LEIBNIZ ALGEBRAS AND GRAPHS

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ABSTRACT. We consider a Leibniz algebra $\mathfrak{L} = \mathfrak{I} \oplus \mathfrak{V}$ over an arbitrary base field $\mathbb{F}$, being $\mathfrak{I}$ the ideal generated by the products $[x,x], x \in \mathfrak{L}$. This ideal has a fundamental role in the study presented in our paper. A basis $\mathcal{B} = \{v_i\}_{i\in I}$ of $\mathfrak{L}$ is called multiplicative if for any $i, j \in I$ we have that $[v_i, v_j] \in \mathbb{F}v_k$ for some $k \in I$. We associate an adequate graph $\Gamma(\mathfrak{L}, \mathcal{B})$ to $\mathfrak{L}$ relative to $\mathcal{B}$. By arguing on this graph we show that $\mathfrak{L}$ decomposes as a direct sum of ideals, each one being associated to one connected component of $\Gamma(\mathfrak{L}, \mathcal{B})$. Also the minimality of $\mathfrak{L}$ and the division property of $\mathfrak{L}$ are characterized in terms of the weak symmetry of the defined subgraphs $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{I}})$ and $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{V}})$.

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1. INTRODUCTION

Leibniz algebras were presented by Bloh [5], who called them the D-algebras. After two decades, Loday introduced them in [18] with the name of Leibniz algebras because it was Gottfried W. Leibniz who discovered the *Leibniz rule* for differentiation of functions. Recently, the structure of the Leibniz algebras has been considered in the frameworks of low dimensional algebras, nilpotence and related problems [1, 2, 3, 6, 10, 16, 17]. The inner structure of Leibniz algebras admitting a multiplicative basis $\mathcal{B}$ has been recently studied in [7], it is focussed the characterization of the $\mathcal{B}$-semisimplicity and of the $\mathcal{B}$-simplicity of the algebra.

An interesting problem in graph theory and in abstract algebra consists in characterizing the structure of an algebraic object by the properties satisfied for some graph associated with it (see for instance [4, 12, 13, 14]). The paper [8] is devoted to the study of the structure of linear spaces by associating an adequate graph.

The main goal of the present paper is to use properties of graphs to study Leibniz algebras $\mathfrak{L}$ with multiplicative bases $\mathcal{B}$ in order to obtain results about their algebraic structure. Given a Leibniz algebras $\mathfrak{L}$, the ideal $\mathfrak{I}$ generated by the products $[x,x]$ with $x \in \mathfrak{L}$ plays a relevant role in our study, so we have to handle specifically the ideas of [8]. By arguing on the associate graph we show that a Leibniz algebra decomposes as a direct sum of ideals, each one being associated to one connected component of the referred graph. In the last part of the work, the authors approach the minimality property of this class of algebras by using the subgraphs $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{I}})$ and $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{V}})$ related with $\mathfrak{I}$ and the remain part of $\mathfrak{L}$, respectively.

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The paper is organized as follows. In Section 2 we introduce the (directed) graph associated with $\mathfrak{L}$ admitting a multiplicative basis $\mathcal{B}$, denoted as $\Gamma(\mathfrak{L}, \mathcal{B})$. By using this graph we prove that $\mathfrak{L}$ decomposes as a direct sum

$$\mathfrak{L} = \bigoplus_k \mathcal{I}_k$$

of ideals with a multiplicative basis contained in $\mathcal{B}$, each one being associated with one connected component of $\Gamma(\mathfrak{L}, \mathcal{B})$. In the next section we discuss the relation among the previous decompositions of $\mathfrak{L}$ given by different choices of bases of $\mathfrak{L}$. Finally, in Section 4 we relate the weak symmetry of two concrete subgraphs with some properties of $\mathfrak{L}$. The minimality of $\mathfrak{L}$ and the division property of $\mathcal{B}$ are characterized. It is shown that $\mathfrak{L}$ is minimal if and only if $\mathcal{B}$ is of division if and only if the two refereed graphs are weak symmetric. All of the Leibniz algebras considered are of arbitrary dimension and over an arbitrary base field $\mathbb{F}$.

Definition 1.1. A *Leibniz algebra* $\mathfrak{L}$ is a vector space over a field $\mathbb{F}$ endowed with a bilinear product $[\cdot, \cdot]$ satisfying the *Leibniz identity*

$$[[y, z], x] = [[y, x], z] + [y, [z, x]],$$

for any $x, y, z \in \mathfrak{L}$.

Clearly, Lie algebras are examples of Leibniz algebras. For any $x \in \mathfrak{L}$, consider the adjoint map $\mathrm{ad}_x : \mathfrak{L} \to \mathfrak{L}$ defined by $\mathrm{ad}_x(y) = [y, x]$, with $y \in \mathfrak{L}$. Observe that Leibniz identity is equivalent to assert that ad_x is a derivation for any $x \in \mathfrak{L}$. A *subalgebra* S of a Leibniz algebra $\mathfrak{L}$ is a vector subspace of $\mathfrak{L}$ such that $[S, S] \subset S$, and an *ideal* I of $\mathfrak{L}$ is a subalgebra such that $[I, \mathfrak{L}] + [\mathfrak{L}, I] \subset I$.

The ideal $\mathfrak{I}$ generated by $\{[x, x] : x \in \mathfrak{L}\}$ plays an important role in the theory since it determines the (possible) non-Lie character of the Leibniz algebra $\mathfrak{L}$. From the Leibniz identity, this ideal satisfies

$$[\mathfrak{L}, \mathfrak{I}] = 0. \tag{1}$$

Observe that we can write

$$\mathfrak{L} = \mathfrak{I} \oplus \mathfrak{V}$$

where $\mathfrak{V}$ is a linear complement of $\mathfrak{I}$ in $\mathfrak{L}$ (actually, $\mathfrak{V}$ is isomorphic as linear space to $\mathfrak{L}/\mathfrak{I}$, the so called corresponding Lie algebra of $\mathfrak{L}$). Hence, by taking $\mathcal{B}_{\mathfrak{I}}$ and $\mathcal{B}_{\mathfrak{V}}$ bases of $\mathfrak{I}$ and $\mathfrak{V}$, respectively, we get

$$\mathcal{B} = \mathcal{B}_{\mathfrak{I}} \mathbin{\dot{\cup}} \mathcal{B}_{\mathfrak{V}}$$

a basis of $\mathfrak{L}$.

Definition 1.2. A decomposition of a Leibniz algebra $\mathfrak{L}$ as the direct sum of linear subspaces

$$\mathfrak{L} = \bigoplus_{j \in J} \mathfrak{L}_j$$

is *orthogonal* if $[\mathfrak{L}_j, \mathfrak{L}_k] = \{0\}$ for any two different elements $j, k \in J$.

For an arbitrary algebra A over a base field $\mathbb{F}$, a basis $\mathcal{B} = \{v_i\}_{i \in I}$ of A is *multiplicative* if for any $i, j \in I$ we have that $v_i v_j \in \mathbb{F} v_k$ for some $k \in I$ (see [9]). In the particular case of Leibniz algebras, as in [7], the relation (1) implies the following characterization.

Definition 1.3. A basis $\mathcal{B} = \mathcal{B}_{\mathfrak{I}} \dot\cup \mathcal{B}_{\mathfrak{V}}$ of a Leibniz algebra $\mathfrak{L}$, where we denote

$$\mathcal{B}_{\mathfrak{I}} := \{e_k\}_{k\in K}, \quad \mathcal{B}_{\mathfrak{V}} := \{u_j\}_{j\in J},$$

is called *multiplicative* if:

i. For any $k \in K$ and $j \in J$ we have $[e_k, u_j] \in \mathbb{F}e_i$ for some $i \in K$.
ii. For any $j, k \in J$ we have either $[u_j, u_k] \in \mathbb{F}u_l$ for some $l \in J$, or $[u_j, u_k] \in \mathbb{F}e_i$ for some $i \in K$.

Example 1.1. (see [19]) Consider a (non-Lie) Leibniz algebra $\mathfrak{L}$ over a field $\mathbb{F}$ with characteristic different of 2 and with the multiplicative basis $\mathcal{B} = \mathcal{B}_{\mathfrak{I}} \dot\cup B_{\mathfrak{V}}$, where $\mathcal{B}_{\mathfrak{I}} = \{p, q\}$ and $\mathcal{B}_{\mathfrak{V}} = \{e, h, f\}$, defined by the following multiplication:

$$[e, h] = -[h, e] = 2e, \quad [h, f] = -[f, h] = 2f, \quad [e, f] = -[f, e] = h,$$

$$[p, h] = -[q, e] = p, \quad [p, f] = -[q, h] = q,$$

where are omitted products equal to zero.

Example 1.2. The infinite-dimensional Leibniz algebra $\mathfrak{L} = \mathfrak{I} \oplus \mathfrak{V}$ given in [7, Example 1.2], over a base field $\mathbb{F}$ with characteristic different to 2, admits a multiplicative basis $\mathcal{B} = \mathcal{B}_{\mathfrak{I}} \dot\cup \mathcal{B}_{\mathfrak{V}}$, where

$$\mathcal{B}_{\mathfrak{I}} = \{e_n : n \in \mathbb{N}\}$$

is a basis of $\mathfrak{I}$ (we denote by $\mathbb{N}$ the set of non-negative integers) and the set

$$\mathcal{B}_{\mathfrak{V}} = \{u_a, u_b, u_c, u_d\}$$

is a basis of $\mathfrak{V}$. The non-zero products respect to the elements in the basis $\mathcal{B}$ of $\mathfrak{L}$ are:

$$\begin{aligned}
&[u_b, u_c] = u_a, \quad [u_c, u_b] = -u_a,\\
&[u_d, u_d] = e_0, \quad [e_0, u_d] = e_1,\\
&[e_n, u_a] = e_n, \text{ for } n \geq 2,\\
&[e_n, u_b] = e_{n+1}, \text{ for } n \geq 2,\\
&[e_n, u_c] = (n-2)e_{n-1}, \text{ for } n \geq 3.
\end{aligned}$$

Example 1.3. *Let $\mathfrak{L}$ be the model filiform Leibniz algebra with multiplicative basis $\mathcal{B} = \mathcal{B}_{\mathfrak{I}} \dot\cup \mathcal{B}_{\mathfrak{V}}$, where $\mathcal{B}_{\mathfrak{I}} = \{e_2, \dots, e_n\}$ and $\mathcal{B}_{\mathfrak{V}} = \{e_1\}$ with non-zero products given as*

$$[e_i, e_1] = e_{i+1}, \quad 2 \leq i \leq n-1.$$

Example 1.4. *Let $\mathfrak{L}$ be an n-dimensional Leibniz algebra (with an odd integer $n \geq 4$) given in Theorem 3.4 of* [20] *over the complex numbers such that $\mathfrak{L}/\mathfrak{I}$ is isomorphic to the simple Lie algebra sl_2 with basis $\mathcal{B} := \{e, f, h, x_0, x_1, \dots, x_{n-4}\}$ of $\mathfrak{L}$ such that the non-zero products of $\mathcal{B}$ are*

$$\begin{aligned}
&[e, f] = -[f, e] = h, && [x_k, e] = k(k+3-n)x_{k-1}, \quad 1 \leq k \leq n-4,\\
&[e, h] = -[h, e] = 2e, && [x_k, h] = (n-4-2k)x_k, \quad 0 \leq k \leq n-4,\\
&[h, f] = -[f, h] = 2f, && [x_k, f] = x_{k+1}, \quad 0 \leq k \leq n-5.
\end{aligned}$$

Clearly $\mathcal{B}$ is a multiplicative basis. This example generalizes the Example 1.1 where in that case we take $n = 5$ and denote $p := x_0, q := x_1$.

2. Leibniz algebras admitting a multiplicative basis, and graphs

In this section we relate the concept of Leibniz algebra admitting a multiplicative basis with graphs in order to characterize their structure. We recall a *(directed) graph* is a pair (V, E) where V is a set of vertices and $E \subset V \times V$ a set of (directed) edges connecting the vertices. The reader can found more usual concepts related to graphs in [8]. From now on, we consider directed graphs.

Definition 2.1. Let $\mathfrak{L}$ be a Leibniz algebra admitting a multiplicative basis $\mathcal{B}$. The directed graph associated to $\mathfrak{L}$ relative to $\mathcal{B}$ can be written as $\Gamma(\mathfrak{L}, \mathcal{B}) := (V, E)$, being $V := \mathcal{B}$ and

$$E := \{(v_j, v_k) \in V \times V : \{[v_i, v_j], [v_j, v_i]\} \cap \mathbb{F}^\times v_k \neq \emptyset \text{ for some } v_i \in \mathcal{B}\}.$$

Remark 2.1. Taking into account Definition 1.3 we observe that the directed graph associated to a Leibniz algebra $\mathfrak{L}$ admitting a multiplicative basis $\mathcal{B}$ can be written more precisely as $\Gamma(\mathfrak{L}, \mathcal{B}) = (V, E)$, being $V = \mathcal{B}_\mathfrak{I} \dot\cup \mathcal{B}_\mathfrak{V}$ and

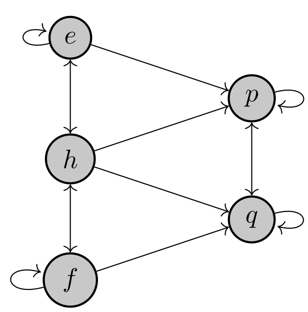
$$\begin{aligned} E = &\{(u_j, e_k) \in V \times V : \{[e_i, u_j]\} \cap \mathbb{F}^\times e_k \neq \emptyset \text{ for some } e_i \in \mathcal{B}_\mathfrak{I}\} \\ &\cup \{(e_i, e_k) \in V \times V : \{[e_i, u_j]\} \cap \mathbb{F}^\times e_k \neq \emptyset \text{ for some } u_j \in \mathcal{B}_\mathfrak{V}\} \\ &\cup \{(u_j, e_i) \in V \times V : \{[u_j, u_k], [u_k, u_j]\} \cap \mathbb{F}^\times e_i \neq \emptyset \text{ for some } u_k \in \mathcal{B}_\mathfrak{V}\} \\ &\cup \{(u_j, u_l) \in V \times V : \{[u_j, u_k], [u_k, u_j]\} \cap \mathbb{F}^\times u_l \neq \emptyset \text{ for some } u_k \in \mathcal{B}_\mathfrak{V}\} \end{aligned}$$

The description presented above of E agrees with Definition 1.3 (first two subsets correspond to item i. and next two subsets correspond to the item ii. of the referred definition).

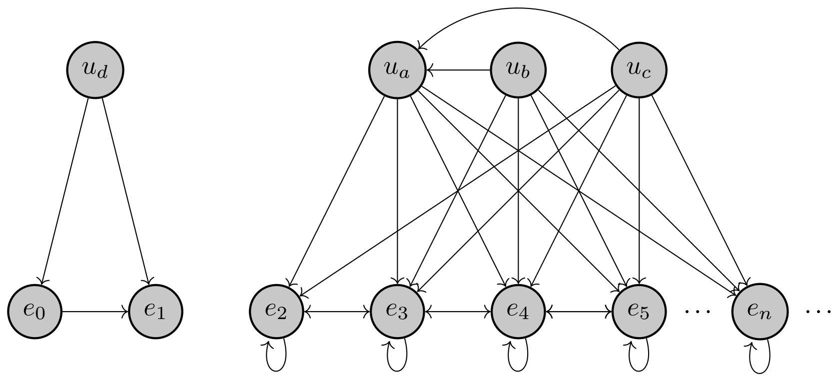
Example 2.1. Consider the Leibniz algebra $\mathfrak{L}$ of the Example 1.1 over a field with characteristic different of 2 and with the basis $\mathcal{B} = \mathcal{B}_\mathfrak{I} \dot\cup B_\mathfrak{V}$, where $\mathcal{B}_\mathfrak{I} = \{p, q\}$, $\mathcal{B}_\mathfrak{V} = \{e, h, f\}$. We have that $V = \{e, h, f, p, q\}$,

$$\begin{aligned} E = &\{(e,p), (h,p), (h,q), (f,q)\} \cup \{(p,p), (p,q), (q,p), (q,q)\} \\ &\cup \emptyset \cup \{(e,e), (e,h), (h,f), (f,f), (h,e), (f,h)\} \end{aligned}$$

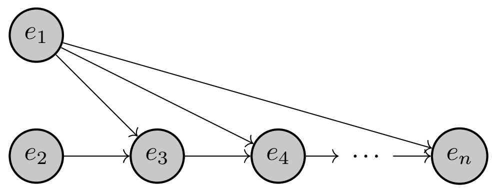
and the associated graph $\Gamma(\mathfrak{L}, \mathcal{B})$ is the following:



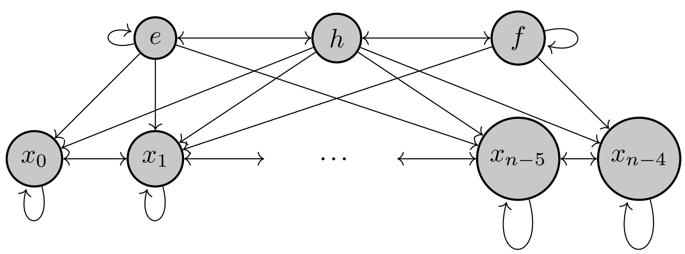
Example 2.2. The associated graph of the Leibniz algebra presented in Example 1.2 over a field with characteristic 0 and with the multiplicative basis $\mathcal{B} = \mathcal{B}_\mathfrak{I} \dot\cup \mathcal{B}_\mathfrak{V}$, where $\mathcal{B}_\mathfrak{I} = \{e_n : n \in \mathbb{N}\}$ and $\mathcal{B}_\mathfrak{V} = \{u_a, u_b, u_c, u_d\}$ is as follows:



Example 2.3. *For the model filiform Leibniz algebra $\mathfrak{L}$ with basis $\mathcal{B} := \{e_1, \dots, e_n\}$ from Example 1.3 the associated graph $\Gamma(\mathfrak{L}, \mathcal{B})$ is*



Example 2.4. *Let $\mathfrak{L}$ be the n-dimensional Leibniz algebra from Example 1.4 over a field with characteristic 0. The associated graph $\Gamma(\mathfrak{L}, \mathcal{B})$ is*



Given two vertices $v_i, v_j \in V$, an *undirected path* from v_i to v_j is an ordered family of vertices $\{v_{i_1}, \dots, v_{i_n}\} \subset V$ with $v_{i_1} = v_i$, $v_{i_n} = v_j$, and such that either $(v_{i_r}, v_{i_{r+1}}) \in E$ or $(v_{i_{r+1}}, v_{i_r}) \in E$, for $1 \leq r \leq n-1$.

We can introduce an equivalence relation in V defined by $v_i \sim v_j$ if and only if either $v_i = v_j$ or there exists an undirected path from v_i to v_j. Then it is said that v_i and v_j are *connected* and the equivalence class of v_i, denoted by $[v_i] \in V/\sim$, corresponds to a connected component $\mathcal{C}_{[v_i]}$ of the graph $\Gamma(\mathfrak{L}, \mathcal{B})$. Then

$$\Gamma(\mathfrak{L}, \mathcal{B}) = \dot{\bigcup_{[v_i] \in V/\sim}} \mathcal{C}_{[v_i]}. \tag{2}$$

We can also associate to any $\mathcal{C}_{[v_i]}$ the linear subspace

$$\mathfrak{L}_{\mathcal{C}_{[v_i]}} := \bigoplus_{v_j \in [v_i]} \mathbb{F} v_j \tag{3}$$

and assert the next result:

Theorem 2.1. *Let $\mathfrak{L}$ be a Leibniz algebra admitting a multiplicative basis $\mathcal{B} = \{v_i\}_{i\in I}$. Then $\mathfrak{L}$ decomposes as the orthogonal direct sum*

$$\mathfrak{L} = \bigoplus_{[v_i]\in V/\sim} \mathfrak{L}_{C_{[v_i]}},$$

where any $\mathfrak{L}_{C_{[v_i]}}$ is an ideal of $\mathfrak{L}$, admitting the set $[v_i] \subset \mathcal{B}$ as multiplicative basis.

Proof. From Equation (2) and Equation (3) we can assert that $\mathfrak{L}$ is the direct sum of the family of linear subspaces $\mathfrak{L}_{C_{[v_i]}}$ with $[v_i] \in V/\sim$, admitting each one the set $[v_i] \subset \mathcal{B}$ as multiplicative basis.

Let us suppose that there exist $v_j \in \mathfrak{L}_{C_{[v_i]}}$ and $v_k, v_l \in \mathcal{B}$ such that

$$\{[v_j, v_k], [v_k, v_j]\} \cap \mathbb{F}^{\times} v_l \neq \emptyset$$

for some $j \in I$. Then (v_j, v_l) and (v_k, v_l) are edges of $C_{[v_i]}$, and then $v_k, v_l \in \mathfrak{L}_{C_{[v_i]}}$. From here we conclude that the direct sum is orthogonal and that $\mathfrak{L}_{C_{[v_i]}}$ is an ideal of $\mathfrak{L}$. $\square$

Example 2.5. The Leibniz algebra of the Example 1.2 decomposes as the orthogonal direct sum

$$\mathfrak{L} = \mathfrak{L}_{C_{[v_d]}} \bigoplus \mathfrak{L}_{C_{[v_a]}}$$

where

$$\mathfrak{L}_{C_{[v_d]}} := \mathbb{F}v_d \oplus \mathbb{F}e_0 \oplus \mathbb{F}e_1, \qquad \mathfrak{L}_{C_{[v_a]}} := \mathbb{F}v_a \oplus \mathbb{F}v_b \oplus \mathbb{F}v_c \oplus \mathbb{F}e_2 \oplus \mathbb{F}e_3 \oplus \cdots \oplus \mathbb{F}e_n \oplus \cdots$$

are ideals of $\mathfrak{L}$ admitting the multiplicative basis in $\mathcal{B}$

$$[v_d] := \{v_d, e_0, e_1\}, \qquad [v_a] := \{v_a, v_b, v_c\} \cup \{e_i\}_{i\geq 2},$$

respectively.

We recall a Leibniz algebra $\mathfrak{L}$ is *simple* if its product is non-zero and its only ideals are $\{0\}$, $\mathfrak{I}$ and $\mathfrak{L}$. It should be noted that this definition agrees with the definition of simple Lie algebra, since in this framework $\mathfrak{I} = \{0\}$.

Corollary 2.1. *If $\mathfrak{L}$ is simple, then any two vertices of $\Gamma(\mathfrak{L}, \mathcal{B})$ are connected.*

Example 2.6. *By Corollary 2.1 the Leibniz algebra of Example 1.2 is not simple because its associated graph has two components as shown in Example 2.2.*

To identify the components of the decomposition given in Theorem 2.1 we only need to focus on the connected components of the associated graph.

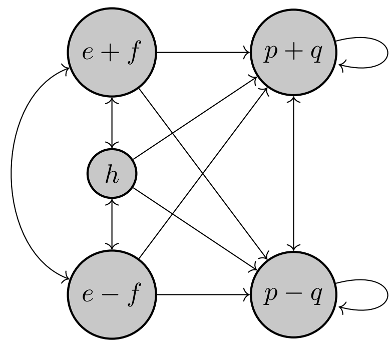
3. Relating the graphs given by different choices of bases

In general, two different multiplicative bases of $\mathfrak{L}$, namely $\mathcal{B}$ and $\mathcal{B}'$, have two different associated graphs, which give rise to two different decompositions of $\mathfrak{L}$ as an orthogonal direct sum of ideals (see Theorem 2.1). For instance, consider the following example:

Example 3.1. *Consider the complex Leibniz algebra $\mathfrak{L}$ of the Example 1.1 with the multiplicative basis $\mathcal{B}' := \{e + f, e - f, h, p + q, p - q\}$. Indeed, the non-zero product in the* *basis $\mathcal{B}'$ are*

$$\begin{aligned}
&[e-f,e+f] = -[e+f,e-f] = 2h,\\
&[e+f,h] = -[h,e+f] = 2(e-f),\\
&[e-f,h] = -[h,e-f] = 2(e+f),\\
&[p+q,h] = -[p+q,e+f] = [p-q,e-f] = p-q,\\
&[p-q,e+f] = [p-q,h] = -[p+q,e-f] = p+q.
\end{aligned}$$

Then the associated graph $\Gamma(\mathfrak{L},\mathcal{B}')$ is



which is clearly different to the one obtained with the multiplicative basis $\mathcal{B}$ presented in Example 2.1.

Next we give a condition under which the graphs associated to $\mathfrak{L}$ and two different multiplicative bases $\mathcal{B}$ and $\mathcal{B}'$ are isomorphic. As a consequence, we establish a sufficient condition under which two decompositions of $\mathfrak{L}$, induced by two different multiplicative bases $\mathcal{B}$ and $\mathcal{B}'$, are equivalent.
We recall that an *automorphism* of a Leibniz algebra $\mathfrak{L}$ is a linear isomorphism $f:\mathfrak{L}\to\mathfrak{L}$ satisfying $[f(x),f(y)] = f([x,y])$, for any $x,y\in\mathfrak{L}$.

Definition 3.1. Let $\mathfrak{L}$ be a Leibniz algebra. Two bases $\mathcal{B} = \{v_i\}_{i\in I}$ and $\mathcal{B}' = \{w_j\}_{j\in J}$ of $\mathfrak{L}$ are *equivalent* if there exists an automorphism $f:\mathfrak{L}\to\mathfrak{L}$ satisfying $f(\mathcal{B}) = \mathcal{B}'$.

We recall that two graphs (V,E) and (V',E') are *isomorphic* if there exists a bijection $f:V\to V'$ such that $(v_i,v_j)\in E$ if and only if $(f(v_i),f(v_j))\in E'$.

Lemma 3.1. *Let $\mathfrak{L}$ be a Leibniz algebra and consider two multiplicative bases $\mathcal{B}$ and $\mathcal{B}'$ of $\mathfrak{L}$. If $\mathcal{B}$ and $\mathcal{B}'$ are two equivalent bases, then the associated graphs $\Gamma(\mathfrak{L},\mathcal{B})$ and $\Gamma(\mathfrak{L},\mathcal{B}')$ are isomorphic.*

Proof. Let us suppose that $\mathcal{B} = \{v_i\}_{i\in I}$ and $\mathcal{B}' = \{w_j\}_{j\in J}$ are two equivalent multiplicative bases of $\mathfrak{L}$. Then, there exists an automorphism $f:\mathfrak{L}\to\mathfrak{L}$ satisfying

$$[f(x),f(y)] = f([x,y]) \tag{4}$$

for any $x,y\in\mathfrak{L}$ and such that $f(\mathcal{B}) = \mathcal{B}'$.

Let us denote by (V,E) and (V',E') the set of vertices and edges of $\Gamma(\mathfrak{L},\mathcal{B})$ and $\Gamma(\mathfrak{L},\mathcal{B}')$, respectively. Taking into account that $V=\mathcal{B}$ and $V'=\mathcal{B}'$, and the fact $f(\mathcal{B}) = \mathcal{B}'$, we have that the map f defines a bijection from V to V'.

Finally, for any $x,y\in V$ we want to show that $(x,y)\in E$ if and only if $(f(x),f(y))\in E'$. We prove that if $(f(x),f(y))\in E'$ then $(x,y)\in E$ by contrapositive. Indeed, for any $x,y\in V$, if $(x,y)\notin E$ then $[x,y]=[y,x]=0$ and so $[f(x),f(y)] = [f(y),f(x)] = 0$, hence, $(f(x), f(y)) \notin E'$. Next we show that $(f(x), f(y)) \in E'$ if $(x, y) \in E$. By Definition 2.1, given $(x, y) \in E$, we have

$$\{[x, z], [z, x]\} \cap \mathbb{F}^{\times} y \neq \emptyset$$

for some $z \in \mathcal{B}$. Then, by Equation (4)

$$\{[f(x), f(z)], [f(z), f(x)]\} \cap \mathbb{F}^{\times} f(y) \neq \emptyset,$$

which means that $(f(x), f(y)) \in E'$. We can conclude that $\Gamma(\mathfrak{L}, \mathcal{B})$ and $\Gamma(\mathfrak{L}, \mathcal{B}')$ are isomorphic by means of f. $\square$

The following concept is taking borrowed from the theory of graded algebras (see for instance [15]).

Definition 3.2. Let $\mathfrak{L}$ be a Leibniz algebra and let

$$\Upsilon := \mathfrak{L} = \bigoplus_{i \in I} \mathfrak{L}_i \quad \text{and} \quad \Upsilon' := \mathfrak{L} = \bigoplus_{j \in J} \mathfrak{L}'_j$$

be two decompositions of $\mathfrak{L}$ as an orthogonal direct sum of ideals. It is said that Υ and Υ' are *equivalent* if there exists an automorphism $f : \mathfrak{L} \to \mathfrak{L}$, and a bijection $\sigma : I \to J$ such that $f(\mathfrak{L}_i) = \mathfrak{L}'_{\sigma(i)}$ for any $i \in I$.

Theorem 3.1. *Let $\mathfrak{L}$ be a Leibniz algebra and consider two multiplicative bases $\mathcal{B} := \{v_i\}_{i \in I}$ and $\mathcal{B}' := \{v'_j\}_{j \in J}$. Consider the following assertions:*

i. *The bases $\mathcal{B}$ and $\mathcal{B}'$ are equivalent.*
ii. *$\Gamma(\mathfrak{L}, \mathcal{B})$ and $\Gamma(\mathfrak{L}, \mathcal{B}')$, the associated graphs to $\mathcal{B}$ and $\mathcal{B}'$ respectively, are isomorphic.*
iii. *The decompositions*

$$\Upsilon := \mathfrak{L} = \bigoplus_{[v_i] \in V/\sim} \mathfrak{L}_{C_{[v_i]}} \qquad \text{and} \qquad \Upsilon' := \mathfrak{L} = \bigoplus_{[v'_j] \in V'/\sim} \mathcal{V}_{C_{[v'_j]}},$$

corresponding to $\mathcal{B}$ and $\mathcal{B}'$ are equivalent.

Then it is satisfied the implication from i. to ii., and the implication from ii. to iii.

Proof. The implication from i. to ii. was proved in Lemma 3.1. Let us prove the implication from ii. to iii. If $f : V \to V'$ defines an isomorphism between the graphs $\Gamma(\mathfrak{L}, \mathcal{B}) = (V, E)$ and $\Gamma(\mathfrak{L}, \mathcal{B}') = (V', E')$, then v is a vertex of $C_{[v]}$ if and only if $f(v)$ is a vertex of $C_{[f(v)]}$, for every $v \in V$. That means $f([v]) = [f(v)]$ for every $v \in V$. Hence, taking into account this observation and Lemma 3.1, we easily get the next result. $\square$

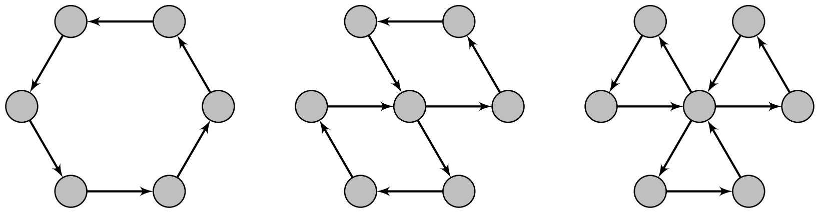
4. Minimality and Weakly Symmetry

Consider a graph (V, E). Given two elements v_i, v_j in V, an ordered family $\{v_{i_1}, \ldots, v_{i_n}\} \subset V$ is called a *directed path from v_i to v_j* if $v_{i_1} = v_i$, $v_{i_n} = v_j$ and $(v_{i_r}, v_{i_{r+1}}) \in E$ for every $1 \leq r \leq n-1$. We also recall that a graph (V, E) is *symmetric* if $(v_i, v_j) \in E$ for every $(v_j, v_i) \in E$. Then, a weaker concept can be introduced as follows:

Definition 4.1. We say that the graph (V, E) is *weakly symmetric* if for every $(v_j, v_i) \in E$ there exists a directed path from v_i to v_j.

In particular, we have that every symmetric graph is weakly symmetric.

Example 4.1. *The following graphs are weakly symmetric:*



Definition 4.2. Given a graph (V, E), we say that two vertices $v_i, v_j \in V$ are *strongly connected* if there exists a directed path from v_i to v_j and viceversa. Additionally, we say that the graph (V, E) is *strongly connected* if any two vertices of V are strongly connected.

In case of Leibniz algebras there no exist edges of type: (e_i, u_j), with $e_i \in \mathcal{B}_{\mathfrak{I}}$, $u_j \in \mathcal{B}_{\mathfrak{V}}$. So given an graph (V, E) associated to a Leibniz algebra $\mathfrak{L}$ we can not apply the notion of weakly symmetric to all graph (V, E) (see Definition 4.1). Due to this fact, we consider two subgraphs, one related with the ideal $\mathfrak{I}$, and the other with the $\mathfrak{V}$ isomorphic to a Lie algebra.

Definition 4.3. Let $\mathfrak{L}$ be a Leibniz algebra admitting a multiplicative basis $\mathcal{B}$. A pair (V', E') is a *subgraph* of the directed graph associated to $\mathfrak{L}$ relative to $\mathcal{B}$, denoted by $\Gamma(\mathfrak{L}, \mathcal{B}) := (V, E)$, if V' is a subset of V and E' is a subset of E such that an edge (v_j, v_k) of E belongs to E' if $v_j, v_k \in V'$.

A subgraph is again a graph, so the concepts presented to graph can be considered to subgraphs also. Let $\mathfrak{L} = \mathfrak{I} \oplus \mathfrak{V}$ be a Leibniz algebra with multiplicative basis $\mathcal{B} = \mathcal{B}_{\mathfrak{I}} \dot{\cup} \mathcal{B}_{\mathfrak{V}}$, being $\mathcal{B}_{\mathfrak{I}} = \{e_i\}_{i \in I}$ and $\mathcal{B}_{\mathfrak{V}} = \{u_j\}_{j \in J}$. In the following, we will consider the two subgraphs $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{I}})$ and $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{V}})$ of the associated graph $\Gamma(\mathfrak{L}, \mathcal{B})$ by removing the edges (u_j, e_i) with $u_j \in \mathcal{B}_{\mathfrak{V}}$ and $e_i \in \mathcal{B}_{\mathfrak{I}}$, where in subgraph $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{I}})$ the vertices are the elements of $\mathcal{B}_{\mathfrak{I}}$ while in subgraph $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{V}})$ the vertices are the elements of $\mathcal{B}_{\mathfrak{V}}$.

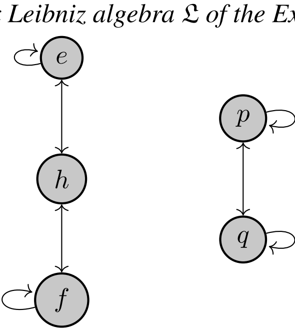
Remark 4.1. More precisely, taking into account Definition 1.3 and Remark 2.1 we observe that the subgraph $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{I}})$ is (V', E') where $V' = \mathcal{B}_{\mathfrak{I}}$ and

$$E' := \{(e_i, e_k) \in V \times V : \{[e_i, u_j]\} \cap \mathbb{F}^{\times} e_k \neq \emptyset \text{ for some } u_j \in \mathfrak{V}\}$$

and the subgraph $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{V}})$ is (V'', E'') where $V'' = \mathcal{B}_{\mathfrak{V}}$ and

$$E'' := \{(u_j, u_l) \in V \times V : \{[u_j, u_k], [u_k, u_j]\} \cap \mathbb{F}^{\times} u_l \neq \emptyset \text{ for some } u_k \in \mathfrak{V}\}.$$

Example 4.2. *For the complex Leibniz algebra $\mathfrak{L}$ of the Example 2.1 we get*



$\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{V}})$ $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{I}})$

At the following, we refer to the smallest ideal of a Leibniz algebra $\mathfrak{L}$ that contains $v \in \mathcal{B}$, denoted by $\mathcal{I}(v)$, as the ideal of $\mathfrak{L}$ generated by v.

Lemma 4.1. *Let $\mathfrak{L}$ be a Leibniz algebra with a multiplicative bases $\mathcal{B} := \{v_i\}_{i\in I}$. Given $v_i, v_j \in \mathcal{B}_K$, we have that $v_j \in \mathcal{I}(v_i)$ if and only if there exists a directed path from v_i to v_j in $\Gamma(\mathfrak{L}, \mathcal{B}_K)$, for whatever $K \in \{\mathfrak{I}, \mathfrak{V}\}$.*

Proof. First we take care of the case $K = \mathfrak{I}$. Suppose $e_j \in \mathcal{I}(e_i)$ for $e_i, e_j \in \mathcal{B}_{\mathfrak{I}}$. We have that $[[[[e_i, u_1], u_2], \dots], u_n] = e_j$, for some $u_1, \dots, u_n \in \mathcal{B}_{\mathfrak{V}}$. From here, by writing

$$w_k := \mathbb{F}[[\cdots[[e_i, u_1], u_2], \cdots], u_k] \cap \mathcal{B}_{\mathfrak{I}},$$

for $k \in \{1, \dots, n-1\}$, we get that $\{e_i, w_1, \dots, w_{n-1}, e_j\}$ is a directed path from e_i to e_j.

Conversely, if $\{e_i, e_1, \dots, e_{n-1}, e_j\}$ is a directed path from e_i to e_j in the subgraph $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{I}})$ then we have successively $0 \neq [e_i, w_1] \in \mathbb{F}e_1$, $0 \neq [e_1, w_2] \in \mathbb{F}e_2, \dots, 0 \neq [e_{n-1}, w_n] \in \mathbb{F}e_j$ for some $w_1, \dots, w_n \in \mathcal{B}_{\mathfrak{V}}$. Hence

$$e_j = \lambda[[\cdots[[e_i, w_1], w_2], \cdots], w_n] \in \mathcal{I}(e_i)$$

for some $\lambda \in \mathbb{F}^{\times}$, as required.

Second, we study the case $K = \mathfrak{V}$. For $u_j \in \mathcal{I}(u_i)$, with for $u_i, u_j \in \mathcal{B}_{\mathfrak{V}}$. Thus, either $[u_i, v_1] \neq 0$ and then $v_1 \in \mathcal{B}_{\mathfrak{V}}$ (otherwise, $v_1 \in \mathcal{B}_{\mathfrak{I}}$ implies $[u_i, v_1] = 0$), or $[v_1, u_i] \neq 0$. In this last case, if $v_1 \in \mathcal{B}_{\mathfrak{I}}$ we get $[v_1, u_i] \in \mathcal{B}_{\mathfrak{I}}$ and consequently $[[v_1, u_i], v_2] \in \mathfrak{I}$. By iterating this process, we will always obtain $[\cdots, [[v_1, u_i], v_2], \cdots, v_k] \in \mathfrak{I}$, what contradicts the fact that $u_j \in \mathcal{B}_{\mathfrak{V}}$. In conclusion, we have again $v_1 \in \mathcal{B}_{\mathfrak{V}}$. A similar argument, shows that also $v_2, \dots, v_n \in \mathcal{B}_{\mathfrak{V}}$. From here, by writing

$$w_k := \mathbb{F}f(\cdots(f(f(u_i, v_1), v_2), \cdots), v_k) \cap \mathcal{B}_{\mathfrak{V}}$$

for $k \in \{1, \dots, n-1\}$, being any $f(x, y) \in \{[x, y], [y, x]\}$, we get that $\{u_i, w_1, \dots, w_{n-1}, u_j\}$ is a directed path from u_i to u_j.

Let us now prove the converse. If $\{u_i, u_1, \dots, u_{n-1}, u_j\}$ is a directed path from u_i to u_j in $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{V}})$ then we have successively $0 \neq [u_i, u_1] \in \mathbb{F}u_1$, $0 \neq [u_1, u_2] \in \mathbb{F}u_2, \dots, 0 \neq [u_{n-1}, u_n] \in \mathbb{F}u_j$ for some $u_1, \dots, u_n \in \mathcal{B}_{\mathfrak{V}}$. Hence

$$u_j = \lambda[[\cdots[[u_i, u_1], u_2], \cdots], u_n] \in \mathcal{I}(u_i)$$

for some $\lambda \in \mathbb{F}^{\times}$, which completes the proof. □

Definition 4.4. Let $\mathfrak{L} = \mathfrak{I} \oplus \mathfrak{V}$ be a Leibniz algebra with a multiplicative basis $\mathcal{B} = \mathcal{B}_{\mathfrak{I}} \dot{\cup} \mathcal{B}_{\mathfrak{V}}$, where $\mathcal{B}_{\mathfrak{I}} = \{e_k\}_{k\in K}$ and $\mathcal{B}_{\mathfrak{V}} = \{u_j\}_{j\in J}$. It is said $\mathcal{B}$ is of *weak division* if

- $0 \neq [e_i, u_j] = \lambda e_k$ implies $e_i \in \mathcal{I}(e_k)$.
- $0 \neq [u_i, u_k] = \lambda u_p$ implies $u_i, u_k \in \mathcal{I}(u_p)$.

Proposition 4.1. *Let $\mathfrak{L}$ be a Leibniz algebra admitting a multiplicative basis $\mathcal{B} = \mathcal{B}_{\mathfrak{I}} \dot{\cup} \mathcal{B}_{\mathfrak{V}}$, where $\mathcal{B}_{\mathfrak{I}} = \{e_k\}_{k\in K}$ and $\mathcal{B}_{\mathfrak{V}} = \{u_j\}_{j\in J}$. In these conditions, $\mathcal{B}$ is of weak division if and only if the subgraph $\Gamma(\mathfrak{L}, \mathcal{B}_K)$ is weakly symmetric, for $K \in \{\mathfrak{I}, \mathfrak{V}\}$.*

Proof. Let us suppose that the multiplicative basis $\mathcal{B}$ is of weak division. First we prove that $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{V}})$ is weakly symmetric. Given an edge (u_i, u_j) of $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{V}})$ we have that there exist $u_k \in \mathcal{B}_{\mathfrak{V}}$ and $\lambda_j \in \mathbb{F}^{\times}$ such that $[u_i, u_k] = \lambda_j u_j$ or $[u_k, u_i] = \lambda_j u_j$. Since $\mathcal{B}$ is of weak division we get $u_i \in \mathcal{I}(u_j)$, and by Lemma 4.1 there exist a direct path from u_j to u_i, so $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{V}})$ is weakly symmetric.

Now we prove that $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{I}})$ is weakly symmetric. Let us take an edge (e_i, e_j) of $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{I}})$. So there exist $u_k \in \mathcal{B}_{\mathfrak{V}}$ such that $[e_i, u_k] = \lambda_j e_j$, for certain $\lambda_j \in \mathbb{F}^{\times}$. Since $\mathcal{B}$ is of weak division we get $e_i \in \mathcal{I}(e_j)$. As in the previous case, by Lemma 4.1 we conclude $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{I}})$ is weakly symmetric.

Conversely, let us suppose that subgraph $\Gamma(\mathfrak{L}, \mathcal{B}_K)$ is weakly symmetric, for $K \in \{\mathfrak{I}, \mathfrak{V}\}$. We want to show that $\mathcal{B}$ is of weak division. Given $e_i, e_k \in \mathcal{B}_{\mathfrak{I}}, u_j \in \mathcal{B}_{\mathfrak{V}}$

such that $0 \neq [e_i, u_j] = \lambda e_k$. Then (e_i, e_k) is an edge of $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{I}})$. Since $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{I}})$ is weak symmetry, there exist directed path from e_k to e_i. Then, by Lemma 4.1 it follows that $e_i \in \mathcal{I}(e_k)$. In a similar way, for $u_i, u_k, u_p \in \mathcal{B}_{\mathfrak{V}}$ such that $0 \neq [u_i, u_k] = \lambda u_p$ we get (u_i, u_p) or (u_k, u_p) are edges of $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{V}})$. Analogously to the previous case, Lemma 4.1 let us assert that $u_i \in \mathcal{I}(u_p)$ or $u_k \in \mathcal{I}(u_p)$. In conclusion, $\mathcal{B}$ is of weak division as required. □

In order to present a simpler exposition of the work, we will introduce a definition of connection both in subgraph $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{I}})$ and subgraph $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{V}})$ inspired in the same concept in all graph $\Gamma(\mathfrak{L}, \mathcal{B})$.

Definition 4.5. *Set $K \in \{\mathfrak{I}, \mathfrak{V}\}$. Given $v_i, v_j \in \mathcal{B}_K$ we say $v_i \sim_K v_j$ if either $v_i = v_j$ or there exists an undirected path from v_i to v_j in subgraph $\Gamma(\mathfrak{L}, \mathcal{B}_K)$, meaning that $\{v_i, u_2, \ldots, u_{n-2}, v_j\}$, with $u_k \in \mathcal{B}_K$ for $k \in \{2, \ldots, n-2\}$.*

Proposition 4.2. *Let $\mathfrak{L}$ be a Leibniz algebra admitting a multiplicative basis $\mathcal{B}$. If $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{I}})$ has one connected component and is weakly symmetric, then any non-zero ideal I of $\mathfrak{L}$ that admits a multiplicative basis contained in $\mathcal{B}$ such that $I \subset \mathfrak{I}$ verifies $I = \mathfrak{I}$.*

Proof. It is enough to prove $\mathfrak{I} \subset I$. Since $I \subset \mathfrak{I}$, we begin taking certain $e_{i_0} \in \mathcal{B}_{\mathfrak{I}}$ such that $e_{i_0} \in I$. By hypothesis, $e_{i_0} \sim_{\mathfrak{I}} e_i$ for any $e_i \in \mathcal{B}_{\mathfrak{I}}$, so there exists a directed path from e_{i_0} to e_i in $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{I}})$. Applying Lemma 4.1 we have $e_i \in \mathcal{I}(e_{i_0}) \subset I$, therefore $\mathfrak{I} \subset I$. □

Proposition 4.3. *Let $\mathfrak{L}$ be a Leibniz algebra admitting a multiplicative basis $\mathcal{B}$. If the subgraph $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{V}})$ has one strongly connected component and is weakly symmetric, then any non-zero ideal I of $\mathfrak{L}$ that admits a multiplicative basis contained in $\mathcal{B}$ such that $I \not\subset \mathfrak{I}$ verifies $I = \mathfrak{L}$.*

Proof. Let us prove $\mathfrak{L} = \mathfrak{I} \oplus \mathfrak{V} \subset I$. Since $I \not\subset \mathfrak{I}$ there exists $u_{j_0} \in \mathcal{B}_{\mathfrak{V}}$ with $u_{j_0} \in I$. By hypothesis, $u_{j_0} \sim_{\mathfrak{V}} u_j$ for any $u_j \in \mathcal{B}_{\mathfrak{V}}$, so there exists a directed path from u_{j_0} to u_j in $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{V}})$. Therefore applying Lemma 4.1 we have $u_j \in \mathcal{I}(u_{j_0}) \subset I$, so consequently

$$\mathfrak{V} \subset I. \tag{5}$$

Finally, since $\mathfrak{I} \subset [\mathfrak{I}, \mathfrak{V}] + [\mathfrak{V}, \mathfrak{V}]$, by Equation (5) we get $\mathfrak{I} \subset I$, which proves $I = \mathfrak{L}$ as desired. □

Definition 4.6. A Leibniz algebra $\mathfrak{L}$ admitting a multiplicative basis $\mathcal{B}$ is *minimal* if its only non-zero ideals admitting a multiplicative basis contained in $\mathcal{B}$ are $\mathfrak{I}$ and $\mathfrak{L}$.

Theorem 4.1. *Let $\mathfrak{L}$ be a Leibniz algebra with multiplicative basis $\mathcal{B} = \mathcal{B}_{\mathfrak{I}} \dot{\cup} \mathcal{B}_{\mathfrak{V}}$. Then the following statements are equivalent:*

i. *$\mathfrak{L}$ is minimal.*
ii. *$\Gamma(\mathfrak{L}, \mathcal{B}_K)$ is weakly symmetric and connected, for $K \in \{\mathfrak{I}, \mathfrak{V}\}$.*
iii. *$\mathcal{B}$ is of weak division and $\Gamma(\mathfrak{L}, \mathcal{B}_K)$ is connected, for $K \in \{\mathfrak{I}, \mathfrak{V}\}$.*

Proof. The equivalence between ii. and iii. is shown by Proposition 4.1. Now, we prove ii. implies i. Let I be a non-zero ideal of $\mathfrak{L}$ admitting a multiplicative basis contained in $\mathcal{B}$. We have two possibilities, either $I \subset \mathfrak{I}$ or $I \not\subset \mathfrak{I}$. In the first case by Proposition 4.2 we infer that $I = \mathfrak{I}$, and in the second case by Proposition 4.3 we have $I = \mathfrak{L}$, consequently $\mathfrak{L}$ is minimal.

It remains to show that i. implies ii. Suppose that $\mathfrak{L}$ is minimal. First we take care of the case $K = \mathfrak{I}$. If $\mathfrak{I} \neq \{0\}$, for $e_i \in \mathcal{B}_{\mathfrak{I}}$ we have that the vector subspace $\oplus_{\substack{e_j \in \mathcal{B}_{\mathfrak{I}} \\ e_j \sim_{\mathfrak{I}} e_i}} \mathbb{F} e_j$ is an ideal of $\mathfrak{L}$. Indeed,

$$[\bigoplus_{\substack{e_j\in\mathcal{B}_{\mathfrak{I}}\\ e_j\sim_{\mathfrak{I}} e_i}} \mathbb{F}e_j, \mathfrak{V}] = \Big[\bigoplus_{\substack{e_j\in\mathcal{B}_{\mathfrak{I}}\\ e_j\sim_{\mathfrak{I}} e_i}} \mathbb{F}e_j, \bigoplus_{v_p\in\mathcal{B}_{\mathfrak{V}}} \mathbb{F}v_p\Big] = \bigoplus_{\substack{e_j\in\mathcal{B}_{\mathfrak{I}}\\ e_j\sim_{\mathfrak{I}} e_i\\ v_p\in\mathcal{B}_{\mathfrak{V}}}} \mathbb{F}[e_j, v_p].$$

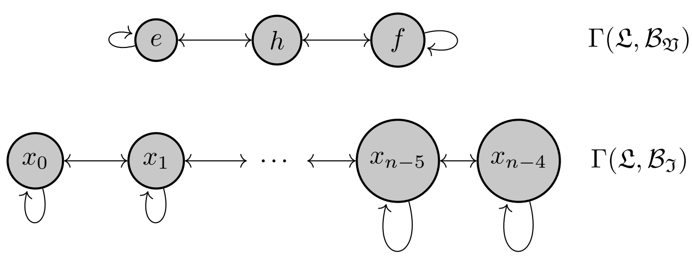
If $0 \neq [e_j, v_p] = \lambda e_q$ then (e_j, e_q) is an edge of $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{I}})$, so $e_q \sim_{\mathfrak{I}} e_j$. Therefore $e_q \sim_{\mathfrak{I}} e_i$, so $e_q \in \oplus_{\substack{e_j\in\mathcal{B}_{\mathfrak{I}}\\ e_j\sim_{\mathfrak{I}} e_i}} \mathbb{F}e_j$ and then $\oplus_{\substack{e_j\in\mathcal{B}_{\mathfrak{I}}\\ e_j\sim_{\mathfrak{I}} e_i}} \mathbb{F}e_j$ is an ideal of $\mathfrak{L}$. Since $\mathfrak{L}$ is minimal we have $\oplus_{\substack{e_j\in\mathcal{B}_{\mathfrak{I}}\\ e_j\sim_{\mathfrak{I}} e_i}} \mathbb{F}e_j = \mathfrak{I}$ and we conclude $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{I}})$ has all its vertices connected.

Let us prove that $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{I}})$ is weakly symmetric. We take an edge (e_i, e_j) in $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{I}})$. By minimality $\mathcal{I}(e_j) = \mathfrak{I}$, so $e_i \in \mathcal{I}(e_j)$. Then by Lemma 4.1 there exists a directed path from e_j to e_i in $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{I}})$, so it is guaranteed the weakly symmetry.

Now we perform the case $K = \mathfrak{V}$. We take $v_j \in \mathcal{B}_{\mathfrak{V}}$ and consider $\mathfrak{I} \oplus (\oplus_{\substack{v_k\in\mathcal{B}_{\mathfrak{V}}\\ v_k\sim_{\mathfrak{V}} v_j}} \mathbb{F}v_k)$ is an ideal of $\mathfrak{L}$. By minimality $\mathfrak{I} \oplus (\oplus_{\substack{v_k\in\mathcal{B}_{\mathfrak{V}}\\ v_k\sim_{\mathfrak{V}} v_j}} \mathbb{F}v_k) = \mathfrak{I} \oplus \mathfrak{V}$. Therefore $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{V}})$ has all its vertices connected.

Let us prove that $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{V}})$ is weakly symmetric. If (v_i, v_j) is an edge in $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{V}})$, by minimality $\mathcal{I}(v_j) = \mathfrak{L}$ and then $v_i \in \mathcal{I}(v_j)$. By Lemma 4.1 there exists a directed path from v_j to v_i, so $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{V}})$ is weakly symmetric, completing the proof. □

Example 4.3. *For the complex n-dimensional Leibniz algebra $\mathfrak{L}$ from Example 1.4 with basis $\mathcal{B} := \{e, f, h, x_0, x_1, \ldots, x_{n-4}\}$ the subgraphs $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{I}})$ and $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{V}})$ are:*



Since the subgraphs $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{I}})$ and $\Gamma(\mathfrak{L}, \mathcal{B}_{\mathfrak{V}})$ are weakly symmetric and connected, applying Theorem 4.1 we have that $\mathfrak{L}$ is minimal.

References

[1] Albeverio, S.; Omirov, B.A.; Rakhimov, I.S.; Varieties of nilpotent complex Leibniz algebras of dimension less than five. *Comm. Algebra* **33** (2005), 5, 1575-1585.

[2] Ayupov, Sh.A.; Omirov, B.A.; On 3-dimensional Leibniz algebras. *Uzbek Math. J.* **1** (1999), 9-14.

[3] Ayupov, Sh.A.; Omirov, B.A.; On some classes of nilpotent Leibniz algebras. *Sib. Math. J.* **42** (2001), 1, 18-29.

[4] Badawi, A; On the annihilator graph of a commutative ring. *Commun. Algebra* **42** (2014), no. 1, 108–121.

[5] Bloh. A.; On a generalization of the concept of Lie algebra. *Dokl. Akad. Nauk SSSR* **165** (1965), no. 3, 471473.

[6] Cabezas, J.M.; Camacho, L.M.; Rodrguez, I.M.; On filiform and 2-filiform Leibniz algebras of maximum length. *J. Lie Theory* **18** (2008), 2, 335-350.

[7] Calderón, A.J.; Leibniz algebras admitting a multiplicative basis. *Bull. Malays. Math. Sci. Soc.* **40** (2017), 679-695.

[8] Calderón, A.J.; Navarro, F.J.; Bilinear maps and graphs. *Discrete Appl. Math.* **263** (2019), 69–78.

[9] Calderón, A.J.; Navarro, F.J.; Arbitrary algebras with a multiplicative basis. *Linear Algebra Appl.* **498** (2016), 1, 106-116.

[10] Casas, J. M.; Corral, N.; On universal central extensions of Leibniz algebras. *Comm. Algebra* **37** (2009), 6, 2104-2120.
[11] Cohn, P.M.; Basic Algebra: Groups, Rings, and Fields. Springer, London (2003)
[12] Chih, T.; Plessas, D; Graphs and their associated inverse semigroups. *Discrete Math.* **340** (2017), no. 10, 2408–2414.
[13] Das, A.; Subspace inclusion graph of a vector space. *Commun. Algebra* **44** (2016), no. 11, 4724–4731.
[14] Das, A.; On non-zero component graph of vector spaces over finite fields. *J. Algebra Appl.* **16** (2017), no. 1, DOI: 10.1142/S0219498817500074
[15] Elduque, A., Kochetov, M.: Gradings on simple Lie algebras. Mathematical Surveys and Monographs, 189. American Mathematical Society, Providence, RI; Atlantic Association for Research in the Mathematical Sciences (AARMS), Halifax, NS, 2013. xiv+336 pp.
[16] Jiang, Q.F.; Classification of 3-dimensional Leibniz algebras. (Chinese) *J. Math. Res. Exposition* **27** (2007), 4, 677-686.
[17] Liu, D., Lin, L.; On the toroidal Leibniz algebras. *Acta Math. Sin. (Engl. Ser.)* **24** (2008), 2, 227-240.
[18] Loday, J.L.; Une version non commutative des algébres de Lie: les algébres de Leibniz. *L'Ens. Math.* **39** (1993), 269-293.
[19] Omirov, B.A.: Conjugacy of Cartan subalgebras of complex finite-dimensional Leibniz algebras. *J. Algebra* **302** (2006), 887-896.
[20] Omirov, B.A.; Rakhimov, I.S.; Turdibaev, R.M.; On description of Leibniz algebras corresponding to sl_2. *Algebr Represent Theor* **16** (2013), 15071519.

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