving the existence of an Extreme Value Law whenever, for a sequence satisfying (2.1), there exists a non-degenerate distribution function such that

$$\mu\left(M_n \leq u_n\right) \underset{n\to\infty}{\longrightarrow} 1 - H(\tau), \tag{2.2}$$

where the convergence is meant at the continuity points of H. Note that on the i.i.d. case, the convergence in (2.1) is equivalent to (2.2), where H is the standard exponential distribution function.

We observe that the inverse functions h_i^{-1} determine the tail of the d.f. F and $u_F = \varphi(\xi_i) = h_i(0)$ is its right endpoint. Moreover, they can be classified according to one of the following three types :

(1) there exists some positive function $g\colon h_i(V_i) \to \mathbb{R}$ such that for all $y \in \mathbb{R}$,
$$\lim_{s\to h_i(0)} \frac{h_i^{-1}(s + yh(s))}{h_i^{-1}(s)} = \mathrm{Exp}(-y);$$
(2) $h_i(0) = \infty$ and there exists $\beta > 0$ such that for all $y > 0$,
$$\lim_{s\to\infty} \frac{h_i^{-1}(sy)}{h_i^{-1}(s)} = y^{-\beta};$$
(3) $h_i(0) = D < \infty$ and there exists $\gamma > 0$ such that for all $y > 0$,
$$\lim_{s\to D} \frac{h_i^{-1}(D - sy)}{h_i^{-1}(D - s)} = y^{\gamma}.$$

The tail of the d.f. F determines the possible three types of limiting Extreme Value Law (given by the classical Extremal Types Theorem − see [LLR83, Section 1.4]), when the sequence of levels $(u_n)_{n\in\mathbb{N}}$ is taken as one parameter linear family like

$$u_n = \frac{y}{a_n} + b_n \tag{2.3}$$

for $y \in \mathbb{R}$, $a_n > 0$. In this case, τ is of one of the following three types expressed as a function of the parameter y given in u_n, for some $\beta, \gamma > 0$, (see [LFF$^+$16, Section 4.2.1]):

(1) $\tau_1(y) = \mathrm{Exp}(-y)$ for $y \in \mathbb{R}$ (Gumbel type);
(2) $\tau_2(y) = y^{-\beta}$ for $y > 0$ (Fréchet type);
(3) $\tau_1(y) = (-y)^{\gamma}$ for $y \leq 0$ (Weibull type).

We recall that the study of the distributional limit of the maximum is intimately related to quantitative recurrence ([Col01, FFT10]), which can be realised by noting that the event $\{M_n \leq u_n\}$ implies that there are no exceedances, *i.e.*, no occurrences of $\{X_i > u_n\}$ for all $i = 0, \dots, n-1$, which could be expressed equivalently by saying that the hitting time to the set $U(u_n) = \{X_0 > u_n\}$ is not less than n.

The study of the extremal behaviour and quantitative recurrence is enhanced by considering point processes that keep track of the orbital visits to the sets $U(u)$.

Definition 2.1. For every $A \subset \mathbb{R}$, $u \in \mathbb{R}$, we define :

$$\mathscr{N}_u(A) := \sum_{i \in A \cap \mathbb{N}_0} \mathbb{1}_{X_i > u}.$$

Let $\mathcal{R}$ denote the set of elements J which can be written as a finite union of subsets $[a, b)$ of $\mathbb{R}_0^+$. We define the Rare Event Point Process (REPP) N_n by :

$$\forall\, J \in \mathcal{R},\ N_n(J) := \mathscr{N}_{u_n}(v_n J) = \sum_{j \in (v_n J) \cap \mathbb{N}_0} \mathbb{1}_{X_j > u_n}$$

with $v_n = 1/\mu\,(X_0 > u_n)$.

We rescale the time in the definition of the REPP because there are less and less exceedances $\{X_i > u_n\}$, on a fixed time period, as the level u_n grows, and we need to compensate it by expanding the time period as the level grows, which creates a concentration of data in a fixed interval. In fact, the rescaling is done so that $\mathbb{E}\,[N_n([0,1))] = \mu\,(X_0 > u_n)\,(\lfloor v_n \rfloor + 1) \underset{n\to\infty}{\longrightarrow} 1$.

In the i.i.d. setting, one can prove that N_n converges weakly to a standard homogeneous Poisson process. However, in the presence of clustering of exceedances, N_n converges weakly to a compound Poisson process. By the time rescaling, the jumps of height 1 occurring in a cluster of size k for the REPP are compressed. Thus this series of jumps looks like a unique jump of size k. Thus the cluster size is asymptotically ruled by the multiplicity distribution of this limit process. Moreover note that $\{\mathscr{N}_{u_n}([0,n)) = 0\} = \{M_n \le u_n\}$, hence the limit distribution of $(M_n)_{n\in\mathbb{N}}$ can be easily recovered from the convergence of the REPP.

We recall here the definition of a compound Poisson process.

Definition 2.2. A compound Poisson process of intensity θ and multiplicity distribution π is a point process defined by

$$N(J) := \int \mathbb{1}_J d\left(\sum_{i=1}^{\infty} D_i \delta_{T_1+\ldots+T_i}\right) = \sum_{i=1}^{\infty} D_i \delta_{T_1+\ldots+T_i}(J) = \sum_{\substack{i=1 \\ T_1+\ldots+T_i \in J}}^{\infty} D_i$$

where $T_1, T_2, \ldots, D_1, D_2, \ldots$ are independent random variables defined on a probability space, $(T_n)_{n\in\mathbb{N}}$ is an i.i.d. sequence with common exponential distribution of mean $1/\theta$ and $(D_n)_{n\in\mathbb{N}}$ is another i.i.d. sequence with common distribution π.

Note that by (2.1) and the definition of the normalising time factor v_n, when there is no clustering, the Poisson process has intensity $\theta = 1$ and $\pi(1) = 1$. On the other hand, in the presence of clustering we clearly have that the expected cluster size $\sum_{j=1}^{\infty} j\pi(j) > 1$, which is compensated by a smaller frequency $\theta < 1$ (so that $\mathbb{E}\,[N_n([0,1))] \underset{n\to\infty}{\longrightarrow} 1$).

The parameter $0 \le \theta \le 1$ is called Extremal Index (EI) and the fact that in most situations it is the reciprocal of the mean cluster size (see [AFF20]) makes it particularly useful to quantify the intensity of clustering. Absence of clustering means $\theta = 1$ and intense clustering means that θ is close to 0.

2.2. **Convergence of Rare Events Point Processes.** The convergence of REPP in the dynamical setting relies heavily on being able to use memory loss properties of the system in a useful way, which was paved from the adjustment and generalisation of classical conditions introduced by Leadbetter (see [LLR83]) to study stationary stochastic processes. These adaptations are particularly sensitive in the case of presence of clustering where the definition of a certain hierarchy of events is rather convenient to write the formulae for the multiplicity distribution and the computation of the EI.

Definition 2.3. Given $q \in \mathbb{N}$ and a level u, we define the following events :

$$U^{(0)}(u) := \{X_0 > u\}$$

$$\mathcal{A}_q^{(0)}(u) := \{X_0 > u, X_1 \le u, \dots, X_q \le u\} = U^{(0)}(u) \cap \bigcap_{i=1}^{q} f^{-i}\left((U^{(0)}(u))^c\right)$$

and for each $\kappa \in \mathbb{N}$

$$U^{(\kappa)}(u) := U^{(\kappa-1)}(u) \setminus \mathcal{A}_q^{(\kappa-1)}(u)$$

$$\mathcal{A}_q^{(\kappa)}(u) := U^{(\kappa)}(u) \cap \bigcap_{i=1}^{q} f^{-i}\left((U^{(\kappa)}(u))^c\right).$$

For $u = u_n$, we write $\mathcal{A}_{q,n}^{(\kappa)} := \mathcal{A}_q^{(\kappa)}(u_n)$ and $U_n^{(\kappa)} := U^{(\kappa)}(u_n)$.
We denote by $U_n^{(0)}(\xi_i)$ the ball $\mathrm{B}\left(\xi_i, h_i^{-1}(u_n)\right)$ for n sufficiently large. It follows that $U_n^{(0)} = \bigcup_{i=1}^{N} U_n^{(0)}(\xi_i)$ and we define

$$U_n^{(\kappa)}(\xi_i) := U_n^{(\kappa)} \cap U_n^{(0)}(\xi_i) \text{ and } \mathcal{A}_{q,n}^{(\kappa)}(\xi_i) := \mathcal{A}_{q,n}^{(\kappa)} \cap U_n^{(0)}(\xi_i).$$

Given $B \in \mathcal{B}$, for some $s \ge 0$ and $\ell \ge 0$, we define :

$$\mathscr{W}_{s,\ell}(B) := \bigcap_{i=\lfloor s \rfloor}^{\lfloor s \rfloor + \max\{\lfloor \ell \rfloor - 1, 0\}} f^{-i}(B^c).$$

$U^{(\kappa)}(u)$ occurs when there are at least $\kappa + 1$ exceedances, the first appears at time 0 and there are less than q units of time between each exceedance (except the $\kappa + 1$-th one) and the succeeding one. $\mathcal{A}_q^{(\kappa)}(u)$ occurs when $U^{(\kappa)}(u)$ does and there is no exceedance up to q units of time after the $\kappa + 1$-th exceedance.

Usually, for every i, $(U_n^{(\kappa)}(\xi_i))_{\kappa \in \mathbb{N}_0}$ is a sequence of nested topological balls and the events $\mathcal{A}_{q,n}^{(\kappa)}(\xi_i)$ are annuli around these topological balls such that $U_n^{(\kappa)}(\xi_i) = \mathcal{A}_{q,n}^{(\kappa)}(\xi_i) \cup U_n^{(\kappa+1)}(\xi_i)$.
$\mathscr{W}_{s,\ell}(B)$ occurs when the event B has not occurred during the time period of length $\lfloor \ell \rfloor$ and starting at time s.

We introduce below two conditions which allow to prove the convergence of the REPP. The first is a condition of asymptotic independence, the second condition restricts the occurrence of events $f^{-i}(\mathcal{A}_{q,n}^{(0)})$ close together in time.

Condition ($Д_q^*$)**.** We say that $Д_q^*$ holds for the sequence $(X_n)_{n \in \mathbb{N}}$ if for any integers $t, \kappa_1, \dots, \kappa_\varsigma, n$ and any $J = \bigcup_{j=2}^{\varsigma} I_j \in \mathcal{R}$ with $I_j = [a_j, b_j)$ and $\inf J \ge t$,

$$\left| \mu\left(\mathcal{A}_{q,n}^{(\kappa_1)} \cap \left(\bigcap_{j=2}^{\varsigma} \mathscr{N}_{u_n}(I_j) = \kappa_j\right)\right) - \mu\left(\mathcal{A}_{q,n}^{(\kappa_1)}\right) \mu\left(\bigcap_{j=2}^{\varsigma} \mathscr{N}_{u_n}(I_j) = \kappa_j\right) \right| \le \gamma(q, n, t)$$

where $\gamma(q, n, t)$ is decreasing in t for each n and there exists a sequence $(t_n)_{n \in \mathbb{N}}$ such that $t_n = o(n)$ and $n\gamma(q, n, t_n) \xrightarrow[n\to\infty]{} 0$.

Condition ($Д'_q$)**.** We say that $Д'_q$ holds for the sequence $(X_n)_{n \in \mathbb{N}}$ if there exists a sequence $(k_n)_{n \in \mathbb{N}}$ such that

- $k_n \xrightarrow[n\to\infty]{} \infty$, $k_n t_n = o(n)$ where t_n is given by condition $Д_q^*$;
- $\lim_{n\to\infty} n \sum_{j=q+1}^{\lfloor n/k_n \rfloor - 1} \mu\left(\mathcal{A}_{q,n}^{(0)} \cap f^{-j}(\mathcal{A}_{q,n}^{(0)})\right) = 0.$

We recall now the main convergence result for the REPP, which also gives formulae for the EI and the multiplicity distribution.

Theorem 2.4 (From [FFT10, FFT13])**.** *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence satisfying $n\mu\left(X_0 > u_n\right) \underset{n\to\infty}{\longrightarrow} \tau$ for some $\tau > 0$. Assume that $Д^*_q$ and $Д'_q$ hold for some $q \in \mathbb{N}_0$. Then the REPP $(N_n)_{n\in\mathbb{N}}$ converges in distribution to a compound Poisson process of intensity θ and an integer valued multiplicity distribution π whenever the following limits exist :*

$$\theta := \lim_{n\to\infty} \frac{\mu\left(\mathcal{A}^{(0)}_{q,n}\right)}{\mu\left(U^{(0)}_n\right)} \quad \text{and} \quad \forall\, \kappa \in \mathbb{N},\ \pi(\kappa) = \lim_{n\to\infty} \frac{\mu\left(\mathcal{A}^{(\kappa-1)}_{q,n}\right) - \mu\left(\mathcal{A}^{(\kappa)}_{q,n}\right)}{\mu\left(\mathcal{A}^{(0)}_{q,n}\right)}.$$

Note that, except for some exceptional cases as given in [AFF20], most of the times we have $\theta^{-1} = \sum_{\kappa\in\mathbb{N}} \kappa\pi(\kappa)$, it is the average cluster size.

The main advantage of the conditions above when compared to the classical ones is the fact that they are easy to check if the systems have nice mixing properties, which are usually captured by the rates of decay of correlations that we define next.

Definition 2.5. (Decay of correlations). Let $\mathscr{C}_1, \mathscr{C}_2$ denote Banach spaces of real valued measurable functions defined on $\mathcal{X}$. We denote the correlation of non-zero functions $\phi \in \mathscr{C}_1$ and $\psi \in \mathscr{C}_2$ w.r.t. a measure μ as

$$\mathrm{Cor}_\mu(\phi, \psi, n) := \frac{1}{\|\phi\|_{\mathscr{C}_1}\|\psi\|_{\mathscr{C}_2}} \left| \int \phi \cdot (\psi \circ f^n)\, \mathrm{d}\mu - \int \phi\, \mathrm{d}\mu \int \psi\, \mathrm{d}\mu \right|.$$

We say that we have decay of correlations, w.r.t. the measure μ, for observables in $\mathscr{C}_1$ against observables in $\mathscr{C}_2$ if we have $\mathrm{Cor}_\mu(\phi, \psi, n) \underset{n\to\infty}{\longrightarrow} 0$ for every $\phi \in \mathscr{C}_1$ and every $\psi \in \mathscr{C}_2$.

Condition $Д^*_q$ is typically very easy to check from most of statements providing summable decay of correlations for the systems in consideration, *i.e.*, $\sum_{n\geq 1} \mathrm{Cor}_\mu(\phi, \psi, n) < \infty$. See [LFF$^+$16, Section 4.4].

For some systems it is possible to prove summable decay of correlations for all φ in some Banach space $\mathscr{C}_1$ against all $L^1(\mu)$ functions ψ. This strong form of decay of correlations, which we will express as summable decay of correlations against L^1, allows us to prove condition $Д'_q$, as well. See for example [LFF$^+$16, Proposition 4.2.13] .

When $N = 1$, *i.e.*, when the observable is maximised at a single point of the phase space ξ, for nice systems (for example, systems with summable decay of correlation against L^1), a dichotomy regarding the extremal behaviour has been proved. See [LFF$^+$16, Theorem 4.3.5]. Namely, either ξ is a non periodic point in which case there is no clustering of exceedances, the extremal index θ is equal to 1 and the N_n converges weakly to a standard Poisson process; or ξ is a periodic point, say of period p, and in that case we have clustering of exceedances, detected by an extremal index less than 1, which, in the case of an absolutely continuous invariant probability measure with a regular density, it is given by the formula:

$$\theta = 1 - |\det Df^p(\xi)|^{-1}.$$

Moreover, the REPP converges to a compound Poisson process of intensity θ and geometric multiplicity distribution π :

$$\pi(\kappa) = \theta(1-\theta)^{\kappa-1}, \qquad \text{for each } \kappa \in \mathbb{N}.$$

Throughout the rest of text, we assume that the dynamical systems considered have summable decay of correlations against L^1 and the observable admits N global maxima lying on the orbit of ξ_1 which is either a non-periodic point or a repelling periodic point.

Moreover, we add other assumptions from which follows an ideal framework for the computations. μ is absolutely continuous, the Lebesgue differentiation theorem holds for the maxima and f is a one-dimensional map. The derivative $\mathrm{D}f(x)$ is defined and non-zero for each point $x = f^j(\xi_1)$ with $j \in \{0, \dots, m_N - 1\}$ in the non-periodic case and for each point x in the orbit of ξ_1, in the periodic case. Finally the observable is such that there exists r_n equal to the radius of each extremal sets $U_n^{(0)}(\xi_i)$ within a constant factor of multiplication depending on i and independent of n.

3. Non-periodic maximal orbits

We start by studying the non-periodic case, i.e. ξ_1 is not periodic. We recall that the observable φ has to achieve N global maxima $\xi_1, \xi_2, \dots, \xi_N$ which are correlated :

$$\forall\, i \in \{1, \dots, N\},\ \xi_i = f^{m_i}(\xi_1)$$

with $0 = m_1 < m_2 < \dots < m_N$. In the applications, almost always, we will choose a dynamical system $(\mathcal{X}, \mathcal{B}, f, \mu)$ with summable decay of correlations of bounded variation functions agains L^1 and maps h_i defining φ on a neighbourhood of each maxima. Note that the maps h_i give the radius of the balls $U_n^{(0)}(\xi_i)$. However we will first define the balls $U_n^{(0)}(\xi_i)$ and then find a corresponding observable φ.

In this case of non-periodicity, it is easy to check that

$$\mathcal{A}_{q,n}^{(\kappa)}(\xi_i) = U_n^{(\kappa)}(\xi_i) \setminus \left(\bigcup_{\ell=i+1}^{N} f^{-m_\ell}(U_n^{(\kappa)}(\xi_\ell)) \right)$$

for n sufficiently large. Note that "for n sufficiently large" will not be specified anymore. It is an asymptotic information following from the continuity of f at each ξ_i and the fact that we only know the expression of φ on the neighbourhoods of maximal points considered.

When N is equal to 1, the exceedances of a high threshold u_n appear scattered through the time line (no clustering) and the multiplicity distribution π always satisfies $\pi(1) = 1$.

For $N \geq 2$, we study several examples of dynamical systems and observable configurations and analyse the possible emerging cluster size distributions.

3.1. **The case of two maximal points and the effect of symmetry of the systems.**

We start by considering that the observable achieves two global maxima at ξ_1 and $\xi_2 = f(\xi_1)$, where ξ_1 is not periodic. Suppose that $U_n^{(0)}$ is defined. Since $N = 2$, it is easy to compute $U_n^{(\kappa)}$ for $\kappa \in \mathbb{N}$:

$$U_n^{(1)} = U_n^{(0)}(\xi_1) \cap f^{-1}(U_n^{(0)}(\xi_2)) \text{ and } \forall\, \kappa \geq 2,\ U_n^{(\kappa)} = \emptyset.$$

Then the multiplicity distribution π and the EI θ are given by :

$$\begin{cases} \pi(1) = \lim\limits_{n\to+\infty} \dfrac{\mu(U_n^{(0)}) - 2\mu(U_n^{(0)}(\xi_1) \cap f^{-1}(U_n^{(0)}(\xi_2)))}{\mu(U_n^{(0)}) - \mu(U_n^{(0)}(\xi_1) \cap f^{-1}(U_n^{(0)}(\xi_2)))} \\ \pi(2) = \lim\limits_{n\to+\infty} \dfrac{\mu(U_n^{(0)}(\xi_1) \cap f^{-1}(U_n^{(0)}(\xi_2)))}{\mu(U_n^{(0)}) - \mu(U_n^{(0)}(\xi_1) \cap f^{-1}(U_n^{(0)}(\xi_2)))} \\ \pi(\kappa) = 0 \quad \text{for all } \kappa \geq 3 \\ \theta = \lim\limits_{n\to+\infty} \left(1 - \dfrac{\mu(U_n^{(0)}(\xi_1) \cap f^{-1}(U_n^{(0)}(\xi_2)))}{\mu(U_n^{(0)})} \right) \end{cases} \tag{3.1}$$

whenever the limits exist. θ^{-1} is the expectation of a distribution on $\{1,2\}$ so θ determines the distribution and must be in $[1/2,1]$. Then we only need to study

$$\alpha_n := \frac{\mu(U_n^{(0)}(\xi_1)\cap f^{-1}(U_n^{(0)}(\xi_2)))}{\mu(U_n^{(0)})}.$$

Note that the case $\theta = 1$ is not interesting as it leads to a distribution π verifying $\pi(1) = 1$, which is a multiplicity distribution that we have already seen in the case $N = 1$.

Regarding the numerator of α_n, we have to compare $U_n^{(0)}(\xi_1)$ and the connected component of ξ_1 in the preimage $f^{-1}(U_n^{(0)}(\xi_2))$, which is equivalent to the comparison between $f(U_n^{(0)}(\xi_1))$ and $U_n^{(0)}(\xi_2)$. Suppose that f is differentiable at ξ_1. We consider an observable such that :

$$\begin{cases} U_n^{(0)}(\xi_1) = \mathrm{B}(\xi_1, r_n) \\ U_n^{(0)}(\xi_2) = \mathrm{B}(\xi_2, \lambda\cdot|\det \mathrm{D}f(\xi_1)|\cdot r_n) \end{cases} \tag{3.2}$$

for some $\lambda \in \mathbb{R}_0^+$ and $(r_n)_{n\in\mathbb{N}}$ vanishing to 0, which we assume to exist. In fact, since $f(U_n^{(0)}(\xi_1))$ is approximately $\mathrm{B}(\xi_2, |\det \mathrm{D}f(\xi_1)|r_n)$, then λ is a relevant parameter for the comparison, depending on whether λ is larger or less than 1.

3.1.1. *Bernoulli maps.* We consider a simple dynamical system given by $\mathcal{X} = \mathbb{S}^1 = [0,1]/(0\sim 1)$, $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ defined by $f(x) = qx \bmod 1$ with q an integer larger or equal to 2 and $\mu = \mathrm{Leb}$ which is an f-invariant probability measure.

Figure/Graph1.png Figure/Graph2.png

FIGURE 3.1. Graphs of $f\colon x\mapsto qx \bmod 1$ with $q = 2$ and $q = 10$.

Proposition 3.1. *Assume that the dynamical system considered is given by $f(x) = qx \bmod 1$ on $\mathbb{S}^1$ with $\mu = \mathrm{Leb}$ and the type of observables (depending on a parameter λ) such that the balls $U_n^{(0)}(\xi_i)$ satisfy (3.2). Then the multiplicity distributions that we get for the limit of the REPP are the distributions π on $\{1,2\}$ such that $1-\frac{1}{q}\le \pi(1) < 1$.*

To be more precise, we obtain $\pi(1) = 1 - \frac{1}{q\lambda}$ with $\lambda \ge 1$.

Remark 3.2. Note that we cannot get $\pi(2) > \frac{1}{q}$ with the system $f(x) = qx \bmod 1$.
To see this, first observe that

$$\frac{\mu(U_n^{(0)}(\xi_1)\cap f^{-1}(U_n^{(0)}(\xi_2)))}{\mu(U_n^{(0)}) - \mu(U_n^{(0)}(\xi_1)\cap f^{-1}(U_n^{(0)}(\xi_2)))} \le \frac{1}{q} \iff (q+1)\mu(U_n^{(0)}(\xi_1)\cap f^{-1}(U_n^{(0)}(\xi_2))) \le \mu(U_n^{(0)}). \tag{3.3}$$

Secondly, the right hand side of (3.3) implies that: $\mu(U_n^{(0)}(\xi_1)\cap f^{-1}(U_n^{(0)}(\xi_2)))\leq \mu(U_n^{(0)}(\xi_1))$ and

$$\mu(U_n^{(0)}(\xi_1)\cap f^{-1}(U_n^{(0)}(\xi_2)))\leq \frac{1}{q}\mu(f^{-1}(U_n^{(0)}(\xi_2)))=\frac{1}{q}\mu(U_n^{(0)}(\xi_2)) \text{ by invariance,} \tag{3.4}$$

then we conclude that

$$\begin{aligned}(q+1)\mu(U_n^{(0)}(\xi_1) &\cap f^{-1}(U_n^{(0)}(\xi_2)))=\\ &= \mu(U_n^{(0)}(\xi_1)\cap f^{-1}(U_n^{(0)}(\xi_2)))+q\mu(U_n^{(0)}(\xi_1)\cap f^{-1}(U_n^{(0)}(\xi_2)))\\ &\leq \mu(U_n^{(0)}(\xi_1))+\mu(U_n^{(0)}(\xi_2))\\ &= \mu(U_n^{(0)}).\end{aligned}$$

Finally, by (3.3), we obtain that $\pi(2):=\lim\frac{\mu(U_n^{(0)}(\xi_1)\cap f^{-1}(U_n^{(0)}(\xi_2)))}{\mu(U_n^{(0)})-\mu(U_n^{(0)}(\xi_1)\cap f^{-1}(U_n^{(0)}(\xi_2)))}$ is less or equal to $1/q$ for all choices of observables achieving two maxima. With the same argument, it is not possible to get $\pi(2)>1/q$ if $\xi_2=f^{m_2}(\xi_1)$ with an integer m_2 not necessarily equal to 1.

A crucial point is that what prevents from improving the result is the first inequality in (3.4). It is due to the fact that μ is translation-invariant and $f(x+1/q)=f(x)$ for every $x\in\mathbb{S}^1$.

3.1.2. *The full quadratic map.* Let $(I,\mathcal{B},f,\mu)$ be the dynamical system where I is the interval $[-1,1]$, $f\colon I\to I$ is the full quadratic map given by $f(x)=1-2x^2$ and μ is an **acip** with density given by :

$$\rho(x)=\frac{\mathrm{d}\mu}{\mathrm{dLeb}}(x)=\frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}. \tag{3.5}$$

The symmetry of f and μ implies that

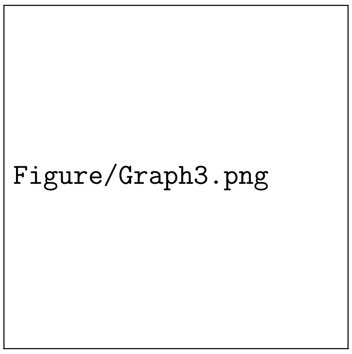

FIGURE 3.2. Graph of the full quadratic map.

$$\mu(U_n^{(0)}(\xi_1)\cap f^{-1}(U_n^{(0)}(\xi_2)))\leq\frac{1}{2}\mu(f^{-1}(U_n^{(0)}(\xi_2))),$$

as for the doubling map $f(x)=2x$ mod 1. Then we cannot get a multiplicity distribution π on $\{1,2\}$ such that $\pi(2)>1/2$. In fact, for the limit of the REPP of the full quadratic map, one obtains the same multiplicity distributions already obtained for the doubling map.

Proposition 3.3. *Assume that the dynamical system considered is given by $f(x)=1-2x^2$ on $[-1,1]$ with an **acip** μ of density given by (3.5) and the type of observables (depending on a parameter λ) such that the balls $U_n^{(0)}(\xi_i)$ satisfy (3.2). Then the multiplicity distributions that we get for the limit of the REPP are the distributions π on $\{1,2\}$ such that $\frac{1}{2}\leq\pi(1)<1$.*

To be more precise, we obtain $\pi(1)=1-\frac{1}{2\lambda}$ with $\lambda\geq 1$.

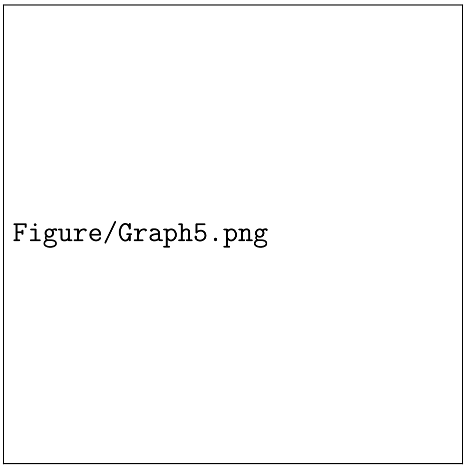

FIGURE 3.3. Graph of f_δ.

3.1.3. *A family of tent maps.* In order to obtain more diversified cluster distributions, we need to break symmetry and consider the following family of tent maps.

$$f_\delta\colon \begin{cases} [0,1] & \to \quad [0,1] \\ \quad x & \mapsto \quad \delta x \, \mathbb{1}_{[0,\delta^{-1}]}(x) + \dfrac{1-x}{1-\delta^{-1}} \, \mathbb{1}_{(\delta^{-1},1]}(x) \end{cases} . \tag{3.6}$$

We consider that $\lambda > 1$. These are uniformly expanding systems for which Lebesgue measure is invariant.

In order to simplify the analysis, we assume that both ξ_1 and ξ_2 lie on $(0, \delta^{-1})$, which means that, in particular, they share the same derivative.

Proposition 3.4. *Let $\delta > 1$. With the type of observables (depending on a parameter λ) such that the balls $U_n^{(0)}(\xi_i)$ satisfy (3.2) and the points ξ_i are in $(0, \delta^{-1})$, the system f_δ gives the multiplicity distributions π on $\{1, 2\}$ such that*

$$1 - \delta^{-1} \leq \pi(1) < 1.$$

Remark 3.5. As in Remark 3.2, we cannot get $\pi(2) > \delta^{-1}$ with the system given by f_δ. However, note that, within the family, we can make $\pi(2)$ arbitrarily close to 1, which means that except for the cases $\pi(1) = 1$ and $\pi(1) = 0$, we can generate any cluster size distribution supported on $\{1, 2\}$.

3.2. **The more general case with an arbitrary finite number of maximal points.** Given an $N \in \mathbb{N}$, we consider the maximal points $\xi_1, \ldots, \xi_N$ such that $\xi_i = f^{i-1}(\xi_1)$, i.e. $m_i = i - 1$, and define the balls:

$$\begin{cases} U_n^{(0)}(\xi_1) = \mathrm{B}(\xi_1, r_n) \\ U_n^{(0)}(\xi_2) = \mathrm{B}\left(\xi_2, \lambda_1 \cdot |\mathrm{D}f(\xi_1)| \cdot r_n\right) \\ U_n^{(0)}(\xi_3) = \mathrm{B}\left(\xi_3, \lambda_1\lambda_2 \cdot |\mathrm{D}f(\xi_1)\mathrm{D}f(\xi_2)| \cdot r_n\right) \\ \qquad\qquad \vdots \\ U_n^{(0)}(\xi_N) = \mathrm{B}\left(\xi_N, \lambda_1\lambda_2 \ldots \lambda_{N-1} \cdot |\mathrm{D}f(\xi_1)\mathrm{D}f(\xi_2) \ldots \mathrm{D}f(\xi_{N-1})| \cdot r_n\right) \end{cases} \tag{3.7}$$

for some $\lambda_1, \ldots, \lambda_{N-1} \in \mathbb{R}_0^+$.

3.2.1. *Application to the family of tent maps.* In this more general setting, the family of tent maps considered earlier can generate quite a diverse scope of finitely supported cluster size distributions.

Proposition 3.6. *Let f_δ be given as in* (3.6)*, for $\delta > 1$ and consider that the observable is such that the balls $U_n^{(0)}(\xi_i)$ satisfy* (3.7)*, with $\lambda_1, \ldots, \lambda_{N-1} \geq 1$, and the points ξ_i are in $(0, \delta^{-1})$, for all $i = 1, \ldots, N$. Then, the cluster size distribution π defined on $\{1, \ldots, N\}$ is such that*

$$\pi_{\{\kappa,\ldots,N\}}(\kappa) = 1 - \delta^{-1}/\lambda_{N-\kappa} \qquad \text{for every } \kappa \in \{1, \ldots, N-1\},$$

where π_A denotes the conditional probability $\frac{\pi(\cdot\cap A)}{\pi(A)}$. Moreover, the EI is given by

$$\theta^{-1} = \sum_{\kappa=1}^{N} \prod_{i=N-\kappa+1}^{N-1} \frac{1}{\lambda_i \delta}.$$

Remark 3.7. Note that $\forall\, \kappa \in \{1, \ldots, N-1\},\ \pi_{\{\kappa,\ldots,N\}}(\kappa) \in [1 - \delta^{-1}, 1)$.

Corollary 3.8. *With the type of observables considered in* (3.7) *(depending on parameters $\lambda_1, \ldots, \lambda_{N-1}$), the family $(f_\delta)_{\delta>1}$ can generate multiplicity distributions π on $\{1, \ldots, N\}$ such that*

$$\forall\, \kappa \in \{1, \ldots, N\},\ \pi(\kappa) > 0.$$

Proof. Given such a distribution on $\{1, \ldots, N\}$, the quantity $\min_{\kappa\in\{1,\ldots,N-1\}} \pi_{\{\kappa,\ldots,N\}}(\kappa)$ is non-zero. Then take $\delta > 1$ such that $\min_{\kappa\in\{1,\ldots,N-1\}} \pi_{\{\kappa,\ldots,N\}}(\kappa) \geq 1 - \delta^{-1}$ and apply Proposition 3.6. □

Numerical results are given in the appendix (see 6.2) where we illustrate the phenomenon with two distributions : a uniform distribution and a binomial distribution. For each case, we highlight the fact that the REPP records clusters whose size is asymptotically ruled by the prescribed distribution and the wait time between each cluster is ruled by an exponential law of mean θ^{-1} where θ is the EI.

3.2.2. *Application to a family of distorted tent maps.* In the previous section, the system is piecewise linear, equipped with Leb and the derivative at the correlated maxima is always equal to δ. We want a system for which the derivative at ξ_i is equal to $\delta_i > 1$, for each $i \in \{1, \ldots, N-1\}$ (the derivative at ξ_N does not affect the multiplicity distribution).

Definition 3.9. Given $\mathbf{\Delta} = (\delta_1, \ldots, \delta_{N-1})$ with $\delta_i > 1$, let $f_{\mathbf{\Delta}}\colon I \longrightarrow I$ denote an interval map satisfying these three properties :

- $f_{\mathbf{\Delta}}$ is a piecewise linear and uniformly expanding map;
- the $f_{\mathbf{\Delta}}$-invariant probability measure is Leb within a constant factor of multiplication;
- there exists $\xi_1 \in I$ such that $|\mathrm{D}f_{\mathbf{\Delta}}(\xi_i)| = \delta_i$ for every $i \in \{1, \ldots, N-1\}$, with $\xi_i := f_{\mathbf{\Delta}}^{i-1}(\xi_1)$ for every $i \in \{1, \ldots, N\}$, and the points ξ_i have two preimages.

We provide a plot of such a map in Figure 3.4. Similarly to Proposition 3.6, we have:

Proposition 3.10. *Let f_Δ be as described above and consider that the observable is such that the balls $U_n^{(0)}(\xi_i)$ satisfy* (3.7)*, with $\lambda_1, \ldots, \lambda_{N-1} \geq 1$, and the points ξ_i are in $(0, \delta^{-1})$,*

Figure/Graph13.png

FIGURE 3.4. Graph of $f_{\boldsymbol{\Delta}}$

for all $i = 1, \ldots, N$. *Then, the cluster size distribution* π *is given by*

$$\begin{cases} \pi(1) &= 1 - \dfrac{1}{\lambda_{N-1}\delta_{N-1}} \\ \pi(\kappa) &= \displaystyle\prod_{i=N-\kappa+1}^{N-1} \frac{1}{\lambda_i\delta_i} - \prod_{i=N-\kappa}^{N-1} \frac{1}{\lambda_i\delta_i} \textit{ for all } \kappa \in \{1, \ldots, N-1\} \\ \pi(N) &= \displaystyle\prod_{i=1}^{N-1} \frac{1}{\lambda_i\delta_i}; \\ \pi(\kappa) &= 0 \textit{ for all } \kappa \geq N+1 \\ \theta^{-1} &= \displaystyle\sum_{\kappa=1}^{N} \prod_{i=N-\kappa+1}^{N-1} \frac{1}{\lambda_i\delta_i} \end{cases}.$$

Remark 3.11. Observe that the distribution π satisfies $\pi_{\{\kappa,\ldots,N\}}(\kappa) = 1 - \delta_{N-\kappa}^{-1}/\lambda_{N-\kappa}$ for every $\kappa \in \{1, \ldots, N-1\}$ and $\lambda \mapsto 1 - \delta_{N-\kappa}^{-1}/\lambda$ is a bijection from $[1, +\infty)$ to $\big[1 - \delta_{N-\kappa}^{-1}, 1\big)$.

3.2.3. *Application to β maps.* In all the previous examples (the Bernoulli maps, the full quadratic map and the tent maps f_δ and $f_{\boldsymbol{\Delta}}$), the preimage of a ball $U_n^{(0)}(\xi_i)$ had at least two connected components. One that contains the point ξ_{i-1} and the others contain the other preimages of ξ_i. The presence of multiple connected components prevents from having the multiplicity distribution for which $\pi(2) = 1$, for example (see Remarks 3.2 and 3.5).

Hence, in order to be able to generate any finitely supported cluster size distribution, we consider a family of systems admitting points with a unique preimage, namely, we consider the family of β maps:

$$T_\beta \colon \begin{cases} \mathbb{S}^1 & \to & \mathbb{S}^1 \\ x & \mapsto & \beta x \bmod 1 \end{cases}, \qquad \text{where } \beta \in (1, 2).$$

For $\beta > 1$, the maps T_β admit an **acip** $\mu = \mu_\beta$ whose density $\rho = \rho_\beta$ w.r.t. Leb is given by

$$x \mapsto \sum_{\substack{n=0 \\ x < T_\beta^n(1)}}^{+\infty} \frac{1}{\beta^n}$$

within a constant factor of multiplication (see [Par60]). Moreover, these systems have decay of correlations against L^1-observables. Now, for β in the interval $(1, 2)$ there is an interval

Figure/Graph8.png

FIGURE 3.5. Graph of T_β.

of points with a unique preimage. Let $\xi_1 \in [0,1]$, $\xi_2 = T_\beta(\xi_1)$, $\dots$, $\xi_N = T_\beta^{N-1}(\xi_1)$ for which the Lebesgue differentiation theorem holds. Let β be sufficiently close to 1 in order to have a point ξ_1 such that the points $\xi_1, \dots, \xi_{N-1}$ belong to the interval $(1-1/\beta, 1/\beta)$ (*i.e.*, $\xi_2, \dots, \xi_N$ have a unique preimage). Let φ be an observable such that the balls $U_n^{(0)}(\xi_i)$ are as in (3.7), with parameters $\lambda_1, \dots, \lambda_{N-1} \geq 1$. As for the previous examples, the ball where the orbit enters first determines the size of the cluster but the way to enter changes by uniqueness of the preimage.

A cluster of size 1 is due to the entrance of the orbit in the set

$$\underbrace{\mathrm{B}\left(\xi_N, \lambda_1\lambda_2\dots\lambda_{N-2}\lambda_{N-1}\beta^{N-1}r_n\right)}_{U_n^{(0)}(\xi_N)} \setminus \underbrace{\mathrm{B}\left(\xi_N, \lambda_1\lambda_2\dots\lambda_{N-2}\beta^{N-2}r_n\right)}_{T_\beta(U_n^{(0)}(\xi_{N-1}))}$$

and the only way to enter in it is to get close to ξ_{N-1} one unit of time before that. Note that in the previous examples, there were connected components of $f^{-1}(U_n^{(0)}(\xi_N))$ other than the component of ξ_{N-1}, which created other possibilities to enter in $U_n^{(0)}(\xi_N) \setminus f(U_n^{(0)}(\xi_{N-1}))$. Then it prevented from having $\pi(1)$ less than a certain positive value as the parameter λ_{N-1} did not affect the proportion between the different connected components of $f^{-1}(U_n^{(0)}(\xi_N))$. That is the reason why it may be expected in the case of a unique preimage that the limit probability to have a cluster of size 1 is equal to 0 when λ_{N-1} is equal to 1.

In the same way, a cluster of size 2 is due to the entrance of the orbit in the set

$$\underbrace{\mathrm{B}\left(\xi_N, \lambda_1\lambda_2\dots\lambda_{N-3}\lambda_{N-2}\beta^{N-2}r_n\right)}_{U_n^{(0)}(\xi_{N-1})} \setminus \underbrace{\mathrm{B}\left(\xi_N, \lambda_1\lambda_2\dots\lambda_{N-3}\beta^{N-3}r_n\right)}_{T_\beta(U_n^{(0)}(\xi_{N-2}))}$$

and the only way to enter in it is to get close to ξ_{N-2} one unit of time before. It is expected that the limit probability to have a cluster of size 2 is equal to 0 when λ_{N-2} is equal to 1, and so on.

Then we can state:

Theorem 3.12. *Let $\beta \in (1,2)$ and $\xi_1 \in (1-1/\beta, 1/\beta)$ such that the points $\xi_1, \ldots, \xi_{N-1}$ belong to $(1-1/\beta, 1/\beta)$ and the Lebesgue differentiation theorem holds for all such points. With the type of observables (depending on parameters $\lambda_1, \ldots, \lambda_{N-1}$) such that the balls $U_n^{(0)}(\xi_i)$ satisfy (3.7), the system T_β gives the multiplicity distributions π on $\{1,\ldots,N\}$ such that*

$$\pi_{\{\kappa,\ldots,N\}}(\kappa) = 1 - 1/\lambda_{N-\kappa}, \qquad \text{for every } \kappa \in \{1,\ldots,N-1\},$$

with $\lambda_1, \ldots, \lambda_{N-1} \geq 1$ and the EI is given by

$$\theta^{-1} = \sum_{\kappa=1}^{N} \prod_{i=N-\kappa+1}^{N-1} \frac{1}{\lambda_i}$$

Corollary 3.13. *Considering the type of observables such that the balls $U_n^{(0)}(\xi_i)$ satisfy (3.7), which depend on $N \in \mathbb{N}$ and the parameters $\lambda_1, \ldots, \lambda_{N-1}$, the family $(T_\beta)_{\beta\in(1,2)}$ generates any multiplicity distributions π of finite support in $\mathbb{N}$.*

4. Clustering patterns

In this section we consider the possible clustering patterns for the exceedances within each cluster. The multiplicity distribution is easy to determine for a certain pattern. With other patterns, it is not as easy and we will need to have a better understanding of the formulae giving the multiplicity distribution π. The idea is simply to consider the set that the orbit has to hit for giving rise to the cluster with the desired size.

We recall that the formulae for the limit cluster size distribution have been reinterpreted in a more probabilistic approach in [AFF20, Section 2.1]. Here, the analysis is more dynamically driven and the orbit structure of maps plays a prominent role.

Before we continue, we would like to stress the role played by the constants λ_i on the magnitude of the exceedances. Namely, observe that if λ_i is larger than 1 then the corresponding exceedance is larger than the previous one, while it it is less than 1, then the opposite occurs.

4.1. Examples of observables.

We start by considering the systems $f_{\mathbf{\Delta}}$ and T_β introduced above. Recall that the observable φ is defined by functions h_i for $i \in \{1,\ldots N\}$ having the same type of behaviour among the three types defined in Section 2.1. For each domain of attraction, we consider the following function h_i :

(I) $h_i^{(1)}(x) = -C\log\left(\dfrac{x}{B_i}\right)$ with $C>0$, $B_i>0$, then $h_i^{(1)}(0)=\infty$ and $\tau(y) = \mathrm{Exp}(-y)$;

(II) $h_i^{(2)}(x) = \left(\dfrac{B_i}{x}\right)^{1/\alpha}$ with $\alpha>0$, $B_i>0$, then $h_i^{(2)}(0)=\infty$ and $\tau(y)=1/y^\alpha$;

(III) $h_i^{(3)}(x) = D - \left(\dfrac{x}{B_i}\right)^{1/\alpha}$ with $D\in\mathbb{R}$, $\alpha>0$, $B_i>0$, then $h_i^{(3)}(0) = D < \infty$ and $\tau(y) = (-y)^\alpha$.

The constants B_i depend on the desired shape of the balls $U_n^{(0)}(\xi_i)$. Here we consider

$$B_i = \lambda_1\lambda_2\ldots\lambda_{i-1}\cdot|\mathrm{D}f(\xi_1)\mathrm{D}f(\xi_2)\ldots\mathrm{D}f(\xi_{i-1})|.$$

The parameters λ_i are larger or equal to 1. Moreover the sequences $(r_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$ depend on the observable :

$h_i^{(1)}$: $r_n = \mathrm{Exp}\Big(-\dfrac{u_n}{C}\Big)$ and $u_n = C\log\left(\dfrac{2n\sum_{i=1}^n \rho(\xi_i)B_i}{\tau}\right)$
then $u_n = y/a_n + b_n$ with $a_n = 1/C$ and $b_n = C\log\left(2n\sum_{i=1}^n\rho(\xi_i)B_i\right)$;

$h_i^{(2)}$: $r_n = \dfrac{1}{u_n^\alpha}$ and $u_n = \left(\dfrac{2n\sum_{i=1}^n \rho(\xi_i)B_i}{\tau}\right)^{1/\alpha}$
then $a_n = 1/(2n\sum_{i=1}^n \rho(\xi_i)B_i)^{1/\alpha}$ and $b_n = 0$;

$h_i^{(3)}$: $r_n = (D - u_n)^\alpha$ and $u_n = D - \left(\dfrac{\tau}{2n\sum_{i=1}^n \rho(\xi_i)B_i}\right)^{1/\alpha}$
then $a_n = (2n\sum_{i=1}^n \rho(\xi_i)B_i)^{1/\alpha}$ and $b_n = D$,

where ρ is the density of the **acip** for the system $f_{\boldsymbol{\Delta}}$ or the system T_β.

For f equal to $f_{\boldsymbol{\Delta}}$ or T_β and $x \in U_n^{(0)}(\xi_1)$, we can easily check that

$$|f^{i-1}(x) - \xi_i| = |f^{i-2}(x) - \xi_{i-1}| \cdot |\mathrm{D}f(\xi_{i-1})|.$$

This gives immediately the following estimates between exceedances within a cluster.

(1) $h_i^{(1)}$: Let x be a point in $U_n^{(0)}(\xi_j)$. If x is sufficiently close to ξ_j, it is sent in $U_n^{(0)}(\xi_i)$ via f^{i-j} (with $i > j$) and the comparison between both exceedances is given by :

$$\varphi(f^{i-j}(x)) - \varphi(x) = C\log(\lambda_j \dots \lambda_{i-1})$$

(2) $h_i^{(2)}$:

$$\frac{\varphi(f^{i-j}(x))}{\varphi(x)} = (\lambda_j \dots \lambda_{i-1})^{1/\alpha}$$

(3) $h_i^{(3)}$:

$$\frac{D - \varphi(f^{i-j}(x))}{D - \varphi(x)} = \frac{1}{(\lambda_j \dots \lambda_{i-1})^{1/\alpha}}$$

4.2. **Occurrence of clustering patterns.** In the examples studied in the previous section, the clusters of size κ have a simple pattern corresponding to an increasing sequence of exceedances, as $\lambda_1, ..., \lambda_{N-1} \geq 1$, and such a cluster appears when the orbit hits the last κ balls $U_n^{(0)}(\xi_{N-\kappa}), ..., U_n^{(0)}(\xi_{N-1})$.

When there exists an i for which $\lambda_i < 1$, the clustering pattern does not necessarily correspond to an increasing sequence of exceedances and the sets $U_n^{(\kappa)}(\xi_i)$ are not easy to compute. It follows that the multiplicity distribution for the limit is difficult to determine. That is why we need a deeper understanding of the meaning of the formulae giving the multiplicity.

4.2.1. *Observations regarding the example of T_β.* Here take $f = T_\beta$. In Figure 4.1 below, we represent the balls $U_n^{(0)}(\xi_i)$. The preimages $f^{-(i-1)}\left(U_n^{(0)}(\xi_i)\right)$ are sets around ξ_1, given by the dotted lines.

Figure/diagram1.jpg

FIGURE 4.1. $U_n^{(0)}(\xi_i)$ and the preimages $f^{-(i-1)}\left(U_n^{(0)}(\xi_i)\right)$.

Using the example given by Figure 4.1, on Figure 4.2 we represent the set $\mathcal{A}_{q,n}^{(0)}$ by encircling its connected components (two components in $U_n^{(0)}(\xi_2)$, two in $U_n^{(0)}(\xi_4)$, one in $U_n^{(0)}(\xi_5)$).

Figure/diagram10.jpg

FIGURE 4.2. The set $\mathcal{A}_{q,n}^{(0)}$

Figure/diagram11.jpg

FIGURE 4.3. The set $\mathcal{A}_{q,n}^{(1)}$

Applying f-invariance to the measures of $f^{-1}(\mathcal{A}_{q,n}^{(0)}(\xi_2))$, $f^{-3}(\mathcal{A}_{q,n}^{(0)}(\xi_4))$ and $f^{-4}(\mathcal{A}_{q,n}^{(0)}(\xi_5))$, we obtain that $\mu(\mathcal{A}_{q,n}^{(0)})$ is equal to the measure of $f^{-1}(\mathcal{A}_{q,n}^{(0)}(\xi_2)) \cup f^{-3}(\mathcal{A}_{q,n}^{(0)}(\xi_4)) \cup f^{-4}(\mathcal{A}_{q,n}^{(0)}(\xi_5))$ (represented by the brace).
Note that this set is a set of points (around ξ_1) whose orbit hits at least one ball among the balls $U_n^{(0)}(\xi_i)$ up to $N-1$ units of time. Moreover it is the largest set among the sets $f^{-(i-1)}\left(U_n^{(0)}(\xi_i)\right)$. Here it actually coincides with $f^{-1}\left(U_n^{(0)}(\xi_2)\right)$.

Using the example given by Figure 4.1, on Figure 4.3, we represent the set $\mathcal{A}_{q,n}^{(1)}$ by encircling its connected components (two components in $U_n^{(0)}(\xi_2)$, two in $U_n^{(0)}(\xi_3)$, one in $U_n^{(0)}(\xi_4)$).
Applying f-invariance to the measures of $f^{-1}(\mathcal{A}_{q,n}^{(1)}(\xi_2))$, $f^{-2}(\mathcal{A}_{q,n}^{(1)}(\xi_3))$ and $f^{-3}(\mathcal{A}_{q,n}^{(1)}(\xi_4))$, we obtain that $\mu(\mathcal{A}_{q,n}^{(1)})$ is equal to the measure of $f^{-1}(\mathcal{A}_{q,n}^{(1)}(\xi_2)) \cup f^{-2}(\mathcal{A}_{q,n}^{(1)}(\xi_3)) \cup f^{-3}(\mathcal{A}_{q,n}^{(1)}(\xi_4))$ (represented by the brace).
Note that this set is a set of points (around ξ_1) whose orbit hits at least two balls among the balls $U_n^{(0)}(\xi_i)$ up to $N-1$ units of time. Moreover it is the second largest set among the sets $f^{-(i-1)}\left(U_n^{(0)}(\xi_i)\right)$. Here it coincides with $f^{-3}\left(U_n^{(0)}(\xi_4)\right)$.

More generally, by using f-invariance of the measure μ, the quantity $\mu(\mathcal{A}_{q,n}^{(\kappa)})$ is equal to the measure of the entrance around ξ_1 (see the diagram on the left in Figure 4.4) corresponding to a cluster of size larger or equal to $\kappa+1$, i.e. a set of points (around ξ_1) whose orbit hits at least $\kappa+1$ balls among the balls $U_n^{(0)}(\xi_i)$. Then the quantity $\pi_n(\kappa) := \left(\mu(\mathcal{A}_{q,n}^{(\kappa-1)}) - \mu(\mathcal{A}_{q,n}^{(\kappa)})\right)/\mu(\mathcal{A}_{q,n}^{(0)})$ is exactly the probability that the orbit enters in the set around ξ_1 corresponding to a cluster of size κ conditionaly to the entrance of the orbit in the set around ξ_1 corresponding to the beginning of a cluster.

Figure/diagram1.jpg Figure/diagram2.jpg

FIGURE 4.4. Cluster size depending on the entrance around ξ_1.

Moreover $\mu(\mathcal{A}_{q,n}^{(\kappa)})$ is the measure of the κ-th largest set among the sets $f^{-(j-1)}(U_n^{(0)}(\xi_j))$ for $j \in \{1, \dots, N\}$. Then, considering

$$U_n^{(0)}(\xi_i) = \mathrm{B}\left(\xi_i, \lambda_1 \dots \lambda_{i-1} \cdot |\mathrm{D}f(\xi_1)\mathrm{D}f(\xi_2)\dots\mathrm{D}f(\xi_{i-1})| \cdot r_n\right)$$

and using the fact that f is piecewise linear, we have:

$$\mu(\mathcal{A}_{q,n}^{(\kappa)}) = \mu\left(\mathrm{B}(\xi_1, \Lambda_\kappa^{(1)} r_n)\right)$$

with $\{\Lambda_0^{(1)} \geq \Lambda_1^{(1)} \geq \dots \geq \Lambda_{N-1}^{(1)}\} = \{1,\ \lambda_1,\ \lambda_1\lambda_2,\ \lambda_1\lambda_2\lambda_3,\ \dots,\ \lambda_1\lambda_2\lambda_3\dots\lambda_{N-1}\}$ and $\Lambda_{N+j}^{(1)} = 0$ for every $j \geq 0$. If the Lebesgue differentiation theorem holds for ξ_1, then

$\mu(\mathcal{A}_{q,n}^{(\kappa)}) \sim 2\rho(\xi_1)\Lambda_\kappa^{(1)} r_n$ and

$$\pi(\kappa) = \frac{\Lambda_{\kappa-1}^{(1)} - \Lambda_\kappa^{(1)}}{\Lambda_0^{(1)}}.$$

Moreover the EI θ is given by $\theta^{-1} = \sum_{\kappa=1}^{N} \kappa\pi(\kappa) = \dfrac{\sum_{\kappa=0}^{N-1} \Lambda_\kappa^{(1)}}{\Lambda_0^{(1)}} = \dfrac{\sum_{\kappa=0}^{N-1} \lambda_1 \dots \lambda_\kappa}{\Lambda_0^{(1)}}$.

In the example of Figure 4.1, we have $N = 5$, $f^{-4}(U_n^{(0)}(\xi_5)) \subset f^{-2}(U_n^{(0)}(\xi_3)) \subset U_n^{(0)}(\xi_1) \subset f^{-3}(U_n^{(0)}(\xi_4)) \subset f^{-1}(U_n^{(0)}(\xi_2))$ and then $\Lambda_1^{(1)} = \lambda_1$, $\Lambda_2^{(1)} = \lambda_1\lambda_2\lambda_3$, $\Lambda_3^{(1)} = 1$, $\Lambda_4^{(1)} = \lambda_1\lambda_2$, $\Lambda_5^{(1)} = \lambda_1\lambda_2\lambda_3\lambda_4$.

Figure/diagram3.jpg

FIGURE 4.5. Diagram showing the different ways to enter in some balls $U_n^{(0)}(\xi_i)$ in this case.

4.2.2. *Observations regarding the example of $f_{\mathbf{\Delta}}$.* Now take $f = f_{\mathbf{\Delta}}$. Here the points ξ_i have two preimages. We will denote by ξ_i' the preimage of ξ_{i+1} other than ξ_i. In this case, hitting a set around ξ_1 (set that we will denote by $E_n^{(1)}$) is not the only way to start a cluster. A cluster can also start when the orbit visits a set around ξ_i' (set that we will denote by $E_n^{(i+1)}$).

As in the case of T_β, f-invariance implies that computing $\pi(\kappa)$ consists in measuring the subset of $\bigcup_{i=1}^{N} E_n^{(i)}$ which leads to clusters of size κ. We consider

$$U_n^{(0)}(\xi_i) = \mathrm{B}\left(\xi_i, \lambda_1 \dots \lambda_{i-1} \cdot |\mathrm{D}f(\xi_1)\mathrm{D}f(\xi_2)\dots\mathrm{D}f(\xi_{i-1})| \cdot r_n\right)$$

and $E_n^{(1)}$, $E_n^{(2)}$, …, $E_n^{(N)}$ the different entrances for the start of a cluster:

- $E_n^{(1)}$ is the connected component of ξ_1 in $\bigcup_{j=1}^{N} f^{-(j-1)}(U_n^{(0)}(\xi_j))$;
- for every $i \in \{2, \dots, N\}$, $E_n^{(i)}$ is the connected component of ξ_{i-1}' in $\bigcup_{j=i}^{N} f^{-(j-i+1)}(U_n^{(0)}(\xi_j))$.

The goal is to measure for each entrance the subset which leads to a cluster of size κ. We write $\pi(\kappa) = \sum_{i=1}^{N} \alpha_i \pi^{(i)}(\kappa)$ where the coefficient α_i is the limit proportion of $E_n^{(i)}$ among the entrances $E_n^{(1)}$, …, $E_n^{(N)}$, *i.e.*,

$$\alpha_i = \lim_{n\to\infty} \frac{\mu(E_n^{(i)})}{\mu(E_n^{(1)}) + \mu(E_n^{(2)}) + \dots + \mu(E_n^{(N)})},$$

and $\pi^{(i)}(\kappa)$ is the probability, conditionaly to an entrance in $E_n^{(i)}$, that the orbit hits the subset of $E_n^{(i)}$ leading to a cluster of size κ. Adapting the ideas of the previous example and using the facts that f is piecewise linear and μ is normalised Leb, we obtain :

$$E_n^{(1)} = \mathrm{B}\left(\xi_1, \Lambda_0^{(1)} r_n\right),$$

$$E_n^{(i)} = \mathrm{B}\left(\xi_i', (1 - r_i)\lambda_1 \dots \lambda_{i-1}\delta_1 \dots \delta_{i-1}\Lambda_0^{(i)} r_n\right),$$

$$\alpha_i = \frac{(1 - r_i)\lambda_1 \dots \lambda_{i-1}\delta_1 \dots \delta_{i-1}\Lambda_0^{(i)}}{\sum_{j=1}^{N} (1 - r_j)\lambda_1 \dots \lambda_{j-1}\delta_1 \dots \delta_{j-1}\Lambda_0^{(j)}},$$

$$\pi^{(i)}(\kappa) = \frac{\Lambda_{\kappa-1}^{(i)} - \Lambda_\kappa^{(i)}}{\Lambda_0^{(i)}},$$

with $\{\Lambda_0^{(i)} \geq \Lambda_1^{(i)} \geq \ldots \geq \Lambda_{N-i}^{(i)}\} = \{1,\ \lambda_i,\ \lambda_i\lambda_{i+1},\ \lambda_i\lambda_{i+1}\lambda_{i+2},\ \ldots,\ \lambda_i\lambda_{i+1}\lambda_{i+2}\ldots\lambda_{N-1}\}$, $\Lambda_{N-i+j}^{(i)} = 0$ for every $j \geq 1$, $r_1 = 0$, $r_j = \delta_{j-1}^{-1}$ for every $j \in \{2, \ldots, N\}$ and by convention $\delta_1 \ldots \delta_{i-1} = 1$ and $\lambda_1 \ldots \lambda_{i-1} = 1$ when $i = 1$.

Note that if $i \geq 2$, the radius of the ball $E_n^{(i)}$ is the one of $\bigcup_{j=i}^N f^{-(j-i)}(U_n^{(0)}(\xi_j))$ affected by a factor $1 - r_i$ which is the inverse of the derivative at ξ'_{i-1}. Moreover, $1 - r_i$ can also be considered as the proportion, in the preimage of a ball around ξ_i, of the connected component of ξ'_{i-1}. This is not the case for $i = 1$ (here $1 - r_1 = 1$) because $E_n^{(1)}$ is exactly the ball $\bigcup_{j=i}^N f^{-(j-i)}(U_n^{(0)}(\xi_j))$. In the next section, we will use these ideas which adapt well to systems which are not necessarily piecewise linear.

5. Periodic maximal orbits

Now, we consider the periodic case. Let $(\mathcal{X}, \mathcal{B}, f, \mu)$ be a dynamical system satisfying with decay of correlations against L^1 and let φ be an observable defined asabove. Given the correlated maxima $\xi_i = f^{m_i}(\xi_1)$, with a repelling p-periodic point ξ_1, and balls $U_n^{(0)}(\xi_i)$, it is easy to check that :

$$\mathcal{A}_{q,n}^{(\kappa)}(\xi_i) = U_n^{(\kappa)}(\xi_i) \setminus \left(\bigcup_{\ell=i+1}^{N+i} f^{-m_\ell}(U_n^{(\kappa)}(\xi_\ell)) \right) \tag{5.1}$$

where $m_{j+N} = m_j + p$ and $\xi_{j+N} = \xi_j$. We assume that $0 = m_1 < m_2 < \ldots < m_N < p$.

When N is equal to 1, the clusters consist of strictly decreasing bulk of exceedances and the multiplicity distribution π is always geometric. The fact that the cluster size is not necessarily equal to 1 is due to the periodicity: when the orbit hits $U_n^{(0)}$, it can return after p units of time. The fact that ξ_1 is repelling means that, once the orbit visits its vicinity, it is repelled farther and farther away from ξ_1 and ends up escaping the periodicity of the point, that is the reason why the exceedances among a cluster are decreasing and the cluster is of finite size. Here the goal is to see how these results are affected when N is larger than 1.

As we will see in the following example, the formula (5.1) does not simplify the computations and we will adapt the ideas of Section 4.2 to the periodic case, for a more direct approach.

5.1. **The doubling map case with period 2.** Here we consider the system given by $f(x) = 2x \bmod 1$ with a periodic point of period 2, ξ_1, and $\xi_2 := f(\xi_1)$.

Consider an observable such that the connected components of $U_n^{(0)}$ are of the shape :

$$U_n^{(0)}(\xi_1) = \mathrm{B}(\xi_1, r_n),\ U_n^{(0)}(\xi_2) = \mathrm{B}\left(\xi_2, 2\lambda\ r_n\right)$$

with $\lambda \geq 1$ and let $J \geq 0$ be the integer such that $\lambda/4^J \geq 1 > \lambda/4^{J+1}$. By computing the sets $U_n^{(\kappa)}(\xi_i)$, explicitely, the multiplicity distribution is given by :

$$\begin{cases} \kappa \in \{1, \ldots, J\} : & \pi(\kappa) = \dfrac{3}{4^\kappa} \\ \kappa = J + 1 : & \pi(J+1) = \dfrac{1}{4^J} - \dfrac{1 + \lambda/2^{2J+1}}{3\lambda} \\ \kappa \geq 1 : & \pi(J+1+\kappa) = \dfrac{1}{2^\kappa} \dfrac{1 + \lambda/2^{2J+1}}{3\lambda} \end{cases} \tag{5.2}$$

More generally, with the system given by $f(x) = qx \bmod 1$ and the orbit of a periodic point of period p, for integers q and p larger or equal to 2, this method becomes cumbersome, as

Figure/Graph6.png

FIGURE 5.1. Graph of $f(x) = 2x$ mod 1 with the orbit of a 2-periodic point.

the sets $U_n^{(\kappa)}(\xi_i)$ are more difficult to determine. On the other hand, the method developed in the next section will turn out to be useful to compute the multiplicity distribution for this type of system.

5.2. **A prototype example with simple combinatorics.** We consider a dynamical system f with decay of correlations against L^1 and admitting an **acip** μ with a density ρ (w.r.t. Leb). We also assume that there exists a repelling p-periodic point ξ_1 such that the Lebesgue differentiation theorem holds for the points $\xi_i = f^{i-1}(\xi_i)$ for every $i \in \{1, \dots, p\}$, ξ_i is the unique preimage of ξ_{i+1} for every $i \in \{1, \dots, p-1\}$ and ξ_1 has two preimages: ξ_p and another point ξ_p'.

Figure/diagram5.jpg

FIGURE 5.2. Diagram showing the unique way to enter in some balls $U_n^{(0)}(\xi_i)$ in this case.

Now consider an observable φ so that the corresponding extremal sets are such that:

$$\begin{cases} U_n^{(0)}(\xi_1) = \mathrm{B}(\xi_1, r_n) \\ U_n^{(0)}(\xi_2) = \mathrm{B}\left(\xi_2, \lambda_1 \cdot d_1 \cdot r_n\right) \\ U_n^{(0)}(\xi_3) = \mathrm{B}\left(\xi_3, \lambda_1\lambda_2 \cdot d_1 d_2 \cdot r_n\right) \\ \qquad\vdots \\ U_n^{(0)}(\xi_p) = \mathrm{B}\left(\xi_p, \lambda_1\lambda_2 \dots \lambda_{p-1} \cdot d_1 d_2 \dots d_{p-1} \cdot r_n\right) \end{cases} \tag{5.3}$$

with $d_i = |\mathrm{D}f(\xi_i)|$ and for some $\lambda_1, \dots, \lambda_{N-1} \in \mathbb{R}_0^+$.

To create a cluster of exceedances when the orbit is not in any of the extremal sets $U_n^{(0)}(\xi_i)$, the orbit has to enter in a set around ξ_p', which corresponds to a topological ball around ξ_p', which in turn will lead to the entrance in $U_n^{(0)}$ after less than p units of time, *i.e.*, a set of points close to ξ_p' whose orbit hits $U_n^{(0)}$ after less than p units of time. This set, denoted by $E_n^{(1)}$, is the connected component containing ξ_p' in $\bigcup_{i=1}^{p} f^{-i}(U_n^{(0)}(\xi_i))$.

Figure/diagram6.jpg

FIGURE 5.3. Cluster size depending on the entrance around ξ'_p.

To understand how the ideas of the non-periodic case can be adapted to the periodic case, we just have to act as though $f^p(\xi_i)$ was not ξ_i, for every $i \in \mathbb{N}$ (see Figure 5.3). Then the goal is to measure the subset in $E_n^{(1)}$ which leads to a cluster of size κ and we have to sort the sets $f^{-i}(U_n^{(0)}(\xi_i)) \cap E_n^{(1)}$ for $i \in \mathbb{N}$. Let r_1 be the limit proportion, in $f^{-1}(\mathrm{B}(\xi_1, x))$, of the connected component containing ξ_p (as x goes to 0). Observe that :

$$\mu\left(f^{-i}(U_n^{(0)}(\xi_i)) \cap E_n^{(1)}\right) \sim (1 - r_1) \cdot \mu\left(f^{-(i-1)}(U_n^{(0)}(\xi_i)) \cap f(E_n^{(1)})\right)$$
$$\sim 2 \cdot (1 - r_1) \cdot \rho(\xi_1) \frac{\lambda_1 \lambda_2 \dots \lambda_q}{(d_1 \dots d_p)^M} r_n$$

with $M \in \mathbb{N}_0$ and $q \in \{0, \dots, p-1\}$ such that $i - 1 = pM + q$. Then we just have to compute the quantities $\lambda_1 \lambda_2 \dots \lambda_q / D^M$ with $D := d_1 \dots d_p$ ($D > 1$ since ξ_1 is repelling) and for $M \in \mathbb{N}_0$, $q \in \{0, \dots, p-1\}$ (by convention, $\lambda_1 \dots \lambda_q = 1$ for $q = 0$).
In the example of Figure 5.3, we have

$$1 \geq \lambda_1 \lambda_2 \geq \frac{1}{D} \geq \frac{\lambda_1 \lambda_2}{D} \geq \frac{1}{D^2} \geq \frac{\lambda_1 \lambda_2}{D^2} \geq \dots \tag{5.4}$$

To include the quantities λ_1 / D^M (for $M \geq 0$) in (5.4), we have to know for which integer J does one of the following bounds holds: $\frac{1}{D^J} \geq \lambda_1 \geq \frac{\lambda_1 \lambda_2}{D^J}$ or $\frac{\lambda_1 \lambda_2}{D^J} \geq \lambda_1 \geq \frac{1}{D^{J+1}}$. To determine this, we have to draw more sets $U_n^{(0)}(\xi_i)$ in the diagram. For instance, if we have $\frac{1}{D^J} \geq \lambda_1 \geq \frac{\lambda_1 \lambda_2}{D^J}$, after sorting, we get:

$$1 \geq \lambda_1 \lambda_2 \geq \frac{1}{D} \geq \frac{\lambda_1 \lambda_2}{D} \geq \dots \geq \frac{\lambda_1 \lambda_2}{D^{J-1}} \geq \frac{1}{D^J} \geq \lambda_1 \geq \frac{\lambda_1 \lambda_2}{D^J} \geq \frac{1}{D^{J+1}} \geq \frac{\lambda_1}{D} \geq \frac{\lambda_1 \lambda_2}{D^{J+1}} \geq$$
$$\geq \frac{1}{D^{J+2}} \geq \frac{\lambda_1}{D^2} \geq \frac{\lambda_1 \lambda_2}{D^{J+2}} \geq \dots$$

Figure/diagram9.jpg

FIGURE 5.4. Multiplicity distribution in the case $N = 3$, $D = 1.1$ and λ_i such that $\lambda_1 = 1.8$, $\lambda_1 \lambda_2 = 1.3$.

It follows that the multiplicity distribution π is given by:

$$\pi(\kappa) = \frac{\Lambda_{\kappa-1}^{(1)} - \Lambda_\kappa^{(1)}}{\Lambda_0^{(1)}}.$$

with $\left\{\Lambda_0^{(1)} \geq \Lambda_1^{(1)} \geq \Lambda_2^{(1)} \geq \dots\right\} = \left\{\frac{\lambda_1 \lambda_2 \dots \lambda_q}{D^M} \;\middle|\; M \in \mathbb{N}_0,\ q \in \{0, \dots, p-1\}\right\}$. Moreover the EI θ is given by:

$$\theta^{-1} = \sum_{\kappa=1}^{\infty} \kappa \pi(\kappa) = \frac{\sum_{\kappa=0}^{\infty} \Lambda_\kappa^{(1)}}{\Lambda_0^{(1)}} = \frac{D}{D-1} \frac{\sum_{q=0}^{N-1} \lambda_1 \dots \lambda_q}{\Lambda_0^{(1)}}.$$

Note that $\pi(\kappa)$ might be zero for some κ. It occurs when $\{\lambda_1 \dots \lambda_{i-1}/D^M \mid M \in \mathbb{N}_0\} \cap \{\lambda_1 \dots \lambda_{j-1}/D^M \mid M \in \mathbb{N}_0\}$ is not empty for different integers i and j in $\{1, \dots, N\}$ and then $\pi = 0$ on an infinite subset of $\mathbb{N}$.

The multiplicity distributions that we get belong to a family of distributions on $\mathbb{N}$, denoted by Π, that we will call *eventually pseudo periodic distributions*:

$$\Pi = \{\text{distribution } \pi \text{ on } \mathbb{N} \mid \exists\, K, p \in \mathbb{N}, \exists\, D > 1, \forall\, \kappa \geq K, \ \pi(\kappa + p) = \pi(\kappa)/D\}.$$

We denote by $\Pi^K_{p,D}$ the set of distributions $\pi \in \Pi$ with rank K, pseudo-period p and attenuation coefficient D and we write $\Pi_{p,D} := \bigcup_{K\in\mathbb{N}} \Pi^K_{p,D}$.

Here the multiplicity distributions that we get are all in $\Pi_{p,D}$ with p the number of maximal points, $D = d_1 \dots d_p$. The rank K, depending on φ, is for instance the integer such that $\Lambda^{(1)}_K = \min\{1, \ \lambda_1, \ \dots, \ \lambda_1 \dots \lambda_{p-1}\}$.

Figure/diagram8.jpg

FIGURE 5.5. Diagram showing the different ways to enter in some balls $U^{(0)}_n(\xi_i)$ in this case.

5.3. **A more general scenario.** After the example of Section 5.2, a possible generalisation consists on allowing the points ξ_{i+1} to have preimages other than ξ_i. As in the non-periodic case, we consider the different entrances $E^{(i)}_n$. More precisely we consider for every $i \in \{1, \dots, p\}$ the factor r_i which is the limit proportion, in $f^{-1}(\mathrm{B}(\xi_i, x))$, of the connected component of ξ_{i-1} (as x tends to 0) with $\xi_0 := \xi_p$. Then $\mu(E^{(i)}_n) \sim 2 \cdot (1 - r_i) \cdot \rho(\xi_i)\mathrm{L}^{(i)}_0 r_n$ where the quantities $\mathrm{L}^{(i)}_\kappa$ play the same role as $\Lambda^{(i)}_\kappa$ and will be defined in (5.6), after we clarify the notations for the balls $U^{(0)}_n(\xi_i)$. Then the probability that the orbit enters in $E^{(i)}_n$ is equal to

$$\alpha_i := \frac{(1 - r_i)\rho(\xi_i)\mathrm{L}^{(i)}_0}{\sum_{j=1}^{N} (1 - r_j)\rho(\xi_j)\mathrm{L}^{(j)}_0},$$

while the probability to start a cluster of size κ conditionaly to an entrance in $E^{(i)}_n$ is equal to

$$\pi^{(i)}(\kappa) = \frac{\mathrm{L}^{(i)}_{\kappa-1} - \mathrm{L}^{(i)}_\kappa}{\mathrm{L}^{(i)}_0}.$$

In the previous cases, the notations were taylored to compare the preimages $f^{-(i-1)}(U^{(0)}_n(\xi_i))$ with $U^{(0)}_n(\xi_1)$. In this periodic case, we have to compare the preimages $f^{-(i-j)}(U^{(0)}_n(\xi_i))$ with $U^{(0)}_n(\xi_j)$ for every j. Then it is not useful to write

$$U^{(0)}_n(\xi_i) = \mathrm{B}\left(\xi_i, \lambda_1\lambda_2 \dots \lambda_{i-1} \cdot d_1 d_2 \dots d_{i-1} \cdot r_n\right).$$

Now, in order to simplify the notation we set

$$\ell_i := \lambda_1\lambda_2 \dots \lambda_{i-1} \cdot d_1 d_2 \dots d_{i-1} \quad \text{so that} \quad U^{(0)}_n(\xi_i) = \mathrm{B}(\xi_i, \ell_i r_n), \tag{5.5}$$

Then the quantities $\mathrm{L}^{(i)}_\kappa$ are defined by :

$$\{\mathrm{L}^{(i)}_0 \geq \mathrm{L}^{(i)}_1 \geq \mathrm{L}^{(i)}_2 \geq \dots\} = \left\{ \frac{\ell_{i+q}}{d_i \dots d_{i+q-1} D^M} \;\middle|\; M \in \mathbb{N}_0, \ q \in \{0, \dots, p-1\} \right\}, \tag{5.6}$$

where $D = d_1 \dots d_p > 1$, $\ell_{j+p} := \ell_j$, $d_{j+p} := d_j$ and by convention $d_i \dots d_{i+q-1} = 1$ when $q = 0$.

Finally the multiplicity distribution π and the EI θ are given by :

$$\pi(\kappa) = \sum_{i=1}^{p} \alpha_i \pi^{(i)}(\kappa) = \sum_{i=1}^{p} \frac{(1-r_i)\rho(\xi_i)\mathrm{L}_0^{(i)}}{\sum_{j=1}^{p}(1-r_j)\rho(\xi_j)\mathrm{L}_0^{(j)}} \frac{\mathrm{L}_{\kappa-1}^{(i)} - \mathrm{L}_{\kappa}^{(i)}}{\mathrm{L}_0^{(i)}}, \tag{5.7}$$

$$\theta^{-1} = \sum_{i=1}^{p} \alpha_i \theta_i^{-1} \quad \text{with } \theta_i^{-1} = \sum_{\kappa=1}^{\infty} \kappa \pi^{(i)}(\kappa) = \frac{\sum_{\kappa=0}^{\infty} \mathrm{L}_{\kappa}^{(i)}}{\mathrm{L}_0^{(i)}} = \frac{1}{\mathrm{L}_0^{(i)}} \frac{D}{D-1} \sum_{q=0}^{p-1} \frac{l_{i+q}}{d_i \dots d_{i+q-1}}.$$

Note that π is a convex combination of multiplicity distributions of $\Pi_{p,D}$ that we introduced in Section 5.2 and it follows that π is also in $\Pi_{p,D}$.

Consequently a simple idea for building examples and getting new multiplicity distributions would consist in considering multiplicity distributions of Section 5.2 and making a convex combination. However the constraint is that these distributions and the weights of the combination must be functions of the same parameters as we can see in the expression in (5.7). Nevertheless this idea will be applied in the proof of Proposition 5.2, below.

5.4. **Illustration and applications.** First we return to the example given in 5.1. More generally, we will determine the multiplicity distribution and the EI for $f(x) = qx \bmod 1$ with $q \geq 2$. Then, assuming the existence of some dynamical systems similar to $f_{\boldsymbol{\Delta}}$, we will classify the limiting cluster size distributions that we could get. Finally, we explain how the method can be redesigned when some points of the orbit are not global maxima and why these ideas can be applied in the non-periodic case.

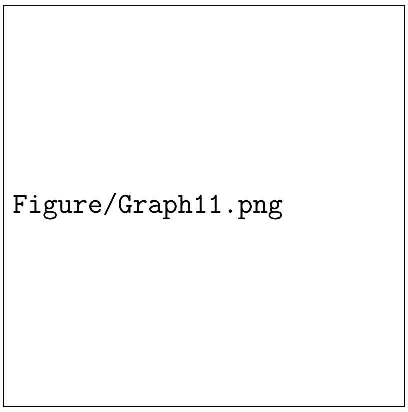

FIGURE 5.6. Graph of $f(x) = 5x \bmod 1$ with the orbit of a 4-periodic point.

We consider again the system given by $f(x) = qx \bmod 1$ with the p-periodic point ξ_1 and $\xi_i := f^{i-1}(\xi_1)$ (for example $\xi_1 := 1/(q^p - 1)$). In this case, $r_i = 1/q$. Considering the notation (5.5), i.e. $U_n^{(0)}(\xi_i) = \mathrm{B}(\xi_i, \ell_i r_n)$, we obtain :

$$\pi(\kappa) = \sum_{i=1}^{p} \frac{(1-1/q)\mathrm{L}_0^{(i)}}{\sum_{j=1}^{p}(1-1/q)\mathrm{L}_0^{(j)}} \frac{\mathrm{L}_{\kappa-1}^{(i)} - \mathrm{L}_{\kappa}^{(i)}}{\mathrm{L}_0^{(i)}} = \frac{\sum_{i=1}^{p} \mathrm{L}_{\kappa-1}^{(i)} - \sum_{i=1}^{p} \mathrm{L}_{\kappa}^{(i)}}{\sum_{j=1}^{p} \mathrm{L}_0^{(j)}}. \tag{5.8}$$

Concerning the example given in 5.1, we find the distribution (5.2) for the limiting cluster size distribution by considering $q = p = 2$, $l_1 = 1$ and $l_2 = 2\lambda$ (see 7.1 in the appendix for the details).

As in Section 3.2.2, we want to find a family of dynamical systems admitting the desired properties, in the periodic case. Now that we have introduced the notion of entrances for computing the multiplicity distribution, we can interpret the family $(f_{\mathbf{\Delta}})_{\mathbf{\Delta}\in(1,\infty)^{N-1}}$ as a family containing the relevant parameters r_i. In Definition (3.9), the second point and the third point imply that for every $(\rho_2,\dots,\rho_N)\in(0,1)^N$, there exists $\mathbf{\Delta}\in(1,\infty)^{N-1}$ such that for $f_{\mathbf{\Delta}}$ the limit proportion r_i is equal to ρ_i ($i\in\{2,\dots,N\}$). Then $r_i=\delta_{i+1}^{-1}$ and we set $r_1=0$. We want the same properties in the periodic case, with a family indexed by the p-tuple of limit proportions r_i.

Definition 5.1. Given $\mathbf{R}=(r_1,\dots,r_p)$ with $r_i\in(0,1)$, $g_{\mathbf{R}}\colon I\longrightarrow I$ denotes an interval map satisfying these three properties :

- it is equipped with an **acip** μ of density ρ;
- it has summable decay of correlations against $L^1(\mu)$;
- there exists a repelling p-periodic point ξ_1 in I such that the limit proportion, in $g_{\mathbf{R}}^{-1}(\mathrm{B}(\xi_i,x))$, of the connected component of ξ_{i-1} is equal to r_i (as x tends to 0) for every $i\in\{1,\dots,p\}$, with $\xi_i:=g_{\mathbf{R}}^{i-1}(\xi_1)$ for every $i\in\{1,\dots,p\}$ and $\xi_0:=\xi_p$. Moreover $g_{\mathbf{R}}$ is differentiable at $\xi_1,\ \dots,\ \xi_p$.

Using the notation introduced in (5.5)) i.e. $U_n^{(0)}(\xi_i)=\mathrm{B}(\xi_i,\ell_i r_n)$, we obtain :

$$\pi(\kappa)=\sum_{i=1}^{p}\frac{(1-r_i)\mathrm{L}_0^{(i)}\rho(\xi_i)}{\sum_{j=1}^{p}(1-r_j)\mathrm{L}_0^{(j)}\rho(\xi_j)}\ \underbrace{\frac{\mathrm{L}_{\kappa-1}^{(i)}-\mathrm{L}_{\kappa}^{(i)}}{\mathrm{L}_0^{(i)}}}_{\pi^{(i)}(\kappa)}. \tag{5.9}$$

Note that this is a convex combination of distributions $\pi^{(i)}$ with non-zero weights which can be adjusted by the parameters $r_i\in(0,1)$. This is what we use in the following statement which illustrates the kind of multiplicity distribution that we could get for the limit of the REPP in the periodic case.

Proposition 5.2. *Assume that such a family $(g_{\mathbf{R}})_{\mathbf{R}\in(0,1)^p}$ exists. Given $p\in\mathbb{N}$ and $a_1,\dots,a_p\in\mathbb{R}_0^+$ such that $a_1+\ldots+a_p<1$, there exist a dynamical system and an observable such that the REPP converges in distribution to a compound Poisson process of multiplicity distribution π satisfying :*

- $\forall\,\kappa\in\{1,\dots,p\},\ \pi(\kappa)=a_\kappa;$
- $\pi\in\Pi_{p,D}^0$ *with* $D=1-a_1-a2-\ldots-a_p$.

For the proof (see 7.2), we use the family $(g_{\mathbf{R}})_{\mathbf{R}\in(0,1)^p}$ with parameters ℓ_i such that for every $j\in\{1,\dots,p\}$, $\pi^{(j)}$ is in $\Pi_{p,D}^0$ and its support is $p\mathbb{N}-j+1=\{pk-j+1\mid k\in\mathbb{N}\}$. Then the difficulty is just to show that the parameters r_i can be chosen to adjust the quantities $\pi^{(j)}(p-j+1)$ and the weights of the convex combination.

Using the notation (5.5, we can also address the case of an observable achieving a maximum value at only N points of the orbit with $N<p$. Namely, consider an observable admitting N global maxima $\xi_1,\ \dots,\ \xi_N$ defined by $\xi_i:=f^{m_i}(\xi_1)$ for every $i\in\{1,\dots,N\}$ ($0=m_1<m_2<\ldots<m_N<p$) with ξ_1 a repelling p-periodic point. We set for every $j\in\{1,\dots,N\}$,

$$U_n^{(0)}(f^{j-1}(\xi_1))=\mathrm{B}\left(f^{j-1}(\xi_1),\ell_j r_n\right)$$

with ℓ_j equal to 0 for $j\notin\{m_1+1,\dots,m_N+1\}$. Then the connected components of $U_n^{(0)}$ are the balls $U_n^{(0)}(\xi_i)$ for $i\in\{1,\dots,N\}$. We need to adjust the definition of the quantities $\mathrm{L}_\kappa^{(i)}$. Namely, for every $i\in\{1,\dots,p\}$,

$$\{\mathrm{L}_0^{(i)}\geq\mathrm{L}_1^{(i)}\geq\mathrm{L}_2^{(i)}\geq\ldots\}=\left\{\frac{\ell_{i+q}}{d_i\ldots d_{i+q-1}D^M}\ \middle|\ M\in\mathbb{N}_0,\ q\in\{0,\dots,p-1\}\text{ s.t. }\ell_{i+q}\neq0\right\}.$$

Note that we consider $\mathrm{L}_\kappa^{(i)}$ for $i \in \{1,\dots,p\}$ and not "$i \in \{1,\dots,N\}$" because there can still be p entrances $E_n^{(1)}$, $\dots$, $E_n^{(p)}$ which are sets around preimages of $f^i(\xi_1)$ other than points of the orbit. Then using the same formula, we obtain multiplicity distributions in $\Pi_{N,D}$ with $D = d_1 \dots d_p > 1$.

We apply this in the appendix 7.3 with an example given in [AFFR16, Section 5.2.1]. It is the good example for explaining how to proceed in some cases where the radii of $U_n^{(0)}(\xi_i)$ are not all proportional to r_n.

Remark 5.3. We remark that the notation introduced in (5.5) is also useful in the non-periodic case, when the global maxima are not defined by $m_{i-1} = i-1$ for all $i \in \{1,\dots,N\}$. Namely, we consider an observable admitting N global maxima ξ_1, $\dots$, ξ_N defined by $\xi_i := f^{m_i}(\xi_1)$ for every $i \in \{1,\dots,N\}$ $(0 = m_1 < m_2 < \dots < m_N)$ with ξ_1 a non-periodic point. We set for every $j \in \{1,\dots,m_N+1\}$,

$$U_n^{(0)}(f^{j-1}(\xi_1)) = \mathrm{B}\left(f^{j-1}(\xi_1), \ell_j r_n\right)$$

with ℓ_j equal to 0 for $j \notin \{m_1+1,\dots,m_N+1\}$ and the quantities $\mathrm{L}_\kappa^{(i)}$ are given by

$$\{\mathrm{L}_0^{(i)} \geq \mathrm{L}_1^{(i)} \geq \mathrm{L}_2^{(i)} \geq \dots\} = \left\{ \frac{\ell_{i+q}}{d_i \dots d_{i+q-1}} \;\middle|\; q \in \{0,\dots,N'-i\} \text{ such that } l_{i+q} \neq 0 \right\}$$

(assuming that $\mathrm{L}_\kappa^{(i)} = 0$ for κ larger or equal to the cardinality of the right hand side) and

$$\pi(\kappa) = \sum_{i=1}^{m_N+1} \frac{(1-\delta_{i-1})\mathrm{L}_0^{(i)}}{\sum_{j=1}^{p}(1-\delta_{j-1})\mathrm{L}_0^{(j)}} \, \frac{\mathrm{L}_{\kappa-1}^{(i)} - \mathrm{L}_\kappa^{(i)}}{\mathrm{L}_0^{(i)}}.$$

We apply this idea in the appendix 7.4 with an example given in [AFFR16, Section 4.3].

6. Multiplicity distributions in the non-periodic case – proofs of the main results and numerical simulations

6.1. **Proofs of Propositions 3.1, 3.3, 3.6 and Theorem 3.12.**

***Proof of Proposition* 3.1**. $\mu(U_n^{(0)}) = 2(1+q\lambda)r_n$.

If $\lambda \geq 1$, then

$$\begin{aligned}\pi(1) &= \lim_{n\to+\infty} \frac{\mu(U_n^{(0)}) - 2\mu(U_n^{(0)}(\xi_1) \cap f^{-1}(U_n^{(0)}(\xi_2)))}{\mu(U_n^{(0)}) - \mu(U_n^{(0)}(\xi_1) \cap f^{-1}(U_n^{(0)}(\xi_2)))} \\ &= \lim_{n\to+\infty} \frac{\mu(U_n^{(0)}) - 2\mu(\mathrm{B}(\xi_1,r_n))}{\mu(U_n^{(0)}) - \mu(\mathrm{B}(\xi_1,r_n))} = \frac{q\lambda-1}{q\lambda} = 1 - \frac{1}{q\lambda}\end{aligned}$$

where $\lambda \mapsto 1 - 1/(q\lambda)$ is a bijection from $[1,+\infty)$ to $[1-1/q,1)$.

If $\lambda \leq 1$, then

$$\pi(1) = \lim_{n\to+\infty} \frac{\mu(U_n^{(0)}) - 2\mu(\mathrm{B}(\xi_1,\lambda r_n))}{\mu(U_n^{(0)}) - \mu(\mathrm{B}(\xi_1,\lambda r_n))} = \frac{1+q\lambda-2\lambda}{1+q\lambda-\lambda} = 1 - \frac{\lambda}{1+(q-1)\lambda}$$

where $\lambda \mapsto 1 - \lambda/(1+(q-1)\lambda)$ is a bijection from $(0,1]$ to $[1-1/q,1)$. □

***Proof of Proposition* 3.3**. First we cannot get $\pi(2) > 1/2$ by symmetry, as explained before Proposition 3.3. Secondly we want to get the multiplicity distributions on $\{1,2\}$ such that $\pi(2) \leq 1/2$.

Note that $\mu(U_n^{(0)}(\xi_1)) = \mu(\mathrm{B}(\xi_1, r_n)) \sim 2\rho(\xi_1)r_n$ and $\mu(U_n^{(0)}(\xi_2)) = \mu(\mathrm{B}(\xi_1, 4|\xi_1|\lambda r_n)) \sim 2\cdot 4\lambda\rho(\xi_2)|\xi_1|r_n$ then $\mu(U_n^{(0)}) \sim 2(\rho(\xi_1)+4\lambda\rho(\xi_2)|\xi_1|)r_n$. Without loss of generality, assume that $\xi_1 \in (0,1)$.

$$U_n^{(0)}(\xi_1) \cap f^{-1}(U_n^{(0)}(\xi_2)) = \left(\max\left\{\xi_1 - r_n, -\sqrt{\xi_1^2 + 2\lambda|\xi_1|r_n}\right\}, \min\left\{\xi_1 + r_n, -\sqrt{\xi_1^2 - 2\lambda|\xi_1|r_n}\right\}\right)$$

On the one hand,

$$\xi_1 + r_n \geq -\sqrt{\xi_1^2 + 2\lambda|\xi_1|r_n} \iff 2\lambda|\xi_1|r_n \leq \underbrace{2|\xi_1|r_n - r_n^2}_{\sim\, 2|\xi_1|r_n}.$$

and

$$\xi_1 - r_n \leq -\sqrt{\xi_1^2 - 2\lambda|\xi_1|r_n} \iff 2\lambda|\xi_1|r_n \leq \underbrace{2|\xi_1|r_n + r_n^2}_{\sim\, 2|\xi_1|r_n}.$$

Note that $\frac{\rho(\xi_1)}{\rho(\xi_2)} = 2|\xi_1|$.

If $\lambda > 1$, then $\mu(U_n^{(0)}(\xi_1) \cap f^{-1}(U_n^{(0)}(\xi_2))) \sim \mu(\mathrm{B}(\xi_1, r_n)) \sim 2\rho(\xi_1)r_n$ and

$$\pi(1) = \frac{4\lambda\rho(\xi_2)|\xi_1| - \rho(\xi_1)}{4\lambda\rho(\xi_2)|\xi_1|} = 1 - \frac{1}{2\lambda}$$

where $\lambda \mapsto 1 - 1/(2\lambda)$ is a bijection from $(1, +\infty)$ to $(1/2, 1)$.

If $\lambda < 1$, then $\mu(U_n^{(0)}(\xi_1) \cap f^{-1}(U_n^{(0)}(\xi_2))) \sim \mu\left(\left(-\sqrt{\xi_1^2 + 2\lambda|\xi_1|r_n}, -\sqrt{\xi_1^2 - 2\lambda|\xi_1|r_n}\right)\right) \sim 2\lambda\rho(\xi_1)r_n$ and

$$\pi(1) = \frac{\rho(\xi_1) + 4\lambda\rho(\xi_2)|\xi_1| - 2\lambda\rho(\xi_1)}{\rho(\xi_1) + 4\lambda\rho(\xi_2)|\xi_1| - \lambda\rho(\xi_1)} = 1 - \frac{\lambda}{1+\lambda}$$

where $\lambda \mapsto 1 - \lambda/(1+\lambda)$ is a bijection from $(0,1)$ to $(1/2, 1)$.

If $\lambda = 1$, then

$$\mu(U_n^{(0)}(\xi_1) \cap f^{-1}(U_n^{(0)}(\xi_2))) = \mu\left(\left(-\sqrt{\xi_1^2 + 2\lambda|\xi_1|r_n}, \xi_1 + r_n\right)\right) \sim (\lambda+1)\rho(\xi_1)r_n = 2\rho(\xi_1)r_n$$

and $\pi(1) = \frac{1}{2}$. □

***Proof of Proposition* 3.6**. In the last proofs, we only had to set a parameter λ larger or equal to 1 (or less or equal to 1). Here, we consider parameters $\lambda_1, \ldots, \lambda_{N-1}$ larger or equal to 1 for this proof. This makes the computations easier because we can check by induction that for all $\kappa \in \mathbb{N}_0$, for all $i \in \{1, \ldots, N\}$,

$$U_n^{(\kappa)}(\xi_i) = \begin{cases} U_n^{(0)}(\xi_i) & \text{if } \kappa \leq N - i \\ \emptyset & \text{if } \kappa > N - i \end{cases}$$

Then $\mathcal{A}_{q,n}^{(\kappa)}$ is equal to $U_n^{(0)}(\xi_{N-\kappa})$ if $\kappa \leq N-1$ and is equal to $\emptyset$ otherwise. The Rare Event Point Process converges to a compound Poisson process with intensity θ and multiplicity distribution π given by :

$$\begin{aligned}
\pi(1) &= 1 - \frac{1}{\lambda_{N-1}\delta} \\
\pi(\kappa) &= \prod_{i=N-\kappa+1}^{N-1} \frac{1}{\lambda_i \delta} - \prod_{i=N-\kappa}^{N-1} \frac{1}{\lambda_i \delta} \text{ for all } \kappa \in \{1, \dots, N-1\} \\
\pi(N) &= \prod_{i=1}^{N-1} \frac{1}{\lambda_i \delta} \\
\pi(\kappa) &= 0 \text{ for all } \kappa \geq N+1 \\
\theta^{-1} &= \sum_{\kappa=1}^{N} \prod_{i=N-\kappa+1}^{N-1} \frac{1}{\lambda_i \delta}
\end{aligned}$$

The distribution π satisfies $\pi_{\{\kappa,\dots,N\}}(\kappa) = 1 - \delta^{-1}/\lambda_{N-\kappa}$ for every $\kappa \in \{1, \dots, N-1\}$ and $\lambda \mapsto 1 - \delta^{-1}/\lambda$ is a bijection from $[1, +\infty)$ to $[1 - \delta^{-1}, 1)$.

Note that the parameter λ_{N-1} defines $\pi(1)$, and if $\pi(1)$ is defined, the parameter λ_{N-2} defines $\pi(2)$ and so on. If all the parameters λ_i are given, then $\pi(1), \dots, \pi(N-1)$ are defined and we get $\pi(N) = 1 - \pi(1) - \dots - \pi(N-1)$. □

***Proof of Theorem* 3.12**. First note that :

$$U_n^{(\kappa)}(\xi_i) = \begin{cases} U_n^{(0)}(\xi_i) & \text{if } \kappa \leq N-i \\ \emptyset & \text{if } \kappa > N-i \end{cases}$$

Then $\mathcal{A}_{q,n}^{(\kappa)}$ is equal to $U_n^{(0)}(\xi_{N-\kappa})$ if $\kappa \leq N-1$, $\emptyset$ otherwise. By linearity of T_β on $(0, 1 - 1/\beta)$, uniqueness of the preimage and T_β-invariance of μ, we have $\mu(U_n^{(0)}(\xi_{i+1})) = \lambda_1 \dots \lambda_i \mu(U_n^{(0)}(\xi_1))$. Then Rare Event Point Process converges to a compound Poisson process with intensity θ and multiplicity distribution π given by :

$$\begin{aligned}
\pi(1) &= 1 - \frac{1}{\lambda_{N-1}} \\
\pi(\kappa) &= \prod_{i=N-\kappa+1}^{N-1} \frac{1}{\lambda_i} - \prod_{i=N-\kappa}^{N-1} \frac{1}{\lambda_i} \text{ for all } \kappa \in \{1, \dots, N-1\} \\
\pi(N) &= \prod_{i=1}^{N-1} \frac{1}{\lambda_i} > 0 \\
\pi(\kappa) &= 0 \text{ for all } \kappa \geq N+1 \\
\theta^{-1} &= \sum_{\kappa=1}^{N} \prod_{i=N-\kappa+1}^{N-1} \frac{1}{\lambda_i}
\end{aligned}$$

The distribution π satisfies $\pi_{\{\kappa,\dots,N\}}(\kappa) = 1 - 1/\lambda_{N-\kappa}$ for every $\kappa \in \{1, \dots, N-1\}$ and $\lambda \mapsto 1 - 1/\lambda$ is a bijection from $[1, +\infty)$ to $[0, 1)$.

Note that the parameter λ_{N-1} defines $\pi(1)$, and if $\pi(1)$ is defined, the parameter λ_{N-2} defines $\pi(2)$ and so on. If all the parameters λ_i are given, then $\pi(1), \dots, \pi(N-1)$ are defined and we get $\pi(N) = 1 - \pi(1) - \dots - \pi(N-1)$. □

6.2. **Numerical simulation results.** Consider the observable φ defined on a neighbourhood of every ξ_i by $\varphi(x) = \frac{1}{|x-\xi_i|} \prod_{j=1}^{i-1} (\lambda_j \delta)$, i.e. $h_i(x) = \frac{1}{x} \prod_{j=1}^{i-1} (\lambda_j \delta)$, and the sequence of levels given by $u_n = \frac{2n}{\tau} \sum_{i=1}^{N} \prod_{j=1}^{i-1} (\lambda_j \delta)$ so that $n\mu\left(X_0 > u_n\right) \underset{n\to\infty}{\longrightarrow} \tau$. We define $v_n = 1/\mu\left(X_0 > u_n\right)$. Note that $r_n = 1/u_n$.

We estimate by empirical averages the EI θ and the quantities $\pi(\kappa)$ for $\kappa \in \{1, \dots, N\}$. For every test i, we compute the time T_i before the first exceedance and we write $\theta_i^{-1} = T_i\tau/n \simeq T_i/v_n$. Then we count the exceedances among the first cluster (the cluster ends when an exceedance is not followed by another during a time period of length N) and we denote the number of exceedances by K_i. Then we obtain estimators $\tilde{\theta}^{-1} = \frac{1}{L}\sum_{i=1}^{L} \theta_i^{-1}$ of θ^{-1} and $\tilde{p}(\kappa) = \frac{1}{L}\sum_{i=1}^{L} K_i \mathbb{1}_{K_i=\kappa}$ of $\pi(\kappa)$ where L is the number of runs.

6.2.1. *Uniform multiplicity distribution.* Take $\tau = 1$, $N = 10$, $n = 1000$, $L = 1000$ and parameters λ_i which lead to a multiplicity distribution which is uniform. There we give the Q-Q plot between the exponential distribution of mean θ^{-1} and the distribution of the $\tilde{\theta_i}^{-1}$, and the histogram of the cluster sizes.

Figure/qqplotunif.png

FIGURE 6.1. Q-Q plot between the exponential distribution of mean θ^{-1} and the distribution of the $\tilde{\theta}_i^{-1}$.

Figure/Histunif.png

FIGURE 6.2. Histogram of the cluster sizes.

6.2.2. *Binomial multiplicity distribution.* Same thing but with parameters which lead to a multiplicity distribution on $\{1, \ldots, 10\}$ which is the distribution of a random variable X such that the law of $X-1$ is the binomial distribution with paramaters $(9, 0.7)$.

Figure/qqplotbin.png

FIGURE 6.3. Q-Q plot between the exponential distribution of mean θ^{-1} and the distribution of the $\tilde{\theta}_i^{-1}$.

Figure/Histbin.png

FIGURE 6.4. Histogram of the cluster sizes.

7. Multiplicity distributions in the periodic case – proofs and case studies

7.1. **Example given in 5.1 using the approach introduced in Section 5.3.** Consider $q = p = 2$, $l_1 = 1$ and $l_2 = 2\lambda$. Using (5.8), the multiplicity distribution is defined by :

$$\pi(\kappa) = \frac{\sum_{i=1}^{2} \mathrm{L}_{\kappa-1}^{(i)} - \sum_{i=1}^{2} \mathrm{L}_{\kappa}^{(i)}}{\sum_{j=1}^{2} \mathrm{L}_0^{(j)}}$$

with $\left\{\mathrm{L}_0^{(1)} \geq \mathrm{L}_1^{(1)} \geq \ldots\right\} = \left\{1/4^M, \lambda/4^M \;\middle|\; M \in \mathbb{N}_0\right\}$ and

$$\left\{\mathrm{L}_0^{(2)} \geq \mathrm{L}_1^{(2)} \geq \ldots\right\} = \left\{1/(2\times 4^M), \lambda/(2\times 4^{M-1}) \;\middle|\; M \in \mathbb{N}_0\right\}.$$

Let $J \geq 0$ be an integer such that $\lambda/4^J \geq 1 > \lambda/4^{J+1}$.

$$\begin{aligned}
\kappa\in\{0,\dots,J\}:\ & \mathrm{L}^{(1)}_{\kappa}=\lambda/4^{\kappa};\\
\kappa\geq 1:\ & \mathrm{L}^{(1)}_{J+2\kappa}=\lambda/4^{J+\kappa};\\
\kappa\geq 0:\ & \mathrm{L}^{(1)}_{J+2\kappa+1}=1/4^{\kappa};\\
\kappa\in\{0,\dots,J+1\}:\ & \mathrm{L}^{(2)}_{\kappa}=\lambda/(2\times 4^{\kappa-1});\\
\kappa\geq 1:\ & \mathrm{L}^{(2)}_{(J+1)+2\kappa}=\lambda/(2\times 4^{J+\kappa});\\
\kappa\geq 0:\ & \mathrm{L}^{(2)}_{(J+1)+2\kappa+1}=1/(2\times 4^{\kappa}),
\end{aligned}$$

then

$$\begin{aligned}
\kappa\in\{0,\dots,J\}:\ & \mathrm{L}^{(1)}_{\kappa}+\mathrm{L}^{(2)}_{\kappa}=\frac{3\lambda}{4^{\kappa}}\\
\kappa\geq 0:\ & \mathrm{L}^{(1)}_{J+2\kappa+1}+\mathrm{L}^{(2)}_{J+2\kappa+1}=\frac{1}{4^{\kappa}}\left(1+\frac{\lambda}{2\times 4^{J}}\right)\\
\kappa\geq 1:\ & \mathrm{L}^{(1)}_{J+2\kappa}+\mathrm{L}^{(2)}_{J+2\kappa}=\frac{\lambda}{4^{J+\kappa}}+\frac{1}{2\times 4^{\kappa-1}}.
\end{aligned}$$

Thus the multiplicity distribution is given by :

$$\begin{aligned}
\kappa\in\{1,\dots,J\}:\quad & \pi(\kappa) &=& \frac{3\lambda/4^{\kappa-1}-3\lambda/4^{\kappa}}{3\lambda}\\
& &=& \frac{3}{4^{\kappa}},\\
\kappa=J+1:\quad & \pi(J+1) &=& \frac{3\lambda/4^{J}-\left(1+\frac{\lambda}{2\times 4^{J}}\right)}{3\lambda}\\
& &=& \frac{1}{4^{J}}-\frac{1+\lambda/2^{2J+1}}{3\lambda},\\
\kappa\geq 0:\quad & \pi((J+1)+2\kappa+1) &=& \frac{\frac{1}{4^{\kappa}}\left(1+\frac{\lambda}{2\times 4^{J}}\right)-\left(\frac{\lambda}{4^{J+\kappa+1}}+\frac{1}{2\times 4^{\kappa}}\right)}{3\lambda}\\
& &=& \frac{1}{2^{2\kappa+1}}\frac{1+\lambda/2^{2J+1}}{3\lambda},\\
\kappa\geq 1:\quad & \pi((J+1)+2\kappa) &=& \frac{\left(\frac{\lambda}{4^{J+\kappa}}+\frac{1}{2\times 4^{\kappa-1}}\right)-\frac{1}{4^{\kappa}}\left(1+\frac{\lambda}{2\times 4^{J}}\right)}{3\lambda}\\
& &=& \frac{1}{2^{2\kappa}}\frac{1+\lambda/2^{2J+1}}{3\lambda},
\end{aligned}$$

we find the same distribution as (5.2).

7.2. **Possible cluster size distributions in the periodic case.**

***Proof of Proposition* 5.2**. Consider the dynamical system given by $g_{\mathbf{R}}$ with $\mathbf{R}$ to be chosen later, and an observable φ such that $U^{(0)}_n(\xi_i)=\mathrm{B}\left(\xi_i,\ell_i r_n\right)$. We want an entrance of the orbit in $E^{(i)}_n$ to be followed by a cluster of size at least $p-i+1$. Then we set $\ell_i=(d_1\dots d_{i-1})$. Then the quantities $\mathrm{L}^{(i)}_{\kappa}$ are defined by :

$$\forall\,\kappa\in\{0,\dots,p-i\},\ \mathrm{L}^{(i)}_{\kappa}=1,$$

$$\forall\,\kappa\in\{1,\dots,p\},\ \forall\,j\in\mathbb{N}_0,\ \mathrm{L}^{(p-i+k+jp)}_{\kappa}=1/D^{j-1},$$

with $D=|\mathrm{D}g_{\mathbf{R}}(\xi_1)\dots\mathrm{D}g_{\mathbf{R}}(\xi_p)|$, and $\pi^{(i)}$ is defined on the support $p\mathbb{N}-i+1$ by :

$$\forall\,j\in\mathbb{N},\ \pi^{(i)}(jp-i+1)=\frac{1}{D^{j-1}}\left(1-\frac{1}{D}\right).$$

Since $\mathrm{L}_0^{(i)} = 1$ for every $i \in \{1, \ldots, p\}$, the multiplicity distribution is defined by :

$$\forall\, \kappa \in \mathbb{N}_0,\ \pi(\kappa) = \sum_{i=1}^{p} \frac{(1 - r_i)\rho(\xi_i)}{\sum_{j=1}^{p} (1 - r_j)\rho(\xi_j)}\, \pi^{(i)}(\kappa).$$

For every $\kappa \in \{1, \ldots, p\}$,

$$\pi(\kappa) = \frac{(1 - r_{p-\kappa+1})\rho(\xi_{p-\kappa+1})}{\sum_{j=1}^{p} (1 - r_j)\rho(\xi_j)}\, \pi^{(p-\kappa+1)}(\kappa) = \frac{(1 - r_{p-\kappa+1})\rho(\xi_{p-\kappa+1})}{\sum_{j=1}^{p} (1 - r_j)\rho(\xi_j)} \left(1 - \frac{1}{D}\right)$$

since $\pi^{(i)}(\kappa) \neq 0 \iff i = p - \kappa + 1$. We want to express D as a function of the parameters r_i. By $g_{\mathbf{R}}$-invariance, the measure of $g_{\mathbf{R}}^{-1}(\mathrm{B}(\xi_i, x))$ is equivalent to $\rho(\xi_i)x$, as x tends to zero, and its connected component containing ξ_{i-1} is of measure equivalent to $\rho(\xi_{i-1})\frac{x}{|\mathrm{D}g_{\mathbf{R}}(\xi_{i-1})|}$. Then $r_i = \frac{\rho(\xi_{i-1})}{\rho(\xi_i)|\mathrm{D}g_{\mathbf{R}}(\xi_{i-1})|}$ and $\frac{1}{D} = r_1 \ldots r_p$.
Now let $a_1, \ \ldots, \ a_p$ be positive real numbers such that $a_1 + \ldots + a_p < 1$ and set $\eta_\kappa := (1 - r_{p-\kappa+1})\rho(\xi_{p-\kappa+1})$. First we want $\pi(\kappa) = a_\kappa$ for every $\kappa \in \{1, \ldots, p\}$, i.e. :

$$\forall\, \kappa \in \{1, \ldots, p\},\ \frac{\eta_\kappa}{\sum_{j=1}^{p} \eta_j} \left(1 - \prod_{j=1}^{p} \left(1 - \frac{\eta_j}{\rho(\xi_{p-j+1})}\right)\right) = a_\kappa.$$

If such real numbers η_κ exist, then $1 - \prod_{j=1}^{p} (1 - \eta_j/\rho(\xi_{p-j+1})) = a_1 + \ldots + a_p$ since $\sum_{\kappa=1}^{p} \frac{\eta_\kappa}{\sum_{j=1}^{p} x_j} = 1$. This implies

$$\forall\, \kappa \in \{1, \ldots, p\},\ \frac{\eta_\kappa}{\sum_{j=1}^{p} \eta_j} = \frac{a_\kappa}{\sum_{j=1}^{p} a_j},$$

and it follows that there exists $\gamma \geq 0$ such that $\eta_\kappa = \gamma a_\kappa$ for every $\kappa \in \{1, \ldots, p\}$. We have to check that this is consistent with the equality $1 - \prod_{j=1}^{p} (1 - \eta_j/\rho(\xi_{p-j+1})) = a_1 + \ldots + a_p$. Let $\psi(\gamma) := 1 - \prod_{j=1}^{p} (1 - \gamma a_j/\rho(\xi_{p-j+1}))$. The parameters r_i have to be chosen in $(0, 1)$, then γ must be in $(0, 1/M)$ with $M := \max\limits_{\kappa \in \{1,\ldots,p\}} \left\{\frac{a_\kappa}{\rho(\xi_{p-\kappa+1})}\right\}$. ψ is continuous as a function of γ, $\psi(0) = 1$ and $\psi(1/M) = 0$, then it follows from the intermediate value theorem that there exists $\gamma \in (0, 1/M)$ such that $\psi(\gamma) = a_1 + \ldots + a_p$. Then we get $\pi(\kappa) = a_\kappa$ for every $\kappa \in \{1, \ldots, p\}$.

The second point follows from the following facts :

- $D = 1/(1 - a_1 - \ldots - a_p)$ by definition of γ;
- $\pi^{(i)} \in \Pi_{p,D}^0$.

□

7.3. **Example given in** [AFFR16] **for the periodic case.** In [AFFR16], the authors illustrate the content with the following example concerning the periodic case.
Consider the dynamical system given by $f(x) = 2x \bmod 1$ on $\mathbb{S}^1$ and the observable :

$$\varphi(x) = \begin{cases} -\log|x - \zeta| & \text{for } x \text{ close to } \zeta \\ |x - f(\zeta)|^{-1/2} & \text{for } x \text{ close to } f(\zeta) \\ |x - f^3(\zeta)|^{-1/2} & \text{for } x \text{ close to } f^3(\zeta) \\ 0 & \text{otherwise} \end{cases}$$

with the 5-periodic point $\zeta := 1/31$. Given τ and $(u_n)_{n\in\mathbb{N}}$ satisfying $n\mathrm{Leb}(X_0 > u_n) \underset{n\to\infty}{\longrightarrow} \tau$, the connected components of $\{\varphi > u_n\}$ are given by :

$$\left\{\begin{array}{lcl} U_n^{(0)}(\zeta) & = & \mathrm{B}(\zeta, \mathrm{Exp}(-u_n)) \\ U_n^{(0)}(f(\zeta)) & = & \mathrm{B}(f(\zeta), 1/u_n^2) \\ U_n^{(0)}(f^3(\zeta)) & = & \mathrm{B}(f^3(\zeta), 1/u_n^2) \end{array}\right. .$$

The problem is that we cannot define the sequence r_n. However note that $\mathrm{Exp}(-u_n) \underset{n\to\infty}{=} o(1/u_n^2)$, meaning that $U_n^{(0)}(\zeta)$ is asymptotically negligible compared with the connected components of $f^{-1}(U_n^{(0)}(f(\zeta)))$ and $f^{-3}(U_n^{(0)}(f^3(\zeta)))$. Then this ball has asymptotically no effect and the multiplicity distribution can be more easily computed by considering the observable :

$$\tilde{\varphi}(x) = \left\{\begin{array}{ll} |x - f(\zeta)|^{-1/2} & \text{for } x \text{ close to } f(\zeta) \\ |x - f^3(\zeta)|^{-1/2} & \text{for } x \text{ close to } f^3(\zeta) \\ 0 & \text{otherwise} \end{array}\right. .$$

Then the theory is applied with $N = 2$, $m_1 = 0$, $m_2 = 2$, $\xi_1 = f(\zeta)$, $\xi_2 = f^2(\xi_1)$, $\ell_1 = \ell_3 = 1$, $\ell_2 = \ell_4 = \ell_5 = 0$ and $r_n = 1/u_n^2$. Concerning the system, the quantities d_i are equal to 2 and $\delta_i = \frac{1}{2}$. Then the quantities $\mathrm{L}_\kappa^{(i)}$ are defined by :

$$\left\{\begin{array}{l} \{\mathrm{L}_0^{(1)} \geq \mathrm{L}_1^{(1)} \geq \ldots\} = \{1/2^{5M}, 1/2^{5M+2} \mid M \in \mathbb{N}_0\} \\ \{\mathrm{L}_0^{(2)} \geq \mathrm{L}_1^{(2)} \geq \ldots\} = \{1/2^{5M+1}, 1/2^{5M+4} \mid M \in \mathbb{N}_0\} \\ \{\mathrm{L}_0^{(3)} \geq \mathrm{L}_1^{(3)} \geq \ldots\} = \{1/2^{5M}, 1/2^{5M+3} \mid M \in \mathbb{N}_0\} \\ \{\mathrm{L}_0^{(4)} \geq \mathrm{L}_1^{(4)} \geq \ldots\} = \{1/2^{5M+2}, 1/2^{5M+4} \mid M \in \mathbb{N}_0\} \\ \{\mathrm{L}_0^{(5)} \geq \mathrm{L}_1^{(5)} \geq \ldots\} = \{1/2^{5M+1}, 1/2^{5M+3} \mid M \in \mathbb{N}_0\} \end{array}\right.$$

$$\left\{\begin{array}{ll} \mathrm{L}_{2\kappa}^{(1)} = 1/2^{5\kappa}, & \mathrm{L}_{2\kappa+1}^{(1)} = 1/2^{5\kappa+2} \\ \mathrm{L}_{2\kappa}^{(2)} = 1/2^{5\kappa+2}, & \mathrm{L}_{2\kappa+1}^{(2)} = 1/2^{5\kappa+4} \\ \mathrm{L}_{2\kappa}^{(3)} = 1/2^{5\kappa}, & \mathrm{L}_{2\kappa+1}^{(3)} = 1/2^{5\kappa+3} \\ \mathrm{L}_{2\kappa}^{(4)} = 1/2^{5\kappa+2}, & \mathrm{L}_{2\kappa+1}^{(4)} = 1/2^{5\kappa+4} \\ \mathrm{L}_{2\kappa}^{(5)} = 1/2^{5\kappa+1}, & \mathrm{L}_{2\kappa+1}^{(5)} = 1/2^{5\kappa+3} \end{array}\right.$$

and we obtain :

$$\left\{\begin{array}{rcl} \pi(2\kappa+1) & = & \dfrac{21}{26}\left(\dfrac{1}{2}\right)^{5\kappa} \\ \pi(2\kappa) & = & \dfrac{67}{13}\left(\dfrac{1}{2}\right)^{5\kappa} \end{array}\right.$$

as in [AFFR16].

7.4. **Example given in** [AFFR16] **for the non-periodic case.** In [AFFR16], the authors illustrate the content with the following example concerning the non-periodic case.

Consider the dynamical system given by $f(x) = 2x \bmod 1$ on $\mathbb{S}^1$ and the observable :

$$\varphi(x) = \left\{\begin{array}{ll} -\log|x - \zeta| & \text{for } x \text{ close to } \zeta \\ |x - f(\zeta)|^{-1/2} & \text{for } x \text{ close to } f(\zeta) \\ |x - f^3(\zeta)|^{-1/2} & \text{for } x \text{ close to } f^3(\zeta) \\ 0 & \text{otherwise} \end{array}\right.$$

with the non-periodic point $\zeta := \sqrt{2}/16$. Given τ and $(u_n)_{n\in\mathbb{N}}$ satisfying $n\text{Leb}(X_0 > u_n) \underset{n\to\infty}{\longrightarrow} \tau$, the connected components of $\{\varphi > u_n\}$ are defined by :

$$\begin{cases} U_n^{(0)}(\zeta) & = & \text{B}(\zeta, \text{Exp}(-u_n)) \\ U_n^{(0)}(f(\zeta)) & = & \text{B}(f(\zeta), 1/u_n^2) \\ U_n^{(0)}(f^3(\zeta)) & = & \text{B}(f^3(\zeta), 1/u_n^2) \end{cases}.$$

As mentionned in 7.3, the ball $U_n^{(0)}(\zeta)$ has asymptotically no effect and we consider :

$$\tilde{\varphi}(x) = \begin{cases} |x - f(\zeta)|^{-1/2} & \text{for } x \text{ close to } f(\zeta) \\ |x - f^3(\zeta)|^{-1/2} & \text{for } x \text{ close to } f^3(\zeta) \\ 0 & \text{otherwise} \end{cases}.$$

Then the theory is applied with $N = 2$, $m_1 = 0$, $m_2 = 2$, $\xi_1 = f(\zeta)$, $\xi_2 = f^2(\xi_1)$, $\ell_1 = \ell_3 = 1$, $\ell_2 = 0$ and $r_n = 1/u_n^2$. Concerning the dynamical system, the quantities d_i are equal to 2, $\delta_0 = 0$ and $\delta_1 = \delta_2 = \frac{1}{2}$. Then the quantities $\text{L}_\kappa^{(i)}$ are defined by :

$$\begin{cases} \text{L}_0^{(1)} = 1, \ \text{L}_1^{(1)} = 1/4 \\ \text{L}_0^{(2)} = 1/2 \\ \text{L}_0^{(3)} = 1 \\ \text{L}_\kappa^{(i)} = 0 \text{ otherwise} \end{cases}$$

and we obtain :

$$\begin{cases} \pi(1) = 6/7 \\ \pi(2) = 1/7 \\ \pi(\kappa) = 0 \text{ for all } \kappa \geq 3 \end{cases}$$

as in [AFFR16].

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Corentin Correia, École normale supérieure (ENS) Paris-Saclay, 4 Avenue des Sciences, 91190 Gif-sur-Yvette, France

Email address: corentin.correia@ens-paris-saclay.fr

Ana Cristina Moreira Freitas, Centro de Matemática & Faculdade de Economia da Universidade do Porto, Rua Dr. Roberto Frias, 4200-464 Porto, Portugal

Email address: amoreira@fep.up.pt

URL: http://www.fep.up.pt/docentes/amoreira/

Jorge Milhazes Freitas, Centro de Matemática & Faculdade de Ciências da Universidade do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal

Email address: jmfreita@fc.up.pt

URL: http://www.fc.up.pt/pessoas/jmfreita/