

On endomorphisms of the direct product of two free groups

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Abstract

We describe the endomorphisms of the direct product of two free groups of finite rank and obtain conditions for which the subgroup of fixed points is finitely generated and we do the same for periodic points. We also describe the automorphisms of such a group and solve the three Whitehead problems for automorphisms, monomorphisms and endomorphisms for groups in this class. We also obtain conditions for an endomorphism to be uniformly continuous when a suitable metric is considered and study infinite fixed points of the extension to the completion of the group.

1 Introduction

The dynamical study of endomorphisms of groups started with the (independent) work of Gersten [12] and Cooper [7], using respectively graph-theoretic and topological approaches. They proved that the subgroup of fixed points $\text{Fix}(\varphi)$ of some fixed automorphism φ of F_n is always finitely generated, and Cooper succeeded on classifying from the dynamical viewpoint the fixed points of the continuous extension of φ to the boundary of F_n . Bestvina and Handel subsequently developed the theory of train tracks to prove that $\text{Fix}(\varphi)$ has rank at most n in [2]. The problem of computing a basis for $\text{Fix}(\varphi)$ had a tribulated history and was finally settled by Bogopolski and Maslakova in 2016 in [3].

This line of research extended early to wider classes of groups. For instance, Paulin proved in 1989 that the subgroup of fixed points of an automorphism of a hyperbolic group is finitely generated [19]. Fixed points were also studied for right-angled Artin groups [21] and lamplighter groups [17].

In this paper we will study endomorphisms of the direct product of two free groups $F_n \times F_m$, with $m, n \geq 2$. We remark that the endomorphisms of free-abelian times free groups $\mathbb{Z}^m \times F_n$ are already studied in [10] and their extension to the completion is studied in [4]. Although the class of free groups is well known, some problems in the product $F_n \times F_m$ are not easily reduced to problems in each factor. In particular, when endomorphisms (and automorphisms) are considered, we have that many endomorphisms are not obtained by applying an endomorphism of F_n to the first component and one of F_m to the second, so some problems arise when the dynamics of an endomorphism in the product is considered. In fact, this class, which is a subclass of the right-angled Artin groups, has some very wild subgroups. For instance, there is a finitely generated subgroup H of $F_2 \times F_2$ for which it is undecidable if a given element $g \in F_2 \times F_2$ belongs to H (see [18]). Surprisingly, endomorphisms of $F_n \times F_m$ have a simple description.

The paper is organized as follows. In Section 2, we present some preliminaries on groups and endomorphisms. In Section 3, we describe the endomorphisms (and automorphisms) of $F_n \times F_m$ and prove that the Whitehead problems for endomorphisms, monomorphisms

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and automorphisms are solvable. We find conditions for an endomorphism to have finitely generated fixed subgroup in Section 4. In Section 5, we do the same for the subgroup of periodic points. Finally, in Section 6, we describe the uniformly continuous endomorphisms when a suitable metric is considered and prove some dynamical properties of the extension of an endomorphism to the completion.

2 Preliminaries

Given a group G , we denote by $\text{End}(G)$ (resp. $\text{Mon}(G)$, $\text{Aut}(G)$) the endomorphism monoid (resp. monomorphism monoid, automorphism group) of G . Given an endomorphism φ , let $\text{Fix}(\varphi) = \{x \in G \mid x\varphi = x\}$ and

$$\text{Per}(\varphi) = \bigcup_{k \geq 1} \text{Fix}(\varphi^k)$$

be the *fixed subgroup* of φ and the *periodic subgroup* of φ , respectively.

We denote by F_n, F_m the free groups of rank n and m and take $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_m\}$ to be the alphabets of F_n and F_m , respectively. We will always assume that $m, n > 1$.

We want to provide conditions for the fixed subgroup to be finitely generated. For that, we will use the following theorem by Anisimov and Seifert.

Theorem 2.1 ([1], Theorem III.2.7). *Let H be a subgroup of a group G . Then $H \in \text{Rat } G$ if and only if H is finitely generated.*

Given two words u and v on a free group, we write $u \wedge v$ to denote the longest common prefix of u and v .

The prefix metric on a free group is defined by

$$d(u, v) = \begin{cases} 2^{-|u \wedge v|} & \text{if } u \neq v \\ 0 & \text{otherwise} \end{cases}.$$

The prefix metric on a free group is in fact an ultrametric and its completion (\hat{F}_n, \hat{d}) is a compact space which can be described as the set of all finite and infinite reduced words on the alphabet $A \cup A^{-1}$. We will denote by ∂F_n the set consisting of only the infinite words and call it the *boundary* of F_n .

We consider the direct product of two free groups $F_n \times F_m$ endowed with the product metric given by taking the prefix metric in each component, i.e.,

$$d((x, y), (z, w)) = \max\{d(x, z), \dots, d(y, w)\}.$$

This metric is also an ultrametric and $\widehat{F_n \times F_m}$ is homeomorphic to $\hat{F}_n \times \hat{F}_m$ by uniqueness of the completion (Theorem 24.4 in [24]).

Also, throughout this paper we will often refer to a matrix and the linear application it defines indistinctively when no confusion arises.

3 Endomorphisms and Automorphisms

In this section, we describe the endomorphisms and automorphisms of $F_n \times F_m$ and solve the Whitehead problems for $F_n \times F_m$.

3.1 Endomorphisms

Consider an endomorphism $\varphi : F_n \times F_m \rightarrow F_n \times F_m$ defined by $(a_i, 1) \mapsto (x_i, y_i)$ and $(1, b_j) \mapsto (z_j, w_j)$ for $i \in [n]$ and $j \in [m]$. We define $X = \{x_i \mid i \in [n]\}$, $Y = \{y_i \mid i \in [n]\}$, $Z = \{z_j \mid j \in [m]\}$ and $W = \{w_j \mid j \in [m]\}$. We say that these sets are *trivial* if they are singletons containing only the empty word and *nontrivial* otherwise.

For φ to be well defined, we must have that, for every $i \in [n]$ and $j \in [m]$, $x_i z_j = z_j x_i$ and $y_i w_j = w_j y_i$. By Proposition 1.3.2 in [15], we have that two words in a free group commute if and only if they are powers of the same word, thus, for every $(i, j) \in [n] \times [m]$, we have

$$x_i = 1 \vee z_j = 1 \vee (\exists u \in F_n \setminus \{1\} \exists p_i, r_j \in \mathbb{Z} \setminus \{0\} : x_i = u^{p_i} \wedge z_j = u^{r_j}) \quad (1)$$

and a similar condition holds for y_i and w_j :

$$y_i = 1 \vee w_j = 1 \vee (v \in F_m \setminus \{1\} \exists q_i, s_j \in \mathbb{Z} \setminus \{0\} : y_i = v^{q_i} \wedge w_j = v^{s_j}) \quad (2)$$

Lemma 3.1. *Suppose X and Z are nontrivial. Then, there is some $1 \neq u \in F_n$ such that $X \cup Z \subseteq \{u^k \mid k \in \mathbb{Z}\}$. Similarly, if Y and W are nontrivial, then, there is some $1 \neq v \in F_m$ such that $Y \cup W \subseteq \{v^k \mid k \in \mathbb{Z}\}$.*

Proof. Just consider $\tilde{X} = X \setminus \{1\}$ and $\tilde{Z} = Z \setminus \{1\}$ which are both finite sets and apply (1). \square

So, we will consider several different cases:

- (I) All sets X, Y, Z and W are nontrivial
- (II) X is the only trivial set
- (III) Y is the only trivial set
- (IV) X and Y are the only trivial sets
- (V) X and Z are the only trivial sets
- (VI) Y and Z are trivial sets
- (VII) X and W are trivial sets.

These cases are sufficient since every other case is analogous to one of the above, by swapping order of the factors (notice we are not assuming any relation between m and n). Indeed, if W is the only trivial set, we have the second case; if Z is the only trivial set, we have the third case; if Z and W are the only trivial sets, we have the fourth case; if Y and W are the only trivial sets, we have the fifth case. If three or more sets are trivial, then we must fall into one of the two last cases.

We define $P = \{p_i \in \mathbb{Z} \mid i \in [n]\}$, $Q = \{q_i \in \mathbb{Z} \mid i \in [n]\}$, $R = \{r_i \in \mathbb{Z} \mid i \in [m]\}$ and $S = \{s_i \in \mathbb{Z} \mid i \in [m]\}$. So, matching the numeration above, φ must have one of the following forms

- (I) $(a_i, 1) \mapsto (u^{p_i}, v^{q_i})$ and $(1, b_j) \mapsto (u^{r_j}, v^{s_j})$, for some $1 \neq u \in F_n$, $1 \neq v \in F_m$ and integers $p_i, q_i, r_j, s_j \in \mathbb{Z}$ for $(i, j) \in [n] \times [m]$, such that $P, Q, R, S \neq 0$.
- (II) $(a_i, 1) \mapsto (1, v^{q_i})$ and $(1, b_j) \mapsto (z_j, v^{s_j})$, for some $1 \neq v \in F_m$, $z_j \in F_n$ and integers $q_i, s_j \in \mathbb{Z}$ for $(i, j) \in [n] \times [m]$, such that $Q, S \neq 0$. We will denote by ϕ the homomorphism from F_m to F_n mapping b_j to z_j .
- (III) $(a_i, 1) \mapsto (u^{p_i}, 1)$ and $(1, b_j) \mapsto (u^{r_j}, w_j)$, for some $1 \neq u \in F_n$, $w_j \in F_m$ and integers $p_i, r_j \in \mathbb{Z}$ for $(i, j) \in [n] \times [m]$, such that $P, R \neq 0$. We will denote by ϕ the endomorphism of F_m mapping b_j to w_j .

- (IV) $(a_i, 1) \mapsto (1, 1)$ and $(1, b_j) \mapsto (z_j, w_j)$, for some $(z_j, w_j) \in F_n \times F_m$. We will denote the component mappings $\phi : F_m \rightarrow F_n$ and $\psi \in \text{End}(F_m)$, defined by $b_j \mapsto z_j$ and $b_j \mapsto w_j$, respectively.
- (V) $(a_i, 1) \mapsto (1, v^{q_i})$ and $(1, b_j) \mapsto (1, v^{s_j})$, for some $1 \neq v \in F_m$, and integers $q_i, s_j \in \mathbb{Z}$ for $(i, j) \in [n] \times [m]$, such that $Q, S \neq 0$.
- (VI) $(a_i, 1) \mapsto (x_i, 1)$ and $(1, b_j) \mapsto (1, w_j)$, for some $x_i \in F_n$ and $w_j \in F_n$ for $(i, j) \in [n] \times [m]$. We will denote the component mappings $\phi \in \text{End}(F_n)$ and $\psi \in \text{End}(F_n)$, defined by $a_i \mapsto x_i$ and $b_j \mapsto w_j$, respectively.
- (VII) $(a_i, 1) \mapsto (1, y_i)$ and $(1, b_j) \mapsto (z_j, 1)$, for some $y_i \in F_m$ and $z_j \in F_n$ for $(i, j) \in [n] \times [m]$. We will denote the component mappings $\phi : F_n \rightarrow F_m$ and $\psi : F_m \rightarrow F_n$, defined by $a_i \mapsto y_i$ and $b_j \mapsto z_j$, respectively.

For $(i, j) \in [n] \times [m]$, we define $\lambda_i : F_n \rightarrow \mathbb{Z}$ as the endomorphism given by $a_k \mapsto \delta_{ik}$ and $\tau_j : F_m \rightarrow \mathbb{Z}$ given by $b_k \mapsto \delta_{jk}$.

3.2 Automorphisms

We are now able to describe the automorphisms of $F_n \times F_m$.

Proposition 3.2. *An endomorphism $\varphi : F_n \times F_m$ is surjective if and only if it is of type VI or type VII such that the component mappings ϕ and ψ are surjective. An endomorphism $\varphi : F_n \times F_m$ is injective if and only if it is of type VI or type VII such that the component mappings ϕ and ψ are injective.*

Proof. Type I endomorphisms are not injective nor surjective. Indeed, take an element $w \in F_n$ such that w is not a power of u and we have that $(w, 1) \notin \text{Im}(\varphi)$. Also, $(a_1 a_2, 1)\varphi = (u^{p_1+p_2}, v^{q_1+q_2}) = (a_2 a_1, 1)\varphi$, so it is not injective.

Type II endomorphisms are not surjective since, taking $w \in F_m$ such that w is not a power of v , we have that $(1, w) \notin \text{Im}(\varphi)$. Also, $(a_1 a_2, 1)\varphi = (1, v^{q_1+q_2}) = (a_2 a_1, 1)\varphi$.

Type III endomorphisms are similar to the previous. Taking $w \in F_n$ such that w is not a power of u , we have that $(w, 1) \notin \text{Im}(\varphi)$. Also, $(a_1 a_2, 1)\varphi = (u^{p_1+p_2}, 1) = (a_2 a_1, 1)\varphi$.

We will now show that a type IV endomorphism is never surjective. Suppose it is. Then, both ϕ and ψ must be surjective. Since F_m is hopfian, then ψ must be an automorphism, so we have that, for $y \in F_m$, $y\psi = 1$ implies that $y = 1$. So, there is no $y \in F_m$ such that $y\phi \neq 1$ and $y\psi = 1$. Also, it is never injective since for every $(x, y) \in F_n \times F_m$, we have that $(1, y)\varphi = (x, y)\varphi$.

Type V endomorphisms are clearly not surjective, since for every $1 \neq u \in F_n$, we have that $(u, 1) \notin \text{Im}(\varphi)$. Also, $(a_1 a_2, 1)\varphi = (1, v^{q_1+q_2}) = (a_2 a_1, 1)\varphi$, so it is not injective.

It is obvious that a type VI endomorphism is surjective (resp. injective) if and only if both ϕ and ψ are surjective (resp. injective) and the same holds for type VII homomorphisms. \square

Corollary 3.3. *$\varphi \in \text{End}(F_n \times F_m)$ is an automorphism if and only if one of the following holds:*

- i. φ is a type VI endomorphism such that both ϕ and ψ are automorphisms.
- ii. $n = m$ and φ is a type VII endomorphism such that both ϕ and ψ are automorphisms.

Corollary 3.4. *If $n \neq m$, then $\text{Aut}(F_n \times F_m) \cong \text{Aut}(F_n) \times \text{Aut}(F_m)$. If $n = m$, then $\text{Aut}(F_n) \times \text{Aut}(F_m) \trianglelefteq \text{Aut}(F_n \times F_m)$ and*

$$\text{Aut}(F_n \times F_m) / (\text{Aut}(F_n) \times \text{Aut}(F_m)) \cong \mathbb{Z}_2.$$

Proof. If $n \neq m$, the claim is obvious from Corollary 3.3.

Suppose now that $n = m$ and denote by $\text{Aut}_6(F_n \times F_m)$ the subgroup of $\text{Aut}(F_n \times F_m)$ whose elements are type VI automorphisms and by $\text{Aut}_7(F_n \times F_m)$ the subset of type VII automorphisms (which is not a subsemigroup of $\text{Aut}(F_n \times F_m)$). We have that $\text{Aut}_6(F_n \times F_m) \cong \text{Aut}(F_n) \times \text{Aut}(F_m)$ is a normal subgroup of $\text{Aut}(F_n \times F_m)$. Indeed, let $\varphi \in \text{Aut}_6(F_n \times F_m)$ and $\theta \in \text{Aut}_7(F_n \times F_m)$ be given by $(x, y)\varphi = (x\phi, y\psi)$ and $(x, y)\theta = (y\alpha, x\beta)$. Then θ^{-1} is given by $(x, y)\theta^{-1} = (y\beta^{-1}, x\alpha^{-1})$ and $(x, y)\theta\varphi\theta^{-1} = (x\beta\psi\beta^{-1}, y\alpha\phi\alpha^{-1})$, so $\theta\varphi\theta^{-1} \in \text{Aut}_6(F_n \times F_m)$. Now, we check that every pair of type VII automorphisms φ, θ belong to the same coset. Indeed, consider $\varphi, \theta \in \text{Aut}_7(F_n \times F_m)$ to be given by $(x, y)\varphi = (y\psi, x\phi)$ and $(x, y)\theta = (y\beta, x\alpha)$. Then, setting $\gamma \in \text{Aut}_6(F_n \times F_m)$ given by $(x, y)\gamma = (x\psi^{-1}\beta, y\phi^{-1}\alpha)$, we have that $(x, y)\varphi\gamma = (y\beta, x\alpha) = (x, y)\theta$, and we are done. \square

Corollary 3.5. *$\text{Aut}(F_n \times F_m)$ is finitely presented.*

Proof. Since $\text{Aut}(F_n)$ and $\text{Aut}(F_m)$ are finitely presentable, then so is $\text{Aut}(F_n) \times \text{Aut}(F_m)$. Also, if we have a group G and a normal subgroup $N \trianglelefteq G$ such that both N and G/N are finitely presentable, then G must also be finitely presentable (see, for example 2.2.4 in [20]). Since \mathbb{Z}_2 is trivially finitely presentable, it follows that $\text{Aut}(F_n \times F_m)$ is finitely presentable. \square

Corollary 3.6. *$F_n \times F_m$ is hopfian but not cohopfian.*

Proof. If $\varphi \in \text{End}(F_n \times F_m)$ is type VII and surjective, then $m = n$ and the component mappings are surjective endomorphisms of F_n . Since F_n is hopfian, we have that the components are injective, so φ is injective. If φ is type VI, then the component mappings ϕ and ψ are surjective endomorphisms of F_n and F_m , respectively, thus injective and we have that φ is injective.

It is not cohopfian since neither F_n nor F_m are cohopfian. Indeed take ϕ and ψ injective and nonsurjective endomorphisms of F_n and F_m respectively and consider the type VI endomorphism $\varphi \in \text{End}(F_n \times F_m)$ defined by taking ϕ and ψ as the component mappings. We have that φ is injective but not surjective. \square

3.3 Whitehead Problems

We start by stating the Whitehead problems for automorphisms, monomorphisms and endomorphisms.

Whitehead problems ($WhP_a(G)$, $WhP_m(G)$, $WhP_e(G)$): Given two elements $u, v \in G$, decide whether there exists an automorphism (resp. monomorphism, endomorphism) ϕ of G such that $u\phi = v$; if so, find one by giving images of generators.

In [8] and [9], the author generalizes the techniques from Whitehead and applies them to partially commutative groups, being able to solve $WhP_a(G)$, when G is partially commutative, which obviously solves $WhP_a(F_n \times F_m)$. The author is not aware of any previous result implying the solution of $WhP_m(F_n \times F_m)$ or $WhP_e(F_n \times F_m)$.

Theorem 3.7. *For $n \geq 2$*

- (i) ([23]) $WhP_a(F_n)$ is solvable
- (ii) ([6]) $WhP_m(F_n)$ is solvable
- (iii) ([16]) $WhP_e(F_n)$ is solvable

Corollary 3.8. *Given $m, n \in \mathbb{N}$, free groups F_n, F_m and elements $u \in F_n, v \in F_m$, the problem of deciding whether there exists a homomorphism $\phi : F_n \rightarrow F_m$ such that $u\phi = v$ and in case it does, finding it, is decidable.*

Proof. If $n = m$, we have the Whitehead problem $WhP_e(F_n)$ for free groups, which is solvable.

If $n < m$, then consider $\psi : F_n \rightarrow F_m$ to be the natural embedding. Then, we can decide if we have an endomorphism $\varphi \in \text{End}(F_m)$ such that $u\psi\varphi = v$. If it exists, then we take $\phi = \psi\varphi$.

Also, if there is some $\phi : F_n \rightarrow F_m$ such that $u\phi = v$, defining $\varphi \in \text{End}(F_m)$ to be the endomorphism that maps the $m - n$ extra elements in the basis of F_m to 1 and the others to their image through ϕ .

In case $n > m$, we proceed in a similar way, extending the codomain instead of the domain. So, take $\psi : F_m \rightarrow F_n$ to be the natural embedding. If there exists a homomorphism $\phi : F_n \rightarrow F_m$ such that $u\phi = v$, then $\phi\psi$ is an endomorphism of F_n such that $u\phi\psi = v\psi$. Now, suppose that there is an endomorphism $\varphi \in \text{End}(F_n)$ such that $u\varphi = v\psi$. Consider $\theta : F_n \rightarrow F_m$ such that $\psi\theta = 1$. It follows that, putting $\phi = \varphi\theta$, we have that $u\phi = v$. \square

Following the proof for free groups monomorphisms in [6] step by step, the same result follows when the free groups have different ranks.

Corollary 3.9. *Given $m, n \in \mathbb{N}$, free groups F_n, F_m and elements $u \in F_n, v \in F_m$, the problem of deciding whether there exists an injective homomorphism $\phi : F_n \rightarrow F_m$ is decidable.*

We observe that, for a word $u \in F_n \setminus \{1\}$, we have that u cannot be a k -th power for $|k| > |u|$, since, putting $u = w\tilde{u}w^{-1}$, we have that $|\tilde{u}| \geq 1$, so $|u^k| = 2|w| + k|\tilde{u}| \geq k$. Also, given a word $w \in F_n$ and $k \in \mathbb{Z}$, it is decidable if there is a word u such that $u^k = w$ and if so, we can compute such a word. Indeed, if $w = v\tilde{u}v^{-1} = u^k$, then $u = v\tilde{u}v^{-1}$ and $\tilde{u}^k = \tilde{w}$, so it is enough to check words u of the form $v\tilde{u}v^{-1}$ with $|\tilde{u}| \leq \frac{|w|}{k}$. Hence, given $u \in F_n$, we can compute $U = \{k \mid \exists \alpha \in F_n : u = \alpha^k\}$.

Proposition 3.10. *$WhP_a(F_n \times F_n)$, $WhP_m(F_n \times F_m)$ and $WhP_e(F_n \times F_m)$ are solvable.*

Proof. Let $(x, y), (z, w) \in F_n \times F_m$. We want to check if there is an endomorphism (resp. monomorphism, automorphism) φ such that $(x, y)\varphi = (z, w)$.

$WhP_a(F_n \times F_m)$ follows directly from Corollary 3.3 and Theorem 3.7 (i).

For $WhP_m(F_n \times F_m)$, we decide if there is a monomorphism of type VI and if not, we check if there is a monomorphism of type VII. For type VI endomorphisms, it follows Proposition 3.2 and Theorem 3.7 (ii). For type VII monomorphisms, we use Proposition 3.2 and Corollary 3.9.

For $WhP_e(F_n \times F_m)$, we will check if there is a type I endomorphism such that $(x, y)\varphi = (z, w)$. If there is, we stop. If not, we check the existence of a type II endomorphism. If there is one, we stop. If not, we check the existence of endomorphisms of a type III endomorphism, and so on.

So, we want to check if, given $(x, y), (z, w) \in F_n \times F_m$, there is a type I endomorphism φ such that $(x, y)\varphi = (z, w)$, i.e., if there are $u \in F_n, v \in F_m, p_i \in \mathbb{Z}, q_i \in \mathbb{Z}, r_j \in \mathbb{Z}$ and $s_j \in \mathbb{Z}$, for $(i, j) \in [n] \times [m]$ such that

$$\begin{cases} z = u^{\sum_{i \in [n]} \lambda_i(x)p_i + \sum_{j \in [m]} \tau_j(y)r_j} \\ w = v^{\sum_{i \in [n]} \lambda_i(x)q_i + \sum_{j \in [m]} \tau_j(y)s_j} \end{cases}$$

We compute the values of $\lambda_i(x), \tau_j(y)$ for $(i, j) \in [n] \times [m]$, $Z = \{k \mid \exists \alpha \in F_n : z = \alpha^k\}$ and $W = \{\ell \mid \exists \alpha \in F_m : w = \alpha^\ell\}$. Then for every $(k, \ell) \in Z \times W$ we see if the linear Diophantine system

$$\begin{cases} k = \sum_{i \in [n]} \lambda_i(x)p_i + \sum_{j \in [m]} \tau_j(y)r_j \\ \ell = \sum_{i \in [n]} \lambda_i(x)q_i + \sum_{j \in [m]} \tau_j(y)s_j \end{cases}$$

has a solution on p_i, q_i, r_j, s_j . If it does, then there is an endomorphism given by the solution of the system together with u, v such that $u^k = z$ and $v^\ell = w$. If not, there is no endomorphism and we check the existence of a type II endomorphism.

To check the existence of such a type II endomorphism, we use Corollary 3.8 to check if there is an endomorphism $\phi : F_m \rightarrow F_n$ such that $z = y\phi$. If there is not, then we stop and see if there is a type III endomorphism. If there is, we compute the values of $\lambda_i(x), \tau_j(y)$ for $(i, j) \in [n] \times [m]$ and $W = \{n \mid \exists \alpha \in F_m : w = \alpha^n\}$. Then for every $k \in W$, we see if the linear Diophantine equation

$$k = \sum_{i \in [n]} \lambda_i(x)q_i + \sum_{j \in [m]} \tau_j(y)s_j$$

has a solution on q_i, s_j . If it does, then there is an endomorphism defined by

$$(\alpha, \beta) \mapsto \left(y\phi, v^{i \in [n]} \lambda_i(\alpha)q_i + \sum_{j \in [m]} \tau_j(\beta)s_j \right),$$

for $v \in F_m$ such that $w = v^k$. If not, there is no endomorphism.

The type III case is entirely analogous to the type II.

Type IV reduces to Corollary 3.8 and Theorem 3.7 (iii).

Type V follows from decidability of the word problem in F_n and the same argument as above for the second component.

Type VI follows from Theorem 3.7 (iii) for F_n and F_m .

Type VII follows from Corollary 3.8. □

4 Fixed subgroup of an endomorphism

In this section, we aim at giving conditions for the fixed subgroup of an endomorphism to be finitely generated. We will deal with every type of endomorphisms one by one. We now state the main result from this section.

Theorem 4.1. *Let $\varphi \in \text{End}(F_n \times F_m)$. If φ is a Type I, II, IV, V, VI or VII endomorphism, then $\text{Fix}(\varphi)$ is finitely generated. If φ is a type III endomorphism such that*

$$\sum_{i \in [n]} \lambda_i(u)p_i \neq 1,$$

then $\text{Fix}(\varphi)$ is finitely generated.

We also provide an example of a type III endomorphism for which $\text{Fix}(\varphi)$ is not finitely generated.

4.1 Type I endomorphisms

We will study the fixed point subgroup of such an endomorphism. To start, we obviously have that a fixed point $(u, v) \in F_n \times F_m$ must be of the form $(u^a, v^b) \in F_n \times F_m$ for some $a, b \in \mathbb{Z}$. For $i \in [n]$, we have that φ is defined by

$$(x, y) \mapsto \left(u^{i \in [n]} \lambda_i(x)p_i + \sum_{j \in [m]} \tau_j(y)r_j, v^{i \in [n]} \lambda_i(x)q_i + \sum_{j \in [m]} \tau_j(y)s_j \right).$$

For $x \in F_n, y \in F_m$, we denote $\sum_{i \in [n]} \lambda_i(x)p_i$ by x^P ; $\sum_{j \in [m]} \tau_j(y)r_j$ by y^R ; $\sum_{i \in [n]} \lambda_i(x)q_i$ by x^Q and $\sum_{j \in [m]} \tau_j(y)s_j$ by y^S . We will keep this notation throughout the paper.

So

$$\begin{aligned}
(u^a, v^b) \in \text{Fix}(\varphi) &\Leftrightarrow \begin{cases} a = \sum_{i \in [n]} \lambda_i(u^a)p_i + \sum_{j \in [m]} \tau_j(v^b)r_j \\ b = \sum_{i \in [n]} \lambda_i(u^a)q_i + \sum_{j \in [m]} \tau_j(v^b)s_j \end{cases} \\
&\Leftrightarrow \begin{cases} a = au^P + bv^R \\ b = au^Q + bv^S \end{cases} \\
&\Leftrightarrow \begin{cases} a(-1 + u^P) + bv^R = 0 \\ au^Q + b(-1 + v^S) = 0 \end{cases}
\end{aligned}$$

So, consider the matrix given by

$$M_\varphi = \begin{bmatrix} -1 + u^P & v^R \\ u^Q & -1 + v^S \end{bmatrix}$$

and we have that $\text{Fix}(\varphi) = \{(u^a, v^b) \in F_n \times F_m \mid (a, b) \in \text{Ker}(M_\varphi)\}$. It is in fact isomorphic to $\text{Ker}(M_\varphi)$ as a subgroup of the free-abelian group \mathbb{Z}^2 , which is finitely generated and has a computable basis. In particular, if $\det(M_\varphi) \neq 0$, we have that the fixed subgroup of φ is trivial.

4.2 Type II endomorphisms

Consider a homomorphism $\phi : F_m \rightarrow F_n$ given by $b_j \mapsto z_j$. Then, we have that φ is defined by

$$(x, y) \mapsto \left(y\phi, v^{\sum_{i \in [n]} \lambda_i(x)q_i + \sum_{j \in [m]} \tau_j(y)s_j} \right).$$

So a fixed point must be of the form $(v^b\phi, v^b)$, for some $b \in \mathbb{Z}$. We have that

$$\begin{aligned}
(v^b\phi, v^b) \in \text{Fix}(\varphi) &\Leftrightarrow b = \sum_{i \in [n]} \lambda_i(v^b\phi)q_i + \sum_{j \in [m]} \tau_j(v^b)s_j \\
&\Leftrightarrow b(-1 + (v\phi)^Q + v^S) = 0
\end{aligned}$$

So, if $(v\phi)^Q + v^S \neq 1$, we have that $\text{Fix}(\varphi)$ is trivial and otherwise, $\text{Fix}(\varphi)$ is given by $\{(v^b\phi, v^b) \in F_n \times F_m \mid b \in \mathbb{Z}\} \cong \mathbb{Z}$.

4.3 Type III endomorphisms

Consider $\phi \in \text{End}(F_m)$ given by $b_j \mapsto w_j$. Then, we have that φ is defined by

$$(x, y) \mapsto \left(u^{\sum_{i \in [n]} \lambda_i(x)p_i + \sum_{j \in [m]} \tau_j(y)r_j}, y\phi \right).$$

So a fixed point must be of the form $(u^a, y) \in F_n \times F_m$, for some $a \in \mathbb{Z}$. We have that

$$\begin{aligned}
(u^a, y) \in \text{Fix}(\varphi) &\Leftrightarrow a = \sum_{i \in [n]} \lambda_i(u^a)p_i + \sum_{j \in [m]} \tau_j(y)r_j \wedge y \in \text{Fix}(\phi) \\
&\Leftrightarrow a(-1 + u^P) + y^R = 0 \wedge y \in \text{Fix}(\phi)
\end{aligned}$$

So, if $\sum_{i \in [n]} \lambda_i(u)p_i = 1$, putting $H = \left\{ y \in \text{Fix}(\phi) \mid \sum_{j \in [m]} \tau_j(y)r_j = 0 \right\} \leq \text{Fix}(\phi)$, we have that

$$\text{Fix}(\varphi) = \{(u^a, y) \mid y \in H, a \in \mathbb{Z}\} \cong \mathbb{Z} \times H$$

and it is finitely generated if and only if H is finitely generated. Notice this might not be the case. Indeed, if $m = 2$, $r_1 = 1$, $r_2 = -1$ and $\phi \in \text{End}(F_2)$ such that $\text{Fix}(\phi) = F_2$, we have that

$$H = \left\{ y \in \text{Fix}(\phi) \mid \sum_{j \in [m]} \tau_j(y) r_j = 0 \right\} = \{y \in F_2 \mid \tau_1(y) = \tau_2(y)\}.$$

The language it defines is not rational, so, by Theorem 2.1, H is not finitely generated.

Although the set of reduced words in H might not be a rational language, we can prove that it is always a context-free language constructing a pushdown automata recognizing

$$H' = \left\{ y \in F_m \mid \sum_{j \in [m]} \tau_j(y) r_j = 0 \right\}.$$

Since the words in $\text{Fix}(\phi)$ form a rational language, and $H = H' \cap \text{Fix}(\phi)$, it follows that H is context-free. To do so, consider $\Gamma = \{z, X, X^{-1}\}$ to be the stack alphabet and z to be the starting symbol. When b_i is read, if r_i is positive, then if the top symbol of the stack has X or z , we add X^{r_i} to the stack. If the stack has at least r_i X^{-1} 's on the top, we remove them and if there are $k < r_i$ symbols X^{-1} at the top, we remove them and add X^{r_i-k} . If r_i is negative, we do as above switching X and X^{-1} . Whenever b_i^{-1} is read, we do the same switching X and X^{-1} . If we read the starting symbol and have nothing to add, we reach a final state. This way, only a finite amount of memory is required, so H' is context-free.

If, on the other hand, $\sum_{i \in [n]} \lambda_i(u) p_i \neq 1$, then putting

$$G = \left\{ y \in \text{Fix}(\phi) \mid (1 - u^P) \text{ divides } y^R \right\} \leq \text{Fix}(\phi)$$

$$\text{Fix}(\varphi) = \left\{ \left(u^{\frac{y^R}{1-u^P}}, y \right) \mid y \in G \right\} \cong G.$$

We now prove that G is always finitely generated. Set

$$H = \left\{ y \in F_m \mid (1 - u^P) \text{ divides } y^R \right\} \leq F_m.$$

We have that $G = \text{Fix}(\phi) \cap H$ and $\text{Fix}(\phi)$ is finitely generated. Now, let $p = u^P - 1$ and consider the automaton $\mathcal{A} = (\mathbb{Z}_{|p|}, 0, \{0\}, \delta)$, where $\delta : Q \times (B \cup B^{-1}) \rightarrow Q$ is defined by $(n, b_i^\varepsilon) \mapsto n + \varepsilon r_i \pmod{|p|}$, for $\varepsilon \in \{-1, 1\}$. It is clear that $L(\mathcal{A}) = H$, since, for $r \in \mathbb{N}$, $\varepsilon_i \in \{-1, 1\}$ and $k_i \in [m]$ for every $i \in [r]$, we have that

$$\begin{aligned} w = b_{k_1}^{\varepsilon_1} \cdots b_{k_r}^{\varepsilon_r} \in L(\mathcal{A}) &\Leftrightarrow \sum_{j \in [m]} \tau_j(w) r_j = \sum_{i \in [r]} \varepsilon_i b_{k_i} \equiv_{|p|} 0 \\ &\Leftrightarrow (1 - u^P) \text{ divides } y^R. \end{aligned}$$

So, by Theorem 2.1, we have that H is finitely generated. By Howson's Theorem ([13]), we have that G is finitely generated.

4.4 Type IV endomorphisms

Consider $\phi : F_m \rightarrow F_n$ defined by $b_j \mapsto z_j$ and $\psi \in \text{End}(F_m)$ given by $b_j \mapsto w_j$. Then $(x, y)\varphi = (y\phi, y\psi)$. In particular, $\text{Fix}(\varphi) = \{(y\phi, y) \in F_n \times F_m \mid y \in \text{Fix}(\psi)\}$ which is finitely generated. In fact, it is isomorphic to $\text{Fix}(\psi)$ which is finitely generated and has bounded rank. Effectively constructing a basis for $\text{Fix}(\varphi)$ depends on the construction of a basis for $\text{Fix}(\psi)$ for which an algorithm is not yet known.

4.5 Type V endomorphisms

In this case, we have that φ is defined by

$$(x, y) \mapsto \left(1, v^{\sum_{i \in [m]} \lambda_i(x)q_i + \sum_{j \in [m]} \tau_j(y)s_j} \right).$$

In particular, we get that a fixed point must be of the form $(1, v^b)$, for some $b \in \mathbb{Z}$. We have that

$$(1, v^b) \in \text{Fix}(\varphi) \Leftrightarrow b = b \sum_{j \in [m]} \tau_j(v)s_j.$$

It follows that, if $v^S = 1$, then $\text{Fix}(\varphi) \cong \mathbb{Z}$, since it is given by $\{(1, v^b) \mid b \in \mathbb{Z}\}$. If $v^S \neq 1$, then $\text{Fix}(\varphi)$ is trivial.

4.6 Type VI endomorphisms

This case reduces to the free group case, since we have that φ is given by $(x, y) \mapsto (x\phi, y\psi)$, for some $\phi \in \text{End}(F_n)$ and $\psi \in \text{End}(F_m)$. It follows that $\text{Fix}(\varphi) = \text{Fix}(\phi) \times \text{Fix}(\psi)$, and so it is finitely generated and computation of a set of generators is dependent on computation of a basis for $\text{Fix}(\phi)$ and $\text{Fix}(\psi)$.

4.7 Type VII endomorphisms

This case is also similar to the previous, since we have that φ is given by $(x, y) \mapsto (y\psi, x\phi)$, for $\phi : F_n \rightarrow F_m$ defined by $a_i \mapsto y_i$ and $\psi : F_m \rightarrow F_n$ defined by $b_j \mapsto z_j$. So a fixed point must be (x, y) such that $x = y\psi$ and $y = x\phi$ and so $x = x\phi\psi$ and $y = y\psi\phi$. So, $\text{Fix}(\varphi) \subseteq \text{Fix}(\phi\psi) \times \text{Fix}(\psi\phi)$. Also, for $x \in \text{Fix}(\phi\psi)$, we have that $(x, x\phi)\varphi = (x\phi\psi, x\phi) = (x, x\phi)$. Similarly, we have that $(y\psi, y)\varphi = (y\psi, y)$ for $y \in \text{Fix}(\psi\phi)$. So, $\text{Fix}(\varphi) = \{(x, x\phi) \mid x \in \text{Fix}(\phi\psi)\} = \{(y\psi, y) \mid y \in \text{Fix}(\psi\phi)\} \cong \text{Fix}(\phi\psi) \cong \text{Fix}(\psi\phi)$.

We remark that this also yields a somewhat trivial proof that for homomorphisms $\phi : F_n \rightarrow F_m$ and $\psi : F_m \rightarrow F_n$, we have that $\text{Fix}(\phi\psi) \cong \text{Fix}(\psi\phi)$.

5 Periodic points

In this section, we aim at giving conditions for the periodic subgroup of an endomorphism to be finitely generated. We will deal with every type of endomorphisms one by one. We now state the main result from this section.

Theorem 5.1. *Let $\varphi \in \text{End}(F_n \times F_m)$. If φ is a Type I, II, IV, V, VI or VII endomorphism, then $\text{Per}(\varphi)$ is finitely generated. If φ is a type III endomorphism such that*

$$\left| \sum_{i \in [n]} \lambda_i(u)p_i \right| \neq 1,$$

then $\text{Per}(\varphi)$ is finitely generated.

We will also prove that, despite the fact that the periodic subgroup is not always finitely generated, the periods of periodic points are bounded.

5.1 Type I endomorphisms

Consider a type I endomorphism given by $(a_i, 1) \mapsto (u^{p_i}, v^{q_i})$ and $(1, b_j) \mapsto (u^{r_j}, v^{s_j})$, for some $1 \neq u \in F_n$, $1 \neq v \in F_m$ and integers $p_i, q_i, r_j, s_j \in \mathbb{Z}$ for $(i, j) \in [n] \times [m]$, such that $P, Q, R, S \neq 0$. We have that

$$(x, y) \mapsto \left(u^{\sum_{i \in [n]} \lambda_i(x)p_i + \sum_{j \in [m]} \tau_j(y)r_j}, v^{\sum_{i \in [n]} \lambda_i(x)q_i + \sum_{j \in [m]} \tau_j(y)s_j} \right).$$

We want to determine the orbit of a point $(x, y) \in F_n \times F_m$. Define sequences in \mathbb{Z} by $a_1(x, y) = x^P + y^R$, $b_1(x, y) = x^Q + y^S$ and $a_{n+1}(x, y) = a_n(x, y)u^P + b_n(x, y)v^R$ and $b_{n+1}(x, y) = a_n(x, y)u^Q + b_n(x, y)v^S$. We have that, for every $k \in \mathbb{N}$, $(x, y)\varphi^k = (u^{a_k(x, y)}, v^{b_k(x, y)})$. Indeed, for $k = 1$, we have that $(x, y)\varphi = (u^{a_1(x, y)}, v^{b_1(x, y)})$. Assume that $(x, y)\varphi^r = (u^{a_r(x, y)}, v^{b_r(x, y)})$, for every $r \leq k$. We have that

$$\begin{aligned} (x, y)\varphi^{k+1} &= (x, y)\varphi^k\varphi = (u^{a_k(x, y)}, v^{b_k(x, y)})\varphi \\ &= (u^{a_k(x, y)u^P + b_k(x, y)v^R}, v^{a_k(x, y)u^Q + b_k(x, y)v^S}) \\ &= (u^{a_{k+1}(x, y)}, v^{b_{k+1}(x, y)}). \end{aligned}$$

Clearly, a periodic point must be of the form (u^a, v^b) . In this case, $x^P = au^P$, $y^R = bv^R$, $x^Q = au^Q$ and $y^S = bv^S$, so putting $a_0 = a$, $b_0 = b$, $a_{n+1} = a_nu^P + b_nv^R$ and $b_{n+1} = a_nu^Q + b_nv^S$, we have that $(u^a, v^b)\varphi^k = (u^{a_k}, v^{b_k})$. Defining the matrix

$$M_\varphi = \begin{bmatrix} u^P & v^R \\ u^Q & v^S \end{bmatrix}$$

and denoting also by M_φ the endomorphism of \mathbb{Z}^2 defined by the matrix, we have that

$$\text{Per}(\varphi) = \{(u^a, v^b) \in F_n \times F_m \mid (a, b) \in \text{Per}(M_\varphi)\} \cong \text{Per}(M_\varphi),$$

which is finitely generated since, for every $n \in \mathbb{N}$, we have that $\text{Per}(\psi)$ is finitely generated for every $\psi \in \text{End}(\mathbb{Z}^n)$.

5.2 Type II endomorphisms

Consider a type II endomorphism defined by

$$(x, y) \mapsto \left(y\phi, v^{\sum_{i \in [n]} \lambda_i(x)q_i + \sum_{j \in [m]} \tau_j(y)s_j} \right).$$

We want to study the orbit of a point $(x, y) \in F_n \times F_m$. We define a sequence $(a_n)_n$ by $a_1(x, y) = x^Q + y^S$, $a_2(x, y) = a_1(x, y)v^S + y\phi^Q$ and for $n > 2$, $a_n(x, y) = a_{n-1}(x, y)v^S + a_{n-2}(x, y)(v\phi)^Q$. We will now prove that, for $k \geq 2$, we have that $(x, y)\varphi^k = (v^{a_{k-1}(x, y)}\phi, v^{a_k(x, y)})$. For $k = 2$, we have that $(x, y)\varphi^2 = (y\phi, v^{x^Q + y^S})\varphi = (v^{a_1(x, y)}\phi, v^{y\phi^Q + a_1(x, y)v^S}) = (v^{a_1(x, y)}\phi, v^{a_2(x, y)})$. Now, assume the statement holds for every $r \leq k$. We have that

$$\begin{aligned} (x, y)\varphi^{k+1} &= (v^{a_{k-1}(x, y)}\phi, v^{a_k(x, y)})\varphi = (v^{a_k(x, y)}\phi, v^{a_{k-1}(x, y)(v\phi)^Q + a_k(x, y)v^S}) \\ &= (v^{a_k(x, y)}\phi, v^{a_{k+1}(x, y)}). \end{aligned}$$

So, a periodic point has the form $(v^b\phi, v^a)$, for $a, b \in \mathbb{Z}$.

Suppose that $v\phi \neq 1$ and consider

$$H = \{(b, a) \in \mathbb{Z}^2 \mid (v^b\phi, v^a) \in \text{Per}(\varphi)\}.$$

Clearly, H is a subgroup of \mathbb{Z}^2 , thus finitely generated and $\text{Per}(\varphi) \cong H$.

Now, if $v\phi = 1$, then a periodic point must be of the form $(1, v^a)$ which is isomorphic to a subgroup of \mathbb{Z} , hence finitely generated.

5.3 Type III endomorphisms

As in the fixed point case, the subgroup of periodic points of a Type III endomorphism is more complex than the subgroup of periodic points of endomorphisms of other types. In particular, it might not be finitely generated, as seen in the fixed point case.

Clearly, a periodic point must be of the form $(u^a, w) \in F_n \times F_m$. We will prove by induction on k that

$$(u^a, w)\varphi^k = \left(u^{a(u^P)^k + \sum_{t=0}^{k-1} (w\phi^t)^R (u^P)^{k-t-1}}, w\phi^k \right). \quad (3)$$

For $k = 1$, we have that $(u^a, w)\varphi = (u^{au^P+w^R}, w\phi)$. Now, assume the statement holds for every $r \leq k$. We have that

$$\begin{aligned} (u^a, w)\varphi^{k+1} &= (u^a, w)\varphi^k \varphi = \left(u^{a(u^P)^k + \sum_{t=0}^{k-1} (w\phi^t)^R (u^P)^{k-t-1}}, w\phi^k \right) \varphi \\ &= \left(u^{\left(a(u^P)^k + \sum_{t=0}^{k-1} (w\phi^t)^R (u^P)^{k-t-1} \right) u^P + (w\phi^k)^R}, w\phi^{k+1} \right) \\ &= \left(u^{a(u^P)^{k+1} + \sum_{t=0}^k (w\phi^t)^R (u^P)^{k-t}}, w\phi^{k+1} \right) \end{aligned}$$

If $u^P = 1$, then put

$$H = \left\{ y \in \text{Per}(\phi) \mid \exists s \in \mathbb{N} : \sum_{t=0}^{s\pi_y-1} (y\phi^t)^R = 0 \right\}.$$

Observe that for $s \in \mathbb{N}$, $y \in \text{Per}(\phi)$, we have that $\sum_{t=0}^{s\pi_y-1} (y\phi^t)^R = s \sum_{t=0}^{\pi_y-1} (y\phi^t)^R$, so

$$H = \left\{ y \in \text{Per}(\phi) \mid \sum_{t=0}^{\pi_y-1} (y\phi^t)^R = 0 \right\},$$

where π_y denotes the period of y . We have that $\text{Per}(\varphi) = \{(u^a, w) \mid w \in H, a \in \mathbb{Z}\} \cong \mathbb{Z} \times H$, and we know that H might not be finitely generated, as seen in the fixed points case.

Moreover, it follows that if (u^a, y) is periodic, then its period is equal to the period of y , which is bounded.

If $u^P = -1$, we will consider two cases. Let $y \in \text{Per}(\varphi)$ with even period π_y and take $a \in \mathbb{Z}$ and $s \in \mathbb{N}$.

We have that

$$(u^a, y)\varphi^{s\pi_y} = \left(u^{a + \sum_{t=0}^{s\pi_y-1} (y\phi^t)^R (-1)^{t+1}}, y \right).$$

So, $(u^a, y) \in \text{Per}(\varphi)$ if and only if there is $s \in \mathbb{N}$, such that $\sum_{t=0}^{s\pi_y-1} (y\phi^t)^R (-1)^{t+1} = 0$. Since,

$$\sum_{t=0}^{s\pi_y-1} (y\phi^t)^R (-1)^{t+1} = s \sum_{t=0}^{\pi_y-1} (y\phi^t)^R (-1)^{t+1}, \text{ then putting}$$

$$H = \left\{ y \in \text{Per}(\phi) \mid \pi_y \equiv_2 0 \wedge \sum_{t=0}^{\pi_y-1} (-1)^{t+1} (y\phi^t)^R = 0 \right\},$$

we have that

$$(u^a, y) \in \text{Per}(\varphi) \Leftrightarrow y \in H.$$

It follows that, in this case, the period of (u^a, y) is the same as the period of y , which is bounded.

Now, let $y \in \text{Per}(\varphi)$ with odd period π_y and take $a \in \mathbb{Z}$ and $s \in \mathbb{N}$. Then

$$(u^a, y)\varphi^{2\pi_y} = \left(u^{a + \sum_{t=0}^{2\pi_y-1} (-1)^{t+1} (y\phi^t)^R}, y \right) = (u^a, y).$$

So, $\text{Per}(\varphi) = \{(u^a, y) \in F_n \times \text{Per}(\varphi) \mid y \in H \vee \pi_y \equiv_2 1\}$. Notice that, if every periodic point of ϕ has odd period, then $\text{Per}(\varphi)$ is finitely generated, since in that case, $\text{Per}(\varphi) \cong \mathbb{Z} \times \text{Per}(\phi)$. However, it might be the case where $\text{Per}(\varphi)$ is not finitely generated. For example, let $m = 2$ and $\phi \in \text{End}(F_2)$ defined by $a \mapsto b$ and $b \mapsto a$. Then $\text{Per}(\phi) = F_m$ and every point has even period. If $r_1 = 1$ and $r_2 = -1$, then

$$\begin{aligned} H &= \{y \in F_2 \mid -y^R + (y\phi)^R = 0\} = \{y \in F_2 \mid -\lambda_1(y) + \lambda_2(y) + \lambda_2(y) - \lambda_1(y) = 0\} \\ &= \{y \in F_2 \mid \lambda_1(y) - \lambda_2(y) = 0\} \end{aligned}$$

which is not finitely generated.

In this case, if (u^a, y) is periodic, then its period is at most $2\pi_y$, which is bounded.

In case $|u^P| \neq 1$, we will show that $\text{Fix}(\varphi)$ is always finitely generated. To do so, we start by proving that if φ is a type III endomorphism such that $|u^P| \neq 1$, then for every $k \in \mathbb{N}$, we have that φ^k is a type III endomorphism such that $|u^P| \neq 1$. Indeed,

$$(a_i, 1)\varphi^k = \left(u^{p_i(u^P)^{k-1}}, 1 \right) \quad \text{and} \quad (1, b_j)\varphi^k = \left(u^{\sum_{t=0}^{k-1} (b_j\phi^t)^R (u^P)^{k-t-1}}, b_j\phi^k \right),$$

so φ^k is a type III endomorphism and

$$\left| \sum_{i \in [n]} \lambda_i(u) p_i (u^P)^{k-1} \right| = |(u^P)^k| \neq 1.$$

So, we know that $\text{Fix}(\varphi^k)$ is finitely generated for every $k \in \mathbb{N}$. We will now show that there is a bound P_φ for the periods of periodic points of φ , which is not obvious since $F_n \times F_m$ does not satisfy the ascending chain condition. To do that, we will show that the period of (u^a, w) is the same as the period of w through ϕ , π_w , which is bounded. This suffices to prove that $\text{Per}(\varphi)$ is finitely generated since, in this case

$$\text{Per}(\varphi) = \bigcup_{k=1}^{P_\varphi} \text{Fix}(\varphi^k),$$

which is a finite union of finitely generated groups.

We have that

$$(u^a, w)\varphi^k = (u^a, w) \Rightarrow k = s\pi_w, \text{ for some } s \in \mathbb{N}.$$

By (3) putting

$$S = \left\{ w \in \text{Per}(\phi) \mid \exists s_w \in \mathbb{N} : (1 - (u^P)^{s_w\pi_w}) \text{ divides } \sum_{t=0}^{s_w\pi_w-1} (w\phi^t)^R (u^P)^{s_w\pi_w-t-1} \right\},$$

So, a periodic point must have the form $(1, v^b)$ for some $b \in \mathbb{Z}$. In this case,

$$(1, v^b)\varphi^k = (1, v^{bv^S(v^S)^{k-1}}) = (1, v^{b(v^S)^k})$$

So, if $v^S \notin \{-1, 1\}$, then, $\text{Per}(\varphi)$ is trivial. If $v^S \in \{-1, 1\}$, then $\text{Per}(\varphi) \cong \mathbb{Z}$.

5.6 Type VI endomorphisms

We have that $(x, y)\varphi^k = (x\phi^k, y\psi^k)$, so $\text{Per}(\varphi) = \text{Per}(\phi) \times \text{Per}(\psi)$.

5.7 Type VII endomorphisms

Consider an endomorphism φ defined by $(x, y)\varphi = (y\psi, x\phi)$, for $\phi : F_n \rightarrow F_m$ and $\psi : F_m \rightarrow F_n$. We will prove by induction that for $k \in \mathbb{N}_0$, $(x, y)\varphi^{2k} = (x(\phi\psi)^k, y(\psi\phi)^k)$ and $(x, y)\varphi^{2k+1} = (y(\psi\phi)^k\psi, x(\phi\psi)^k\phi)$. We have that $(x, y)\varphi = (y\psi, x\phi)$. Now suppose the claim holds for $(x, y)\varphi^r$ for every r up to some k .

We have that

$$\begin{aligned} (x, y)\varphi^{k+1} &= (x, y)\varphi^k\varphi = \begin{cases} (x(\phi\psi)^{\frac{k}{2}}, y(\psi\phi)^{\frac{k}{2}})\varphi, & \text{if } k \text{ is even} \\ (y(\psi\phi)^{\frac{k-1}{2}}\psi, x(\phi\psi)^{\frac{k-1}{2}}\phi)\varphi, & \text{if } k \text{ is odd} \end{cases} \\ &= \begin{cases} (y(\psi\phi)^{\frac{k}{2}}\psi, x(\phi\psi)^{\frac{k}{2}}\phi)\varphi, & \text{if } k+1 \text{ is odd} \\ (x(\phi\psi)^{\frac{k+1}{2}}, y(\psi\phi)^{\frac{k+1}{2}})\varphi, & \text{if } k+1 \text{ is even} \end{cases} \end{aligned}$$

So, the periodic points with even period are given by $\text{Per}(\phi\psi) \times \text{Per}(\psi\phi)$, which is finitely generated.

Observe now that $G = \{x \in \text{Per}(\varphi) \mid \pi_x \equiv_2 1\}$ is a subgroup of $F_n \times F_m$. Indeed, if $\alpha, \beta \in G$, then $(\alpha\beta)\varphi^{\pi_\alpha\pi_\beta} = \alpha\beta$, so $\pi_{\alpha\beta} \mid \pi_\alpha\pi_\beta$ and $\pi_{\alpha-1} = \pi_\alpha$.

If (x, y) is a periodic point with odd period, then there is k such that $(x, y)\varphi^{2k+1} = (x, y)$, thus we have that

$$\begin{aligned} \begin{cases} x = y(\psi\phi)^k\psi \\ y = x(\phi\psi)^k\phi \end{cases} &\Rightarrow \begin{cases} x = x(\phi\psi)^k\phi(\psi\phi)^k\psi \\ y = y(\psi\phi)^k\psi(\phi\psi)^k\phi \end{cases} \Rightarrow \begin{cases} x = x(\phi\psi)^{2k+1} \\ y = y(\psi\phi)^{2k+1} \end{cases} \\ &\Rightarrow \begin{cases} x \in \text{Per}(\phi\psi) \wedge \pi_x \equiv_2 1 \\ y \in \text{Per}(\psi\phi) \wedge \pi_y \equiv_2 1 \end{cases} \end{aligned}$$

Also, taking some $x \in \text{Per}(\phi\psi)$ such that $\pi_x \equiv_2 1$, we have that

$$\left(x, x(\phi\psi)^{\frac{\pi_x-1}{2}}\phi\right)\varphi^{\pi_x} = \left(x(\phi\psi)^{\frac{\pi_x-1}{2}}\phi(\psi\phi)^{\frac{\pi_x-1}{2}}\psi, x(\phi\psi)^{\frac{\pi_x-1}{2}}\phi\right) = \left(x, x(\phi\psi)^{\frac{\pi_x-1}{2}}\phi\right).$$

Similarly, we have that, for $y \in \text{Per}(\psi\phi)$ such that $\pi_y \equiv_2 1$, we have that $\left(y(\psi\phi)^{\frac{\pi_y-1}{2}}, y\right)\varphi^{\pi_y} = \left(y(\psi\phi)^{\frac{\pi_y-1}{2}}, y\right)$. So, the subgroup of periodic points with odd period is isomorphic to the subgroup of periodic points with odd period of $\phi\psi$ and to the subgroup of periodic points with odd period of $\psi\phi$, which is finitely generated since the period is bounded, so it is a finite union of finitely generated groups.

Hence, $\text{Per}(\varphi)$ is finitely generated.

6 Dynamics of infinite points

For $i = 1, 2$, denote by π_i the projections in the i -th component and by φ_i the homomorphism $\varphi\pi_i$. We have that φ is uniformly continuous if and only if both φ_1 and φ_2 are. Also, recall the definition of the sets X, Y, Z, W introduced in Section 4. We will consider $F_n \times F_m$

endowed with the product metric given by taking the prefix metric in each component. Check Section 2 for definitions.

If φ is uniformly continuous, then it admits an extension to the completion $\hat{\varphi} : \widehat{F_n \times F_m} \rightarrow \widehat{F_n \times F_m}$. The dynamics of this extension has been studied for several classes of groups including the free groups (see, for example, [7],[11],[14]), virtually free groups (see [22]) and free-abelian times free groups (see [4]).

A fixed point $\alpha \in \partial(F_n \times F_m)$ is said to be *singular* if α belongs to the topological closure $(\text{Fix}(\varphi))^c$ of $\text{Fix}(\varphi)$. Otherwise α is said to be *regular*. We denote by $\text{Sing}(\hat{\varphi})$ (resp. $\text{Reg}(\hat{\varphi})$) the set of all singular (resp. regular) infinite fixed points of $\hat{\varphi}$.

Definition 6.1. Let $\varphi \in \text{Aut}(F_n \times F_m)$. An infinite fixed point $\alpha \in \text{Fix}(\hat{\varphi})$ is

- an *attractor* if

$$\exists \varepsilon > 0 \forall \beta \in \widehat{F_n \times F_m} (d(\alpha, \beta) < \varepsilon \Rightarrow \lim_{n \rightarrow +\infty} \beta \hat{\varphi}^n = \alpha)$$

- a *repeller* if

$$\exists \varepsilon > 0 \forall \beta \in \widehat{F_n \times F_m} (d(\alpha, \beta) < \varepsilon \Rightarrow \lim_{n \rightarrow +\infty} \beta \hat{\varphi}^{-n} = \alpha)$$

In this section, we obtain conditions for an endomorphism of $F_n \times F_m$ to be uniformly continuous and see how some questions typically studied can be reduced to the free group case.

Proposition 6.2. *Let $\varphi \in \text{End}(F_n \times F_m)$. The following conditions are equivalent:*

1. φ is uniformly continuous.
2. φ is of type IV, VI or VII with uniformly continuous components functions ϕ and ψ .

Proof. It is obvious that 2. \Rightarrow 1.

1. \Rightarrow 2. Suppose that φ is uniformly continuous. We start by proving that $1 \in X \Rightarrow X = 1$. Indeed, if $x_i = 1$ for some $i \in [n]$ and $x_j \neq 1$ for some $j \in [n]$, then for every $\delta > 0$, we have

$$d\left(\left(a_i^{\lceil \log_2(\frac{1}{\delta}) \rceil}, 1\right), \left(a_i^{\lceil \log_2(\frac{1}{\delta}) \rceil} a_j, 1\right)\right) < \delta$$

and

$$d\left(\left(a_i^{\lceil \log_2(\frac{1}{\delta}) \rceil}, 1\right) \varphi_1, \left(a_i^{\lceil \log_2(\frac{1}{\delta}) \rceil} a_j, 1\right) \varphi_1\right) = d(1, x_j) = 1.$$

The same holds for Y, Z and W .

We now prove that we can't have $Y \neq \{1\}$ and $W \neq \{1\}$ (types I, II and V). Suppose we do. Then $1 \notin Y \cup W$. We know that there is some word $1 \neq v \in F_m$ and nonzero exponents q_i and s_j such that $y_i = v^{q_i}$ and $w_j = v^{s_j}$. Take $\delta > 0$ and set $k = \lceil \log_2(\frac{1}{\delta}) \rceil$. We have that

$$d\left(\left(a_i^{ks_j}, b_j^{-kq_i}\right), \left(a_i^{ks_j+1}, b_j^{-kq_i}\right)\right) < \delta$$

and

$$d\left(\left(a_i^{ks_j}, b_j^{-kq_i}\right) \varphi_2, \left(a_i^{ks_j+1}, b_j^{-kq_i}\right) \varphi_2\right) = d(1, v^{q_i}) = 1.$$

The same argument shows that we can't have $X \neq \{1\}$ and $Z \neq \{1\}$, so type III is also done.

It is obvious that an endomorphism of type IV, VI or VII is uniformly continuous if and only if both ϕ and ψ are. \square

Following the proof in [5] step by step, where the following lemma is proved for endomorphisms of a free group, we can actually prove the same result for homomorphisms of free groups with different ranks.

Lemma 6.3. *Let $\phi : F_n \rightarrow F_m$ be a homomorphism. Then ϕ is uniformly continuous if and only if it is either trivial or injective.*

6.1 Type IV uniformly continuous endomorphisms

Let $\varphi : F_n \times F_m \rightarrow F_n \times F_m$ be a type IV uniformly continuous endomorphism. If both component mappings ϕ and ψ are trivial, then φ is trivial. If ϕ is trivial and ψ is injective, then $(u, v)\varphi^r = (1, v\psi^r)$ and φ has essentially the same dynamical behaviour as ψ . If ψ is trivial and ϕ is injective, then $(u, v)\varphi = (v\phi, 1)$ and $(u, v)\varphi^r = (1, 1)$, for $r > 1$.

So, take injective $\phi \in F_m \rightarrow F_n$ and $\psi \in \text{End}(F_m)$ such that $(u, v) \mapsto (v\phi, v\psi)$. We have that $(u, v)\varphi^r = (v\psi^{r-1}\phi, v\psi^r)$, for every $r > 0$. By uniqueness of extension, we have that $\hat{\varphi}$ is given by $(u, v) \mapsto (v\hat{\phi}, v\hat{\psi})$. As above, we have that $(u, v)\hat{\varphi}^r = (v\hat{\psi}^{r-1}\hat{\phi}, v\hat{\psi}^r)$.

So

$$\text{Fix}(\hat{\varphi}) = \{(v\hat{\phi}, v) \mid v \in \text{Fix}(\hat{\psi})\}.$$

and

$$\text{Per}(\hat{\varphi}) = \{(v\hat{\psi}^{\pi_v-1}\hat{\phi}, v) \mid v \in \text{Per}(\hat{\psi})\},$$

where π_v denotes the period of v .

Proposition 6.4. *Let φ be a type IV uniformly continuous endomorphism of $F_n \times F_m$, with $m, n > 1$. Then, $\text{Sing}(\hat{\varphi}) = \{(v\hat{\phi}, v) \mid v \in \text{Sing}(\hat{\psi})\}$. Consequently, $\text{Reg}(\hat{\varphi}) = \{(v\hat{\phi}, v) \mid v \in \text{Reg}(\hat{\psi})\}$.*

Proof. We start by showing that $\text{Sing}(\hat{\varphi}) \subseteq \{(v\hat{\phi}, v) \mid v \in \text{Sing}(\hat{\psi})\}$. Take some $(v\hat{\phi}, v) \in (\text{Fix}(\varphi))^c$ with $v \in \text{Fix}(\hat{\psi})$. Then, for every $\varepsilon > 0$, the open ball of radius ε centered in $(v\hat{\phi}, v)$ contains an element $(u_\varepsilon\phi, u_\varepsilon) \in \text{Fix}(\varphi)$, with $u_\varepsilon \in \text{Fix}(\psi)$. Notice that $d(v, u_\varepsilon) \leq d((v\hat{\phi}, v), (u_\varepsilon\phi, u_\varepsilon)) < \varepsilon$, thus $v \in (\text{Fix}(\psi))^c$.

For the reverse inclusion, take some $v \in \text{Sing}(\hat{\psi})$. As above, we know that for every $\varepsilon > 0$, there is some $u_\varepsilon \in B(v; \varepsilon) \cap \text{Fix}(\psi)$. Notice that, since $\hat{\phi}$ is uniformly continuous, for every $\varepsilon > 0$, there is some δ_ε such that, for all $x, y \in \widehat{F}_m$ such that $d(x, y) < \delta_\varepsilon$, we have that $d(x\hat{\phi}, y\hat{\phi}) < \varepsilon$. We want to prove that $(v\hat{\phi}, v) \in (\text{Fix}(\varphi))^c$, by showing that, for every $\varepsilon > 0$, the ball centered in $(v\hat{\phi}, v)$ with radius ε contains a fixed point of φ . So, let $\varepsilon > 0$ and consider $\delta = \min\{\delta_\varepsilon, \varepsilon\}$. We have that $(u_\delta\hat{\phi}, u_\delta) \in B((v\hat{\phi}, v); \varepsilon)$ since, by definition of u_δ , we have that $d(v, u_\delta) < \delta \leq \varepsilon$ and also, $d(v, u_\delta) < \delta_\varepsilon$ means that $d(v\hat{\phi}, u_\delta\hat{\phi}) < \varepsilon$. \square

Two infinite fixed points $(u\hat{\phi}, u)$ and $(v\hat{\phi}, v)$ are in the same $\text{Fix}(\varphi)$ -orbit if and only if there is some (finite) fixed point $(w\hat{\phi}, w)$ such that $(v\hat{\phi}, v) = (w\hat{\phi}, w)(u\hat{\phi}, u)$, i.e., if and only if u and v are in the same $\text{Fix}(\psi)$ -orbit. It is known that $\text{Reg}(\hat{\psi})$ has finitely many $\text{Fix}(\psi)$ -orbits (see [22]), so $\text{Reg}(\hat{\varphi})$ has finitely many $\text{Fix}(\varphi)$ -orbits.

Moreover, if ψ is an automorphism, then every regular infinite fixed point v of $\hat{\psi}$ must be either an attractor or a repeller (see [22]) and a singular infinite fixed point can never be an attractor nor a repeller (see [11]).

Proposition 6.5. *Let $v \in \text{Reg}(\hat{\psi})$. Then $(v\hat{\phi}, v)$ is an attractor if and only if v is an attractor for $\hat{\psi}$. Moreover, if $\alpha \in \text{Sing}(\hat{\varphi})$, then α is not an attractor.*

Proof. Let $(v\hat{\phi}, v) \in \text{Reg}(\hat{\varphi})$ such that v is an attractor through ψ . We have that $\hat{\phi}$ is uniformly continuous, so, for every $\varepsilon > 0$, there is some $\delta_\varepsilon > 0$ such that, for every $x, y \in \widehat{F}_m$ such that $d(x, y) < \delta_\varepsilon$ we have that $d(x\hat{\phi}, y\hat{\phi}) < \varepsilon$. Also, let $\tau > 0$ such that

$$\forall \beta \in \widehat{F}_m (d(v, \beta) < \tau \Rightarrow \lim_{n \rightarrow +\infty} \beta\hat{\psi}^n = v).$$

We will prove that, for every $x \in \widehat{F}_n$, $y \in B(v; \tau)$, we have $(x, y)\hat{\varphi}^n \rightarrow (v\hat{\phi}, v)$. Let $\varepsilon > 0$ and take $\delta = \min\{\varepsilon, \delta_\varepsilon\}$. Take $N \in \mathbb{N}$ such that for every $n > N$ we have that $d(v, y\hat{\psi}^n) < \delta$, which also implies that $d(v\hat{\phi}, y\hat{\psi}^n\hat{\phi}) < \varepsilon$. Thus, for every $n > N + 1$, we have that $(x, y)\hat{\varphi}^n = (y\hat{\psi}^{n-1}\hat{\phi}, v\hat{\psi}^n) \in B((v\hat{\phi}, v); \varepsilon)$. Thus, $(x, y)\hat{\varphi}^n \rightarrow (v\hat{\phi}, v)$.

Hence, $(v\hat{\phi}, v)$ is an attractor and the set of points it attracts is given by $\widehat{F}_n \times \mathcal{A}_{v, \hat{\psi}}$, where $\mathcal{A}_{v, \hat{\psi}}$ denotes the set of points v attracts through ψ .

Now, we prove that, if $(w\hat{\phi}, w) \in \text{Fix}(\hat{\varphi})$ is an attractor, then w is an attractor for ψ and that completes the proof. Indeed, take $\varepsilon > 0$ such that

$$\forall (x, y) \in \widehat{F}_n \times \widehat{F}_m (d((w\hat{\phi}, w), (x, y)) < \varepsilon \Rightarrow \lim_{n \rightarrow +\infty} (x, y)\hat{\varphi}^n = (w\hat{\phi}, w)).$$

For every $y \in B(w; \varepsilon)$, we have that $d((w\hat{\phi}, w), (w\hat{\phi}, y)) < \varepsilon$, so

$$\lim_{n \rightarrow +\infty} (y\psi^{n-1}\hat{\phi}, y\psi^n) = \lim_{n \rightarrow +\infty} (w\hat{\phi}, y)\hat{\varphi}^n = (w\hat{\phi}, w),$$

thus $y\psi^n \rightarrow w$. □

6.2 Type VI uniformly continuous endomorphisms

Let $\varphi : F_n \times F_m \rightarrow F_n \times F_m$ be a type VI uniformly continuous endomorphism and take $\phi \in \text{End}(F_n)$, $\psi \in \text{End}(F_m)$ such that $(u, v) \mapsto (u\phi, v\psi)$.

By uniqueness of extension, we have that $\hat{\varphi}$ is given by $(u, v) \mapsto (u\hat{\phi}, v\hat{\psi})$. We then have that $\text{Fix}(\hat{\varphi}) = \text{Fix}(\hat{\phi}) \times \text{Fix}(\hat{\psi})$, $\text{Sing}(\hat{\varphi}) = \text{Sing}(\hat{\phi}) \times \text{Sing}(\hat{\psi})$ and $\text{Reg}(\hat{\varphi}) = \text{Fix}(\hat{\phi}) \times \text{Reg}(\hat{\psi}) \cup \text{Reg}(\hat{\phi}) \times \text{Fix}(\hat{\psi})$. There is no hope of finding a finiteness condition that holds in general, since if $n = 2$ and ϕ is the identity mapping, then $\text{Sing}(\hat{\phi})$ is uncountable. In this case, both $\text{Reg}(\hat{\varphi})$ and $\text{Sing}(\hat{\varphi})$ are uncountable.

Now suppose that both ϕ and ψ are automorphisms, so φ is also an automorphism. We will now describe the attractors and repellers of $\hat{\varphi}$.

Proposition 6.6. *An infinite fixed point $\alpha = (u, v)$, where $u \in \text{Fix}(\hat{\phi})$ and $v \in \text{Fix}(\hat{\psi})$ is an attractor (resp. repeller) if and only if u and v are attractors (resp. repellers) for $\hat{\phi}$ and $\hat{\psi}$, respectively.*

Proof. Let $\alpha = (u, v)$ be an infinite fixed point, where $u \in \text{Fix}(\hat{\phi})$ and $v \in \text{Fix}(\hat{\psi})$. Clearly if $u \in \text{Fix}(\hat{\phi})$ and $v \in \text{Fix}(\hat{\psi})$ are attractors, then, $(u, v) \in \text{Fix}(\hat{\varphi})$ is an attractor. Indeed, in that case, there are $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that

$$\forall x \in \widehat{F}_n, \left(d(u, x) < \varepsilon_1 \Rightarrow \lim_{n \rightarrow +\infty} x\hat{\phi}^n = u \right)$$

and

$$\forall y \in \widehat{F}_m, \left(d(v, y) < \varepsilon_2 \Rightarrow \lim_{n \rightarrow +\infty} y\hat{\psi}^n = v \right).$$

Thus, taking $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, we have that

$$\begin{aligned} \forall (x, y) \in \widehat{F}_n \times \widehat{F}_m (d((u, v), (x, y)) < \varepsilon &\Rightarrow d(u, x) < \varepsilon \wedge d(v, y) < \varepsilon \\ &\Rightarrow \lim_{n \rightarrow +\infty} x\hat{\phi}^n = u \wedge \lim_{n \rightarrow +\infty} y\hat{\psi}^n = v \\ &\Rightarrow \lim_{n \rightarrow +\infty} (x, y)\hat{\varphi}^n = (u, v). \end{aligned}$$

Conversely, suppose w.l.o.g that u is not an attractor for $\hat{\phi}$. Then, for every $\varepsilon > 0$, there is some $x_\varepsilon \in \widehat{F}_n$ such that $d(u, x_\varepsilon) < \varepsilon$ but $x_\varepsilon\hat{\phi} \not\rightarrow u$. In this case, we have that, for every $\varepsilon > 0$, $d((u, v), (x_\varepsilon, v)) < \varepsilon$ and $(x_\varepsilon, v)\hat{\varphi}^n = (x_\varepsilon\hat{\phi}^n, v) \not\rightarrow (u, v)$.

The repellers case is analogous. □

6.3 Type VII uniformly continuous endomorphisms

Let $\varphi : F_n \times F_m \rightarrow F_n \times F_m$ be a type VII uniformly continuous endomorphism and take $\phi : F_n \rightarrow F_m$, $\psi : F_m \rightarrow F_n$ such that $(u, v) \mapsto (v\psi, u\phi)$.

By uniqueness of extension, we have that $(x, y)\hat{\varphi} = (y\hat{\psi}, x\hat{\phi})$. As done in the finite case, we have that an infinite fixed point must be $(x, y) \in \widehat{F_n \times F_m}$ such that $x = y\hat{\psi}$ and $y = x\hat{\phi}$ and so $x = x\hat{\phi}\hat{\psi}$ and $y = y\hat{\psi}\hat{\phi}$. So, $\text{Fix}(\hat{\varphi}) \subseteq \text{Fix}(\hat{\phi}\hat{\psi}) \times \text{Fix}(\hat{\psi}\hat{\phi})$. Also, for $x \in \text{Fix}(\hat{\phi}\hat{\psi})$, we have that $(x, x\hat{\phi})\hat{\varphi} = (x\hat{\phi}\hat{\psi}, x\hat{\phi}) = (x, x\hat{\phi})$. Similarly, we have that $(y\hat{\psi}, y)\hat{\varphi} = (y\hat{\psi}, y)$ for $y \in \text{Fix}(\hat{\psi}\hat{\phi})$. So, $\text{Fix}(\hat{\varphi}) = \{(x, x\hat{\phi}) \mid x \in \text{Fix}(\hat{\phi}\hat{\psi})\} = \{(y\hat{\psi}, y) \mid y \in \text{Fix}(\hat{\psi}\hat{\phi})\}$.

Notice that, by uniqueness of extension, we have that $\hat{\phi}\hat{\psi} = \widehat{\phi\psi}$ and $\hat{\psi}\hat{\phi} = \widehat{\psi\phi}$.

Proposition 6.7. *Let φ be a type VII uniformly continuous endomorphism of $F_n \times F_m$, with $m, n > 1$. Then, $\text{Sing}(\hat{\varphi}) = \{(x, x\hat{\phi}) \mid x \in \text{Sing}(\hat{\phi}\hat{\psi})\}$. Consequently, $\text{Reg}(\hat{\varphi}) = \{(x, x\hat{\phi}) \mid x \in \text{Reg}(\hat{\phi}\hat{\psi})\}$.*

Proof. We start by showing that $\text{Sing}(\hat{\varphi}) \subseteq \{(x, x\hat{\phi}) \mid x \in \text{Sing}(\hat{\phi}\hat{\psi})\}$. Take some $(x, x\hat{\phi}) \in (\text{Fix}(\varphi))^c$ with $x \in \text{Fix}(\hat{\phi}\hat{\psi})$. Then, for every $\varepsilon > 0$, the open ball of radius ε centered in $(x, x\hat{\phi})$ contains an element $(y_\varepsilon, y_\varepsilon\phi) \in \text{Fix}(\varphi)$, with $y_\varepsilon \in \text{Fix}(\phi\psi)$. Notice that $d(x, y_\varepsilon) \leq d((x, x\hat{\phi}), (y_\varepsilon, y_\varepsilon\phi)) < \varepsilon$, thus $x \in (\text{Fix}(\phi\psi))^c$.

For the reverse inclusion, take some $x \in \text{Sing}(\hat{\phi}\hat{\psi})$. As above, we know that for every $\varepsilon > 0$, there is some $y_\varepsilon \in B(x; \varepsilon) \cap \text{Fix}(\phi\psi)$. Notice that, since $\hat{\phi}$ is uniformly continuous, for every $\varepsilon > 0$, there is some δ_ε such that, for all $x, y \in \widehat{F_n}$ such that $d(x, y) < \delta_\varepsilon$, we have that $d(x\hat{\phi}, y\hat{\phi}) < \varepsilon$. We want to prove that $(x, x\hat{\phi}) \in (\text{Fix}(\varphi))^c$, by showing that, for every $\varepsilon > 0$, the ball centered in $(x, x\hat{\phi})$ with radius ε contains a fixed point of φ . So, let $\varepsilon > 0$ and consider $\delta = \min\{\delta_\varepsilon, \varepsilon\}$. We have that $(y_\delta, y_\delta\phi) \in B((x, x\hat{\phi}); \varepsilon)$ since, by definition of y_δ , we have that $d(x, y_\delta) < \delta \leq \varepsilon$ and also, $d(x, y_\delta) < \delta_\varepsilon$ means that $d(x\hat{\phi}, y_\delta\hat{\phi}) < \varepsilon$. \square

As done in the Type IV case, observing that two infinite fixed points $(x, x\hat{\phi})$ and $(y, y\hat{\phi})$ are in the same $\text{Fix}(\varphi)$ -orbit if and only if x and y are in the same $\text{Fix}(\phi\psi)$ -orbit, we have that $\text{Reg}(\hat{\varphi})$ has finitely many $\text{Fix}(\varphi)$ -orbits.

Proposition 6.8. *Suppose that $m = n$ and both ϕ and ψ are automorphisms. Let $x \in \text{Fix}(\hat{\phi}\hat{\psi})$. Then $x \in \text{Sing}(\hat{\phi}\hat{\psi})$ if and only if $x\hat{\phi} \in \text{Sing}(\hat{\psi}\hat{\phi})$. Consequently, $x \in \text{Reg}(\hat{\phi}\hat{\psi})$ if and only if $x\hat{\phi} \in \text{Reg}(\hat{\psi}\hat{\phi})$.*

Proof. Let $x \in \text{Sing}(\hat{\phi}\hat{\psi})$. Then $x\hat{\phi} \in \text{Fix}(\hat{\psi}\hat{\phi})$.

Since $\hat{\phi}$ and $\hat{\phi}^{-1}$ are uniformly continuous, we have that for every $\varepsilon > 0$, there are some $\delta_\varepsilon, \delta'_\varepsilon$ such that for every $z, w \in \widehat{F_n}$, we have

$$d(z, w) < \delta_\varepsilon \Rightarrow d(z\hat{\phi}, w\hat{\phi}) < \varepsilon$$

and

$$d(z, w) < \delta'_\varepsilon \Rightarrow d(z\hat{\phi}^{-1}, w\hat{\phi}^{-1}) < \varepsilon.$$

Now, let $\varepsilon > 0$. We want to prove that there is some $y \in \text{Fix}(\psi\phi) \cap B(x\hat{\phi}; \varepsilon)$. Since $x \in \text{Sing}(\hat{\phi}\hat{\psi})$, then there is some y_ε such that $y_\varepsilon \in \text{Fix}(\phi\psi) \cap B(x; \delta_\varepsilon)$, so

$$y_\varepsilon\phi \in \text{Fix}(\psi\phi) \cap B(x\hat{\phi}; \varepsilon).$$

Now, suppose $x \in \text{Reg}(\hat{\phi}\hat{\psi})$. There is some $\varepsilon > 0$ such that $B(x; \varepsilon) \cap \text{Fix}(\phi\psi) = \emptyset$. We will now prove that $B(x\hat{\phi}; \delta'_\varepsilon) \cap \text{Fix}(\psi\phi) = \emptyset$. Suppose there is some $y \in B(x\hat{\phi}; \delta'_\varepsilon) \cap \text{Fix}(\psi\phi)$. Then, $y\phi^{-1} \in B(x; \varepsilon)$. But

$$y\phi^{-1}(\phi\psi) = y\psi = y\psi\phi\phi^{-1} = y\phi^{-1},$$

so $y\phi^{-1} \in B(x; \varepsilon) \cap \text{Fix}(\phi\psi)$, which is absurd. \square

Proposition 6.9. *Suppose that $m = n$ and both ϕ and ψ are automorphisms. Let $x \in \text{Reg}(\hat{\phi}\hat{\psi})$. Then $(x, x\hat{\phi})$ is an attractor (resp. repeller) if and only if x is an attractor (resp. repeller) for $\hat{\phi}\hat{\psi}$. Moreover, if $\alpha \in \text{Sing}(\hat{\phi})$, then α is not an attractor nor a repeller.*

Proof. Let $(x, x\hat{\phi}) \in \text{Reg}(\hat{\phi}\hat{\psi})$ be such that x is an attractor through $\hat{\phi}\hat{\psi}$. Then there is some τ such that

$$\forall \beta \in \hat{F}_n(d(x, \beta) < \tau \Rightarrow \lim_{n \rightarrow +\infty} \beta(\hat{\phi}\hat{\psi})^n = x). \quad (4)$$

Also, since $\hat{\phi}$, $\hat{\phi}^{-1}$ and $\hat{\psi}$ are uniformly continuous, we have that for every $\varepsilon > 0$, there is some δ_ε , such that for every $z, w \in \hat{F}_n$, we have

$$d(z, w) < \delta_\varepsilon \Rightarrow d(z\hat{\phi}, w\hat{\phi}) < \varepsilon \wedge d(z\hat{\phi}^{-1}, w\hat{\phi}^{-1}) < \varepsilon \wedge d(z\hat{\psi}, w\hat{\psi}) < \varepsilon.$$

We start by proving that $x\hat{\phi}$ is an attractor for $\hat{\psi}\hat{\phi}$ by proving that

$$\forall \beta \in \hat{F}_n(d(x\hat{\phi}, \beta) < \delta_\tau \Rightarrow \lim_{n \rightarrow +\infty} \beta(\hat{\psi}\hat{\phi})^n = x\hat{\phi}). \quad (5)$$

Indeed, if $d(x\hat{\phi}, \beta) < \delta_\tau$ then $d(x\hat{\phi}\hat{\phi}^{-1}, \beta\hat{\phi}^{-1}) = d(x, \beta\hat{\phi}^{-1}) < \tau$, so

$$\lim_{n \rightarrow +\infty} \beta(\hat{\psi}\hat{\phi})^{n-1}\hat{\psi} = \lim_{n \rightarrow +\infty} \beta\hat{\phi}^{-1}(\hat{\phi}\hat{\psi})^n = x.$$

By continuity of $\hat{\phi}$, we have that $\lim_{n \rightarrow +\infty} \beta(\hat{\psi}\hat{\phi})^{n-1}\hat{\psi}\hat{\phi} = \lim_{n \rightarrow +\infty} \beta(\hat{\psi}\hat{\phi})^n = x\hat{\phi}$.

We want to prove that $(x, x\hat{\phi})$ is an attractor for $\hat{\phi}$. Let $\tau' = \min\{\tau, \delta_\tau\}$. We will prove that, for every $(z, w) \in B((x, x\hat{\phi}); \tau')$, we have $(z, w)\hat{\phi}^n \rightarrow (x, x\hat{\phi})$. Let $\varepsilon > 0$ and take $\delta = \min\{\varepsilon, \delta_\varepsilon\}$. By (4), we can take $N \in \mathbb{N}$ such that for every $n > N$ we have that

$$d(x, z(\hat{\phi}\hat{\psi})^n) < \delta, \quad \text{and so} \quad d(x\hat{\phi}, z(\hat{\phi}\hat{\psi})^n\hat{\phi}) < \varepsilon \quad (6)$$

and by (5), we can take $N' \in \mathbb{N}$ such that for every $n > N'$ we have that

$$d(x\hat{\phi}, w(\hat{\psi}\hat{\phi})^n) < \delta, \quad \text{and so} \quad d(x\hat{\phi}\hat{\psi}, w(\hat{\psi}\hat{\phi})^n\hat{\psi}) = d(x, w(\hat{\psi}\hat{\phi})^n\hat{\psi}) < \varepsilon. \quad (7)$$

Thus, for every $n > 2 \max\{N, N'\} + 1$, we have that, if n is even, $(z, w)\hat{\phi}^n = (z(\hat{\phi}\hat{\psi})^{\frac{n}{2}}, w(\hat{\psi}\hat{\phi})^{\frac{n}{2}}) \in B(x, x\hat{\phi})$ by the first parts of both (6) and (7); if n is odd, then $(z, w)\hat{\phi}^n = (w(\hat{\psi}\hat{\phi})^{\frac{n-1}{2}}\hat{\psi}, z(\hat{\phi}\hat{\psi})^{\frac{n-1}{2}}\hat{\phi}) \in B(x, x\hat{\phi})$ by the second parts of both (6) and (7).

Finally, we only have to prove that, if $(x, x\hat{\phi})$ is an attractor, for $\hat{\phi}$, then x is an attractor for $\hat{\phi}\hat{\psi}$. Suppose then that there is some $\tau > 0$ such that

$$\forall (z, w) \in \widehat{F_n \times F_n} \left(d((z, w), (x, x\hat{\phi})) < \tau \Rightarrow \lim_{n \rightarrow +\infty} (z, w)\hat{\phi}^n = (x, x\hat{\phi}) \right).$$

We will prove that

$$\forall y \in \hat{F}_n \left(d(x, y) < \tau \Rightarrow \lim_{n \rightarrow +\infty} y(\hat{\phi}\hat{\psi})^n = x \right)$$

which is equivalent to

$$\forall y \in \hat{F}_n \left(d(x, y) < \tau \Rightarrow \forall \varepsilon > 0 \exists N \in \mathbb{N} : \forall n > N d(y(\hat{\phi}\hat{\psi})^n, x) < \varepsilon \right)$$

Let $y \in \hat{F}_n$ be such that $d(x, y) < \tau$ and $\varepsilon > 0$. We know that $d((y, x\hat{\phi}), (x, x\hat{\phi})) < \tau$, there is $N \in \mathbb{N}$ such that for every $n > N$, $d((y, x\hat{\phi})\hat{\phi}^n, (x, x\hat{\phi})) < \varepsilon$. Let $n > N$. Then

$$d\left((y, x\hat{\phi})\hat{\phi}^{2n}, (x, x\hat{\phi})\right) = d\left((y(\hat{\phi}\hat{\psi})^n, x\hat{\phi}(\hat{\psi}\hat{\phi})^n), (x, x\hat{\phi})\right) < \varepsilon,$$

so $d(y(\hat{\phi}\hat{\psi})^n, x) < \varepsilon$.

The repellers case is analogous noticing that $(x, y)\hat{\phi}^{-1} = (y\hat{\psi}^{-1}, x\hat{\phi}^{-1})$ and so

$$\text{Reg}(\hat{\phi}^{-1}) = \left\{ (x, x\hat{\psi}^{-1}) \mid x \in \text{Reg}(\hat{\psi}^{-1}\hat{\phi}^{-1}) \right\} = \left\{ (x, x\hat{\psi}^{-1}) \mid x \in \text{Reg}\left((\hat{\phi}\hat{\psi})^{-1}\right) \right\}.$$

□

Corollary 6.10. *Let φ be a type VII automorphism of $F_n \times F_n$. Then $\alpha \in \text{Reg}(\hat{\varphi})$ is either an attractor or a repeller.*

Proof. Let φ be a type VII automorphism of $F_n \times F_n$ and take $\alpha \in \text{Reg}(\hat{\varphi})$. From Proposition 6.7, we have that $\alpha = (x, x\hat{\phi})$, for some $x \in \text{Reg}(\hat{\phi}\hat{\psi})$. We know, from [22], that x is either an attractor or a repeller. The result follows directly from Proposition 6.9. □

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