Geršgorin location of zeros for perturbed Chebyshev polynomials of second kind

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January 24, 2021

I dedicate this paper to the memory of Manuel Rogério de Jesus da Silva (1941-2018), Full Professor of my department, who taught me the foundations of Numerical Analysis and first told me about Geršrgorin circles theorems.

Abstract:
We consider some perturbed of the Chebyshev polynomials of second kind obtained by modifying one of its recurrence coefficients at an arbitrary order. By applying *Geršgorin circle theorems* to Jacobi matrices associated to perturbed Chebyshev polynomials of second kind, we obtain some locations of their zeros.

Key words: Perturbed Chebyshev polynomials; zeros; Jacobi matrices; Geršgorin circle theorems; *Mathematica*®.

2010 Mathematics Subject Classification: 33C45, 33D45, 42C05, 65F15, 65F50.

1 Introduction

It is well known that a polynomial sequence $\{P_n(x)\}_{n\geq 0}$ is *orthogonal* if and only if it satisfies a recurrence relation of order two (see (3)-(4)) defined by two sequences of coefficients $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_{n+1}\}_{n\geq 0}$, the so-called *recurrence coefficients*. If we modify, at the order r, the β-coefficient by translation, $\beta_r \to \beta_r+\mu_r$, or the γ-coefficient by dilation, $\gamma_r \to \lambda_r\gamma_r$, by means of some parameters μ_r or λ_r, we obtain a *perturbed orthogonal sequence*. We remark that μ_r and λ_r are not supposed to take small values, they are parameters that can be symbolic or numerical. In the last years, several authors have worked about perturbed orthogonal polynomials often taking as study case the perturbed Chebyshev sequence of second kind due to the simplicity of its recurrence coefficients which are constant if we consider monic polynomials (see (5)). With no attempt of completion, we cite [3, 4, 16, 17, 18, 20, 21]. Zeros of orthogonal polynomials, and in particular zeros of Chebyshev families, constitutes an important topic, because they are used in several methods in numerical analysis and have applications in applied sciences [11, 14, 19, 21]. With respect to properties of zeros of perturbed polynomials see in special [1, 2, 15, 20]. Furthermore, these sequences have some applications, in which their zeros play a role [10, 22]. I have also studied these perturbed sequences in the papers [5, 6, 7, 8], where the reader can find further references about this subject. The present article is one more contribution to the knowledge of the endless properties of these polynomials.

Is is well known that $[-1, 1]$ is the smallest interval that contains the set of zeros of the four Chebyshev sequences [19]. It is an interesting question to know how the above perturbations change the location of zeros. In this work, we furnish an answer to this question by applying the *Geršgorin circle theorems* [12, 13, 23] to the Jacobi matrices associated to perturbed Chebyshev polynomials of second kind [4, 19]. Then, we obtain some locations for their zeros, and also a refined location for one extremal zero in the translation case. Those results by *Geršgorin* [12, 13, 23] provide a simple way to determine certain circles or intervals whose union contains the eigenvalues of any $n \times n$ complex matrix. Jacobi matrices (8) are defined from the recurrence coefficients of an orthogonal polynomial sequence in such a way that $P_n(x)$ is the characteristic polynomial of J_n (9); in other words, the eigenvalues of J_n are the zeros of $P_n(x)$ [4].

This article is organized as follows. After this introduction, in Section 2, we define the perturbed Chebyshev polynomials of second kind by translation and by dilation, and we give some of their properties. In Section 3, we consider the Jacobi matrices associated to perturbed Chebyshev polynomials of second kind, and we compute their Geršgorin disks and sets. We begin by fixing the order r of perturbation, then for each r, we should consider separately some initial values of the degree n, thereafter we obtain results valid for all the upper values of n. In the next section, we apply the *Geršgorin circle theorems* and we obtain, in each case, a location for the set of zeros depending on the values of the parameters of perturbation μ_r or λ_r. Furthermore, in the translation case, doing a diagonal similarity transformation, we get a more precise location for one extremal zero that belong to a neighborhood of μ_r. Also, we present a definition for the sharpness of Geršgorin locations. In Section 5, in order to illustrate this work, we give some symbolic and numerical results, and we show some graphical representations. We finish this article with some conclusions about the locations obtained.

2 Perturbed Chebyshev polynomials of second kind

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its topological dual space. We denote by $\langle u, p\rangle$ the effect of the form $u \in \mathcal{P}'$ on the polynomial $p \in \mathcal{P}$. In particular, $\langle u, x^n\rangle := (u)_n \, , n \geq 0$, represent the *moments* of u. A form u is said *regular* [17] if and only if there exists a polynomial sequence $\{P_n\}_{n\geq 0}$, such that

$$\begin{aligned} \langle u, P_n P_m\rangle &= 0 \ , \ n \neq m \ , \ n, m \geq 0 \ , && (1)\\ \left\langle u, P_n^2\right\rangle = k_n &\neq 0 \ , \ n \geq 0 \ . && (2) \end{aligned}$$

Consequently $\deg P_n = n$, $n \geq 0$, and any P_n can be taken monic, then $\{P_n\}_{n\geq 0}$ is called a *monic orthogonal polynomial sequence* (MOPS) with respect to u. The sequence $\{P_n\}_{n\geq 0}$ is regularly orthogonal with respect to u [17] if and only if there are two sequences of coefficients $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_{n+1}\}_{n\geq 0}$, with $\gamma_{n+1} \neq 0$, $n \geq 0$, such that $\{P_n\}_{n\geq 0}$ satisfies the following initial conditions and linear recurrence relation of order 2

$$P_0(x) = 1, \quad P_1(x) = x - \beta_0, \tag{3}$$

$$P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad n \geq 2. \tag{4}$$

Furthermore, the recurrence coefficients $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_{n+1}\}_{n\geq 0}$ satisfy

$$\beta_n = \frac{\langle u, xP_n^2(x)\rangle}{k_n} \quad , \quad \gamma_{n+1} = \frac{k_{n+1}}{k_n}, \quad n \geq 0.$$

As usual, we suppose that, $\beta_n = 0$, $\gamma_{n+1} = 0$ and $P_n(x) = 0$, for all $n < 0$.

A form $u \in \mathcal{P}'$ is *positive definite* if and only if $< u, p >> 0$, $\forall p \in \mathcal{P}$: $p(x) \geq 0$, $p \not\equiv 0$, $x \in \mathbb{R}$. A form $u \in \mathcal{P}'$ is *symmetric* if and only if $(u)_{2n+1} = 0$, $n \geq 0$. A polynomial sequence $\{P_n\}_{n\geq 0}$ is symmetric if and only if $P_n(-x) = (-1)^n P_n(x)$, $n \geq 0$; then $P_{2n+1}(0) = 0$, and zeros of $P_n(x)$ are symmetric with respect to the origin. If $\{P_n\}_{n\geq 0}$ is a MOPS with respect to u, the symmetry of $\{P_n\}_{n\geq 0}$ is equivalent to $\beta_n = 0$, $n \geq 0$, and the positive definiteness of u is equivalent to β_n, $\gamma_{n+1} \in \mathbb{R}$, $\gamma_{n+1} > 0$, $n \geq 0$ [4]. If $\{P_n\}_{n\geq 0}$ is a symmetric MOPS, then $P_{2n}(0) \neq 0$, $n \geq 0$.

In this work, we study the monic Chebyshev sequence of second kind $\{P_n(x)\}_{n\geq 0}$ with the following recurrence coefficients

$$\beta_n = 0 \quad , \quad \gamma_{n+1} = \frac{1}{4} \ , \ n \geq 0 \ . \tag{5}$$

We consider the r-th perturbed Chebyshev sequence of second kind by translation, noted by $\{P_n^t(\mu_r; r)(x)\}_{n\geq 0}$, with parameter $\mu_r \in \mathbb{R}$, $\mu_r \neq 0$, and order $r \geq 0$, defined by the following recurrence coefficients

$$\beta_r^t = \mu_r \ , \ \beta_n^t = 0, \ n \neq r \ ; \ \gamma_{n+1}^t = \frac{1}{4} \ , \ n \geq 0 \ . \tag{6}$$

When $r = 0$, we recover the so-called *co–recursive* sequence [3, 4], for which only the initial polynomial $P_1(x)$ is perturbed becoming $P_1^t(\mu_0; 0)(x) = x - \beta_0 - \mu_0 = x - \mu_0$. Also, we consider the r-th perturbed Chebyshev sequence of second kind by dilation, noted by $\{P_n^d(\lambda_r; r)(x)\}_{n\geq 0}$, with parameter $\lambda_r \in \mathbb{R}$, $\lambda_r \neq 0$, $\lambda_r \neq 1$, and order $r \geq 1$, defined by the following recurrence coefficients

$$\beta_n^d = 0 \ ; \ \gamma_r^d = \frac{\lambda_r}{4} \ , \ \gamma_{n+1}^d = \frac{1}{4}, \ n \neq r-1 \ , \ n \geq 0 \ , \ r \geq 1 \ . \tag{7}$$

Then $\{P_n^t(\mu_r; r)(x)\}_{n\geq 0}$ is nonsymmetric and positive definite, and $\{P_n^d(\lambda_r; r)(x)\}_{n\geq 0}$ is symmetric; in addition, if $\lambda_r > 0$, then the last sequence is also positive definite. It is well known that in the positive definite case all zeros are distinct real numbers and satisfy some *interlacing* and *monotonicity* properties [4]. Perturbed Chebyshev polynomials by translation, and by dilation with $\lambda_r > 0$, satisfy these properties as well. In both cases, perturbed Chebyshev polynomials are semi-classical [18], which ensure that their zeros are simple [17, p.123]. We shall see that in the case of dilation, with $\lambda_r < 0$, there are some conjugate pairs of complex zeros. We pointed out in [6], that the monic Chebyshev sequences of first $\{T_n\}_{n\geq 0}$, third $\{V_n\}_{n\geq 0}$, and fourth $\{W_n\}_{n\geq 0}$ kinds [19] can be naturally considered as perturbed of the sequence of second kind $\{P_n\}_{n\geq 0}$ as follows: $T_n(x) = P_n^d\,(2;1)\,(x),\ V_n(x) = P_n^t\left(\frac{1}{2};0\right)(x),\ W_n(x) = P_n^t\left(-\frac{1}{2};0\right)(x)$.

3 Geršgorin disks and sets of Jacobi matrices

3.1 Geršgorin disks of Jacobi matrices associated to monic orthogonal polynomials

Let us recall some results by Geršgorin to locate the eigenvalues of any matrix $A_n = (a_{ij})_{1\leq i,j\leq n} \in \mathbb{C}^{n\times n}$ [12, 13, 23]. *Geršgorin disks* (G-disks) are defined by [1]

$$\mathcal{D}_i^{(n)} = \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq r_i^{(n)} \right\} \ ,\ r_i^{(n)} = \sum_{j=1,\ j\neq i}^{n} |a_{ij}| \ ,\ \ 1 \leq i \leq n \ .$$

Geršgorin set (G-set) is the union of G-disks, i.e.,

$$\mathcal{D}^{(n)} = \cup_{i=1}^{n} \mathcal{D}_i^{(n)} \ ,\ n \geq 1 \ .$$

Geršgorin circle theorem [12] states that the set of eigenvalues of A_n, the so-called *spectrum* of A_n, $\sigma(A_n)$, satisfies $\sigma(A_n) \subset \mathcal{D}^{(n)}$. Furthermore, if there are m G-disks disjoints from the remainders ones, then their union contains exactly m eigenvalues of A_n. In particular, if there is one G-disk disjoint from the others, then it contains an unique eigenvalue.

Let us consider the Jacobi matrix associated with $P_n(x)$ given by (3)-(4), taking α_n such that $\alpha_n^2 = \gamma_n$, $n \geq 1$,

$$J_n = \begin{pmatrix} \beta_0 & \alpha_1 & & & & \\ \alpha_1 & \beta_1 & \alpha_2 & & & \\ & \alpha_2 & \beta_2 & \alpha_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \alpha_{n-2} & \beta_{n-2} & \alpha_{n-1} \\ & & & & \alpha_{n-1} & \beta_{n-1} \end{pmatrix} . \tag{8}$$

It is well know that $P_n(x)$ is the characteristic polynomial of J_n [4], that is,

$$P_n(x) = \det(xI_n - J_n) \ ,\ n \geq 1 \ , \tag{9}$$

[1] these are the row G-disks; columns G-disks are obtained exchanging i with j.

where I_n denotes the identity matrix of order n. Thus zeros of $P_n(x)$ are eigenvalues of J_n, and we can apply the above cited results by Geršgorin to J_n in order to find a location for the zeros of $P_n(x)$. G-disks of J_n are

$$\begin{aligned}
&\mathcal{D}_1^{(1)} = \{\beta_0\} \ . \\
&\mathcal{D}_1^{(n)} = \{z \in \mathbb{C} : |z - \beta_0| \leq |\alpha_1|\} \ , \ n \geq 2 \ ; \\
&\mathcal{D}_i^{(n)} = \{z \in \mathbb{C} : |z - \beta_{i-1}| \leq |\alpha_{i-1}| + |\alpha_i|\} \ , \ 2 \leq i \leq n-1 \ , \ n \geq 2 \ ; \\
&\mathcal{D}_n^{(n)} = \{z \in \mathbb{C} : |z - \beta_{n-1}| \leq |\alpha_{n-1}|\} \ , \ n \geq 2 \ .
\end{aligned}$$

3.2 Geršgorin disks and sets of Jacobi matrices associated to Chebyshev polynomials of second kind

From (5), we see that the Jacobi matrix of the Chebyshev polynomial of second kind $P_n(x)$, for $n \geq 1$, is

$$\mathcal{J}_n = \begin{pmatrix} 0 & 1/2 & & & \\ 1/2 & 0 & 1/2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1/2 & 0 & 1/2 \\ & & & & 1/2 & 0 \end{pmatrix} .$$

The corresponding G-disks and G-sets are

$$\mathcal{D}_1^{(1)} = \mathcal{D}^{(1)} = \{0\} \ , \ n = 1 \ . \quad \mathcal{D}_1^{(2)} = \mathcal{D}_2^{(2)} = \mathcal{D}^{(2)} = [-\frac{1}{2}, \frac{1}{2}] \ , \ n = 2 \ . \tag{10}$$

$$\mathcal{D}_1^{(n)} = [-\frac{1}{2}, \frac{1}{2}] \ ; \ \mathcal{D}_i^{(n)} = [-1, 1] \ , \ 2 \leq i \leq n-1 \ ; \ \mathcal{D}_n^{(n)} = [-\frac{1}{2}, \frac{1}{2}] \ , \ n \geq 3 \ . \tag{11}$$

$$\mathcal{D}^{(n)} = [-1, 1] \ , \ n \geq 3 \ . \tag{12}$$

We consider that $\mathcal{J}_n$ is the matrix constituted by the first n rows and n columns of an infinite dimensional Jacobi matrix, noted $\mathcal{J}$, corresponding to the sequence $\{P_n\}_{n\geq 1}$.

3.3 Geršgorin disks and sets of Jacobi matrices associated to perturbed Chebyshev polynomials of second kind

For $n \geq 1$, let us denote the Jacobi matrices associated to $P_n^t(\mu_r; r)(x)$ and $P_n^d(\lambda_r; r)(x)$ by $J_n^t(\mu_r; r)$ and $J_n^d(\lambda_r; r)$, the corresponding G-disks by $\mathcal{D}_i^{t(n)}(\mu_r; r)$ and $\mathcal{D}_i^{d(n)}(\lambda_r; r)$, with radius $r_i^{t(n)}(\mu_r; r)$ and $r_i^{d(n)}(\lambda_r; r)$, for $1 \leq i \leq n$, and the G-sets by

$$\mathcal{D}^{t(n)}(\mu_r; r) = \cup_{i=1}^n \mathcal{D}_i^{t(n)}(\mu_r; r) \quad , \quad \mathcal{D}^{d(n)}(\lambda_r; r) = \cup_{i=1}^n \mathcal{D}_i^{d(n)}(\lambda_r; r) \ ,$$

respectively. From (5), (6) and (7), we obtain the following infinite dimensional Jacobi matrices

$$J^t(\mu_r;r) = \begin{pmatrix} 0 & \frac{1}{2} & & & & & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & & & & & \\ & \ddots & \ddots & \ddots & & & & & \\ & & \frac{1}{2} & \mu_r & \frac{1}{2} & & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & \frac{1}{2} & 0 & \frac{1}{2} & & \\ & & & & & \ddots & \ddots & \ddots & \end{pmatrix}, \begin{matrix} \\ \\ \\ \leftarrow \textit{row } r+1 \\ \\ \\ \\ \end{matrix} \tag{13}$$

$$J^d(\lambda_r;r) = \begin{pmatrix} 0 & \frac{1}{2} & & & & & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & & & & & \\ & \ddots & \ddots & \ddots & & & & & \\ & & \frac{1}{2} & 0 & \frac{\sqrt{\lambda_r}}{2} & & & & \\ & & & \frac{\sqrt{\lambda_r}}{2} & 0 & \frac{1}{2} & & & \\ & & & & \ddots & \ddots & \ddots & & \\ & & & & & \frac{1}{2} & 0 & \frac{1}{2} & \\ & & & & & & \ddots & \ddots & \ddots \end{pmatrix}, \begin{matrix} \\ \\ \\ \leftarrow \text{row } r \\ \leftarrow \text{row } r+1 \\ \\ \\ \\ \end{matrix} \tag{14}$$

such that $J_n^t(\mu_r;r)$ and $J_n^d(\lambda_r;r)$ are the matrices constituted by the first n rows and n columns of them, for $n \geq 1$. For $r \geq 1$ and $1 \leq n \leq r$, $J_n^t(\mu_r;r) \equiv \mathcal{J}_n$. For $r \geq 2$ and $1 \leq n \leq r-1$, $J_n^d(\lambda_r;r) \equiv \mathcal{J}_n$. If μ_r, $\lambda_r \in \mathbb{R}$ and $\lambda_r > 0$, then these symmetrical tridiagonal Jacobi matrices are real, it is well know that their eigenvalues are real and distinct [13], thus G-disks and G-sets are intervals of real numbers. If $\lambda_r < 0$, then $J^d(\lambda_r;r)$ has two imaginary complex entries equal to $\frac{\sqrt{\lambda_r}}{2} = \frac{\sqrt{|\lambda_r|}}{2}i$, it has conjugate pairs of complex eigenvalues and G-disks and G-sets are disks in the complex plane.

3.3.1 Translation case

Perturbation by translation modify only the row of order $r+1$ of $\mathcal{J}$. Consequently it modifies only the G-disk of order $i = r+1$ that now has center μ_r and whose radius depends if $n = r+1$ or if $n \geq r+2$, and also if $r = 0$ or if $r \geq 1$; it is given as follows

$$n = r+1 \Longrightarrow \left\{ r = 0 \Rightarrow r_1^{t(1)}(\mu_0;0) = 0 \;;\; r \geq 1 \Rightarrow r_{r+1}^{t(r+1)}(\mu_r;r) = \frac{1}{2} \right\},$$

$$n \geq r+2 \Longrightarrow \left\{ r = 0 \Rightarrow r_1^{t(n)}(\mu_0;0) = \frac{1}{2} \;;\; r \geq 1 \Rightarrow r_{r+1}^{t(n)}(\mu_r;r) = 1 \right\}.$$

Next, we shall explicit G-disks and G-sets for $J_n^t(\mu_r;r)$, $n \geq 1$, for different values of the order r of perturbation. For $r \geq 0$, the i-th G-disk of $J_n^t(\mu_r;r)$, for $i \neq r+1$, coincide with the one of $\mathcal{J}_n$, that is,

$$\mathcal{D}_i^{t(n)}(\mu_r;r)=\mathcal{D}_i^{(n)}\ ,\ i\neq r+1\ ,\ 1\leq i\leq n\ ,\ n\geq 1\ .$$

For $r=0$, the G-disk of $J_1^t(\mu_0;0)$ and the first G-disk of $J_n^t(\mu_0;0)$ are

$$\mathcal{D}_1^{t(1)}(\mu_0;0)=\{\mu_0\}\ ,\ n=1\ \ ;\ \mathcal{D}_1^{t(n)}(\mu_0;0)=[\mu_0-\frac{1}{2},\mu_0+\frac{1}{2}]\ ,\ n\geq 2\ .$$

For $r\geq 1$, the $(r+1)$-th G-disk of $J_n^t(\mu_r;r)$, for $n\geq r+1$, is

$$\mathcal{D}_{r+1}^{t(r+1)}(\mu_r;r)=[\mu_r-\frac{1}{2},\mu_r+\frac{1}{2}]\ ,\ n=r+1\ ;$$
$$\mathcal{D}_{r+1}^{t(n)}(\mu_r;r)=[\mu_r-1,\mu_r+1]\ ,\ n\geq r+2\ .$$

G-sets of $\{J_n^t(\mu_r;r)\}_{n\geq 1}$ are the followings.
For $r=0$,

$$\mathcal{D}^{t(1)}(\mu_0;0)=\{\mu_0\},\ n=1\ ;\ \mathcal{D}^{t(2)}(\mu_0;0)=[-\frac{1}{2},\frac{1}{2}]\cup[\mu_0-\frac{1}{2},\mu_0+\frac{1}{2}]\ ,\ n=2\ ;$$
$$\mathcal{D}^{t(n)}(\mu_0;0)=[-1,1]\cup[\mu_0-\frac{1}{2},\mu_0+\frac{1}{2}]\ ,\ n\geq 3\ .$$

For $r=1$,

$$\mathcal{D}^{t(1)}(\mu_1;1)=\{0\}\ ,\ n=1\ ;\ \mathcal{D}^{t(2)}(\mu_1;1)=[-\frac{1}{2},\frac{1}{2}]\cup[\mu_1-\frac{1}{2},\mu_1+\frac{1}{2}]\ ,\ n=2\ ;$$
$$\mathcal{D}^{t(3)}(\mu_1;1)=[-\frac{1}{2},\frac{1}{2}]\cup[\mu_1-1,\mu_1+1]\ ,\ n=3\ ;$$
$$\mathcal{D}^{t(n)}(\mu_1;1)=[-1,1]\cup[\mu_1-1,\mu_1+1]\ ,\ n\geq 4\ .$$

For $r=2$,

$$\mathcal{D}^{t(1)}(\mu_2;2)=\{0\}\ ,\ n=1\ ;\ \mathcal{D}^{t(2)}(\mu_2;2)=[-\frac{1}{2},\frac{1}{2}]\ ,\ n=2\ ;$$
$$\mathcal{D}^{t(3)}(\mu_2;2)=[-1,1]\cup[\mu_2-\frac{1}{2},\mu_2+\frac{1}{2}]\ ,\ n=3\ ;$$
$$\mathcal{D}^{t(n)}(\mu_2;2)=[-1,1]\cup[\mu_2-1,\mu_2+1]\ ,\ n\geq 4\ .$$

For $r\geq 3$,

$$\mathcal{D}^{t(1)}(\mu_r;r)=\{0\}\ ,\ n=1\ ;\ \mathcal{D}^{t(2)}(\mu_r;r)=[-\frac{1}{2},\frac{1}{2}]\ ,\ n=2\ ;$$
$$\mathcal{D}^{t(k)}(\mu_r;r)=[-1,1]\ ,\ n=k\ ,\ k=3(1)r\ ;$$
$$\mathcal{D}^{t(r+1)}(\mu_r;r)=[-1,1]\cup[\mu_r-\frac{1}{2},\mu_r+\frac{1}{2}]\ ,\ n=r+1\ ;$$
$$\mathcal{D}^{t(n)}(\mu_r;r)=[-1,1]\cup[\mu_r-1,\mu_r+1]\ ,\ n\geq r+2\ .$$

3.3.2 Dilation case

Perturbation by dilation modify only the rows of orders r and $r+1$ of $\mathcal{J}$. Consequently it modifies only the G-disks of orders $i = r$ and $i = r+1$ whose radius depend if $n = r+1$ or if $n \geq r+2$, and also if $r = 1$ or if $r \geq 2$; they are given as follows

$$n = r+1 \Longrightarrow \quad \begin{cases} r = 1 \Rightarrow \left\{ r_1^{d(2)}(\lambda_1; 1) = r_2^{d(2)}(\lambda_1; 1) = \dfrac{\sqrt{|\lambda_1|}}{2} \right\}, \\ r \geq 2 \Rightarrow \left\{ r_r^{d(r+1)}(\lambda_r; r) = \dfrac{1+\sqrt{|\lambda_r|}}{2} \ , \ r_{r+1}^{d(r+1)}(\lambda_r; r) = \dfrac{\sqrt{|\lambda_r|}}{2} \right\} \end{cases} ;$$

$$n \geq r+2 \Longrightarrow \quad \begin{cases} r = 1 \Rightarrow \left\{ r_1^{d(n)}(\lambda_1; 1) = \dfrac{\sqrt{|\lambda_1|}}{2} \ ; \ r_2^{d(n)}(\lambda_1; 1) = \dfrac{1+\sqrt{|\lambda_1|}}{2} \right\}, \\ r \geq 2 \Rightarrow \left\{ r_r^{d(n)}(\lambda_r; r) = r_{r+1}^{d(n)}(\lambda_r; r) = \dfrac{1+\sqrt{|\lambda_r|}}{2} \right\} \end{cases} .$$

Next, we shall explicit G-disks and G-sets for $J_n^d(\lambda_r; r)$, $n \geq 1$, for different values of the order r of perturbation. For $r \geq 1$, the i-th G-disk of $J_n^d(\lambda_r; r)$, for $i \neq r$ and $i \neq r+1$, coincide with the one of $\mathcal{J}_n$, that is,

$$\mathcal{D}_i^{d(n)}(\lambda_r; r) = \mathcal{D}_i^{(n)} \ , \ i \neq r \ , \ i \neq r+1 \ , \ 1 \leq i \leq n \ , \ n \geq 1 \ .$$

We begin by giving results for $\lambda_r > 0$, $\lambda_r \neq 1$.
For $r \geq 1$ and $n = 1$, the G-disk of $J_1^d(\lambda_r; r)$ is $\mathcal{D}_1^{d(1)}(\lambda_r; r) = \{0\}$.
For $r = 1$ and $n \geq 2$, the first G-disk of $J_n^d(\lambda_r; r)$ is

$$\mathcal{D}_1^{d(n)}(\lambda_1; 1) = [-\frac{\sqrt{\lambda_1}}{2}, \frac{\sqrt{\lambda_1}}{2}] \ .$$

For $r \geq 2$ and $n \geq r+1$, the r-th G-disk of $J_n^d(\lambda_r; r)$ is

$$\mathcal{D}_r^{d(n)}(\lambda_r; r) = [-\frac{1+\sqrt{\lambda_r}}{2}, \frac{1+\sqrt{\lambda_r}}{2}] \ .$$

For $r \geq 1$ and $n = r+1$, the $(r+1)$-th G-disk of $J_n^d(\lambda_r; r)$ is

$$\mathcal{D}_{r+1}^{d(r+1)}(\lambda_r; r) = [-\frac{\sqrt{\lambda_r}}{2}, \frac{\sqrt{\lambda_r}}{2}] \ .$$

For $r \geq 1$ and $n \geq r+2$, the $(r+1)$-th G-disk of $J_n^d(\lambda_r; r)$ is

$$\mathcal{D}_{r+1}^{d(n)}(\lambda_r; r) = [-\frac{1+\sqrt{\lambda_r}}{2}, \frac{1+\sqrt{\lambda_r}}{2}] \ .$$

For $\lambda_1 > 0$, $\lambda_1 \neq 1$, the G-sets of $\{J_n^d(\lambda_r; r)\}_{n \geq 1}$ are the followings.

For $r = 1$,

$$\mathcal{D}^{d(1)}(\lambda_1; 1) = \{0\} \ , \ n = 1 \ ; \ \mathcal{D}^{d(2)}(\lambda_1; 1) = [-\frac{\sqrt{\lambda_1}}{2}, \frac{\sqrt{\lambda_1}}{2}] \ , \ n = 2 \ ;$$
$$\mathcal{D}^{d(3)}(\lambda_1; 1) = [-\frac{1+\sqrt{\lambda_1}}{2}, \frac{1+\sqrt{\lambda_1}}{2}] \ , \ n = 3 \ ;$$
$$\mathcal{D}^{d(n)}(\lambda_1; 1) = [-1, 1] \cup [-\frac{1+\sqrt{\lambda_1}}{2}, \frac{1+\sqrt{\lambda_1}}{2}] \ , \ n \geq 4 \ .$$

For $r = 2$,

$$\mathcal{D}^{d(1)}(\lambda_2; 2) = \{0\} \ , \ n = 1 \ ; \ \mathcal{D}^{d(2)}(\lambda_2; 2) = [-\frac{1}{2}, \frac{1}{2}] \ , \ n = 2 \ ;$$
$$\mathcal{D}^{d(3)}(\lambda_2; 2) = \mathcal{D}^{d(4)}(\lambda_2; 2) = [-\frac{1+\sqrt{\lambda_2}}{2}, \frac{1+\sqrt{\lambda_2}}{2}] \ , \ n = 3 \ , \ n = 4 \ ;$$
$$\mathcal{D}^{d(n)}(\lambda_2; 2) = [-1, 1] \cup [-\frac{1+\sqrt{\lambda_2}}{2}, \frac{1+\sqrt{\lambda_2}}{2}] \ , \ n \geq 5 \ .$$

For $r \geq 3$,

$$\mathcal{D}^{d(1)}(\lambda_r; r) = \{0\} \ , \ n = 1 \ ; \ \mathcal{D}^{d(2)}(\lambda_r; r) = [-\frac{1}{2}, \frac{1}{2}] \ , \ n = 2 \ ;$$
$$\mathcal{D}^{d(k)}(\lambda_r; r) = [-1, 1] \ , \ n = k \ , \ k = 3(1)r \ ;$$
$$\mathcal{D}^{d(n)}(\lambda_r; r) = [-1, 1] \cup [-\frac{1+\sqrt{\lambda_r}}{2}, \frac{1+\sqrt{\lambda_r}}{2}] \ , \ n \geq r+1 \ .$$

G-disks and G-sets for $\lambda_r < 0$, $r \geq 1$, are easily obtained from the preceding ones. In each case of r, the intervals $\{0\}$, $[-\frac{1}{2}, \frac{1}{2}]$, $[-1, 1]$, $[-\frac{\sqrt{\lambda_r}}{r}, \frac{\sqrt{\lambda_r}}{r}]$ and $[-\frac{1+\sqrt{\lambda_r}}{r}, \frac{1+\sqrt{\lambda_r}}{r}]$ are replaced by the following disks in the complex plane centered at the origin, $\{(0,0)\}$, $\{z \in \mathbb{C} : |z| \leq \frac{1}{2}\}$, $\{z \in \mathbb{C} : |z| \leq 1\}$, $\left\{z \in \mathbb{C} : |z| \leq \frac{\sqrt{|\lambda_r|}}{2}\right\}$, and $\left\{z \in \mathbb{C} : |z| \leq \frac{1+\sqrt{|\lambda_r|}}{2}\right\}$, respectively.

4 Geršgorin location of zeros of perturbed Chebyshev polynomials of second kind

4.1 Translation case

Proposition 4.1 *For the rth-perturbed by translation case with $\mu_r \in \mathbb{R}$, $\mu_r \neq 0$, $r \geq 0$, we have*

- *For $r = 0$ and $n = 1$, $\mathcal{D}^{t(1)}(\mu_0; 0) = \{\mu_0\}$.*
- *For $r = 0$ or $r = 1$, and $n = 2$, the following holds.*

  - If $-1 \leq \mu_r < 0$, then $\mathcal{D}^{t(n)}(\mu_r; r) = [\mu_r - \frac{1}{2}, \frac{1}{2}]$ (it varies from $[-\frac{3}{2}, \frac{1}{2}]$ to $[-\frac{1}{2}, \frac{1}{2}]$).

- *If* $0 < \mu_r \leq 1$, *then* $\mathcal{D}^{t(n)}(\mu_r; r) = [-\frac{1}{2}, \frac{1}{2} + \mu_r]$ *(it varies from* $[-\frac{1}{2}, \frac{1}{2}]$ *to* $[-\frac{1}{2}, \frac{3}{2}]$*).*

- *If* $\mu_r < -1$ *or* $\mu_r > 1$, *then* $\mathcal{D}^{t(n)}(\mu_r; r) = [-\frac{1}{2}, \frac{1}{2}] \cup [\mu_r - \frac{1}{2}, \mu_r + \frac{1}{2}]$.

  As $[-\frac{1}{2}, \frac{1}{2}] \cap [\mu_r - \frac{1}{2}, \mu_r + \frac{1}{2}] = \emptyset$, *there is one zero of* $P_n^t(\mu_r; r)(x)$ *in* $[\mu_r - \frac{1}{2}, \mu_r + \frac{1}{2}]$, *the other one is in* $[-\frac{1}{2}, \frac{1}{2}]$.

- *For* $r = 0$ *and* $n \geq 3$, *or for* $r \geq 2$ *and* $n = r + 1$, *the following holds.*

  - *If* $-\frac{3}{2} \leq \mu_r \leq -\frac{1}{2}$, *then* $\mathcal{D}^{t(n)}(\mu_r; r) = [\mu_r - \frac{1}{2}, 1]$ *(it varies from* $[-2, 1]$ *to* $[-1, 1]$*).*
  - *If* $-\frac{1}{2} \leq \mu_r \leq \frac{1}{2}$, *then* $\mathcal{D}^{t(n)}(\mu_r; r) = [-1, 1]$.
  - *If* $\frac{1}{2} \leq \mu_r \leq \frac{3}{2}$, *then* $\mathcal{D}^{t(n)}(\mu_r; r) = [-1, \frac{1}{2} + \mu_r]$ *(it varies from* $[-1, 1]$ *to* $[-1, 2]$*).*
  - *If* $\mu_r < -\frac{3}{2}$ *or* $\mu_r > \frac{3}{2}$, *then* $\mathcal{D}^{t(n)}(\mu_r; r) = [-1, 1] \cup [\mu_r - \frac{1}{2}, \mu_r + \frac{1}{2}]$.

    As $[-1, 1] \cap [\mu_r - \frac{1}{2}, \mu_r + \frac{1}{2}] = \emptyset$, *there is one zero of* $P_n^t(\mu_r; r)(x)$ *in* $[\mu_r - \frac{1}{2}, \mu_r + \frac{1}{2}]$, *the other* $(n - 1)$ *ones are in* $[-1, 1]$.

- *For* $r = 1$ *and* $n = 3$, *the following holds.*

  - *If* $-\frac{3}{2} \leq \mu_r \leq -\frac{1}{2}$, *then* $\mathcal{D}^{t(n)}(\mu_r; r) = [\mu_r - 1, \frac{1}{2}]$ *(it varies from* $[-\frac{5}{2}, \frac{1}{2}]$ *to* $[-\frac{3}{2}, \frac{1}{2}]$*).*
  - *If* $-\frac{1}{2} \leq \mu_r \leq \frac{1}{2}$, *then* $\mathcal{D}^{t(n)}(\mu_r; 1) = [\mu_r - 1, \mu_r + 1]$ *(it varies from* $[-\frac{3}{2}, \frac{1}{2}]$ *to* $[-\frac{1}{2}, \frac{3}{2}]$*).*
  - *If* $\frac{1}{2} \leq \mu_r \leq \frac{3}{2}$, *then* $\mathcal{D}^{t(n)}(\mu_r; r) = [-\frac{1}{2}, 1 + \mu_r]$ *(it varies from* $[-\frac{1}{2}, \frac{3}{2}]$ *to* $[-\frac{1}{2}, \frac{5}{2}]$*).*
  - *If* $\mu_r < -\frac{3}{2}$ *or* $\mu_r > \frac{3}{2}$, *then* $\mathcal{D}^{t(n)}(\mu_r; r) = [-\frac{1}{2}, \frac{1}{2}] \cup [\mu_r - 1, \mu_r + 1]$.

    As $[-\frac{1}{2}, \frac{1}{2}] \cap [\mu_r - 1, \mu_r + 1] = \emptyset$, *there is one zero of* $P_n^t(\mu_r; r)(x)$ *in* $[\mu_r - 1, \mu_r + 1]$, *the other 2 ones are in* $[-\frac{1}{2}, \frac{1}{2}]$.

- *For* $r = 1$ *and* $n \geq 4$, *or for* $r \geq 2$ *and* $n \geq r + 2$, *the following holds.*

  - *If* $-2 \leq \mu_r < 0$, *then* $\mathcal{D}^{t(n)}(\mu_r; r) = [\mu_r - 1, 1]$ *(it varies from* $[-3, 1]$ *to* $[-1, 1]$*).*
  - *If* $0 < \mu_r \leq 2$, *then* $\mathcal{D}^{t(n)}(\mu_r; r) = [-1, 1 + \mu_r]$ *(it varies from* $[-1, 1]$ *to* $[-1, 3]$*).*
  - *If* $\mu_r < -2$ *or* $\mu_r > 2$, *then* $\mathcal{D}^{t(n)}(\mu_r; r) = [-1, 1] \cup [\mu_r - 1, \mu_r + 1]$.

    As $[-1, 1] \cap [\mu_r - 1, \mu_r + 1] = \emptyset$, *there is one zero of* $P_n^t(\mu_r; r)(x)$ *in* $[\mu_r - 1, \mu_r + 1]$, *the other* $(n - 1)$ *ones are in* $[-1, 1]$.

In the last proposition, in the cases there is one extremal zero isolated in a G-interval centered at the parameter of perturbation μ_r, it is possible to improve its location, reducing the radius of that interval, as stated by next result.

Let $P = P^t(r, p)$ be an infinite diagonal matrix whose diagonal entries are $P_{r+1,r+1} = p$, $p \neq 0$, and $P_{ii} = 1$, $i \neq r + 1$. We consider the following similarity transformation that does not change the eigenvalues of $J^t(\mu_r;r)$

$$P^t(\mu_r;r;p) = P^{-1}.J^t(\mu_r;r).P = \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & \frac{1}{2} & 0 & \frac{p}{2} & & & \\ & & \frac{1}{2p} & \mu_r & \frac{1}{2p} & & \\ & & & \frac{p}{2} & 0 & \frac{1}{2} & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix} \leftarrow \text{row } r+1. \tag{15}$$

We shall note by $P^{t(n)}(\mu_r;r;p)$ the matrix constituted by the first n rows and n columns of $P^t(\mu_r;r;p)$ and its G-set by $\mathcal{P}^{t(n)}(\mu_r;r;p)$.

Proposition 4.2 *For the rth-perturbed by translation case with $\mu_r \in \mathbb{R}$, $\mu_r \neq 0$, $r \geq 0$, we have*

- *For $r = 0$ or $r = 1$, and $n = 2$, the following holds.*

  $\mathcal{P}^{t(n)}(\mu_r;r;p) = G_1 \cup G_2$, $G_1 = \left[-\frac{p}{2}, \frac{p}{2}\right]$, $G_2 = \left[\mu_r - \frac{1}{2p}, \mu_r + \frac{1}{2p}\right]$.

  Let $B(\mu_r) = -\mu_r + s$, and $D(\mu_r) = \mu_r + s$, with $s = \sqrt{\mu_r^2 - 1}$.

  If $\mu_r < -1$ and $p \in [1, B(\mu_r)[$, then $\left|\xi_1^{t(n)} - \mu_r\right| < \dfrac{1}{2B(\mu_r)} = \mathcal{O}\left(\dfrac{1}{4|\mu_r|}\right)$.

  If $\mu_r > 1$ and $p \in [1, D(\mu_r)[$, then $\left|\xi_n^{t(n)} - \mu_r\right| < \dfrac{1}{2D(\mu_r)} = \mathcal{O}\left(\dfrac{1}{4|\mu_r|}\right)$.

- *For $r = 0$ and $n \geq 3$, or for $r \geq 2$ and $n = r + 1$, the following holds.*

  $\mathcal{P}^{t(n)}(\mu_r;r;p) = G_1 \cup G_2$, $G_1 = \left[-\frac{p+1}{2}, \frac{p+1}{2}\right]$, $G_2 = \left[\mu_r - \frac{1}{2p}, \mu_r + \frac{1}{2p}\right]$.

  Let $B(\mu_r) = -s_1 + \sqrt{s_1^2 - 4}$ with $s_1 = 1 + 2\mu_r$.

  Let $D(\mu_r) = -s_3 + \sqrt{s_3^2 - 4}$ with $s_3 = 1 - 2\mu_r$.

  If $\mu_r < -\frac{3}{2}$, and $p \in [1, B(\mu_r)[$, then $\left|\xi_1^{t(n)} - \mu_r\right| < \dfrac{1}{B(\mu_r)} = \mathcal{O}\left(\dfrac{1}{4|\mu_r|}\right)$.

  If $\mu_r > \frac{3}{2}$ and $p \in [1, D(\mu_r)[$, then $\left|\xi_n^{t(n)} - \mu_r\right| < \dfrac{1}{D(\mu_r)} = \mathcal{O}\left(\dfrac{1}{4|\mu_r|}\right)$.

- *For $r = 1$ and $n = 3$, the following holds.*

  $\mathcal{P}^{t(n)}(\mu_r;r;p) = G_1 \cup G_2$, $G_1 = \left[-\frac{p}{2}, \frac{p}{2}\right]$, $G_2 = \left[\mu_r - \frac{1}{p}, \mu_r + \frac{1}{p}\right]$.

  Let $B(\mu_r) = -\mu_r + s$ and $D(\mu_r) = \mu_r + s$, with $s = \sqrt{\mu_r^2 - 2}$.

If $\mu_r < -\frac{3}{2}$ *and* $p \in [1, B(\mu_r)[$*, then* $\left|\xi_1^{t(n)} - \mu_r\right| < \dfrac{1}{B(\mu_r)} = \mathcal{O}\left(\dfrac{1}{2|\mu_r|}\right)$.

If $\mu_r > \frac{3}{2}$ *and* $p \in [1, D(\mu_r)[$*, then* $\left|\xi_n^{t(n)} - \mu_r\right| < \dfrac{1}{D(\mu_r)} = \mathcal{O}\left(\dfrac{1}{2|\mu_r|}\right)$.

- *For* $r = 1$ *and* $n \geq 4$*, or for* $r \geq 2$ *and* $n \geq r+2$*, the following holds.*

  $\mathcal{P}^{t(n)}(\mu_r; r; p) = G_1 \cup G_2\ ,\ G_1 = \left[-\dfrac{p+1}{2}, \dfrac{p+1}{2}\right]\ ,\ G_2 = \left[\mu_r - \dfrac{1}{p}, \mu_r + \dfrac{1}{p}\right].$

  Let $B(\mu_r) = \frac{1}{2}\left(-s_1 + \sqrt{s_1^2 - 8}\right)$*, with* $s_1 = 1 + 2\mu_r$.

  Let $D(\mu_r) = \frac{1}{2}\left(-s_3 + \sqrt{s_3^2 - 8}\right)$*, with* $s_3 = 1 - 2\mu_r$.

  If $\mu_r < -2$*, then* $\left|\xi_1^{t(n)} - \mu_r\right| < \dfrac{1}{B(\mu_r)} = \mathcal{O}\left(\dfrac{1}{2|\mu_r|}\right)$.

  If $\mu_r > 2$*, then* $\left|\xi_n^{t(n)} - \mu_r\right| < \dfrac{1}{D(\mu_r)} = \mathcal{O}\left(\dfrac{1}{2|\mu_r|}\right)$.

In all cases, $G_1 \cap G_2 = \emptyset$*,* G_2 *contains a unique (one extremal) zero and the others zeros belong to* G_1*.*

Proof. In all cases, we take $p > 1$, in order to assure that the radius of G_2 will be smaller than the one provided by Proposition 4.1. Let us demonstrate the first case, other ones are similar. From (15), we obtain the G-intervals: $G_1 = \left[-\frac{p}{2}, \frac{p}{2}\right]$ and $G_2 = \left[\mu_r - \frac{1}{2p}, \mu_r + \frac{1}{2p}\right]$. If $\mu_r < -1$, in order to $G_1 \cap G_2 = \emptyset$, we must have $\mu_r + \dfrac{1}{2p} < -\dfrac{p}{2} \Leftrightarrow p^2 + 2\mu_r p + 1 < 0 \Leftrightarrow p \in]A(\mu_r), B(\mu_r)[$, where $A(\mu_r) = -\mu_r - s = 1/B(\mu_r)$, $B(\mu_r) = -\mu_r + s$, and $s = \sqrt{\mu_r^2 - 1}$. It is easy to see that $B(\mu_r) > 1$, thus $A(\mu_r) < 1$, then we must take $p \in [1, B(\mu_r)[$. Now applying the *Geršgorin circle theorem*, we conclude that G_2 contains a unique zero, the extremal zero $\xi_1^{t(n)}$, that satisfies $\left|\xi_1^{t(n)} - \mu_r\right| \leq \dfrac{1}{2p}$. Taking p as big as possible, we get the best location of $\xi_1^{t(n)}$, obtaining $\left|\xi_1^{t(n)} - \mu_r\right| < \dfrac{1}{2B(\mu_r)} = \mathcal{O}\left(\dfrac{1}{4|\mu_r|}\right)$. The other zeros belong to G_1; in fact by Proposition 4.1, they belong to $[-\frac{1}{2}, \frac{1}{2}] \subset G_1$. ∎

In all cases of the last proposition, we observe that one extremal zero is located in a neighborhood of the parameter μ_r of perturbation. We remark that μ_r, for $r \geq 1$, is not a zero of $P_n^t(\mu_r, r)(x)$, $n \geq 0$. For $r = 0$, μ_0 is only zero of $P_1^t(\mu_0, 0)(x) = x - \mu_0$.

Note that the matrix $P^t(\mu_r; r; p)$ is not symmetric, consequently row and column G-sets are different. Nevertheless, if we consider column G-sets with $0 < p < 1$, we obtain exactly the same conclusions.

Definition 4.3 *For the rth-perturbed by translation case with $\mu_r \in \mathbb{R}$, $\mu_r \neq 0$, $r \geq 0$,* a measure of the sharpness $\mathcal{S}^{t(n)}(\mu_r; r)$ *of the Geršgorin location is defined as follows, where* $\{\xi_i^{t(n)}(\mu_r; r)\}_{i=1}^n$ *notes the zeros of* $P_n^t(\mu_r; r)(x)$, $n \geq 1$.

- **Case 1**

  If the G-set is a unique interval $\mathcal{D}^{t(n)}(\mu_r; r) = [a, b]$ *given by Proposition 4.1, then*

$$\mathcal{S}^{t(n)}(\mu_r; r) = \left(b - a - \left(\xi_n^{t(n)}(\mu_r; r) - \xi_1^{t(n)}(\mu_r; r)\right)\right) /2 \ .$$

  Observe that $[\xi_1^{t(n)}(\mu_r; r), \xi_n^{t(n)}(\mu_r; r)]$ *is the smallest interval that contains all zeros.*

  In this case, we say will that the Geršgorin location is sharp if and only if

$$\lim_{n\to+\infty} \mathcal{S}^{t(n)}(\mu_r; r) = 0.$$

- **Case 2**

  If the G-set is the union of two disjoint intervals

$$\mathcal{D}^{t(n)}(\mu_r; r) = [-b, b] \cup [\mu_r - \epsilon, \mu_r + \epsilon] \ ,$$

  with b given by Proposition 4.1, and ϵ given by Proposition 4.2, then

$$\mathcal{S}_{int}^{t(n)}(\mu_r; r) = b - b' \quad , \quad \mathcal{S}_{out}^{t(n)}(\mu_r; r) = \epsilon - \epsilon_o \ ,$$

  where

$$b' = \max\left\{\left|\xi_{k_i}^{t(n)}(\mu_r; r)\right|, \left|\xi_{k_f}^{t(n)}(\mu_r; r)\right|\right\} \ , \ \epsilon_o = \left|\mu_r - \xi_{k_o}^{t(n)}(\mu_r; r)\right| ,$$

  with $k_o = 1$, $k_i = 2$ and $k_f = n$, if the zero outside of $[-b, b]$ is $\xi_1^{t(n)}(\mu_r; r)$; or $k_i = 1$, $k_f = n - 1$ and $k_o = n$, if the zero outside of $[-b, b]$ is $\xi_n^{t(n)}(\mu_r; r)$.

  Observe that $[-b', b']$ is the smallest interval centered at the origin that contains all zeros belonging to $[-b, b]$, and $[\mu_r - \epsilon_o, \mu_r + \epsilon_o]$ is the smallest interval centered at μ_r that contains the extremal zero outside of $[-b, b]$.

  In this case, we will say that the Geršgorin location is sharp in $[-b, b]$ and in $[\mu_r - \epsilon, \mu_r + \epsilon]$ if and only if

$$\lim_{n\to+\infty} \mathcal{S}_{int}^{t(n)}(\mu_r; r) = 0 \text{ and } \lim_{n\to+\infty} \mathcal{S}_{out}^{t(n)}(\mu_r; r) = 0 \ .$$

Example 4.4 *As particular cases of the Proposition 4.1, we recover the well known location of zeros of the Chebyshev families of third and fourth kinds [19].*

$$V_1(x) \ , \ V_2(x) \ : \quad \mathcal{D}^{t(1)}\left(\frac{1}{2}; 0\right) = \left\{\frac{1}{2}\right\} \ , \ n = 1 \ ; \ \mathcal{D}^{t(2)}\left(\frac{1}{2}; 0\right) = [-\frac{1}{2}, 1] \ , \ n = 2 \ ;$$

$$V_n(x) \ , \ n \geq 3 \ : \quad \mathcal{D}^{t(n)}\left(\frac{1}{2}; 0\right) = [-1, 1] \ , \ n \geq 3 \ .$$

$$W_1(x) \ , \ W_2(x) \ : \quad \mathcal{D}^{t(1)}\left(-\frac{1}{2}; 0\right) = \left\{-\frac{1}{2}\right\}, n = 1 \ ; \ \mathcal{D}^{t(2)}\left(-\frac{1}{2}; 0\right) = [-1, \frac{1}{2}], n = 2 \ ;$$

$$W_n(x) \ , \ n \geq 3 \ : \quad \mathcal{D}^{t(n)}\left(-\frac{1}{2}; 0\right) = [-1, 1] \ , \ n \geq 3 \ .$$

In both cases, the sharpness satisfies

$$\lim_{n\to+\infty} \mathcal{S}^{t(n)}\left(\frac{1}{2};0\right) = 0 = \lim_{n\to+\infty} \mathcal{S}^{t(n)}\left(-\frac{1}{2};0\right),$$

because (see [4, 19])

$$\lim_{n\to+\infty} \xi_1^{t(n)}\left(\frac{1}{2};0\right) = -1 = \lim_{n\to+\infty} \xi_1^{t(n)}\left(-\frac{1}{2};0\right),$$

$$\lim_{n\to+\infty} \xi_n^{t(n)}\left(\frac{1}{2};0\right) = 1 = \lim_{n\to+\infty} \xi_n^{t(n)}\left(-\frac{1}{2};0\right).$$

Then, we conclude that the Gešrgorin location is sharp.

Proposition 4.5 *In the case 2 of the Definition 4.3, the Geřgorin location is sharp in* $[-b,b]=[-1,1]$.

Proof. In [9], we state the following interlacing property between the zeros of perturbed Chebyshev polynomials of second kind by translation $\{\xi_i^{t(n+r+1)}(\mu_r;r)\}_{i=1}^{n+r+1}$ and the zeros $\{\xi_j^{(n+r+1)}\}_{j=1}^{n+r+1}$ of Chebyshev polynomials of second kind:

- If $\mu_r > 0$, then

$$\xi_j^{(n+r+1)} \le \xi_j^{t(n+r+1)} \le \xi_{j+1}^{(n+r+1)} \ ,\ j=1(1)n+r\ ;\ \xi_{n+r+1}^{t(n+r+1)} > \xi_{n+r+1}^{(n+r+1)}.$$

- If $\mu_r < 0$, then

$$\xi_1^{t(n+r+1)} < \xi_1^{(n+r+1)}\ ;\ \xi_j^{(n+r+1)} \le \xi_{j+1}^{t(n+r+1)} \le \xi_{j+1}^{(n+r+1)},\ j=1(1)n+r\ .$$

Now, observe that $\lim_{n\to+\infty} \xi_1^{(n+r+1)} = -1$, and $\lim_{n\to+\infty} \xi_{n+r+1}^{(n+r+1)} = 1$.

■

4.2 Dilation case

Proposition 4.6 *For the* r*th-perturbed by dilation case with* $\lambda_r \in \mathbb{R}$, $\lambda_r \neq 0$, $\lambda_r \neq 1$, $r \ge 1$, *we have*

- *For* $r=1$, *and* $n=2$ *or* $n=3$, *the following holds.*

  - *If* $\lambda_1 < -1$, *then*

$$\mathcal{D}^{d(2)}(\lambda_1;1) \supset \{z\in\mathbb{C} : |z| \le \frac{1}{2}\}\ ,\ \mathcal{D}^{d(3)}(\lambda_1;1) \supset \{z\in\mathbb{C} : |z| \le 1\}\ .$$

  - *If* $-1 \le \lambda_1 < 0$, *then*

$$\mathcal{D}^{d(2)}(\lambda_1;1) \subset \{z\in\mathbb{C} : |z| \le \frac{1}{2}\}\ ,\ \mathcal{D}^{d(3)}(\lambda_1;1) \subset \{z\in\mathbb{C} : |z| \le 1\}\ .$$

  - *If* $0 < \lambda_1 < 1$, *then* $\mathcal{D}^{d(2)}(\lambda_1;1) \subset [-\frac{1}{2},\frac{1}{2}]$, $\mathcal{D}^{d(3)}(\lambda_1;1) \subset [-1,1]$.

  - *If* $\lambda_1 > 1$, *then* $\mathcal{D}^{d(2)}(\lambda_1;1) \supset [-\frac{1}{2},\frac{1}{2}]$, $\mathcal{D}^{d(3)}(\lambda_1;1) \supset [-1,1]$.

- *For* $r=2$*, and* $n=3$ *or* $n=4$*, the following holds.*

  - *If* $\lambda_2 < -1$*, then* $\mathcal{D}^{d(3)}(\lambda_2;2) = \mathcal{D}^{d(4)}(\lambda_2;2) \supset \{z \in \mathbb{C} : |z| \le 1\}$.

  - *If* $-1 \le \lambda_2 < 0$*, then* $\mathcal{D}^{d(3)}(\lambda_2;2) = \mathcal{D}^{d(4)}(\lambda_2;2) \subset \{z \in \mathbb{C} : |z| \le 1\}$.

  - *If* $0 < \lambda_2 < 1$*, then* $\mathcal{D}^{d(3)}(\lambda_2;2) = \mathcal{D}^{d(4)}(\lambda_2;2) \subset [-1,1]$.

  - *If* $\lambda_2 > 1$*, then* $\mathcal{D}^{d(3)}(\lambda_2;2) = \mathcal{D}^{d(4)}(\lambda_2;2) \supset [-1,1]$.

- *For* $r=1$ *and* $n \ge 4$*, or for* $r=2$ *and* $n \ge 5$*, or for and* $r \ge 3$ *and* $n \ge r+1$*, the following holds.*

  - *If* $\lambda_r < -1$*, then* $\mathcal{D}^{d(n)}(\lambda_r;r) = \left\{ z \in \mathbb{C} : |z| \le \frac{1+\sqrt{|\lambda_r|}}{2} \right\} \supset \{z \in \mathbb{C} : |z| \le 1\}$.

  - *If* $-1 \le \lambda_r < 0$*, then* $\mathcal{D}^{d(n)}(\lambda_r;r) = \{z \in \mathbb{C} : |z| \le 1\}$.

  - *If* $0 < \lambda_r < 1$*, then* $\mathcal{D}^{d(n)}(\lambda_r;r) = [-1,1]$.

  - *If* $\lambda_r > 1$*, then* $\mathcal{D}^{d(n)}(\lambda_r;r) = [-\frac{1+\sqrt{\lambda_r}}{2}, \frac{1+\sqrt{\lambda_r}}{2}] \supset [-1,1]$.

Definition 4.7 *For the* r*th-perturbed by dilation case with* $\lambda_r \in \mathbb{R}$*,* $\lambda_r \ne 0$*,* $\lambda_r \ne 1$*,* $r \ge 1$*, a measure of the sharpness of the Geršgorin location is defined by*

$$\mathcal{S}^{d(n)}(\lambda_r;r) = r^{d(n)}(\lambda_r;r) - \max\left\{ \left|\xi_i^{d(n)}(\lambda_r;r)\right| : i = 1(1)n \right\},$$

where $\{\xi_i^{d(n)}(\lambda_r;r)\}_{i=1}^n$ *notes the zeros of* $P_n^d(\lambda_r;r)(x)$*,* $n \ge 1$*. We will say that the Geršgorin location is sharp if and only if*

$$\lim_{n\to+\infty} \mathcal{S}^{d(n)}(\lambda_r;r) = 0 \ .$$

Observe that, in this case, the sharpness measures the difference between the radius of the G-set and of the smallest circle centered at the origin that contains all zeros.

Example 4.8 *As a particular case of the Proposition 4.6, we recover the well known location of zeros for the Chebyshev family of first kind*

$$T_1(x)\ ,\ T_2(x)\ :\qquad \mathcal{D}^{d(1)}(2;1) = \{0\}\ ,\ n=1\ ;\ \mathcal{D}^{d(2)}(2;1) = [-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]\ ,\ n=2\ ;$$
$$T_n(x)\ ,\ n \ge 3\ :\qquad \mathcal{D}^{d(n)}(2;1) = [-\frac{1+\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2}] \supset [-1,1]\ ,\ n \ge 3\ .$$

The sharpness satisfies

$$\lim_{n\to+\infty} \mathcal{S}^{d(n)}(2;1) = \frac{\sqrt{2}-1}{2} \approx 0.21\ ,$$

because (see [4, 19])

$$r^{d(n)}(2;1) = \frac{1+\sqrt{2}}{2},\ n \geq 3\ ;\ \lim_{n\to+\infty} \max\left\{\left|\xi_i^{d(n)}(2;1)\right| : i = 1(1)n\right\} = 1.$$

Then, we conclude that the Geřgorin location is not sharp.

5 Symbolic, numerical, and graphical results

In this section, we give some symbolic, numerical, and graphical results, obtained with the software *Mathematica*®, in order to exemplify and illustrate the main results of this work. To get concrete results, we should fix the order r of perturbation and the degree n. Taking, for example, $r = 5$ and $n = 17$ in the translation case; and $r = 6$ and $n = 18$ in the dilation case, and using the recurrence relation (3)-(4) with the recurrence coefficients (6) and (7), we compute recursively the polynomials $P^t_{17}(5;\mu_5)(x)$ and $P^t_{18}(6;\lambda_6)(x)$ presented next, with symbolic parameters of perturbation μ_5 and λ_6. We have used the command *Expand* to get these polynomials in the canonical basis.

$$\begin{aligned} P^t_{17}(5;\mu_5)(x) &= x^{17} - \mu_5 x^{16} - 4x^{15} + \frac{7\mu_5}{2}x^{14} + \frac{105}{16}x^{13} - \frac{79\mu_5}{16}x^{12} - \frac{91}{16}x^{11} \\ &+ \frac{115\mu_5}{32}x^{10} + \frac{715}{256}x^9 - \frac{367\mu_5}{256}x^8 - \frac{99}{128}x^7 + \frac{157\mu_5}{512}x^6 + \frac{231}{2048}x^5 \\ &- \frac{129\mu_5}{4096}x^4 - \frac{15}{2048}x^3 + \frac{9\mu_5}{8192}x^2 + \frac{9}{65536}x\ . \end{aligned} \tag{16}$$

$$\begin{aligned} P^d_{18}(6;\lambda_6)(x) &= x^{18} - \frac{1}{4}(\lambda_6+16)x^{16} + \frac{1}{8}(7\lambda_6+53)x^{14} - \frac{1}{64}(79\lambda_6+376)x^{12} \\ &+ \frac{1}{256}(230\lambda_6+771)x^{10} - \frac{(367\lambda_6+920)}{1024}x^8 + \frac{(157\lambda_6+305)}{2048}x^6 \\ &- \frac{3(43\lambda_6+67)}{16384}x^4 + \frac{9(2\lambda_6+3)}{65536}x^2 - \frac{1}{262144}\ . \end{aligned}$$

From (16), it is easy to realise that

$$P^t_{17}(5;-\mu_5)(-x) = -P^t_{17}(5;\mu_5)(x)\ .$$

In fact, $P^t_{2n+1}(r;-\mu_r)(-x) = -P^t_{2n+1}(r;\mu_r)(x)$, $r \geq 0$, $n \geq 0$, (see [7, Prop. 4.4]). This implies that,

$$P^t_{2n+1}(r;\mu_r)(\xi) = 0 \iff P^t_{2n+1}(r;-\mu_r)(-\xi) = 0\ , r \geq 0,\ n \geq 0\ . \tag{17}$$

This result will be cited in Figure 1 above for interpreting some numerical values of zeros.

Fixing the parameters of perturbation, taking, for example, $\mu_5 = 1$ and $\lambda_6 = -2$, we obtain polynomials that can be traced and study numerically.

$$\begin{aligned} P^t_{17}(5;1)(x) &= x^{17} - x^{16} - 4x^{15} + \frac{7x^{14}}{2} + \frac{105x^{13}}{16} - \frac{79x^{12}}{16} - \frac{91x^{11}}{16} + \frac{115x^{10}}{32} \\ &+ \frac{715x^9}{256} - \frac{367x^8}{256} - \frac{99x^7}{128} + \frac{157x^6}{512} + \frac{231x^5}{2048} - \frac{129x^4}{4096} - \frac{15x^3}{2048} \\ &+ \frac{9x^2}{8192} + \frac{9x}{65536} \,. \end{aligned}$$

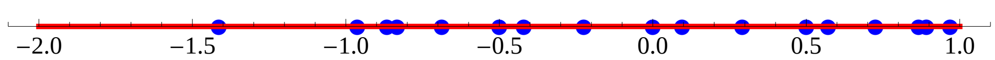
$$\begin{aligned} P^d_{18}(6;-2)(x) &= x^{18} - \frac{7x^{16}}{2} + \frac{39x^{14}}{8} - \frac{109x^{12}}{32} + \frac{311x^{10}}{256} - \frac{93x^8}{512} - \frac{9x^6}{2048} + \frac{57x^4}{16384} \\ &- \frac{9x^2}{65536} - \frac{1}{262144} \,. \end{aligned}$$

We compute their zeros with the command *NSolve*. They are

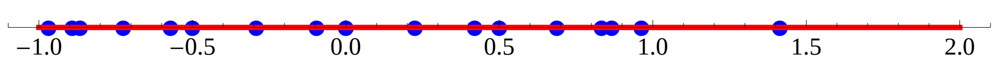
$$\begin{aligned} \{\xi^{t(17)}_k(5;1)\}_{k=1}^{17} &= \{-0.968526, -0.891216, -0.866025, -0.725445, -0.571048, -0.5, \\ &-0.291687, -0.0951584, 0., 0.225188, 0.420171, 0.5, 0.687401, \\ &0.833122, 0.866025, 0.963003, 1.4142\}\,. \\ \{\xi^{d(18)}_k(6;-2)\}_{k=1}^{18} &= \{-0.967703, -0.877088 - 0.0130199I, -0.877088 + 0.0130199I, \\ &-0.719435, -0.535283 - 0.053851I, -0.535283 + 0.053851I, \\ &-0.27156, 0. - 0.13711I, 0. + 0.13711I, 0. - 0.338328I, \\ &0. + 0.338328I, 0.27156, 0.535283 - 0.053851I, \\ &0.535283 + 0.053851I, 0.719435, 0.877088 - 0.0130199I, \\ &0.877088 + 0.0130199I, 0.967703\}\,. \end{aligned}$$

Next, we present two Figures with graphical representations of zeros and Gešrgorin locations for $P^t_{17}(5;\mu_5)(x)$ and $P^d_{18}(6;\lambda_6)(x)$, with values of parameters μ_5 and λ_6 corresponding to each case of the propositions of the preceding section. Also, we give the measures of sharpness of G-sets, numerical values of extremal zeros, and some comments.

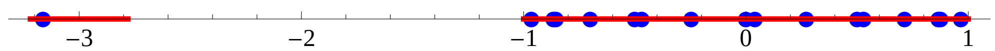
Figure 1: llustration of Propositions 4.1 and 4.2. **Zeros** and **G-sets** of some perturbed **by translation** of the Chebyshev polynomial of second kind of order $\mathbf{r = 5}$ and degree $\mathbf{n = 17}$, $\mathbf{P^t_{17}(5; \mu_5)(x)}$, for $\mu_\mathbf{5} = \mathbf{-1, 1, -3, 3}$. Remark the symmetry of zeros with respect to the origin, when $\mu_5 \to -\mu_5$ due to (17).



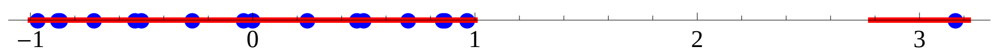
$\mathbf{-2 \leq \mu_5 = -1 < 0}$, $\mathcal{S} \approx \mathbf{0.31}$. The G-set is $[\mathbf{-2, 1}]$.
$\xi^\mathbf{t}_\mathbf{1} = \mathbf{-1.4142}$, $\xi^\mathbf{t}_\mathbf{2} = \mathbf{-0.963003}$, ... , $\xi^\mathbf{t}_\mathbf{17} = \mathbf{0.968526}$



$\mathbf{0 < \mu_5 = 1 \leq 2}$, $\mathcal{S} \approx \mathbf{0.31}$. The G-set is $[\mathbf{-1, 2}]$.
$\xi^\mathbf{t}_\mathbf{1} = \mathbf{-0.968526}$, ... , $\xi^\mathbf{t}_\mathbf{16} = \mathbf{0.963003}$, $\xi^\mathbf{t}_\mathbf{17} = \mathbf{1.4142}$

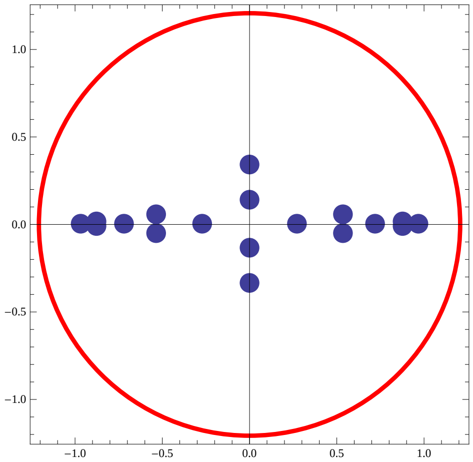


$\mathbf{\mu_5 = -3 < -2}$, $\mathcal{S}_\mathbf{out} \approx \mathbf{0.16}$, $\mathcal{S}_\mathbf{int} \approx \mathbf{0.033}$
The **extremal zero** $\xi^\mathbf{t}_\mathbf{1} = \mathbf{-3.16228}$ at the left is in the G-interval $[\mathbf{-3.219},\ \mathbf{-2.781}]$.
All other zeros are in the G-interval $[\mathbf{-1, 1}]$. $\xi^\mathbf{t}_\mathbf{2} = \mathbf{-0.964978}$, ... , $\xi^\mathbf{t}_\mathbf{17} = \mathbf{0.966836}$



$\mathbf{\mu_5 = 3 > 2}$, $\mathcal{S}_\mathbf{int} \approx \mathbf{0.033}$, $\mathcal{S}_\mathbf{out} \approx \mathbf{0.16}$
The **extremal zero** $\xi^\mathbf{t}_\mathbf{17} = \mathbf{3.16228}$ at the right is in the G-interval $[\mathbf{2.780},\ \mathbf{3.220}]$.
All other zeros are in the G-interval $[\mathbf{-1, 1}]$. $\xi^\mathbf{t}_\mathbf{1} = \mathbf{-0.966836}$, ... , $\xi^\mathbf{t}_\mathbf{16} = \mathbf{0.964978}$

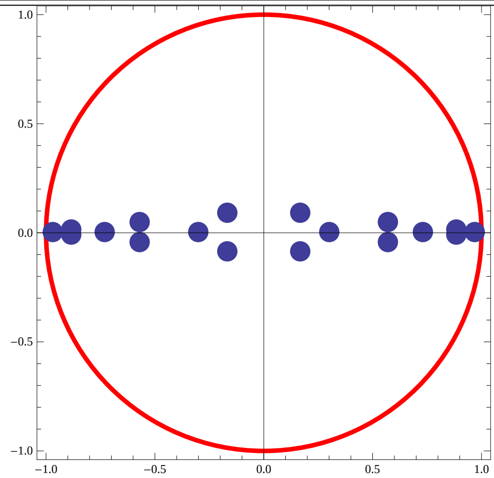
Figure 2: llustration of Proposition 4.6. **Zeros** and **G-sets** for some perturbed **by dilation** of the Chebyshev polynomial of second kind of order $\mathbf{r = 6}$ and degree $\mathbf{n = 18}$, $\mathbf{P^d_{18}(6;\lambda_6)(x)}$, for $\lambda_6 = \mathbf{-2, -0.75, 0.75, 5}$. For $\mathbf{1} \neq \lambda_6 > \mathbf{0}$, all zeros are real. For $\lambda_6 < \mathbf{0}$, there are some pairs of conjugate complex zeros. Real zeros are symmetric with respect to the origin.



$\lambda_6 = \mathbf{-2 < -1}$, $\mathcal{S} \approx \mathbf{0.24}$

All real zeros are in $[\mathbf{-1, 1}]$ [9].

The G-set is $\left\{\mathbf{z} \in \mathbb{C} : |\mathbf{z}| \leq \frac{\mathbf{1+\sqrt{|-2|}}}{\mathbf{2}} = \mathbf{1.21}\right\}$



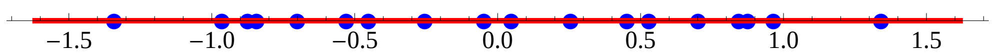
$\mathbf{-1} \leq \lambda_6 = \mathbf{-0.75 < 0}$, $\mathcal{S} \approx \mathbf{0.031}$

The G-set is the unit disk.

$\mathbf{0} < \lambda_6 = \mathbf{0.75 < 1}$, $\mathcal{S} \approx \mathbf{0.021}$

The G-set is $[\mathbf{-1, 1}]$.

The extremal zeros are : $\xi^{\mathbf{d}}_{\mathbf{1}} = \mathbf{-0.978712}$, $\xi^{\mathbf{d}}_{\mathbf{18}} = -\xi^{\mathbf{d}}_{\mathbf{1}}$.



$\lambda_6 = \mathbf{5 > 1}$, $\mathcal{S} \approx \mathbf{0.28}$

The G-set is $[-\frac{\mathbf{1+\sqrt{5}}}{\mathbf{2}}, \frac{\mathbf{1+\sqrt{5}}}{\mathbf{2}}] \approx [\mathbf{-1.62, 1.62}]$

Both extremal zeros are outside $[\mathbf{-1, 1}]$ [9] : $\xi^{\mathbf{d}}_{\mathbf{1}} = \mathbf{-1.34162}$, $\xi^{\mathbf{d}}_{\mathbf{18}} = -\xi^{\mathbf{d}}_{\mathbf{1}}$.

All other zeros are inside $[\mathbf{-1, 1}]$.

6 Conclusion

We remark that *Geršgorin circles theorems* were applied to the matrices $J_n^t(\mu_r; r)$ and to $J_n^d(\lambda_r; r)$, for r and n fixed, giving constant results for r and n sufficiently big, depending in almost cases on the parameters of perturbation μ_r and λ_r. All initial conditions are made explicit.

The methodology applied in this article constitutes a starting point for treating more complicated perturbed orthogonal sequences and geting circles or intervals locations for their zeros.

Symbolic, numerical and graphical results such as we showed in the previous section are useful in the investigation on this topic, because they can serve as a verification tool, as a way to get negative answers, to formulate conjectures or to make some discoveries, and therefore allow to direct some aspects of the theoretical study. For example, we notice the existence of pairs of zeros very close. This fact will be treated in a forthcoming work [9] about interlacing properties and locations of extremal zeros of perturbed Chebyshev polynomials of second kind in terms of zeros of Chebyshev polynomials.

It is worthy to note that *Geršgorin circles theorems* are not intended to furnish estimates for the eigenvalues of a matrix, but only some locations of them, that can be sharp or not.

Funding: The author was partially supported by CMUP (UID/MAT/00144/2013), which is funded by FCT (Portugal) with national (MEC) and European structural funds through the programs FEDER, under the partnership agreement PT2020.

Conflicts of Interest: The author declares no conflicts of interest.

References

[1] K. Castillo, F. Marcellán, J. Rivero, On co-polynomials on the real line, J. Math. Anal. App., (2015) 427(1) 469-483.

[2] K. Castillo, Monotonicity of zeros for a class of polynomials including hypergeometric polynomials, Appl. Math. Comp., (2015) 266, 183-193.

[3] T. S. Chihara, On co-recursive orthogonal polynomials, Proc. Amer. Math. Soc. 8 (1957) 899-905.

[4] T. S. Chihara, An Introduction to Orthogonal Polynomials, Mathematics and its Applications, Vol. 13. Gordon and Breach Science Publishers, New York-London-Paris, 1978.

[5] Z. da Rocha, A general method for deriving some semi-classical properties of perturbed second degree forms: the case of the Chebyshev form of second kind. J. Comput. Appl. Math., 296 (2016) 677-689.

[6] Z. da Rocha, On the second order differential equation satisfied by perturbed Chebyshev polynomials, J. Math. Anal., 7(1) (2016) 53-69.

[7] Z. da Rocha, On connection coefficients of some perturbed of arbitrary order of the Chebyshev polynomials of second kind, J. Differ. Equ. Appl., 25:1 (2019) 97-118.

[8] Z. da Rocha, Common points between perturbed Chebyshev polynomials of second kind, Math. Comput. Sci., (2020) 1-9.

[9] Z. da Rocha, Some properties of zeros of perturbed Chebyshev polynomials of second kind, (in preparation).

[10] W. Erb, Accelerated Landweber methods based on co-dilated orthogonal polynomials, Numer Algor (2015) 68: 229-260.

[11] W. Gautschi: Orthogonal Polynomials: Computation and Approximation. Numerical Mathematics and Scientific Computation. Oxford Science Publications. Oxford University Press, New York (2004).

[12] S. Geršgorin, (1931) Über die Abgrenzung der Eigenwerte einer Matrix, Izv. Akad. Nauk SSSR Ser. Mat. 1, 749-754.

[13] R. A. Horn, C. R. Johnson, Matrix Analysis - Second edition, Cambridge University Press, Cambridge, 2013.

[14] M. E. Ismail, Classical and Quantum Orthogonal Polynomials in One Variable. Encyclopedia of Mathematics and its Applications, 98. Cambridge University Press, Cambridge, 2005.

[15] E. Leopold, The extremal zeros of a perturbed orthogonal polynomials systems, J. Comp. Appl. Math., 98 (1998) 99-120.

[16] F. Marcellán, J.S. Dehesa and A. Ronveaux, On orthogonal polynomials with perturbed recurrence relations, J. Comput. Appl. Math. 30 (1990) 203–212.

[17] P. Maroni, Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classiques (in French) [An algebraic theory of orthogonal polynomials. Applications to semi–classical orthogonal polynomials]. In C. Brezinski et al. Eds., Orthogonal Polynomials and their Applications (Erice, 1990), IMACS Ann. Comput. Appl. Math., 9, Baltzer, Basel, (1991), 95-130.

[18] P. Maroni, Tchebychev forms and their perturbed as second degree forms, Ann. Numer. Math., 2 (1-4) (1995), 123–143.

[19] J. C. Mason, D. C. Handscomb, Chebyshev Polynomials, Chapman & Hall/CRC, Boca Raton, FL, 2003.

[20] A. Ronveaux, A. Zargi, E. Godoy, Fourth-order differential equations satisfied by the generalized co-recursive of all classical orthogonal polynomials. A study of their distribution of zeros, J. Comput. Appl. Math., 59 (1995), 295-328.

[21] G. Szegö, Orthogonal Polynomials, fourth edition, Amer. Math. Soc., Colloq. Publ., vol. 23, Providence, Rhode Island, 1975.

[22] H.A. Slim, On co-recursive orthogonal polynomials and their application to potential scattering, J. Math. Anal. Appl. 136 (1988) 1-19.

[23] Richard S. Varga, Geršgorin and His Circles, Springer Series in Computational Mathematics, 36, Springer-Verlag Berlin, 2004.