A GENERAL CAYLEY CORRESPONDENCE AND HIGHER TEICHMÜLLER SPACES

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Abstract. We introduce a new class of $\mathfrak{sl}_2$-triples in a complex simple Lie algebra $\mathfrak{g}$, which we call magical. Such an $\mathfrak{sl}_2$-triple canonically defines a real form and various decompositions of $\mathfrak{g}$. Using this decomposition data, we explicitly parameterize special connected components of the moduli space of Higgs bundles on a compact Riemann surface X for an associated real Lie group, hence also of the corresponding character variety of representations of $\pi_1 X$ in the associated real Lie group. This recovers known components when the real group is split, Hermitian of tube type, or $\mathrm{SO}_{p,q}$ with $1 < p \leqslant q$, and also constructs previously unknown components for the quaternionic real forms of E_6, E_7, E_8 and F_4. The classification of magical $\mathfrak{sl}_2$-triples is shown to be in bijection with the set of Θ-positive structures in the sense of Guichard–Wienhard, thus the mentioned parameterization conjecturally detects all examples of higher Teichmüller spaces. Indeed, we discuss properties of the surface group representations obtained from these Higgs bundle components and their relation to Θ-positive Anosov representations, which indicate that this conjecture holds.

Contents

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1. Introduction

In this paper we introduce a new framework for special components in moduli spaces of Higgs bundles. Via the nonabelian Hodge correspondence these components are the analogs of higher Teichmüller spaces in character varieties of surface group representations. The framework unifies previously described constructions for various types of real Lie groups, namely split real groups, Hermitian groups of tube type, and $\mathrm{SO}(p, q)$, and establishes the existence of new Teichmüller-like spaces for quaternionic exceptional real Lie groups.

Given a closed orientable surface Σ, of genus $g \geqslant 2$, and a reductive Lie group G, the character variety $\mathrm{Hom}(\pi_1\Sigma, \mathrm{G})/\mathrm{G}$ parameterizes conjugacy classes of reductive representations. Recall that the Teichmüller space $\mathcal{T}(\Sigma)$ of complex structures on Σ is realized as the set of conjugacy classes of *Fuchsian representations* $\rho : \pi_1\Sigma \to \mathrm{PSL}_2\mathbb{R}$. Moreover, $\mathcal{T}(\Sigma)$ defines an open and closed subset of $\mathrm{Hom}(\pi_1\Sigma, \mathrm{PSL}_2\mathbb{R})/\mathrm{PSL}_2\mathbb{R}$ consisting entirely of discrete and faithful representations. In the general setting, where $\mathrm{PSL}_2\mathbb{R}$ is replaced by a reductive group G, there is a class of representations (introduced by Labourie [56] and since studied by many authors; see [42, 51, 39]) called *Anosov representations* which generalize many features of Fuchsian representations. These representations define *open* subsets of the character variety consisting entirely of discrete and faithful representations, with many interesting geometric and dynamical properties. Unlike $\mathcal{T}(\Sigma) \subset \mathrm{Hom}(\pi_1\Sigma, \mathrm{PSL}_2\mathbb{R})/\mathrm{PSL}_2\mathbb{R}$, the Anosov loci are not necessarily closed, so do not automatically define connected components. In cases where they do constitute such components, the components are called *higher Teichmüller spaces* [70, 61].

One way of constructing Anosov representations is to post-compose a lift of a representation in $\mathcal{T}(\Sigma)$ with a homomorphism $\iota_e : \mathrm{SL}_2\mathbb{R} \to \mathrm{G}$. Up to conjugation, such homomorphisms are labeled by nilpotent elements e in the Lie algebra of G. When G is a complex simple Lie group, there is a (unique, up to conjugation) special homomorphism $\iota_e : \mathrm{SL}_2\mathbb{C} \to \mathrm{G}$, called *principal*, and the restriction of ι_e to $\mathrm{SL}_2\mathbb{R}$ is contained in the split real form $\mathrm{G}^{\mathbb{R}} \subset \mathrm{G}$, [55]. In [48], Hitchin used this to define connected components of $\mathcal{X}(\Sigma, \mathrm{G}^{\mathbb{R}})$ containing $\iota_e(\mathcal{T}(\Sigma))$ — now called *Hitchin components*. Labourie then showed all representations in Hitchin components are Anosov [56]. Other examples of connected components consisting of entirely of Anosov representations arise from so-called *maximal representations* into Hermitian Lie groups [14].

Recently, Guichard–Labourie–Wienhard [41] developed a refinement of the Anosov condition which aims to characterize all higher Teichmüller spaces. Roughly, a parabolic subgroup $\mathrm{P}_\Theta \subset \mathrm{G}^{\mathbb{R}}$ of a real Lie group $\mathrm{G}^{\mathbb{R}}$ has a Θ*-positive structure* if triples of pairwise disjoint transverse points in $\mathrm{G}^{\mathbb{R}}/\mathrm{P}_\Theta$ admit a cyclic order. For such pairs $(\mathrm{G}^{\mathbb{R}}, \mathrm{P}_\Theta)$, it is possible to define a set of Θ*-positive Anosov representations*. This set is open and conjectured to be closed [43]. The Θ-positive structures have been classified, leading to a list of possible higher Teichmüller spaces, which includes all the examples mentioned above as well as two other possible families.

The Hitchin components were discovered in [48] using the *nonabelian Hodge correspondence*, which defines a homeomorphism between the character variety $\mathcal{X}(\Sigma, \mathrm{G})$ and the moduli space $\mathcal{M}(X, \mathrm{G})$ of polystable G-Higgs bundles on a Riemann surface X, with underlying surface Σ. In particular, using Higgs bundles, Hitchin parameterized the Hitchin component by a vector space of holomorphic differentials. The spirit of the current paper is similar, and Higgs bundles will be our main focus. Due to the transcendental nature of this correspondence, it is very difficult to characterize the notions of Anosov representations and Θ-positive structures in terms of Higgs bundles so we develop in this paper a new Lie theoretic notion, called *magical* $\mathfrak{sl}_2$*-triples* in a complex Lie algebra $\mathfrak{g}$, which is adapted to the language of Higgs bundles.

In one of our main results, we classify all such magical $\mathfrak{sl}_2$-triples, and confirm that this classification establishes a bijection between them and Θ-positive structures. Furthermore we prove properties about the resulting Higgs bundles and find new connected components in moduli spaces $\mathcal{M}(X, \mathrm{G}^{\mathbb{R}})$ where $\mathrm{G}^{\mathbb{R}}$ is a real Lie group determined by a magical $\mathfrak{sl}_2$-triple. We call these components *Cayley components* (see Definition 7.3) because the construction generalizes a similarly named construction in the case where $\mathrm{G}^{\mathbb{R}}$ is a Hermitian group of tube type. Using the nonabelian Hodge correspondence to translate our results into statements about character varieties, we show that these components contain open sets of Θ-positive Anosov representations and hence should described new higher Teichmüller spaces.

We now give slightly more detailed statements of our results, starting with a description of the magical $\mathfrak{sl}_2$-triples.

Let $\mathfrak{g}$ be a complex simple Lie algebra and $e \in \mathfrak{g}$ be a nonzero nilpotent element. By the Jacobson–Morozov theorem, e can be completed to a triple $\{f, h, e\}$ which generates a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2\mathbb{C}$. This defines a bijective correspondence between conjugacy classes of nonzero nilpotents and conjugacy classes of $\mathfrak{sl}_2\mathbb{C}$-subalgebras. Using the decomposition of $\mathfrak{g}$ as an $\mathfrak{sl}_2\mathbb{C}$-module, we define a *vector space* involution

$$\sigma_e : \mathfrak{g} \to \mathfrak{g},$$

which is $+\mathrm{Id}$ on the trivial $\mathfrak{sl}_2\mathbb{C}$-representation, $-\mathrm{Id}$ on the nonzero highest weight spaces and $-\mathrm{Id}$ on f (see §2.1 for details). We call the $\mathfrak{sl}_2$-triple $\{f, h, e\} \subset \mathfrak{g}$ *magical* if σ_e is a *Lie algebra involution*.

One of our main results is that magical $\mathfrak{sl}_2$-triples determine components of character varieties which conjecturally describe all higher Teichmüller components. The character varieties in which these occur are determined by canonical real forms $\mathfrak{g}^{\mathbb{R}}$ associated to magical triples $\{f,h,e\}$ (see Definition 2.10).

Theorem A (Theorem 8.8)**.** *Let* G *be a complex simple Lie group with Lie algebra* $\mathfrak{g}$ *and* $\{f,h,e\}\subset\mathfrak{g}$ *be a magical* $\mathfrak{sl}_2$*-triple with canonical real form* $\mathrm{G}^{\mathbb{R}}\subset\mathrm{G}$*. Let* Σ *be a closed orientable surface of genus* $g\geqslant 2$ *and* $\mathcal{X}(\Sigma,\mathrm{G}^{\mathbb{R}})$ *be the* $\mathrm{G}^{\mathbb{R}}$*-character variety. Then, there exists a nonempty open and closed subset*

$$\mathcal{P}_e(\Sigma,\mathrm{G}^{\mathbb{R}})\subset\mathcal{X}(\Sigma,\mathrm{G}^{\mathbb{R}}),$$

which contains $\iota_e(\mathcal{T}(\Sigma))$ *and does not contain representations which factor through compact subgroups. Moreover, the centralizer of any representation* $\rho\in\mathcal{P}_e(\Sigma,\mathrm{G}^{\mathbb{R}})$ *is compact. In particular, there is no proper parabolic subgroup* $\mathrm{P}^{\mathbb{R}}$ *such that* $\rho:\pi_1\Sigma\to\mathrm{P}^{\mathbb{R}}\to\mathrm{G}^{\mathbb{R}}$*.*

As mentioned above, the sets $\mathcal{P}_e(\Sigma,\mathrm{G}^{\mathbb{R}})$ are constructed by applying the nonabelian Hodge correspondence to Cayley components in the moduli space $\mathcal{M}(X,\mathrm{G}^{\mathbb{R}})$ of $\mathrm{G}^{\mathbb{R}}$-Higgs bundles. Briefly, a $\mathrm{G}^{\mathbb{R}}$-Higgs bundle on a Riemann surface X is a pair $(\mathcal{E},\varphi)$, where $\mathcal{E}$ is a holomorphic principal bundle on X and φ (the Higgs field) is a holomorphic section of an associated vector bundle twisted by the holomorphic tangent bundle K of X (see §5.1 for more details). We will also consider the moduli space $\mathcal{M}_L(X,\mathrm{G}^{\mathbb{R}})$ of L-twisted Higgs bundles, where the twisting line bundle K is replaced by a line bundle L.

The Cayley components in $\mathcal{M}(X,\mathrm{G}^{\mathbb{R}})$ are constructed from the Lie theoretic data of a magical $\mathfrak{sl}_2$-triple. In addition to the real form $\mathfrak{g}^{\mathbb{R}}$, each magical $\mathfrak{sl}_2$-triple $\{f,h,e\}\subset\mathfrak{g}$ defines a real form $\mathfrak{g}^{\mathbb{R}}_{\mathcal{C}}$ of the centralizer $\mathfrak{g}_0$ of the semisimple element h (see Definition 2.13). We call $\mathfrak{g}^{\mathbb{R}}_{\mathcal{C}}$ the *Cayley real form*. We also show that a magical $\mathfrak{sl}_2$-triple $\{f,h,e\}$ is principal (see Proposition 4.5) in a simple subalgebra $\mathfrak{g}(e)$ where $\mathfrak{g}(e)$ is the semisimple part of the centralizer of the centralizer of $\{f,h,e\}$. This defines a decomposition of the Cayley real form (see Proposition 4.8) as

$$\mathfrak{g}^{\mathbb{R}}_{\mathcal{C}}=\mathfrak{g}^{\mathbb{R}}_{0,ss}\oplus\mathbb{R}^{\mathrm{rk}(\mathfrak{g}(e))},$$

where $\mathfrak{g}^{\mathbb{R}}_{0,ss}$ either zero or a simple real Lie algebra, and hence leads to a real Lie group

$$\mathrm{G}^{\mathbb{R}}_{\mathcal{C}}=\mathrm{G}^{\mathbb{R}}_{0,ss}\times(\mathbb{R}^+)^{\mathrm{rk}(\mathfrak{g}(e))},\tag{1.1}$$

which we call the Cayley group. This additional structure imposed by the existence of a magical $\mathfrak{sl}_2$-triple leads to a concrete description of these new connected components in terms of moduli spaces associated to the Cayley real form.

Theorem B (Theorem 7.1)**.** *Let* G *be a complex simple Lie group with Lie algebra* $\mathfrak{g}$*, and* $\{f,h,e\}\subset\mathfrak{g}$ *be a magical* $\mathfrak{sl}_2$*-triple with canonical real form* $\mathrm{G}^{\mathbb{R}}$*. Let* $\mathfrak{g}(e)\subset\mathfrak{g}$ *be the semisimple part of the centralizer of the centralizer of* $\{f,h,e\}$ *and* $\mathrm{G}^{\mathbb{R}}_{\mathcal{C}}=\mathrm{G}^{\mathbb{R}}_{0,ss}\times(\mathbb{R}^+)^{\mathfrak{g}(e)}$ *be the Cayley group. Let* X *be a compact Riemann surface of genus* $g\geqslant 2$ *with canonical bundle* K*, and let* $\mathcal{M}(X,\mathrm{G}^{\mathbb{R}})$ *be the moduli space of* $\mathrm{G}^{\mathbb{R}}$*-Higgs bundles over* X*. Then there is a positive integer* m_c *and a well-defined injective, open and closed map*

$$\Psi_e:\mathcal{M}_{K^{m_c+1}}(X,\mathrm{G}^{\mathbb{R}}_{0,ss})\times\bigoplus_{j=1}^{\mathrm{rk}(\mathfrak{g}(e))}H^0(K^{l_j+1})\longrightarrow\mathcal{M}(X,\mathrm{G}^{\mathbb{R}}),\tag{1.2}$$

where $\{l_j\}$ *are the exponents of* $\mathfrak{g}(e)$ *and* $\mathcal{M}_{K^{m_c+1}}(X,\mathrm{G}^{\mathbb{R}}_{0,ss})$ *is the moduli space of* K^{m_c+1}*-twisted* $\mathrm{G}^{\mathbb{R}}_{0,ss}$*-Higgs bundles. Furthermore, every Higgs bundle in the image of* Ψ_e *has nowhere vanishing Higgs field.*

Remark 1.1. The connected components in the image of Ψ_e are the Cayley components. The integer m_c and the exponents of $\mathfrak{g}(e)$ come from the decomposition of $\mathfrak{g}$ as an $\mathfrak{sl}_2\mathbb{C}$-module. Namely, as an $\mathfrak{sl}_2\mathbb{C}$-module, $\mathfrak{g} = W_0 \oplus W_{2m_c} \oplus \bigoplus_{j=1}^{\mathrm{rk}(\mathfrak{g}(e))} W_{2l_j}$, where W_{2k} is a direct sum of a certain number of copies of the unique irreducible $\mathfrak{sl}_2\mathbb{C}$-representation of dimension $2k+1$. See Lemma 5.7 for more details.

The map Ψ_e is a moduli space version of the global Slodowy slice map for Higgs bundles constructed in [16]. However, it is nontrivial to show that when $\{f, h, e\}$ is magical the Slodowy map descends to an injective map on moduli spaces. Our proof relies on our third main result, namely the classification of magical $\mathfrak{sl}_2$-triples given in Theorem 3.1.

Theorem C (Theorem 3.1 and Proposition 4.1)**.** *Let $\mathfrak{g}$ be a simple complex Lie algebra and let $\mathfrak{g}^{\mathbb{R}} \subset \mathfrak{g}$ be a real form. Then $\mathfrak{g}^{\mathbb{R}}$ is the canonical real form associated to a magical $\mathfrak{sl}_2$-triple if and only if it is one of the following:*

(1) $\mathfrak{g}$ *is any type and* $\mathfrak{g}^{\mathbb{R}}$ *is its split real form;*
(2) $\mathfrak{g}$ *has type* $\mathrm{A}_{2n-1},\ \mathrm{B}_n,\ \mathrm{C}_n,\ \mathrm{D}_n,\ \mathrm{D}_{2n}$*, or* E_7 *and* $\mathfrak{g}^{\mathbb{R}}$ *is Hermitian of tube type, i.e.* $\mathfrak{g}^{\mathbb{R}}$ *is one of the following:*
  (a) $\mathfrak{su}_{n,n}$*,*
  (b) $\mathfrak{so}_{2,p}$ (with $2+p=2n+1$),
  (c) $\mathfrak{sp}_{2n}\mathbb{R}$,
  (d) $\mathfrak{so}_{2,p}$ (with $2+p=n$),
  (e) $\mathfrak{so}^*_{4n}$*, or*
  (f) the real form of E_7 *of Hermitian type;*
(3) $\mathfrak{g}$ *has type* B_n *or* D_n *and* $\mathfrak{g}^{\mathbb{R}}$ *is* $\mathfrak{so}_{p,q}$ *with* $1<p<q$ *and* $p+q=2n+1$ *or* $p+q=2n$*;*
(4) $\mathfrak{g}$ *has type* $\mathrm{E}_6,\ \mathrm{E}_7,\ \mathrm{E}_8$ *or* F_4 *and* $\mathfrak{g}^{\mathbb{R}}$ *is its quaternionic real form.*

For this classification result, we use the correspondence between nilpotents in classical Lie algebras and partitions, and use classification data of Doković [20, 21] for exceptional Lie algebras.

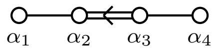
While a magical $\mathfrak{sl}_2$-triple $\{f, h, e\} \subset \mathfrak{g}$ defines a canonical real form $\mathfrak{g}^{\mathbb{R}} \subset \mathfrak{g}$, the fact that magical $\mathfrak{sl}_2$-triples are normal $\mathfrak{sl}_2$-triples means that the nilpotent elements f and e are not contained $\mathfrak{g}^{\mathbb{R}}$. We can obtain real $\mathfrak{sl}_2$-triples using the Cayley transform (which identifies the Poincaré disk with the upper half-plane) to define a bijection between normal $\mathfrak{sl}_2$-triples and so-called Cayley $\mathfrak{sl}_2$-triples (see §2.4). The Cayley transform, denoted by $\{\hat{f}, \hat{h}, \hat{e}\}$, of a magical $\mathfrak{sl}_2$-triple has each of its generators in the canonical real form $\mathfrak{g}^{\mathbb{R}}$. We define a Cayley triple $\{\hat{f}, \hat{h}, \hat{e}\} \subset \mathfrak{g}^{\mathbb{R}}$ in a real Lie algebra to be magical if it is associated, under the Cayley transform, to a magical triple in $\mathfrak{g}$ whose canonical real form is $\mathfrak{g}^{\mathbb{R}}$ (see Definition 2.16). This allows us to relate magical triples to the Guichard–Wienhard notion of Θ-positivity. Recall that a nilpotent element $\hat{e} \in \mathfrak{g}^{\mathbb{R}}$ determines a parabolic subgroup $\mathrm{P}^{\mathbb{R}}_{\hat{e}} \subset \mathrm{G}^{\mathbb{R}}$.

Theorem D (Theorem 8.14)**.** *Let* $\mathrm{G}^{\mathbb{R}}$ *be a real simple Lie group. A pair* $(\mathrm{G}^{\mathbb{R}}, \mathrm{P}^{\mathbb{R}}_{\Theta})$ *admits a* Θ*-positive structure if and only if there is a magical Cayley triple* $\{\hat{f}, \hat{h}, \hat{e}\} \subset \mathfrak{g}^{\mathbb{R}}$ *such that* $(\mathrm{G}^{\mathbb{R}}, \mathrm{P}^{\mathbb{R}}_{\Theta}) = (\mathrm{G}^{\mathbb{R}}, \mathrm{P}^{\mathbb{R}}_{\hat{e}})$*. In particular, there are four such families*

(1) $\mathrm{G}^{\mathbb{R}}$*-split and* $\mathrm{P}^{\mathbb{R}}_{\Theta}$ *is the Borel subgroup.*
(2) $\mathrm{G}^{\mathbb{R}}$ *is a Hermitian group of tube type and* $\mathrm{P}^{\mathbb{R}}_{\Theta}$ *is the maximal parabolic associated to the Shilov boundary.*
(3) $\mathrm{G}^{\mathbb{R}}$ *is locally isomorphic to* $\mathrm{SO}_{p,q}$ *and* $\mathrm{P}^{\mathbb{R}}_{\Theta}$ *stabilizes an isotropic flag of the form*

$$\mathbb{R} \subset \mathbb{R}^2 \subset \cdots \subset \mathbb{R}^{p-1} \subset \mathbb{R}^{q+1} \subset \cdots \subset \mathbb{R}^{p+q-1} \subset \mathbb{R}^{p+q}.$$

(4) $\mathrm{G}^{\mathbb{R}}$ *is a quaternionic real form of* E_6*,* E_7*,* E_8 *or* F_4*, so that its restricted root system is that of* F_4*, and* $\Theta = \{\alpha_1, \alpha_2\}$*, where*

F_4 :


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To further relate the open and closed sets $\mathcal{P}_e(\Sigma, \mathrm{G}^{\mathbb{R}})$ from Theorem A with Θ-positivity, we prove each of the sets $\mathcal{P}_e(\Sigma, \mathrm{G}^{\mathbb{R}})$ contains an open set of Θ-positive Anosov representations. As above, let $\mathfrak{g}(e) \subset \mathfrak{g}$ be the semisimple part of the centralizer of the centralizer of a magical $\mathfrak{sl}_2$-triple $\{f, h, e\}$. The Lie algebra $\mathfrak{g}(e)$ defines a split subalgebra $\mathfrak{g}(e)^{\mathbb{R}} \subset \mathfrak{g}^{\mathbb{R}}$ of the canonical real form. Let $\mathrm{G}(e)^{\mathbb{R}} \subset \mathrm{G}^{\mathbb{R}}$ be the connected subgroup with Lie algebra $\mathfrak{g}(e)^{\mathbb{R}}$. One special property of magical $\mathfrak{sl}_2$-triples is that their $\mathrm{G}^{\mathbb{R}}$-centralizer $\mathrm{C}^{\mathbb{R}}$ is compact. By construction, the groups $\mathrm{G}(e)^{\mathbb{R}}$ and $\mathrm{C}^{\mathbb{R}}$ commute, so we can form a $\mathrm{G}^{\mathbb{R}}$-representation by multiplying a $\mathrm{G}(e)^{\mathbb{R}}$-representation with a $\mathrm{C}^{\mathbb{R}}$-representation. In Proposition 8.11 we show that the sets $\mathcal{P}_e(\Sigma, \mathrm{G}^{\mathbb{R}})$ contain representations of the form

$$\rho = \rho_{Hit} * \rho_{\mathrm{C}^{\mathbb{R}}} : \pi_1\Sigma \to \mathrm{G}^{\mathbb{R}}, \tag{1.3}$$

where $\rho_{Hit} : \pi_1\Sigma \to \mathrm{G}(e)^{\mathbb{R}}$ is a $\mathrm{G}(e)^{\mathbb{R}}$-Hitchin representation and $\rho_{\mathrm{C}^{\mathbb{R}}} : \pi_1\Sigma \to \mathrm{C}^{\mathbb{R}}$ is any representation. This allows us to prove the following theorem.

Theorem E (Theorem 8.21)**.** *Let* G *be a simple complex Lie group with Lie algebra* $\mathfrak{g}$*. Let* $\{f, h, e\} \subset \mathfrak{g}$ *be a magical* $\mathfrak{sl}_2$*-triple with canonical real form* $\mathrm{G}^{\mathbb{R}} \subset \mathrm{G}$*. Then the set of representations* $\rho_{Hit} * \rho_{\mathrm{C}^{\mathbb{R}}}$ *from* (1.3) *are* Θ*-positive Anosov representations. In particular, each of the sets* $\mathcal{P}_e(\Sigma, \mathrm{G}^{\mathbb{R}}) \subset \mathcal{X}(\Sigma, \mathrm{G}^{\mathbb{R}})$ *from Theorem A contains a nonempty open set of* Θ*-positive Anosov representations.*

Remark 1.2. In the case that the magical $\mathfrak{sl}_2$-triple $\{f, h, e\} \subset \mathfrak{g}$ defines a principal $\mathfrak{sl}_2\mathbb{C}$-subalgebra this theorem is due to Labourie [56] and Fock-Goncharov [27], and for Hermitian groups of tube type it is due to [14]. It is expected that the sets $\mathcal{P}_e(\Sigma, \mathrm{G}^{\mathbb{R}})$ correspond exactly to the sets of Θ-positive Anosov representations in all cases.

Our results follow on from several projects focussed on the enumeration or understanding of distinguished components in the moduli spaces of Higgs bundles, and hence in character varieties for surface groups.

Given that Higgs bundles consist of an underlying principal bundle together with a Higgs field, it is clear that the topological type of the principal bundle is an invariant of connected components in the moduli space. Similarly, for the character varieties, the topological type of the associated flat bundle is also an invariant of the components. For complex reductive Lie groups [58, 31], and also for compact groups [62], the components are fully classified by these topological invariants. The count is more complicated for $\mathrm{G}^{\mathbb{R}}$-Higgs bundles, where $\mathrm{G}^{\mathbb{R}}$ is a non-compact real form. Indeed, this was already evident in Goldman's component count for $\mathrm{G}^{\mathbb{R}}$-character varieties where $\mathrm{G}^{\mathbb{R}}$ is a finite cover of $\mathrm{PSL}_2\mathbb{R}$ [35], and in Hitchin's Higgs-bundle version for $\mathrm{SL}_2\mathbb{R}$ [46]. Since then the enumeration and study of connected components in Higgs bundle moduli space for other non-compact real forms has been extensively pursued, for example see [4, 5, 6, 48, 3, 10, 30, 32, 40, 13, 33].

From one perspective, the starting point for the present work is the description of the Hitchin components in [48]. In particular, Hitchin proved Theorems A and B above for the case in which the magical $\mathfrak{sl}_2$-triple $\{f, h, e\} \subset \mathfrak{g}$ defines a principal $\mathfrak{sl}_2\mathbb{C}$-subalgebra, i.e. Case (1) of the Theorems C and D. Indeed, in this case, the first factor in (1.1) equals the center of the group (which is finite), and Hitchin's description is recovered exactly by the map (1.2). The opposite extreme, where the second factor in (1.1) is $H^0(K^2)$, occurs in Case (2) of the classification theorems. In this case Theorems A and B recover results for $G^{\mathbb{R}}$-Higgs bundles when $G^{\mathbb{R}}$ is of Hermitian tube type (see [11] and [7]). In particular, the moduli space $\mathcal{M}_{K^{m_c+1}}(X, G^{\mathbb{R}}_{0,ss}) \times H^0(K^2)$ has $m_c = 1$, and is then exactly the moduli space of K^2-twisted Higgs bundles for the Cayley partner to $G^{\mathbb{R}}$, i.e. the space which describes components with maximal Toledo invariant. The third case in Theorems C and D includes the case investigated in [3] for $G^{\mathbb{R}} = SO_{p,q}$, in which case the map (1.2) recovers the description of the 'exotic' components identified in [3], but now adds the remaining locally isomorphic groups.

From a slightly different perspective, our results relate to a program initiated by Hitchin to count connected components by a Morse-theoretic method. Described more fully in §7.5, the method is based on a proper function $F : \mathcal{M}(G^{\mathbb{R}}) \to \mathbb{R}$ defined by the L^2-norm of the Higgs field, and exploits the fact that proper functions attain their minima on closed sets. The locus of local minima thus has at least as many components as the full moduli space. Obvious minima of F, where the Higgs field is identically zero, lie on components detected by the topological invariants of principal bundles. The existence of other components — including the ones we study in this paper — is detected by more subtle local minima. In §7.5 we identify such minima coming from the components in the image of (1.2) and use this to enumerate the components.

We end this introduction with some open questions and a conjecture. The first unresolved issue is whether the components identified by Theorems A and B are indeed the components of Θ-positive representations conjectured in [43]. We also expect that our results exhaust the list of such components.

Conjecture 1.3. *Under the nonabelian Hodge correspondence between $\mathcal{M}(X, G^{\mathbb{R}})$ and $\mathcal{X}(\Sigma, G^{\mathbb{R}})$, the Cayley components in $\mathcal{M}(X, G^{\mathbb{R}})$ correspond to the higher Teichmüller components in $\mathcal{X}(\Sigma, G^{\mathbb{R}})$. Moreover, the list of Cayley components completes the classification of connected components in $\mathcal{M}(X, G^{\mathbb{R}})$ where $G^{\mathbb{R}}$ is any simple Lie group*

As pointed out in Remark 1.2, we expect the first part of the conjecture to be answered in the affirmative. The existence of as-yet-undetected components is not yet ruled out in all cases. One way to settle the second part of the conjecture would be to complete an analysis of local minima of the Hitchin function, though this seems daunting especially in the case of the exceptional groups.

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2. Nilpotents and magical $\mathfrak{sl}_2$-triples

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra and G be a connected complex Lie group with Lie algebra $\mathfrak{g}$. For background on nilpotents we mostly follow [17].

2.1. Nilpotents and $\mathfrak{sl}_2\mathbb{C}$-triples.

An element $e \in \mathfrak{g}$ is called *nilpotent* if the corresponding adjoint map

$$\mathrm{ad}_e : \mathfrak{g} \to \mathfrak{g}$$

is a nilpotent endomorphism. The nilpotent elements of $\mathfrak{g}$ form a G-invariant cone consisting of finitely many G-orbits. In fact, there is a unique nilpotent orbit which is open and dense in the nilpotent cone, and elements in this orbit are called *principal nilpotents*. For example, when $\mathrm{G} = \mathrm{SL}_n\mathbb{C}$, nilpotent orbits are in bijection with partitions of n by the Jordan decomposition theorem. In this case, a principal nilpotent is conjugate to a full Jordan block.

By the Jacobson–Morozov theorem, every nonzero nilpotent element $e \in \mathfrak{g}$ can be completed to a triple of nonzero elements $\{f, h, e\} \subset \mathfrak{g}$ satisfying

$$[h, e] = 2e, \qquad [h, f] = -2f \qquad \text{and} \qquad [e, f] = h. \tag{2.1}$$

Moreover, if $\{f, h, e\}$ and $\{f', h, e\}$ are two such triples, then $f = f'$. A triple $\{f, h, e\}$ of nonzero elements verifying the bracket relations (2.1) will be called an $\mathfrak{sl}_2$-*triple* and the subalgebra $\langle f, h, e\rangle \subset \mathfrak{g}$ will be called the $\mathfrak{sl}_2\mathbb{C}$-*subalgebra associated to* $\{f, h, e\}$. This defines a bijection between conjugacy classes of nilpotents and conjugacy classes of $\mathfrak{sl}_2\mathbb{C}$-subalgebras

$$\{e \in \mathfrak{g} \text{ nonzero nilpotent}\}/\mathrm{G} \xleftrightarrow{1-1} \{\phi : \mathfrak{sl}_2\mathbb{C} \to \mathfrak{g}\}/\mathrm{G}. \tag{2.2}$$

An $\mathfrak{sl}_2$-triple $\{f, h, e\}$ defines two decompositions of $\mathfrak{g}$. One as an $\mathfrak{sl}_2\mathbb{C}$-module, namely

$$\mathfrak{g} = \bigoplus_{j=0}^{M} W_j, \tag{2.3}$$

where W_j is isomorphic to a direct sum of n_j copies (with $n_j \geqslant 0$) of the unique $(j+1)$-dimensional $\mathfrak{sl}_2\mathbb{C}$-representation. By $\mathfrak{sl}_2$-*data* of $\{f, h, e\}$ we will mean the collection of pairs of non-negative integers (j, n_j) such that, for each $j > 0$, the multiplicity n_j of W_j is positive (so we consider the pair $(0, n_0)$ part of the $\mathfrak{sl}_2$-data even if $n_0 = 0$). Another decomposition of $\mathfrak{g}$ determined by $\{f, h, e\}$ is given by ad_h-weight spaces,

$$\mathfrak{g} = \bigoplus_{j=-l}^{l} \mathfrak{g}_j, \tag{2.4}$$

where $\mathfrak{g}_j = \{x \in \mathfrak{g} \mid \mathrm{ad}_h(x) = jx\}$. Note that $\mathrm{ad}_e : \mathfrak{g}_j \to \mathfrak{g}_{j+2}$ and $\mathrm{ad}_f : \mathfrak{g}_j \to \mathfrak{g}_{j-2}$. The subalgebra $\bigoplus_{j\geqslant 0} \mathfrak{g}_j$ is a parabolic subalgebra determined by the nilpotent e.

Remark 2.1. A nilpotent $e \in \mathfrak{g}$ is called *even* if ad_h only has even eigenvalues, i.e., if $\mathfrak{g}_j = 0$ for all j odd. The $\mathfrak{sl}_2\mathbb{C}$-subalgebra $\langle f, h, e\rangle \subset \mathfrak{g}$, for an even nilpotent e, defines a subgroup of the adjoint group of G which is isomorphic to $\mathrm{PSL}_2\mathbb{C}$.

The centralizer $\ker(\mathrm{ad}_e) = V(e) = V \subset \mathfrak{g}$ of e decomposes into a direct sum of highest weight spaces of each W_j

$$V = \bigoplus_{j\geqslant 0} V_j, \tag{2.5}$$

where $V_j = W_j \cap \mathfrak{g}_j$. We have the following proposition (see Lemmas 3.4.5 and 3.7.3 of [17]).

Proposition 2.2. *The subspace $V \subset \mathfrak{g}$ is a subalgebra such that $V_0 = W_0$ is a reductive subalgebra and $\bigoplus_{j>0} V_j$ is a nilpotent subalgebra. In addition, for each j, k, the subspace $W_j \cap \mathfrak{g}_k \subset \mathfrak{g}$ is preserved by bracketing with W_0.*

Remark 2.3.

(1) Note that $V_0 = W_0 \subset \mathfrak{g}$ is the Lie subalgebra which centralizes the $\mathfrak{sl}_2\mathbb{C}$-subalgebra $\langle f, h, e\rangle$. We will often denote this subalgebra by $\mathfrak{c} = W_0 \subset \mathfrak{g}$.
(2) The affine space

$$f + V \subset \mathfrak{g}$$

is a slice of the adjoint action of G on $\mathfrak{g}$ through the nilpotent f, which is usually called a *Slodowy slice* [69]. Note that $\mathfrak{c}$ preserves the Slodowy slice.

2.2. **Magical $\mathfrak{sl}_2$-triples.** Let $\{f, h, e\} \subset \mathfrak{g}$ be an $\mathfrak{sl}_2$-triple. Note that

$$\mathfrak{g} = \bigoplus_{j=0}^{M} \bigoplus_{k=0}^{j} W_j \cap \mathfrak{g}_{j-2k}$$

and that $W_j \cap \mathfrak{g}_{j-2k} = \mathrm{ad}_f^k(V_j)$. Consider the map $\sigma_e : \mathfrak{g} \to \mathfrak{g}$ defined by the linear extension of

$$\sigma_e(x) = \begin{cases} x & \text{if } x \in V_0 \\ (-1)^{k+1}x & \text{if } x \in \mathrm{ad}_f^k(V_j) \text{ for some } 0 \leqslant k \leqslant j \text{ and } j > 0. \end{cases} \tag{2.6}$$

This defines a vector space involution of $\mathfrak{g}$ with $\sigma_e|_{V_j} = -\mathrm{Id}$ for $j > 0$. On the given $\mathfrak{sl}_2$-triple, we have $\sigma_e(f) = -f$, $\sigma_e(h) = h$ and $\sigma_e(e) = -e$.

Definition 2.4. *An $\mathfrak{sl}_2$-triple $\{f, h, e\}$ will be called* magical *if the involution $\sigma_e : \mathfrak{g} \to \mathfrak{g}$ defined by (2.6) is a Lie algebra involution. We will also refer to a nilpotent element $e \in \mathfrak{g}$ as* magical *if it belongs to a magical $\mathfrak{sl}_2$-triple.*

Remark 2.5. Note that if $\{f, h, e\} \subset \mathfrak{g}$ is magical and contained in a reductive subalgebra $\mathfrak{g}' \subset \mathfrak{g}$, then $\{f, h, e\}$ is magical in the subalgebra $\mathfrak{g}'$.

We will classify magical nilpotents in §3 and by (2.2) this will be equivalent to classifying magical $\mathfrak{sl}_2$-triples. Although the terminology was not used, Kostant showed that a principal $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ is magical; see [55, Lemma 6.4A] and [48, Proposition 6.1]. A key feature of principal $\mathfrak{sl}_2$-triples is that the subalgebra $\mathfrak{g}_0$ is a Cartan subalgebra. We now generalize this. For an $\mathfrak{sl}_2\mathbb{C}$-triple $\{f, h, e\}$, let $Z_{2m_j} = W_{2m_j} \cap \mathfrak{g}_0$. Thus, we have a decomposition of $\mathfrak{g}_0$ as a $\mathfrak{c}$-module

$$\mathfrak{g}_0 = \mathfrak{c} \oplus \bigoplus_{j=1}^{M} Z_{2m_j}. \tag{2.7}$$

Proposition 2.6. *If $\{f, h, e\}$ is a magical $\mathfrak{sl}_2$-triple, then $[Z_{2m_i}, Z_{2m_j}] \subset \mathfrak{c}$ for all m_i, m_j, and $[Z_{2m_i}, Z_{2m_j}] = 0$ if $m_i \neq m_j$.*

Before giving the proof we recall some facts about $\mathfrak{sl}_2\mathbb{C}$-representation theory. Consider the decomposition (2.3) of $\mathfrak{g}$. The Lie bracket defines a morphism of $\mathfrak{sl}_2\mathbb{C}$-representations:

$$[\ ,\ ] : W_{2m_i} \otimes W_{2m_j} \to W_0 \oplus \bigoplus_{k=1}^{M} W_{2m_k}.$$

According to the Clebsch–Gordan formula, the tensor product $W_{2m_i} \otimes W_{2m_j}$ decomposes as a direct sum of irreducible representations

$$W_{2m_i} \otimes W_{2m_j} \cong \bigoplus_{l=0}^{2\min(m_i,m_j)} \left(S^{2m_i+2m_j-2l}\right)^{\oplus n_i n_j}, \tag{2.8}$$

where S^d is the d^{th}-symmetric product of the standard $\mathfrak{sl}_2\mathbb{C}$-representation W_1. The projection onto the summand $(S^{2m_i+2m_j-2l})^{\oplus n_i n_j}$ is given by contracting l-times with the volume form on $\mathbb{C}^2$. If we represent S^{2d} as homogeneous polynomials in z_1, z_2 of degree $2d$, the elements $x \in Z_{2m_i}$ and $y \in Z_{2m_j}$ are multiples of $z_1^{m_i} z_2^{m_i}$ and $z_1^{m_j} z_2^{m_j}$, respectively. Moreover, since the volume form is skew-symmetric, contracting $(2l+1)$-times $z_1^{m_i} z_2^{m_i}$ with $z_1^{m_j} z_2^{m_j}$ gives zero. Thus, the projection of the bracket $[x, y]$ to Z_{2m_k} is zero when $m_i + m_j = m_k + 1 \mod 2$.

Proof of Proposition 2.6. Suppose $\{f, h, e\}$ is magical. Let $x \in Z_{2m_i}$, $y \in Z_{2m_j}$ and write $[x, y] = z_0 + \sum z_k$, where $z_0 \in \mathfrak{c}$ and $z_k \in Z_{2m_k}$. Note that $\sigma_e(z_0) = z_0$ and $\sigma_e(z_k) = (-1)^{m_k+1} z_k$. By assumption, we have $\sigma_e([x, y]) = [\sigma_e(x), \sigma_e(y)]$, thus

$$z_0 + \sum (-1)^{m_k+1} z_k = (-1)^{m_i+m_j}\left(z_0 + \sum z_k\right).$$

In particular, if $m_i + m_j = m_k \mod 2$ then $z_k = 0$. It follows, by the above discussion, that $z_k = 0$ for all $k > 0$. Thus,

$$[Z_{2m_i}, Z_{2m_j}] \subset \mathfrak{c}$$

for all m_i, m_j. Moreover, by Schur's Lemma, the projection of the bracket $[x, y]$ to W_0 is zero unless the decomposition of $W_{2m_i} \otimes W_{2m_j}$ has the trivial representation W_0 as a summand. But by (2.8) this only happens if $m_i = m_j$, completing the proof. □

By Proposition 2.6, a magical $\mathfrak{sl}_2$-triple $\{f, h, e\}$ defines a Lie algebra involution $\theta_e : \mathfrak{g}_0 \to \mathfrak{g}_0$

$$\theta_e(x) = \begin{cases} x & \text{if } x \in W_0 \\ -x & \text{if } x \in \bigoplus_{j=1}^{M} Z_{2m_j}. \end{cases} \tag{2.9}$$

Remark 2.7. Note that θ_e and $\sigma_e|_{\mathfrak{g}_0}$ are different since $\theta_e(h) = -h$ and $\sigma_e(h) = h$.

2.3. **The canonical real form associated to a magical nilpotent.** In this section we mainly follow [1, §3]. A *real form* of the complex Lie group G is defined to be the fixed point set G^τ of an anti-holomorphic involution

$$\tau : \mathrm{G} \to \mathrm{G}.$$

We will sometimes refer to the involution τ itself as a real form. Note that even though G is connected, the real form G^τ may not be connected. For example, $\mathrm{SO}_{p,q} \subset \mathrm{SO}_{p+q}\mathbb{C}$ is a real form which has two components whenever p or q is nonzero. If the fixed point set $\mathrm{G}^\tau \subset \mathrm{G}$ is compact, the real form is said to be *compact*. Such real forms exist and are unique up to conjugation.

A holomorphic involution $\sigma : \mathrm{G} \to \mathrm{G}$ is called a *Cartan involution for a real form* τ if $\sigma\tau = \tau\sigma$ and, in addition, $\sigma\tau$ is a compact real form of G. Given a real form τ, a Cartan involution σ for τ exists and is unique up to conjugation by the identity component $(\mathrm{G}^\tau)^0 \subset \mathrm{G}^\tau$. Conversely, given a holomorphic involution σ, there exists a real form τ, unique up to conjugation by $(\mathrm{G}^\sigma)^0$, such that σ is a Cartan involution for τ.

The following proposition will be useful (cf. [1, Theorem 3.13]).

Proposition 2.8. *Let* $\mathrm{G}' \subset \mathrm{G}$ *be a reductive subgroup. If* $\sigma : \mathrm{G} \to \mathrm{G}$ *is a holomorphic involution of* G *with* $\sigma(\mathrm{G}') = \mathrm{G}'$ *and* $\tau_{\mathrm{G}'}$ *is a real form of* G' *such that* $\sigma|_{\mathrm{G}'}$ *a Cartan involution for* $\tau_{\mathrm{G}'}$*, then there exists a real form* $\tau : \mathrm{G} \to \mathrm{G}$ *with* σ *a Cartan involution for* τ *such that* $\tau|_{\mathrm{G}'} = \tau_{\mathrm{G}'}$*. Conversely, if* $\tau : \mathrm{G} \to \mathrm{G}$ *is a real form of* G *with* $\tau(\mathrm{G}') = \mathrm{G}'$ *and* $\sigma_{\mathrm{G}'}$ *is a Cartan involution for* $\tau|_{\mathrm{G}'}$*, then there exists a Cartan involution* $\sigma : \mathrm{G} \to \mathrm{G}$ *for* τ *such that* $\sigma|_{\mathrm{G}'} = \sigma_{\mathrm{G}'}$*.*

An involution $\alpha : \mathrm{G} \to \mathrm{G}$ induces an involution $\alpha : \mathfrak{g} \to \mathfrak{g}$, and the Lie algebra of the fixed-point group G^α is the fixed-point subalgebra $\mathfrak{g}^\alpha$. Moreover, if $\alpha : \mathrm{G} \to \mathrm{G}$ is holomorphic or anti-holomorphic then $\alpha : \mathfrak{g} \to \mathfrak{g}$ is complex linear or conjugate-linear respectively. In the latter case, $\mathfrak{g}^\alpha$ is a real form of $\mathfrak{g}$, i.e., $\mathfrak{g}^\alpha \otimes \mathbb{C} \cong \mathfrak{g}$.

Remark 2.9. An involution of the Lie algebra $\mathfrak{g}$ does *not* always integrate to an involution of the group G. However, every inner involution of $\mathfrak{g}$ integrates to G. Also, when G is an adjoint group or simply connected, every Lie algebra involution integrates to G. Whenever we are dealing with Lie algebra involutions, *we will always assume* G *is a Lie group for which the involution integrates.*

Now fix a real form τ of G and let σ be a Cartan involution for τ. Denote the fixed-point groups by $\mathrm{G}^\mathbb{R} = \mathrm{G}^\tau$ and $\mathrm{H} = \mathrm{G}^\sigma$. Then

$$\mathrm{H}^\mathbb{R} = \mathrm{H} \cap \mathrm{G}^\mathbb{R}$$

is a maximal compact subgroup of both $\mathrm{G}^\mathbb{R}$ and H. Furthermore, the associated Lie algebra involution $\sigma : \mathfrak{g}^\mathbb{R} \to \mathfrak{g}^\mathbb{R}$ defines an $\mathrm{H}^\mathbb{R}$-invariant decomposition of $\mathfrak{g}^\mathbb{R}$ into ± 1-eigenspaces

$$\mathfrak{g}^\mathbb{R} = \mathfrak{h}^\mathbb{R} \oplus \mathfrak{m}^\mathbb{R},$$

called a *Cartan decomposition*. The associated H-invariant decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ will be referred to as the *complexified Cartan decomposition*.

Now we go back to our setting. Since the definition of a magical $\mathfrak{sl}_2$-triple involves a complex linear involution of $\mathfrak{g}$, there is a canonical real form of $\mathfrak{g}$ associated to each such triple.

Definition 2.10. *Let* $\{f, h, e\} \subset \mathfrak{g}$ *be a magical* $\mathfrak{sl}_2$*-triple and* $\sigma_e : \mathfrak{g} \to \mathfrak{g}$ *be the associated Lie algebra involution. Let* $\tau_e : \mathfrak{g} \to \mathfrak{g}$ *be a real form such that* σ_e *is a Cartan involution* (2.6)*. The Lie algebra* $\mathfrak{g}^\mathbb{R} = \mathfrak{g}^{\tau_e}$ *will be called* the canonical real form of $\mathfrak{g}$ associated to $\{f, h, e\}$.

Remark 2.11. The $\mathfrak{sl}_2\mathbb{C}$-subalgebra $\mathfrak{s} = \langle f, h, e\rangle$ spanned by the magical $\mathfrak{sl}_2$-triple is σ_e-stable. Moreover, $\sigma_e|_\mathfrak{s}$ is a Cartan involution for the conjugate linear involution $\tau_\mathfrak{s} : \mathfrak{s} \to \mathfrak{s}$ defined by

$$\tau_\mathfrak{s}(h) = -h \qquad \tau_\mathfrak{s}(e) = f \qquad \text{and} \qquad \tau_\mathfrak{s}(f) = e\ . \tag{2.10}$$

Since $\mathfrak{s}^{\tau_\mathfrak{s}}$ is isomorphic to $\mathfrak{sl}_2\mathbb{R}$, we can choose the canonical real form $\tau_e : \mathfrak{g} \to \mathfrak{g}$ such that the magical $\mathfrak{sl}_2\mathbb{C}$-subalgebra defines a subalgebra of $\mathfrak{g}^\mathbb{R}$ isomorphic to $\mathfrak{sl}_2\mathbb{R}$.

Definition 2.12. *Let* $\{f, h, e\} \subset \mathfrak{g}$ *be a magical* $\mathfrak{sl}_2$*-triple and* $\sigma_e : \mathfrak{g} \to \mathfrak{g}$ *be the associated Lie algebra involution* (2.6)*. Let* $\tau_e : \mathfrak{g} \to \mathfrak{g}$ *be a real form such that* σ_e *is a Cartan involution. Let* G *be a connected complex Lie group with Lie algebra* $\mathfrak{g}$ *such that* σ_e *integrates to an involution* $\sigma_e : \mathrm{G} \to \mathrm{G}$ *and let* $\tau_e : \mathrm{G} \to \mathrm{G}$ *be the anti-holomorphic involution integrating* τ_e*. We define the* canonical real form $\mathrm{G}^\mathbb{R}$ of G associated to e *to be the fixed-point group* $\mathrm{G}^{\tau_e} \subset \mathrm{G}$*.*

The Lie algebra of the canonical real form $\mathrm{G}^\mathbb{R}$ is the canonical real form $\mathfrak{g}^\mathbb{R}$ of Definition 2.10. The complex linear Lie algebra involution $\theta_e : \mathfrak{g}_0 \to \mathfrak{g}_0$ defined in (2.9) also associates a real form to a magical $\mathfrak{sl}_2$-triple.

Definition 2.13. *Let $\{f,h,e\}$ be a magical $\mathfrak{sl}_2$-triple, $\mathfrak{g}_0$ be the centralizer of h and $\theta_e:\mathfrak{g}_0\to\mathfrak{g}_0$ be the Lie algebra involution from (2.9). Let $\tau_0:\mathfrak{g}_0\to\mathfrak{g}_0$ be a real form, such that θ_e is a Cartan involution for τ_0. The Lie algebra $\mathfrak{g}_0^{\tau_0}\subset\mathfrak{g}_0$ will be called the* Cayley real form of $\mathfrak{g}_0$ associated to e*, and denoted by $\mathfrak{g}_{\mathcal{C}}^{\mathbb{R}}$.*

Remark 2.14. Note that $\theta_e|_{\mathfrak{c}}=\sigma_e|_{\mathfrak{c}}=\mathrm{Id}:\mathfrak{c}\to\mathfrak{c}$ is a Cartan involution for a compact real form $\tau_{\mathfrak{c}}$ of $\mathfrak{c}$. Thus, by Proposition 2.8, we can assume that the canonical real form $\tau_e:\mathfrak{g}\to\mathfrak{g}$ and the Cayley real form $\tau_0:\mathfrak{g}\to\mathfrak{g}$ are such that $\tau_e|_{\mathfrak{c}}=\tau_{\mathfrak{c}}=\tau_0|_{\mathfrak{c}}$. In particular, the centralizer $\mathfrak{c}^{\tau_{\mathfrak{c}}}$ of the $\mathfrak{sl}_2\mathbb{R}$-subalgebra $\mathfrak{s}^{\tau_e}\subset\mathfrak{g}^{\tau_e}$ is compact (where $\mathfrak{s}=\langle f,h,e\rangle$).

2.4. **Real nilpotents and the Sekiguchi correspondence.** The classification of magical $\mathfrak{sl}_2$-triples will use the classification of nilpotent elements in real Lie algebras and the Sekiguchi correspondence. Fix a real form $\tau:\mathrm{G}\to\mathrm{G}$, a Cartan involution $\sigma:\mathrm{G}\to\mathrm{G}$ for τ and write $\mathrm{G}^{\mathbb{R}}=\mathrm{G}^\tau$, $\mathrm{H}=\mathrm{G}^\sigma$ and $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$ for the complexified Cartan decomposition. In this section, we will refer to $\mathfrak{sl}_2$-triples in $\mathfrak{g}$ as $\mathfrak{sl}_2\mathbb{C}$-triples, to distinguish them from $\mathfrak{sl}_2\mathbb{R}$-triples in $\mathrm{G}^{\mathbb{R}}$, which will also appear.

The *Sekiguchi correspondence* gives a one-to-one correspondence between $\mathrm{G}^{\mathbb{R}}$-conjugacy classes of nilpotents in $\mathfrak{g}^{\mathbb{R}}$ and H-conjugacy classes of nilpotents in $\mathfrak{m}$:

$$\{\hat{e}\in\mathfrak{g}^{\mathbb{R}}\text{ nonzero nilpotent}\}/\mathrm{G}^{\mathbb{R}}\xleftrightarrow{1-1}\{e\in\mathfrak{m}\text{ nonzero nilpotent}\}/\mathrm{H}. \tag{2.11}$$

It was proven independently in [64] and [19].

We now describe the correspondence in more detail and refer the reader to [17, Chapter 9] and [1, §6.1] for further details. The Jacobson–Morozov theorem also holds over $\mathbb{R}$. Namely, every nonzero nilpotent $\hat{e}\in\mathfrak{g}^{\mathbb{R}}$ can be completed to an $\mathfrak{sl}_2\mathbb{R}$-triple $\{\hat{f},\hat{h},\hat{e}\}$, such that $\hat{f},\hat{h},\hat{e}\in\mathfrak{g}^{\mathbb{R}}\setminus\{0\}$ satisfy the bracket relations (2.1). Moreover, this defines a bijection on conjugacy classes

$$\{\hat{e}\in\mathfrak{g}^{\mathbb{R}}\text{ nonzero nilpotent}\}/\mathrm{G}^{\mathbb{R}}\xleftrightarrow{1-1}\{\phi:\mathfrak{sl}_2\mathbb{R}\to\mathfrak{g}^{\mathbb{R}}\}/\mathrm{G}^{\mathbb{R}}.$$

Following [17, Chapter 9.4], an $\mathfrak{sl}_2\mathbb{R}$-triple $\{\hat{f},\hat{h},\hat{e}\}\subset\mathfrak{g}^{\mathbb{R}}$ is called a *Cayley triple* if $\sigma(\hat{f})=-\hat{e}$, $\sigma(\hat{e})=-\hat{f}$ and $\sigma(\hat{h})=-\hat{h}$. Using Proposition 2.8, one can show that every $\mathfrak{sl}_2\mathbb{R}$-triple is $(\mathrm{G}^{\mathbb{R}})^0$-conjugate to a Cayley triple. On the other hand, an $\mathfrak{sl}_2\mathbb{C}$-triple $\{f,h,e\}$ is called a *normal triple* if $\sigma(f)=-f$, $\sigma(h)=h$ and $\sigma(e)=-e$. Note that every magical $\mathfrak{sl}_2\mathbb{C}$-triple is a normal triple with respect to the Cartan involution (2.6).

The *Cayley transform* (which identifies the upper half-plane with the Poincaré disc) defines a bijection between Cayley triples in $\mathfrak{g}^{\mathbb{R}}$ and normal triples in $\mathfrak{g}$ by

$$\begin{aligned}\gamma:\ \text{Cayley triples}&\longrightarrow\text{Normal triples}\\ \{\hat{f},\hat{h},\hat{e}\}&\longmapsto\{\tfrac{1}{2}(\hat{f}+\hat{e}-i\hat{h}),i(\hat{e}-\hat{f}),\tfrac{1}{2}(\hat{f}+\hat{e}+i\hat{h})\},\end{aligned}$$

with inverse given by

$$\begin{aligned}\gamma^{-1}:\ \text{Normal triples}&\longrightarrow\text{Cayley triples}\\ \{f,h,e\}&\longmapsto\{\tfrac{1}{2}(f-e+ih),i(f+e),\tfrac{1}{2}(f-e-ih)\}.\end{aligned}\tag{2.12}$$

We will refer to both γ and γ^{-1} as the Cayley transform. For the proof of the following, see for instance [17, Theorem 9.5.1].

Proposition 2.15. *The Cayley transform provides the bijective correspondence of the Sekiguchi correspondence (2.11).*

Definition 2.16. *Let $\mathfrak{g}^{\mathbb{R}}$ be a real form of $\mathfrak{g}$ with Cartan involution σ. A Cayley triple $\{\hat{f}, \hat{h}, \hat{e}\} \subset \mathfrak{g}^{\mathbb{R}}$ is* magical *if its Cayley transform $\gamma(\{\hat{f}, \hat{h}, \hat{e}\}) \subset \mathfrak{g}$ is magical and, moreover, $\mathfrak{g}^{\mathbb{R}}$ is the canonical real form of $\gamma(\{\hat{f}, \hat{h}, \hat{e}\})$. A nilpotent $\hat{e} \in \mathfrak{g}^{\mathbb{R}}$ will be called* magical *if it belongs to a magical Cayley triple.*

Let $\{\hat{f}, \hat{h}, \hat{e}\} \subset \mathfrak{g}^{\mathbb{R}}$ be a Cayley triple and $\mathfrak{c}^{\mathbb{R}} \subset \mathfrak{g}^{\mathbb{R}}$ be its centralizer. Similarly, let $\mathfrak{c} \subset \mathfrak{g}$ be the centralizer of its Cayley transform $\{\gamma(\hat{f}), \gamma(\hat{h}), \gamma(\hat{e})\} \subset \mathfrak{g}$. It is straightforward to check that $\mathfrak{c}^{\mathbb{R}} \otimes \mathbb{C} = \mathfrak{c}$.

Recall that $V(\gamma(\hat{e})) = \ker(\mathrm{ad}_{\gamma(\hat{e})}) \subset \mathfrak{g}$ denotes the centralizer of the nilpotent $\gamma(\hat{e}) \in \mathfrak{g}$.

Proposition 2.17. *Let $\{\hat{f}, \hat{h}, \hat{e}\} \subset \mathfrak{g}^{\mathbb{R}}$ be a Cayley triple. Then $\{\hat{f}, \hat{h}, \hat{e}\}$ is magical if and only if $\mathfrak{c}^{\mathbb{R}} \subset \mathfrak{h}^{\mathbb{R}}$ and $\dim(\mathfrak{h} \cap V(\gamma(\hat{e}))) = \dim(\mathfrak{c})$.*

Proof. If $\{\hat{f}, \hat{h}, \hat{e}\} \subset \mathfrak{g}^{\mathbb{R}}$ is magical, then $\mathfrak{c}^{\mathbb{R}} \otimes \mathbb{C} = \mathfrak{c} \subset \mathfrak{h}$ and $V(\gamma(\hat{e})) \cap \mathfrak{h} = \mathfrak{c}$ by the Definition 2.4. Conversely, if $\mathfrak{c}^{\mathbb{R}} \subset \mathfrak{h}^{\mathbb{R}}$ and $\dim(\mathfrak{h} \cap V(\gamma(\hat{e}))) = \dim(\mathfrak{c})$, then the Cartan involution σ satisfies (2.6). Indeed, σ is a Lie algebra involution which preserves $V(\gamma(\hat{e}))$. Moreover, σ equals Id on $\mathfrak{c}$, equals $-$Id on the nontrivial highest weight spaces, and also $\sigma(\gamma(\hat{f})) = -\gamma(\hat{f})$. □

The first point of Proposition 2.17 says that the centralizer of a magical Cayley triple is compact. For the dimension of $\mathfrak{h} \cap V(\gamma(\hat{e}))$ we will use the following result.

Proposition 2.18. [54, Proposition 5] *The dimension of $\mathfrak{h} \cap V(\gamma(\hat{e}))$ is given by*

$$\dim(\mathfrak{h} \cap V(\gamma(\hat{e}))) = \frac{1}{2}\Big(\dim(V(\gamma(\hat{e}))) + \dim(\mathfrak{h}) - \dim(\mathfrak{m})\Big).$$

3. Classification of magical $\mathfrak{sl}_2$-triples

In this section we classify (conjugacy classes of) magical $\mathfrak{sl}_2$-triples in complex simple Lie algebras $\mathfrak{g}$. For classical Lie algebras, we use a classification of nilpotents using signed Young diagrams and, for exceptional Lie algebras, we use results of Doković in [20, 21].

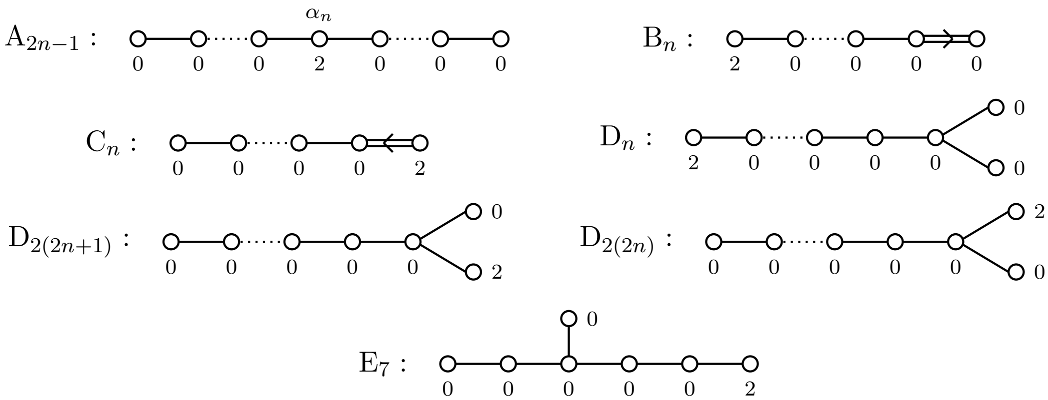
3.1. **The classification theorem.** There is a complete invariant of conjugacy classes of nilpotent elements of $\mathfrak{g}$ (and hence of $\mathfrak{sl}_2$-triples) called the weighted Dynkin diagram. We briefly recall how this works and refer the reader to [17, §3.5] for more details. Recall that the Dynkin diagram of $\mathfrak{g}$ is a diagram associated to a Cartan subalgebra $\mathfrak{a} \subset \mathfrak{g}$ and a choice of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_{\mathrm{rk}\,\mathfrak{g}}\} \subset \mathfrak{a}^*$. Its nodes are labeled by the simple roots α_i.

Consider an $\mathfrak{sl}_2$-triple $\{f, h, e\} \subset \mathfrak{g}$. Since h is semisimple, there exists a Cartan subalgebra $\mathfrak{a} \subset \mathfrak{g}$ containing h. Furthermore, we may choose a set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_{\mathrm{rk}\,\mathfrak{g}}\} \subset \mathfrak{a}^*$ so that $\alpha_i(h) \geqslant 0$ for all i. In fact, the properties of $\mathfrak{sl}_2$-representation theory imply that $\alpha_i(h) \in \{0, 1, 2\}$. The *weighted Dynkin diagram* associated to the $\mathfrak{sl}_2$-triple $\{f, h, e\} \subset \mathfrak{g}$ is defined to be the Dynkin diagram of $(\mathfrak{g}, \mathfrak{a}, \Pi)$, where the node associated to the simple root α_i is labeled by the integer $\alpha_i(h)$. Note that an $\mathfrak{sl}_2$-triple is even (see Remark 2.1) if and only if every node is labeled with either a 0 or a 2. It turns out that if two $\mathfrak{sl}_2$-triples in $\mathfrak{g}$ have the same weighted Dynkin diagram, then they are conjugate. However, not every Dynkin diagram whose nodes have labels in $\{0, 1, 2\}$ is the weighted Dynkin diagram of an $\mathfrak{sl}_2$-triple.

Here is one of the cornerstones of this paper: the classification of magical $\mathfrak{sl}_2$-triples.

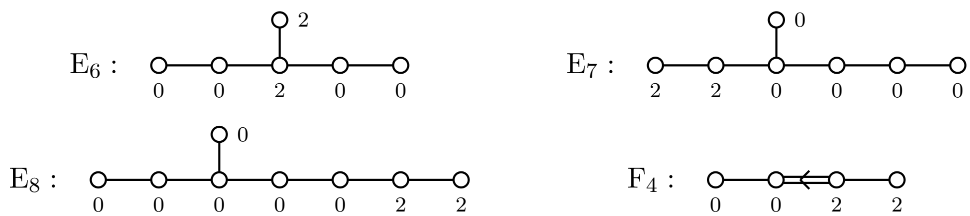
Theorem 3.1. *Let $\mathfrak{g}$ be a simple complex Lie algebra. Then an $\mathfrak{sl}_2$-triple $\{f, h, e\} \subset \mathfrak{g}$ is magical if and only if the associated weighted Dynkin diagram is one of the following:*

(1) $\mathfrak{g}$ *is any type and every node is labeled with a 2;*
(2) $\mathfrak{g}$ *has type* A_{2n-1}, B_n, C_n, D_n, D_{2n}, *or* E_7 *with weighted Dynkin diagrams*



(3) $\mathfrak{g}$ *has type* B_n *or* D_n *with weighted Dynkin diagrams*

where $1 < p < n-1$ *for* B_n *and* $1 < p < n-2$ *for* D_n*;*
(4) $\mathfrak{g}$ *has type* E_6, E_7, E_8 *or* F_4 *with weighted Dynkin diagrams*



The following is an immediate corollary.

Corollary 3.2. *Every magical* $\mathfrak{sl}_2$*-triple* $\{f,h,e\} \subset \mathfrak{g}$ *is even. In particular, if* $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ *is the* ± 1*-eigenspace of the Lie algebra involution* (2.6)*, then* $\mathfrak{c} = \ker(\mathfrak{h} \xrightarrow{\mathrm{ad}_f} \mathfrak{m})$*, and moreover* $\mathrm{ad}_f(\mathfrak{m}) \xrightarrow{\mathrm{ad}_f} \mathrm{ad}_f^2(\mathfrak{m})$ *is an isomorphism.*

3.2. **The proof.** We now prove Theorem 3.1. Let $\mathfrak{g}^{\mathbb{R}} \subset \mathfrak{g}$ be a real form of a complex simple Lie algebra and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be a complexified Cartan decomposition. Let $\{f,h,e\} \subset \mathfrak{g}$ be a normal $\mathfrak{sl}_2\mathbb{C}$-triple and $\mathfrak{c} \subset \mathfrak{g}$ its $\mathfrak{g}$-centralizer. We will classify (conjugacy classes of) magical $\mathfrak{sl}_2$-triples of $\mathfrak{g}$ among the normal ones. This will be done via the corresponding real notions of Definition 2.16 by the Sekiguchi correspondence and using Propositions 2.17 and 2.18. We will actually prove the theorem by classifying (conjugacy classes of) magical nilpotents in $\mathfrak{g}$ (see (2.2)).

We start with the exceptional case. In [20] and [21], Doković computes the dimensions $\dim(\mathfrak{h} \cap \mathfrak{c})$ and $\dim(\mathfrak{h} \cap V(e))$ for all real forms $\mathfrak{g}^{\mathbb{R}}$ of simple exceptional Lie algebras. By Proposition 2.17, the normal triple $\{f,h,e\}$ is magical if and only if these dimensions are both equal to the dimension of $\mathfrak{c}^{\mathbb{R}} \subset \mathfrak{h}^{\mathbb{R}}$.

Proof of Theorem 3.1 for exceptional Lie algebras. For $\mathfrak{g}^{\mathbb{R}} \subset \mathfrak{g}$ a real form of inner type, the nilpotent orbits (thus the conjugacy classes of nilpotents) are listed in tables VI-XV of [20]. The first column of each table lists the associated weighted Dynkin diagram of $\mathfrak{g}$, the fourth column lists the dimension of $\mathfrak{h} \cap V(e)$, the fifth column lists the dimension of $\mathfrak{h} \cap \mathfrak{c}$, and the last column lists the isomorphism class of $\mathfrak{c}^{\mathbb{R}}$. For the two outer real forms of $\mathfrak{e}_6$, the weighted Dynkin diagram is column 1 of Tables VI and VII of [21], while the dimensions of $\mathfrak{h} \cap V(e)$ and $\mathfrak{h} \cap \mathfrak{c}$ are columns 9 and 10 of Table VI and columns 12 and 13 of Table VII.

Table 1 of Section 9 summarizes this information for inner real forms of $\mathfrak{g}$; note that the real forms $\mathfrak{f}_4^{-20}$ and $\mathfrak{e}_6^{-14}$ do not admit magical nilpotents. For the two outer real forms of $\mathfrak{e}_6$, there is only one magical nilpotent. Namely, the real form $\mathfrak{e}_6^{-26}$ has no magical nilpotents and there is one magical nilpotent in the split real form $\mathfrak{e}_6^6$ (Table VII row 20 of [21]). In this case, the weighted Dynkin diagram is that of Case (1) of Theorem 3.1 and $\mathfrak{c}^{\mathbb{R}} = 0$. □

We now move to the case of real forms of classical Lie algebras. Conjugacy classes of nilpotent endomorphisms of $\mathbb{C}^n$ are in bijective correspondence with *partitions* of n. Namely, if $n = \sum_{i=1}^n r_i$ is a partition of n, with $r_i \geqslant 0$ (the multiplicity of i), then the nilpotent endomorphism associated to this partition is

$$e = \begin{pmatrix} J_1^{\oplus r_1} & & \\ & \ddots & \\ & & J_n^{\oplus r_n} \end{pmatrix}, \tag{3.1}$$

where J_i is the standard $i \times i$ Jordan block.

The following proposition classifies conjugacy classes of nilpotents in $\mathfrak{sl}_n\mathbb{C}$, $\mathfrak{so}_n\mathbb{C}$ and $\mathfrak{sp}_{2m}\mathbb{C}$. For a proof, see [17, Chapter 5.1].

Proposition 3.3. *Let* G *be a connected complex simple Lie group with Lie algebra* $\mathfrak{g}$.

- *For* $\mathfrak{g} = \mathfrak{sl}_n\mathbb{C}$, G*-conjugacy classes of nilpotents are in bijective correspondence with partitions of* n.
- *For* $\mathfrak{g} = \mathfrak{so}_{2n+1}\mathbb{C}$, G*-conjugacy classes of nilpotents are in bijective correspondence with partitions of* $2n+1 = \sum_{i=1}^{2n+1} r_i \cdot i$, *where* r_i *is even whenever* i *is even.*
- *For* $\mathfrak{g} = \mathfrak{sp}_{2n}\mathbb{C}$, G*-conjugacy classes of nilpotents are in bijective correspondence with partitions of* $2n = \sum_{i=1}^{2n} r_i \cdot i$, *where* r_i *is even whenever* i *is odd.*
- *For* $\mathfrak{g} = \mathfrak{so}_{2n}\mathbb{C}$, G*-conjugacy classes of nilpotents are in bijective correspondence with partitions of* $2n = \sum_{i=1}^{2n} r_i \cdot i$, *where* r_i *is even whenever* i *is even, except that there are two classes associated to partitions which have* $r_i = 0$ *for all* i *odd.*

Note that the above proposition is independent of the choice of G under the given conditions, since the any two choices are related by a quotient by central elements.

Given a partition $n = \sum_{i=1}^n r_i \cdot i$, define the *dual partition* by $n = \sum_{j=1}^n s_j$, where $s_j = \sum_{i=j}^n r_i$. The following proposition describes the centralizer of a nilpotent and the centralizing subalgebra of an associated $\mathfrak{sl}_2\mathbb{C}$-subalgebra; see [17, Chapter 6.1].

Proposition 3.4. *Let* $\mathfrak{g}$ *be* $\mathfrak{sl}_n\mathbb{C}$, $\mathfrak{so}_n\mathbb{C}$ *or* $\mathfrak{sp}_{2m}\mathbb{C}$. *Let* $e \in \mathfrak{g}$ *be a nilpotent element with corresponding partition* $n = \sum_{i=1}^n r_i \cdot i$ *and dual partition* $n = \sum_{j=1}^n s_j$, *with* $2m = n$ *for* $\mathfrak{sp}_{2m}\mathbb{C}$. *Finally, let* $V(e) = \ker(\mathrm{ad}_e) \subset \mathfrak{g}$ *be the centralizer of* e *and* $\mathfrak{c}$ *be the centralizer of an associated* $\mathfrak{sl}_2\mathbb{C}$*-subalgebra. Then,*

$\mathfrak{g}$	$\mathfrak{sl}_n\mathbb{C}$	$\mathfrak{so}_n\mathbb{C}$	$\mathfrak{sp}_{2m}\mathbb{C}$
$\dim(V(e))$	$\sum_{j=1}^n s_j^2 - 1$	$\frac{1}{2}(\sum_{j=1}^n s_j^2 - \sum_{i-odd} r_i)$	$\frac{1}{2}(\sum_{j=1}^n s_j^2 - \sum_{i-odd} r_i)$
$\mathfrak{c}$	$\mathfrak{s}(\bigoplus_{i=1}^n \mathfrak{gl}_{r_i}\mathbb{C})$	$\bigoplus_{i-even} \mathfrak{sp}_{r_i}\mathbb{C} \oplus \bigoplus_{i-odd} \mathfrak{so}_{r_i}\mathbb{C}$	$\bigoplus_{i-odd} \mathfrak{sp}_{r_i}\mathbb{C} \oplus \bigoplus_{i-even} \mathfrak{so}_{r_i}\mathbb{C}$

The different noncompact real forms $\mathfrak{g}^{\mathbb{R}} \subset \mathfrak{g}$ of the Lie algebras $\mathfrak{sl}_n\mathbb{C}$, $\mathfrak{so}_n\mathbb{C}$, $\mathfrak{sp}_{2m}\mathbb{C}$ are described in Table 2 of Section 9. We follow [17, Chapter 9.3] for the classification of nilpotents in these real forms. In $\mathfrak{sl}_n\mathbb{R}$ and $\mathfrak{su}^*_{2m}$, such classification can be phrased in terms of partitions. For the remaining real forms in the mentioned table, it can be phrased in terms of signed Young diagrams. Recall that partitions of n are described by Young diagrams. We will use the convention that the Young diagram associated to a partition $n = \sum_{i=1}^n r_i \cdot i$ has r_i rows of length i. A *signed Young diagram* is a Young diagram in which each box is decorated with a $+$ or $-$ sign and these signs alternate along each row. The *signature* of a signed Young diagram is (p, q) if there are p plus signs and q minus signs. Given a signed Young diagram, for each sub-diagram of rows of length i, let p_i denote the number of rows with leftmost box labeled $+$ and q_i denote the number of of rows with leftmost box labeled $-$. The following proposition collects a set of propositions proved in Section 9.3 of [17].

Proposition 3.5. *The classification of conjugacy classes of nilpotent elements in classical real Lie algebras reads as follows:*

- $\mathrm{SL}_n\mathbb{R}$*-conjugacy classes of nilpotents in* $\mathfrak{sl}_n\mathbb{R}$ *are in 1-1 correspondence with partitions* $n = \sum_{i=1}^n r_i \cdot i$*, except that there are two orbits associated to partitions with* $r_i = 0$ *for all* i *odd. The centralizer of an associated* $\mathfrak{sl}_2\mathbb{R}$*-subalgebra is isomorphic to* $\mathfrak{s}(\bigoplus_{i=1}^n \mathfrak{gl}_{r_i}\mathbb{R})$.
- SU^*_{2m}*-conjugacy classes of nilpotents in* $\mathfrak{su}^*_{2m}$ *are in 1-1 correspondence with partitions* $m = \sum_{i=1}^m r_i \cdot i$*. The centralizer of an associated* $\mathfrak{sl}_2\mathbb{R}$*-subalgebra is isomorphic to* $\mathfrak{s}(\bigoplus_{i=1}^m \mathfrak{u}^*_{2r_i})$.
- $\mathrm{SU}_{p,q}$*-conjugacy classes of nilpotents in* $\mathfrak{su}_{p,q}$ *are in 1-1 correspondence with signed Young diagrams of signature* (p, q)*. The centralizer of an associated* $\mathfrak{sl}_2\mathbb{R}$*-subalgebra is isomorphic to* $\mathfrak{s}(\bigoplus_{i=1}^n \mathfrak{u}_{p_i,q_i})$.
- $\mathrm{SO}_{p,q}$*-conjugacy classes of nilpotents in* $\mathfrak{so}_{p,q}$*are in 1-1 correspondences with signed Young diagrams of signature* (p, q) *where even rows occur with even multiplicity and have their leftmost boxes labeled with* $+$*, except that there are two orbits for diagrams in which all rows have even length. The centralizer of an associated* $\mathfrak{sl}_2\mathbb{R}$*-subalgebra is isomorphic to* $\bigoplus_{i-even} \mathfrak{sp}_{p_i+q_i}\mathbb{R} \oplus \bigoplus_{i-odd} \mathfrak{so}_{p_i,q_i}$.
- SO^*_{2m}*-conjugacy classes of nilpotents in* $\mathfrak{so}^*_{2m}$ *are in 1-1 correspondence with signed Young diagrams of size* m *and any signature in which rows with odd length have their leftmost boxes labeled with a* $+$*. The centralizer of an associated* $\mathfrak{sl}_2\mathbb{R}$*-subalgebra is isomorphic to* $\bigoplus_{i-even} \mathfrak{sp}_{2p_i,2q_i}\mathbb{R} \oplus \bigoplus_{i-odd} \mathfrak{so}^*_{2(p_i+q_i)}$.
- $\mathrm{Sp}_{2m}\mathbb{R}$*-conjugacy classes of nilpotents in* $\mathfrak{sp}_{2m}\mathbb{R}$ *are in 1-1 correspondence with signed Young diagrams of size* $2m$ *of any signature where odd rows occur with even multiplicity and have their leftmost boxes labeled with* $+$*. The centralizer of an associated* $\mathfrak{sl}_2\mathbb{R}$*-subalgebra is isomorphic to* $\bigoplus_{i-odd} \mathfrak{sp}_{p_i+q_i}\mathbb{R} \oplus \bigoplus_{i-even} \mathfrak{so}_{p_i,q_i}$.
- $\mathrm{Sp}_{2p,2q}$*-conjugacy classes of nilpotents in* $\mathfrak{sp}_{2p,2q}$ *are in 1-1 correspondence with signed Young diagrams of signature* (p, q) *in which even rows have their leftmost boxes labeled* $+$*. The centralizer of an associated* $\mathfrak{sl}_2\mathbb{R}$*-subalgebra is isomorphic to* $\bigoplus_{i-odd} \mathfrak{sp}_{2p_i,2q_i} \oplus \bigoplus_{i-even} \mathfrak{so}^*_{2(p_i+q_i)}$.

Remark 3.6. For the classical Lie algebras other than $\mathfrak{su}^*_{2m}$, $\mathfrak{so}^*_{2m}$, $\mathfrak{sp}_{2p,2q}$, the partition of the associated nilpotent orbit in $\mathfrak{g}$ corresponds to Young diagram obtained by forgetting the signs. For $\mathfrak{su}^*_{2m}$, $\mathfrak{so}^*_{2m}$, $\mathfrak{sp}_{2p,2q}$, the partition of the associated nilpotent orbit in $\mathfrak{g}$ corresponds to the Young diagram obtained doubling every row and forgetting the signs.

We now classify magical nilpotent elements for classical real forms in terms of signed Young diagrams and partitions.

Theorem 3.7. *Let* $\mathfrak{g}^{\mathbb{R}}$ *be a real form of a classical Lie complex simple Lie algebra* $\mathfrak{g}$*. A nilpotent* $\hat{e} \in \mathfrak{g}^{\mathbb{R}}$ *is magical if and only if it is one of the following cases:*

(1) $\mathfrak{g}^{\mathbb{R}} \cong \mathfrak{sl}_n\mathbb{R}$ *and the associated Young diagram has one row of length* n*,*
(2) $\mathfrak{g}^{\mathbb{R}} \cong \mathfrak{so}_{p,p+1}$ *or* $\mathfrak{g}^{\mathbb{R}} \cong \mathfrak{so}_{p+1,p}$ *and the signed Young diagram has one row of length* $2p+1$*,*
(3) $\mathfrak{g}^{\mathbb{R}} \cong \mathfrak{so}_{p,p}$ *and the signed Young diagram has one row of length* $2p-1$ *and one row of length* 1*,*
(4) $\mathfrak{g}^{\mathbb{R}} \cong \mathfrak{sp}_{2m}\mathbb{R}$ *and the signed Young diagram has one row of length* $2m$*,*
(5) $\mathfrak{g}^{\mathbb{R}} = \mathfrak{su}_{m,m}$*,* $\mathfrak{so}^*_{4m}$ *,*$\mathfrak{sp}_{2m}\mathbb{R}$ *and the signed Young diagram has* m*-rows of length* 2 *and the leftmost boxes are either all labeled* $+$ *or all labeled* $-$*,*
(6) $\mathfrak{g}^{\mathbb{R}} \cong \mathfrak{so}_{p,q}$ *and the signed Young diagram has one row of length* $2\min\{p,q\}-1$ *and* $(|q-p|+1)$*-rows of length* 1*, where the labels of the length* 1 *row are the same and opposite the label of the leftmost box of the row of length* $2\min\{p,q\}-1$*.*

Remark 3.8. In first four cases, $\mathfrak{g}^{\mathbb{R}}$ is split and we have the principal nilpotent. Case (5) corresponds to Lie algebras which are Hermitian of tube type, and the same holds in (6) if $p=2$ or $q=2$.

Proof. Let $\mathfrak{g}^{\mathbb{R}}$ be a real form of a classical complex simple Lie algebra $\mathfrak{g}$ and $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$ be a Cartan decomposition. By Propositions 2.17 and 2.18 a nilpotent $\hat{e}\in\mathfrak{g}^{\mathbb{R}}$ is magical if and only if the centralizer of an associated $\mathfrak{sl}_2\mathbb{R}$-subalgebra $\mathfrak{c}^{\mathbb{R}}$ is compact and

$$2\dim(\mathfrak{c}^{\mathbb{R}}\otimes\mathbb{C}) - \dim(V(\gamma(\hat{e}))) - \dim(\mathfrak{h}) + \dim(\mathfrak{m}) \tag{3.2}$$

vanishes. Now we use Proposition 3.5 together with this criterion to detect magical nilpotents in $\mathfrak{g}^{\mathbb{R}}$.

For $\mathfrak{g}^{\mathbb{R}}=\mathfrak{sl}_n\mathbb{R}$, $\mathfrak{c}^{\mathbb{R}}=\mathfrak{s}(\bigoplus_{i=1}^n \mathfrak{gl}_{r_i}\mathbb{R})$. So $\mathfrak{c}^{\mathbb{R}}$ is compact if and only if the partition is $n=1\cdot n$, i.e. the corresponding Young diagram has just one row of length n. So we are left with this corresponding nilpotent $\hat{e}$. In this case, $\mathfrak{c}^{\mathbb{R}}=0$. Moreover, the dual partition is $n=n\cdot 1$, so Proposition 3.4, together with Table 2 of Section 9, show that $-\dim(V(\gamma(\hat{e})))-\dim(\mathfrak{h})+\dim(\mathfrak{m})=-n+1+n-1=0$. Hence (3.2) equals to zero, so $\hat{e}\in\mathfrak{sl}_n\mathbb{R}$ is magical, proving (1).

The remaining cases will be dealt with by a similar argument.

For $\mathfrak{g}^{\mathbb{R}}=\mathfrak{su}^*_{2m}$, $\mathfrak{c}^{\mathbb{R}}=\mathfrak{s}(\bigoplus_{i=1}^m \mathfrak{u}^*_{2r_i})$ is compact if and only if $r_m=1$ and $r_i=0$, for $i\neq m$, so that $\mathfrak{c}^{\mathbb{R}}=\mathfrak{su}^*_2=\mathfrak{su}_2$. We are then left with the nilpotent in $\hat{e}\in\mathfrak{su}^*_{2m}$ whose corresponding nilpotent (under the Cayley transform) in $\mathfrak{g}=\mathfrak{sl}_{2m}\mathbb{C}$ is given by the partition $2m=2\cdot m$. Its dual partition is $2m=\sum_{j=1}^{2m}s_j$, with $s_j=2$ for $1\leqslant j\leqslant m$ and $s_j=0$ otherwise. Then, Proposition 3.4 shows that (3.2) equals $6-6m$. Hence the nilpotent $\hat{e}\in\mathfrak{su}^*_{2m}$ can only be magical if $m=1$. But $\mathfrak{su}^*_2$ is compact, and thus has no nonzero nilpotent elements (recall that magical nilpotents are nonzero by definition), so $\hat{e}$ is not magical. We conclude that $\mathfrak{su}^*_{2m}$ does not admit magical nilpotents.

Consider now the case of $\mathfrak{g}^{\mathbb{R}}=\mathfrak{su}_{p,q}$. Then $\mathfrak{c}^{\mathbb{R}}\cong\mathfrak{s}(\bigoplus_{i=1}^{p+q}\mathfrak{u}_{p_i,q_i})$, which is compact if and only if $p_i=0$ or $q_i=0$ for each i. The associated nilpotent in $\mathfrak{sl}_{p+q}\mathbb{C}$ corresponds to the partition $p+q=\sum_{i=1}^{p+q}r_i\cdot i$, where $r_i=p_i+q_i$. By Proposition 3.4, we see that (3.2) is given by

$$2\sum_{i=1}^{p+q}r_i^2-\sum_{i=1}^{p+q}s_i^2-(q-p)^2, \tag{3.3}$$

with $p+q=\sum_{i=1}^{p+q}r_i$ the corresponding dual partition. We want to understand when (3.3) vanishes.

First assume $r_1 = 0$. Using $s_i = r_i + s_{i+1}$ twice, (3.3) can be rewritten as

$$-4\sum_{i=2}^{p+q-1} r_i s_{i+1} - \sum_{i=3}^{p+q} s_i^2 - (q-p)^2.$$

If $r_i \neq 0$ for some $i > 2$, then this expression is strictly negative, therefore the corresponding partition does not correspond to a magical nilpotent in $\mathfrak{su}_{p,q}$. If r_2 is the only nonzero r_i, the previous expression equals $-(q-p)^2$, hence (3.3) vanishes if and only if $p = q$. So the nonzero nilpotent determined by that partition is magical, and corresponds to Case (5) for $\mathfrak{g}^{\mathbb{R}} = \mathfrak{su}_{m,m}$.

Now suppose $r_1 \neq 0$. Since the Jordan block J_1 is a 1×1 zero matrix, a nilpotent $\hat{e} \in \mathfrak{su}_{p,q}$ with $r_1 \neq 0$ is contained in a subalgebra isomorphic to $\mathfrak{su}_{p-r_1,q}$ (in case $r_1 = p_1$) or $\mathfrak{su}_{p,q-r_1}$ (in case $r_1 = q_1$). In this subalgebra, $\hat{e}$ has no r_1-term. If it is magical, then by the above argument we must have $r_i = 0$ for $i > 2$ and $q - p = \pm r_1$. Thus, (3.3) is given by

$$2r_1^2 + 2r_2^2 - r_1^2 - 2r_1r_2 - r_2^2 - r_2^2 - r_1^2 = -2r_1r_2.$$

This is zero if and only if $r_1 = 0$ or $r_2 = 0$, but we are assuming $r_1 \neq 0$ and if $r_2 = 0$ then $\hat{e}$ is the zero nilpotent. So there are no magical nilpotents in $\mathfrak{su}_{p,q}$ other than the one detected in the previous paragraph.

Consider now $\mathfrak{g}^{\mathbb{R}} = \mathfrak{so}_{p,q}$. By Proposition 3.5, $\mathfrak{c}^{\mathbb{R}} \cong \bigoplus_{i-even} \mathfrak{sp}_{p_i+q_i}\mathbb{R} \oplus \bigoplus_{i-odd} \mathfrak{so}_{p_i,q_i}$. This is compact if and only if $p_i + q_i = 0$ for i even and either $p_i = 0$ or $q_i = 0$ for i odd. The partition of the associated nilpotent in $\mathfrak{so}_{p+q}\mathbb{C}$ is $p + q = \sum_{i=1}^{p+q} r_i \cdot i$, where $r_i = p_i + q_i$. Then we see that twice the quantity (3.2) equals to

$$2\sum_{i=1}^{p+q} r_i^2 - \sum_{i=1}^{p+q} r_i - \sum_{i=1}^{p+q} s_i^2 + p + q - (q-p)^2. \tag{3.4}$$

First assume that $r_1 = 0$. Using again that $s_i = r_i + s_{i+1}$, (3.4) can be rewritten as

$$-4\sum_{i=2}^{p+q-1} r_i s_{i+1} - \sum_{i=3}^{p+q} s_i^2 - \sum_{i=1}^{p+q} r_i + p + q - (q-p)^2.$$

If only one r_k is nonzero, then $p + q = r_k \cdot k$ and $(q-p)^2 = r_k^2$, therefore (3.4) simplifies to $r_k(k-1)(r_k-1)$. Since $r_1 = 0$, this is zero if and only if $r_k = 1$ and thus $k = p + q$ and k is odd. This proves that Case (2) of the theorem is a magical nilpotent for $\mathfrak{so}_{p,p+1}$ if the left most box is labeled $-$ and $\mathfrak{so}_{p+1,p}$ if the left most box is labeled $+$. Still assuming $r_1 = 0$, if at least two r_i are nonzero, then $p + q = \sum_{j=1}^{p+q-1} s_j - 2s_2$ and $-4\sum_{i=2}^{p+q-1} r_i s_{i+1} + 2s_2 < 0$. Such a nilpotent is not a magical one because

$$-4\sum_{i=2}^{p+q-1} r_i s_{i+1} - \sum_{i=3}^{p+q} s_i^2 - \sum_{i=1}^{p+q} r_i + p + q - (q-p)^2 \leqslant -4\sum_{i=2}^{p+q-1} r_i s_{i+1} + 2s_2 - \sum_{i=1}^{p+q} r_i - (q-p)^2 < 0.$$

Now assume $r_1 \neq 0$. As in the $\mathfrak{su}_{p,q}$-case, the nilpotent $\hat{e}$ is contained in a subalgebra isomorphic to $\mathfrak{so}_{p-r_1,q}$ or $\mathfrak{so}_{p,q-r_1}$ and has no r_1-term. If $\hat{e}$ is magical, by the above argument, the partition must be of the form $p+q = r_1 \cdot 1 + 1 \cdot (2\min\{p,q\} - 1)$. Since the signature of the signed Young diagram is (p,q), if the left most box of the row of length $2\min\{p,q\} - 1$ is labeled $+$, then each row of length 1 is labeled $-$ and vice versa. This means that $(q-p)^2 = (1-r_1)^2$. In this case (3.4) is given by

$$2r_1^2 + 2 - r_1 - 1 - (r_1+1)^2 - (2\min\{p,q\} - 2) \cdot 1 + r_1 + 2\min\{p,q\} - 1 - (1-r_1)^2.$$

This expression always vanishes, proving Case (6) of the theorem.

For $\mathfrak{g}^{\mathbb{R}} = \mathfrak{so}^*_{2m}$, we have that $\mathfrak{c}^{\mathbb{R}} \cong \bigoplus_{i-even} \mathfrak{sp}_{2p_i,2q_i}\mathbb{R} \oplus \bigoplus_{i-odd} \mathfrak{so}^*_{2(p_i+q_i)}$ and this is compact if and only if $p_i + q_i = 0$ for all i odd and either $p_i = 0$ or $q_i = 0$ for every i even. So we are left with nilpotents $\hat{e} \in \mathfrak{so}^*_{2m}$ whose partition of the corresponding nilpotent in $\mathfrak{so}_{2m}\mathbb{C}$ is $2m = \sum_{i=1}^m (2r_i) \cdot i$, with $r_i = p_i + q_i$ verifying these conditions. Then, twice the quantity (3.2) is given by

$$2\sum_{i=1}^m (2r_i)^2 + 2\sum 2r_i - \sum_{i=1}^m s_i^2 - 2m, \tag{3.5}$$

where $s_j = \sum_{i=j}^n 2r_j$. Since $r_1 = 0$ and $s_{j+1} = 2r_j + s_j$, (3.5) is given by

$$-4\sum_{i=2}^{m-1} 2r_i s_{i+1} + 2\sum_{i=1}^m 2r_i - \sum_{i=3}^m s_i^2 - 2m.$$

If r_i is nonzero for $i > 2$, then the above expression is negative and the nilpotent is not magical. If r_2 is the only nonzero r_i, then (3.5) equals $4r_2 - 2m$. This is zero if and only if $r_2 = \frac{m}{2}$ with $\frac{m}{2}$ and integer. In such case the nilpotent $\hat{e}$ is magical, proving the part of Case (5) regarding $\mathfrak{so}^*_{2m}$.

Consider now $\mathfrak{g}^{\mathbb{R}} = \mathfrak{sp}_{2m}\mathbb{R}$. Then $\mathfrak{c}^{\mathbb{R}} \cong \bigoplus_{i-odd} \mathfrak{sp}_{p_i+q_i}\mathbb{R} \oplus \bigoplus_{i-even} \mathfrak{so}_{p_i,q_i}$, so it is compact if and only if $p_i + q_i = 0$ for i odd and either $p_i = 0$ or $q_i = 0$ for all i even, so we are left with nilpotents under these conditions. Write $2m = \sum_{i=1}^{2m} r_i \cdot i$ be the partition of the associated nilpotents (by the Cayley transform) in $\mathfrak{sp}_{2m}\mathbb{C}$, where $r_i = p_i + q_i$ satisfy the previous constrains. By Proposition 3.4, twice the quantity (3.2) equals

$$2\sum_{i=1}^{2m} r_i^2 - 2\sum_{i=1}^{2m} r_i - \sum_{i=1}^{2m} s_i + 2m. \tag{3.6}$$

Since $r_1 = 0$, (3.6) can be rewritten as

$$-4\sum_{i=2}^{2m-2} r_i s_{i+1} - \sum_{i=3}^{2m} s_i^2 - 2\sum_{i=1}^n r_i + 2m.$$

Similarly to the previous cases, if at least two r_i are nonzero, then this expression is negative so the corresponding nilpotents are not magical. If $2m = r_k \cdot k$, then (3.6) is given by $(2-k)r_k^2 - 2r_k + 2m$, which is zero if and only if $k = 2$ or $k = 2m$. This proves Case (4) and completes the proof of Case (5).

Finally, let us consider $\mathfrak{g}^{\mathbb{R}} = \mathfrak{sp}_{2p,2q}$. Then $\mathfrak{c}^{\mathbb{R}} \cong \bigoplus_{i-odd} \mathfrak{sp}_{2p_i,2q_i} \oplus \bigoplus_{i-even} \mathfrak{so}^*_{2(p_i+q_i)}$ is compact if and only if $p_i + q_i = 0$ for every i even and either $p_i = 0$ or $q_i = 0$ for all i odd. Let $2p + 2q = \sum_{i=1}^{p+q} 2r_i \cdot i$ be the partitions of the associated nilpotents in $\mathfrak{sp}_{2p+2q}\mathbb{C}$, where each $r_i = p_i + q_i$ verifies the previous conditions. Then, twice the number (3.2) is given by

$$2\sum_{i=1}^m (2r_i)^2 + \sum_{i=1}^m 2r_i - \sum_{i=1}^m s_i^2 - 2p - 2q - 4(q-p)^2, \tag{3.7}$$

where $s_j = \sum_{i=j}^n 2r_j$. If $r_1 = 0$, then (3.7) can be rewritten as

$$\sum_{i=1}^m 2r_i - 4\sum_{i=2}^{m-1} 2r_i s_{i+1} - \sum_{i=3}^m s_i^2 - 2p - 2q - 4(q-p)^2.$$

This expression is always negative, hence no magical nilpotents arise with $r_1 = 0$. As in previous cases, if $r_1 \neq 0$, then the nilpotent must lie in a subalgebra isomorphic to $\mathfrak{sp}_{2p-2r_1,2q}$ or $\mathfrak{sp}_{2p,2q-2r_1}$ and have $r_1 = 0$ in that subalgebra. Moreover, if $\{f, h, e\}$ is magical in $\mathfrak{sp}_{2p,2q}$, then it is magical in the subalgebra (see Remark 2.5). But the previous argument says that there is no magical nilpotent in such setting. Hence $\mathfrak{sp}_{2p,2q}$ does not admit any magical nilpotents, and this completes the proof. □

It remains to translate the partition classification of Theorem 3.7 into the weighted Dynkin diagram classification of Theorem 3.1. The algorithm for doing this is the following. Let $\mathfrak{a} \subset \mathfrak{g}$ be the Cartan subalgebra of diagonal matrices and choose the simple roots such that the positive root spaces correspond to upper triangular matrices. Given a (signed) Young diagram from Theorem 3.7 let $e \in \mathfrak{g}$ be the associated nilpotent given by (3.1). Complete e to an $\mathfrak{sl}_2\mathbb{C}$-triple $\{f, h, e\}$ such that h is a diagonal matrix. Finally, conjugate $\{f, h, e\}$ so that the eigenvalues of h are decreasing and compute $\alpha_i(h)$. We will sketch this process for one case; see [17, Chapters 3.6, 5.3] for more details.

Completing a nilpotent $e \in \mathfrak{sl}_n\mathbb{C}$ to an $\mathfrak{sl}_2\mathbb{C}$-triple $\{f, h, e\}$ with h diagonal requires doing this for each Jordan block J_k. Such an $\mathfrak{sl}_2\mathbb{C}$-triple $\{f, h, e\}$ is given by

$$\left\{ \begin{pmatrix} 0 & & & & \\ \mu_1 & 0 & & & \\ & \ddots & \ddots & & \\ & & & \mu_{k-1} & 0 \end{pmatrix}, \begin{pmatrix} k-1 & & & \\ & k-3 & & \\ & & \ddots & \\ & & & 1-k \end{pmatrix}, \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \right\},$$

where $\mu_j = j(k-j)$. Consider now Case (6) of Theorem 3.7 with $p \leqslant q$. Then the resulting semisimple element h is diagonal with entries $\{2p-2, 2p-4, \ldots, 2-2p, 0, \ldots, 0\}$. Rearranging the matrix so that the eigenvalues are decreasing, yields

$$\{2p-2, 2p-4, \ldots, 2, 0, \ldots, 0, -2, -4, \ldots, 2-2p\}.$$

Using [17, Lemma 5.3.3] when $p+q$ is odd and [17, Lemma 5.3.4] when $p+q$ is even, we conclude that the associated weighted Dynkin diagram is given by Case (3) of Theorem 3.1. We leave the remaining cases to the reader.

4. Explicit data and real forms for magical $\mathfrak{sl}_2$-triples

Associated to a given magical $\mathfrak{sl}_2$-triple $\{f, h, e\} \subset \mathfrak{g}$, we explicitly exhibit, in this section, the $\mathfrak{sl}_2\mathbb{C}$-module and ad_h-weight space decompositions (2.3) and (2.4), the associated canonical real form of $\mathfrak{g}$ of Definition 2.10 and the Cayley real form of $\mathfrak{g}_0$ of Definition 2.13. We also show that the magical $\mathfrak{sl}_2$-triple $\{f, h, e\} \subset \mathfrak{g}$ arises from a principal $\mathfrak{sl}_2$-triple in a simple subalgebra $\mathfrak{g}(e) \subset \mathfrak{g}$, where $\mathfrak{g}(e)$ is the semisimple part of the $\mathfrak{g}$-centralizer of the centralizer of $\{f, h, e\}$. Finally, we deduce the detailed Lie theoretic information for magical $\mathfrak{sl}_2$-triples in Case (4) of Theorem 3.1.

4.1. $\mathfrak{sl}_2$-data.

Recall from Definition 2.10 that a magical triple $\{f, h, e\}$ of $\mathfrak{g}$ determines a canonical real form $\mathfrak{g}^{\mathbb{R}}$ via the involution $\sigma_e : \mathfrak{g} \to \mathfrak{g}$. Note that the canonical real form of a magical $\mathfrak{sl}_2$-triple follows from Theorem 3.7 for the classical cases and [20] for the exceptional cases.

Proposition 4.1. *The canonical real forms $\mathfrak{g}^{\mathbb{R}} \subset \mathfrak{g}$ associated to magical $\mathfrak{sl}_2$-triples are given as follows:*

(1) In Case (1) of Theorem 3.1, $\mathfrak{g}^{\mathbb{R}}$ is the split real form

$\mathfrak{g}$	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
$\mathfrak{g}^{\mathbb{R}}$	$\mathfrak{sl}_{n+1}\mathbb{R}$	$\mathfrak{so}_{n,n+1}$	$\mathfrak{sp}_{2n}\mathbb{R}$	$\mathfrak{so}_{n,n}$	$\mathfrak{e}_6^6$	$\mathfrak{e}_7^7$	$\mathfrak{e}_8^8$	$\mathfrak{f}_4^4$	$\mathfrak{g}_2^2$

(2) In Case (2) of Theorem 3.1, $\mathfrak{g}^{\mathbb{R}}$ is the Hermitian Lie algebra of tube type given by

$\mathfrak{g}$	A_{2n-1}	B_n	C_n	D_n	D_{2n}	E_7
$\mathfrak{g}^{\mathbb{R}}$	$\mathfrak{su}_{n,n}$	$\mathfrak{so}_{2,2n-1}$	$\mathfrak{sp}_{2n}\mathbb{R}$	$\mathfrak{so}_{2,2n-2}$	$\mathfrak{so}^*_{4n}$	$\mathfrak{e}_7^{-25}$

(3) *In Case (3) of Theorem 3.1 with* $\mathfrak{g} = \mathfrak{so}_N\mathbb{C}$, $\mathfrak{g}^{\mathbb{R}} \cong \mathfrak{so}_{p,N-p}$.
(4) *In Case (4) of Theorem 3.1,* $\mathfrak{g}^{\mathbb{R}}$ *is the quaternionic real form of* $\mathfrak{g}$

$\mathfrak{g}$	E_6	E_7	E_8	F_4
$\mathfrak{g}^{\mathbb{R}}$	$\mathfrak{e}_6^2$	$\mathfrak{e}_7^{-5}$	$\mathfrak{e}_8^{-24}$	$\mathfrak{f}_4^4$

Let $\mathfrak{a} \subset \mathfrak{g}$ be a Cartan subalgebra and denote the root space decomposition by

$$\mathfrak{g} = \mathfrak{a} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha,$$

where $\Delta \subset \mathfrak{a}^*\backslash\{0\}$ is the set of roots and $\mathfrak{g}_\alpha = \{y \in \mathfrak{g} \mid \mathrm{ad}_x(y) = \alpha(x)y,\ \forall\ x \in \mathfrak{a}\}$ is the root space of $\alpha \in \Delta$. Choosing a set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_{\mathrm{rk}\,\mathfrak{g}}\} \subset \Delta$ splits the roots into positive and negative roots $\Delta = \Delta^+ \sqcup \Delta^-$, where Δ^+ (resp. Δ^-) consists of roots $\alpha = \sum_{i=1}^{\mathrm{rk}\,\mathfrak{g}} a_i\alpha_i$ with $a_i \in \mathbb{Z}_{\geqslant 0}$ (resp. $a_i \in \mathbb{Z}_{\leqslant 0}$) for all i.

Let $\{f, h, e\}$ be an $\mathfrak{sl}_2\mathbb{C}$-triple, with $h \in \mathfrak{a}$ and $\alpha_i(h) \geqslant 0$ for all $\alpha_i \in \Pi$. The element h acts on a root space $\mathfrak{g}_\alpha$ with weight $\sum_{i=1}^{\mathrm{rk}\,\mathfrak{g}} a_i\alpha_i(h)$, where $\alpha = \sum_{i=1}^{\mathrm{rk}\,\mathfrak{g}} a_i\alpha_i$. Thus, the ad_h-weight space decomposition (2.4) of $\mathfrak{g}$ is given by

$$\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j,$$

where $\mathfrak{g}_j$ is a direct sum of root spaces $\mathfrak{g}_\alpha$ with $\alpha = \sum_{i=1}^{\mathrm{rk}\,\mathfrak{g}} a_i\alpha_i$ and $j = \sum_{i=1}^{\mathrm{rk}\,\mathfrak{g}} a_i\alpha_i(h)$ if $j \neq 0$, and $\mathfrak{g}_0$ is the direct sum of $\mathfrak{a}$ and the set of root spaces $\mathfrak{g}_\alpha$ with $\alpha = \sum_{i=1}^{\mathrm{rk}\,\mathfrak{g}} a_i\alpha_i$ such that $0 = \sum_{i=1}^{\mathrm{rk}\,\mathfrak{g}} a_i\alpha_i(h)$, i.e.

$$\mathfrak{g}_0 = \mathfrak{a} \oplus \bigoplus_{\alpha(h)=0} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{g}_j = \bigoplus_{\alpha(h)=j} \mathfrak{g}_\alpha,\ j \neq 0. \tag{4.1}$$

We record the Lie algebra $\mathfrak{g}_0$ of a magical nilpotent. This follows immediately from the weighted Dynkin diagram classification of Theorem 3.1.

Proposition 4.2. *The subalgebra* $\mathfrak{g}_0 \subset \mathfrak{g}$ *associated to a magical* $\mathfrak{sl}_2$*-triple in* $\mathfrak{g}$ *is described as follows:*

(1) *In Case (1) of Theorem 3.1,* $\mathfrak{g}_0 \cong \mathbb{C}^{\oplus\,\mathrm{rk}\,\mathfrak{g}}$.
(2) *In Case (2) of Theorem 3.1,*

$\mathfrak{g}$	A_{2n-1}	B_n	C_n	D_n	D_{2n}	E_7
$\mathfrak{g}_0$	$\mathfrak{sl}_n\mathbb{C} \oplus \mathfrak{sl}_n\mathbb{C} \oplus \mathbb{C}$	$\mathfrak{so}_{2n-1}\mathbb{C} \oplus \mathbb{C}$	$\mathfrak{sl}_n\mathbb{C} \oplus \mathbb{C}$	$\mathfrak{so}_{2n-2}\mathbb{C} \oplus \mathbb{C}$	$\mathfrak{sl}_{2n}\mathbb{C} \oplus \mathbb{C}$	$\mathfrak{e}_6 \oplus \mathbb{C}$

(3) *In Case (3) of Theorem 3.1 with* $\mathfrak{g} = \mathfrak{so}_N\mathbb{C}$, *then* $\mathfrak{g}_0 = \mathbb{C}^{p-1} \oplus \mathfrak{so}_{N-2p+2}\mathbb{C}$.
(4) *In Case (4) of Theorem 3.1,*

$\mathfrak{g}$	E_6	E_7	E_8	F_4
$\mathfrak{g}_0$	$\mathfrak{sl}_3\mathbb{C} \oplus \mathfrak{sl}_3\mathbb{C} \oplus \mathbb{C}^2$	$\mathfrak{sl}_6\mathbb{C} \oplus \mathbb{C}^2$	$\mathfrak{e}_6 \oplus \mathbb{C}^2$	$\mathfrak{sl}_3\mathbb{C} \oplus \mathbb{C}^2$

The $\mathfrak{sl}_2$-module decomposition $\mathfrak{g} = \bigoplus_j W_j$ from (2.3) can be deduced from the ad_h-weight space decomposition. Namely,

$$n_j = \dim(\mathfrak{g}_j) - \dim(\mathfrak{g}_{j+2}). \tag{4.2}$$

Recall that the $\mathfrak{sl}_2$-data of a magical nilpotent is determined by the collection of pairs of non-negative integers $\{(m_j, n_j)\}_{j=0}^M$ such that, for each $j \geqslant 1$, the multiplicity n_{2m_j} of W_{2m_j} is positive. Thus, the $\mathfrak{sl}_2\mathbb{C}$-module decomposition of a magical nilpotent is given by

$$\mathfrak{g} = \mathfrak{c} \oplus \bigoplus_{j=1}^M W_{2m_j}.$$

Proposition 4.3. *The $\mathfrak{sl}_2$-data of a magical $\mathfrak{sl}_2$-triple $\{f, h, e\}$ is given as follows:*

(1) In Case (1) of Theorem 3.1 the set of $\{m_j\}$ is given by

$\mathrm{A}_n : \{0, 1, 2, \dots, n\}$	$\mathrm{B}_n : \{0, 1, 3, \dots, 2n-1\}$	$\mathrm{C}_n : \{0, 1, 3, \dots, 2n-1\}$
$\mathrm{D}_n : \{0, 1, 3, \dots, 2n-3, n-1\}$	$\mathrm{E}_6 : \{0, 1, 4, 5, 7, 8, 11\}$	$\mathrm{E}_7 : \{0, 1, 5, 7, 9, 11, 13, 17\}$
$\mathrm{E}_8 : \{0, 1, 7, 11, 13, 17, 19, 23, 29\}$	$\mathrm{F}_4 : \{0, 1, 5, 7, 11\}$	$\mathrm{G}_2 : \{0, 1, 5\}$

For all cases, $n_0 = 0$ and $n_{2m_j} = 1$, with the exception that $n_{4n-2} = 2$ for D_{2n}.

(2) In Case (2) of Theorem 3.1, $\{m_j\} = \{0, 1\}$ and n_0 and n_2 are given as follows:

$\mathfrak{g}$	A_{2n-1}	B_n	C_n	D_n	D_{2n}	E_7
n_0	n^2-1	$2n^2-5n+3$	$\frac{n(n-1)}{2}$	$2n^2-7n+6$	$n(2n+1)$	52
n_2	n^2	$2n-1$	$\frac{n(n+1)}{2}$	$2n-2$	$n(2n-1)$	27

(3) In Case (3) of Theorem 3.1, we have $\{m_j\} = \{0, 1, 3, \dots, 2p-3, p-1\}$ and

$$n_0 = \frac{(N-2p)(N-2p+1)}{2} \quad \text{and} \quad n_{2m_j} = \begin{cases} N-2p+2 & p \text{ even and } m_j = p-1 \\ N-2p+1 & p \text{ odd and } m_j = p-1 \\ 1 & \text{otherwise,} \end{cases}$$

where $N = 2n+1$ in type B_n and $N = 2n$ in type D_n.

(4) In Case (4) of Theorem 3.1, $\{m_j\} = \{0, 1, 3, 5\}$, $n_2 = 1$, $n_{10} = 1$ and n_0 and n_6 are given as follows:

$\mathfrak{g}$	E_6	E_7	E_8	F_4
n_0	8	21	52	3
n_6	8	14	26	5

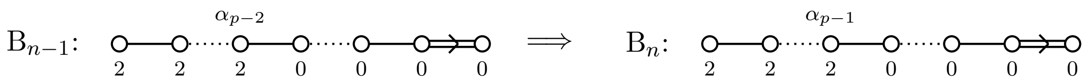
Proof. For Case (1), all the nodes of the Dynkin diagram have label 2, and hence the nilpotent $e \in \mathfrak{g}$ is principal. We have $n_0 = 0$ since the $\mathfrak{g}$-centralizer of a principal $\mathfrak{sl}_2\mathbb{C}$ is trivial. The integers m_j with $n_{2m_j} > 0$ are the exponents of $\mathfrak{g}$ (see [17, Chapter 4]).

For Case (2), there is one root α_M with label 2 and all other roots are labeled 0. Moreover, if $\sum a_i \alpha_i$ is the expression of the highest root, then $a_M = 1$. Thus, the ad_h-weight space decomposition is $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2$ and the $\mathfrak{sl}_2$-module decomposition is $\mathfrak{g} = W_0 \oplus W_2$. We have $\dim(\mathfrak{g}_0) = n_0 + n_2$ and $\dim(\mathfrak{g}) = 3n_2 + n_0$, hence $n_2 = \frac{\dim(\mathfrak{g}) - \dim(\mathfrak{g}_0)}{2}$.

We compute the cases of A_{2n-1} and leave the rest to the reader. For the A_{2n-1} weighted Dynkin diagram we have $\mathfrak{g}_0 = \mathfrak{sl}_{n-1}\mathbb{C} \oplus \mathbb{C} \oplus \mathfrak{sl}_{n-1}\mathbb{C}$. Hence,

$$n_2 = \tfrac{(4n^2-1)-(2n^2-2)-1}{2} = n^2 \qquad \text{and} \qquad n_0 = \dim(\mathfrak{g}) - 3n_2 = n^2 - 1.$$

For Case (3), B_n and D_n are similar. We will prove the B_n-case and leave D_n to the reader. The proof is by induction, showing that



The base case was proven in Case (2). Let $\alpha = \sum_{j=1}^{n} a_j\alpha_j$ be a positive root in B_n. The root space of α is in $\mathfrak{g}_{2\sum_{j=1}^{p-1} a_j}$, the $2\sum_{j=1}^{p-1} a_j$-eigenspace of ad_h.

The set of roots with $a_1 = 0$ defines a subsystem of type B_{n-1}, with corresponding subalgebra $\mathfrak{so}_{2n-1}\mathbb{C} \subset \mathfrak{g}$. On the other hand, there are $2n-1$ positive roots in B_n with $a_1 \neq 0$, namely

$$\left\{\beta_i = \sum_{j=1}^{i} \alpha_j\right\}_{i\in\{1,\ldots,n\}} \cup \quad \left\{\gamma_i = \beta_n + \sum_{k=i}^{n} \alpha_k\right\}_{i\in\{n,\ldots,2\}}.$$

We have

$$\mathfrak{g}_{\beta_i} \subset \begin{cases} \mathfrak{g}_{2i} & i \leqslant p-2 \\ \mathfrak{g}_{2p-2} & p-1 \leqslant i \leqslant n \end{cases} \quad \text{and} \quad \mathfrak{g}_{\gamma_i} \subset \begin{cases} \mathfrak{g}_{2p-2} & p \leqslant i \leqslant n \\ \mathfrak{g}_{2p-2+2(p-i)} & 2 \leqslant i \leqslant p-1 \end{cases}.$$

In particular, for $j \geqslant 0$ we have

$$\dim(\mathfrak{g}_{2j}) = \begin{cases} \dim(\mathfrak{g}_{2j} \cap \mathfrak{so}_{2n-1}\mathbb{C}) + 3 + 2n - 2p & j = p-1 \\ \dim(\mathfrak{g}_{2j} \cap \mathfrak{so}_{2n-1}\mathbb{C}) + 1 & \text{otherwise} \end{cases}.$$

Set $\theta_{2m_j} = \dim(\mathfrak{g}_{2m_j} \cap \mathfrak{so}_{2n-1}\mathbb{C}) - \dim(\mathfrak{g}_{2m_j+2} \cap \mathfrak{so}_{2n-1}\mathbb{C})$. Using (4.2) we have

$$n_{2m_j} = \theta_{2m_j} + \begin{cases} 2n-2p+2 & m_j = p-1 \\ 2p-2n-2 & m_j = p-2 \\ 1 & m_j = 2p-3 \\ 0 & \text{otherwise} \end{cases}.$$

The result for B_{n-1} gives the values of θ_{2m_j}. Thus, we have $n_0 = (2n+1-2p)(n-p+1)$, $n_{2m_j} = 1$ for $m_j \in \{1, 3, \ldots, 2p-3\} \setminus \{p-1\}$ and

$$n_{2p-2} = \begin{cases} 2n-2p+3 & p \text{ even} \\ 2n-2p+2 & p \text{ odd} \end{cases}.$$

Finally, we refer to the diagrams in section 9.2 to prove Case (4). In these diagrams, the circles denote the positive roots and the integer labels correspond the ad_h-eigenvalue on the root space. For E_8, we have $\dim(\mathfrak{g}_0) = 2 + \dim(\mathfrak{e}_6) = 80$, and

$$\dim(\mathfrak{g}_{10}) = 1, \quad \dim(\mathfrak{g}_8) = 1, \quad \dim(\mathfrak{g}_6) = 27, \quad \dim(\mathfrak{g}_4) = 27, \quad \dim(\mathfrak{g}_2) = 28.$$

Thus, the nonzero n_{2m_j}'s are $n_{10} = 1$, $n_6 = 26$, $n_2 = 1$ and $n_0 = 52$. This settles the E_8 case, and the other ones, E_6, E_7 and F_4, are left to the reader. □

4.2. **The centralizer $\mathfrak{c}$ and its centralizer.** The next description of the centralizer $\mathfrak{c}$ of a magical $\mathfrak{sl}_2$-triple $\{f, h, e\} \subset \mathfrak{g}$ follows, for classical Lie algebras, from the partition classification of magical nilpotents of Theorem 3.7 and [17, Theorem 6.1.3]. For the exceptional Lie algebras, $\mathfrak{c}$ is the complexification of the last column in the tables of [20].

Proposition 4.4. *The centralizer $\mathfrak{c} \subset \mathfrak{g}$ of a magical $\mathfrak{sl}_2$-triple is given as follows:*

(1) In Case (1) of Theorem 3.1, $\mathfrak{c} = 0$;
(2) In Case (2) of Theorem 3.1,

$\mathfrak{g}$	A_{2n-1}	B_n	C_n	D_n	D_{2n}	E_7
$\mathfrak{c}$	$\mathfrak{sl}_n\mathbb{C}$	$\mathfrak{so}_{2n-2}\mathbb{C}$	$\mathfrak{so}_n\mathbb{C}$	$\mathfrak{so}_{2n-3}\mathbb{C}$	$\mathfrak{sp}_{2n}\mathbb{C}$	$\mathfrak{f}_4$

(3) *In Case (3) of Theorem 3.1 with* $\mathfrak{g} = \mathfrak{so}_N\mathbb{C}$, $\mathfrak{c} \cong \mathfrak{so}_{N-2p+1}\mathbb{C}$*;*
(4) *In Case (4) of Theorem 3.1,*

$\mathfrak{g}$	E_6	E_7	E_8	F_4
$\mathfrak{c}$	$\mathfrak{sl}_3\mathbb{C}$	$\mathfrak{sp}_6\mathbb{C}$	$\mathfrak{f}_4$	$\mathfrak{so}_3\mathbb{C}$

We now show that a magical $\mathfrak{sl}_2$-triple $\{f, h, e\} \subset \mathfrak{g}$ arises from a principal $\mathfrak{sl}_2$-triple in a simple subalgebra $\mathfrak{g}(e) \subset \mathfrak{g}$.

Proposition 4.5. *Let* $\{f, h, e\} \subset \mathfrak{g}$ *be a magical* $\mathfrak{sl}_2$*-triple and* $\mathfrak{c} \subset \mathfrak{g}$ *be the centralizer of* $\{f, h, e\}$*. Then the centralizer of* $\mathfrak{c}$ *is the direct sum* $\mathfrak{z}(\mathfrak{c}) \oplus \mathfrak{g}(e)$*, where* $\mathfrak{z}(\mathfrak{c})$ *is the center of* $\mathfrak{c}$ *and* $\mathfrak{g}(e)$ *is a simple subalgebra* $\mathfrak{g}(e) \subset \mathfrak{g}$ *such that* $\{f, h, e\}$ *is a principal* $\mathfrak{sl}_2$*-subalgebra of* $\mathfrak{g}(e)$*. The subalgebra* $\mathfrak{z}(\mathfrak{c}) \oplus \mathfrak{g}(e)$ *is given by*

- *For Case (1) of Theorem 3.1,* $\mathfrak{g}(e) \cong \mathfrak{g}$ *and* $\mathfrak{z}(\mathfrak{c}) = 0$*.*
- *For Case (2) of Theorem 3.1,* $\mathfrak{g}(e) \cong \{f, h, e\}$ *and* $\mathfrak{z}(\mathfrak{c}) = \{0\}$*, unless* $\mathfrak{g} \cong \mathfrak{so}_5\mathbb{C}$ *in which case* $\mathfrak{z}(\mathfrak{c}) = \mathfrak{c} \cong \mathbb{C}$*.*
- *For Case (3) of Theorem 3.1,* $\mathfrak{g}(e) \cong \mathfrak{so}_{2p-1}\mathbb{C} \subset \mathfrak{so}_N\mathbb{C}$ *and* $\mathfrak{z}(\mathfrak{c}) = 0$*, unless* $\mathfrak{g} \cong \mathfrak{so}_{2p+1}\mathbb{C}$ *in which case* $\mathfrak{z}(\mathfrak{c}) = \mathfrak{c} \cong \mathbb{C}$*.*
- *For Case (4) of Theorem 3.1.* $\mathfrak{g}(e) \cong Lie(\mathrm{G}_2)$ *and* $\mathfrak{z}(\mathfrak{c}) = 0$*.*

Proof. We first identify $\mathfrak{g}(e)$ and show it centralizes $\mathfrak{c}$. In the Cases (1), there is nothing to prove since $\mathfrak{c} = 0$. In Case (2), there is also nothing to prove.

For Case (3) of Theorem 3.1, the magical nilpotent $e \in \mathfrak{so}_N\mathbb{C}$ corresponds to the Young diagram with one row of length $2p-1$ and $N-2p+1$-rows of length 1, by Case (6) of Theorem 3.7. This corresponds to principally embedding e in $\mathfrak{so}_{2p-1}\mathbb{C}$ followed by the embedding $\mathfrak{so}_{2p-1}\mathbb{C} \subset \mathfrak{so}_N\mathbb{C}$. In this case, the centralizer $\mathfrak{c}$ of $\{f, h, e\}$ is isomorphic to $\mathfrak{so}_{N-2p+1}\mathbb{C}$. The centralizer of $\mathfrak{g}(e) = \mathfrak{so}_{2p-1}\mathbb{C}$ is also isomorphic to $\mathfrak{so}_{N-2p+1}\mathbb{C}$ and contains the centralizer of $\{f, h, e\}$. Thus $\mathfrak{c}$ centralizes $\mathfrak{g}(e)$

For Case (4) of Theorem 3.1, we use the classification of nilpotents by theory of Bala–Carter (see [17, §8]). Very briefly, G-conjugacy classes of nilpotents in $\mathfrak{g}$ are in bijective correspondence with G-conjugacy classes of pairs $(\mathfrak{l}, \mathfrak{p}_\mathfrak{l})$. Here, $\mathfrak{l} \subset \mathfrak{g}$ is the Levi factor of a parabolic subalgebra of $\mathfrak{g}$ and $\mathfrak{p}_\mathfrak{l} \subset \mathfrak{l}$ is the parabolic subalgebra of $\mathfrak{l}$ associated to a so-called *distinguished* nilpotent of the semisimple part $[\mathfrak{l}, \mathfrak{l}]$ of $\mathfrak{l}$, i.e. one which does not belong to any proper Levi subalgebra of $[\mathfrak{l}, \mathfrak{l}]$. In particular, the principal nilpotent in $[\mathfrak{l}, \mathfrak{l}]$ is distinguished and corresponds to the Borel subalgebra of $\mathfrak{l}$.

In the tables of [17, §8.4], the label of the nilpotent has the form $X_N(a_i)$, where X_N is the type of the associated Levi $\mathfrak{l}$ and a_i is the number of simple roots in a Levi of $\mathfrak{p}_\mathfrak{l}$. The notation $X_N(a_0) = X_N$ is used and, in this case, the associated distinguished nilpotent of $\mathfrak{l}$ is principal. The labels of the weighted Dynkin diagrams of the magical nilpotents from Case (4) of Theorem 3.1 are B_3 for $\mathfrak{g} = \mathfrak{f}_4$ and D_4 for $\mathfrak{g} = \mathfrak{e}_6,\ \mathfrak{e}_7,\ \mathfrak{e}_8$. Thus, the magical nilpotent in F_4 arises from the principal nilpotent in $\mathfrak{so}_7\mathbb{C} \subset \mathfrak{f}_4$ and the magical nilpotents in type E_i arise from a principal nilpotent in $\mathfrak{so}_8\mathbb{C} \subset \mathfrak{e}_i$ for $i = 6, 7, 8$.

Now, a principal nilpotent in $\mathfrak{so}_7\mathbb{C}$ or $\mathfrak{so}_8\mathbb{C}$ is induced by a principal nilpotent in a subalgebra isomorphic of type G_2[1], $Lie(\mathrm{G}_2) \subset \mathfrak{so}_7\mathbb{C} \subset \mathfrak{so}_8\mathbb{C}$. More precisely, for a principal $\mathfrak{sl}_2$-triple

[1]We use the notation $Lie(\mathrm{G}_2)$ for the Lie algebra of G_2 since $\mathfrak{g}_2$ denotes the weight 2 space of ad_h.

$\{f, h, e\} \subset \mathfrak{so}_7\mathbb{C} \subset \mathfrak{so}_8\mathbb{C}$, the $\mathfrak{sl}_2\mathbb{C}$-module decomposition is

$$W_2 \oplus W_6 \oplus W_{10},$$

where the multiplicity n_6 of W_6 is 1 for $\mathfrak{so}_7\mathbb{C}$ and 2 for $\mathfrak{so}_8\mathbb{C}$, and

$$Lie(\mathrm{G}_2) \cong W_2 \oplus W_{10}.$$

Recall that the magical $_2$-triple in in $\mathfrak{g} = \mathfrak{f}_4,\ \mathfrak{e}_6,\ \mathfrak{e}_7,\ \mathfrak{e}_8$ of Case (4) of Theorem 3.1, induces the $\mathfrak{sl}_2\mathbb{C}$-module decomposition

$$\mathfrak{g} = W_0 \oplus W_2 \oplus W_6 \oplus W_{10},$$

and we have $\mathfrak{g}(e) = W_2 \oplus W_{10} \cong Lie(\mathrm{G}_2)$.

To complete the proof we claim that $\mathfrak{c}$ centralizes $W_2 \oplus W_{10}$. We have $W_2 = \{f, h, e\}$ and hence $\mathfrak{c}$ commutes with W_2. The multiplicity of n_{10} is 1. Hence $Z_{10} = W_{10} \cap \mathfrak{g}_0$ is 1-dimensional and $\mathfrak{c}$ acts by a character on Z_{10}. But $\mathfrak{c}$ has no nontrivial characters by Proposition 4.4. The space W_{10} is generated by the action of W_2 on Z_{10}, so $\mathfrak{c}$ centralizes $\mathfrak{g}(e) = W_2 \oplus W_{10}$.

Finally we argue that $\mathfrak{z}(\mathfrak{c}) \oplus \mathfrak{g}(e)$ is equal to the centralizer of $\mathfrak{c}$. By Proposition 4.4, $\mathfrak{g}_0 = \mathbb{C}^{\mathrm{rk}(\mathfrak{g}(e))} \oplus \mathfrak{g}_{0,ss}$ and $\mathfrak{c} \subset \mathfrak{g}_{0,ss}$ is the complexification of the maximal compact subalgebra of $\mathfrak{g}_{0,ss}$. By construction $\mathfrak{g}(e) \cap \mathfrak{g}_0 = \mathbb{C}^{\mathrm{rk}(\mathfrak{g}(e))}$. Since $\mathfrak{g}_{0,ss}$ has a trivial center, the only elements in $\mathfrak{g}_{0,ss}$ which centralize $\mathfrak{c}$ is the center of $\mathfrak{c}$. From Proposition 4.4, $\mathfrak{z}(\mathfrak{c}) = 0$ except when $\mathfrak{c} = \mathfrak{so}_2\mathbb{C} = \mathbb{C}$. It follows that intersection of the centralizer of $\mathfrak{c}$ with $\mathfrak{g}_0$ is $\mathfrak{g}(e) \oplus \mathfrak{z}(\mathfrak{c})$. Let x be an arbitrary element of the centralizer of $\mathfrak{c}$ and write $x = \sum x_{2j}$ for $x_{2j} \in \mathfrak{g}_{2j}$. Since $[\mathfrak{c}, \mathfrak{g}_j] \subset \mathfrak{g}_j$ we must have $[x_j, \mathfrak{c}] = 0$ for all j. For $j > 0$, we have $[\mathfrak{c}, \mathrm{ad}_f^j x_j] = 0$ and $\mathrm{ad}_f^j x_j \in \mathfrak{g}_0 \cap (\mathfrak{g}(e))$, and, for $j < 0$, we have $[\mathfrak{c}, \mathrm{ad}_e^j(x_j)] = 0$ and $\mathrm{ad}_e^j x_j \in \mathfrak{g}_0 \cap (\mathfrak{g}(e))$. Since $\{f, h, e\} \subset \mathfrak{g}(e)$, we conclude that $x_j \in \mathfrak{g}(e)$ for all $j \neq 0$. Hence, $\mathfrak{z}(\mathfrak{c}) \oplus \mathfrak{g}(e)$ is the centralizer of $\mathfrak{c}$. □

The following proposition is immediate from Propositions 4.3 and 4.5 (and proof).

Proposition 4.6. *Let $\{f, h, e\} \subset \mathfrak{g}$ be a magical $\mathfrak{sl}_2$-triple and $\mathfrak{g} = \bigoplus_{j=0}^{M} W_{2m_j}$ be the $\mathfrak{sl}_2$-module decomposition.*

- *For Case (3) of Theorem 3.1, we have*

$$\mathfrak{g}(e) \cong \mathfrak{so}_{2p-1}\mathbb{C} = \begin{cases} \displaystyle\bigoplus_{j=1}^{p-1} W_{4j-2} & p \text{ odd} \\ \displaystyle (W_{2p-2} \cap \mathfrak{g}(e)) \oplus \bigoplus_{j=1, j \neq \frac{p}{2}}^{p-1} W_{4j-2} & p \text{ even} \end{cases}$$

- *For Case (4) of Theorem 3.1, $\mathfrak{g}(e) \cong Lie(\mathrm{G}_2) = W_2 \oplus W_{10}$.*

Finally, we prove the following lemma which will be useful in the next section.

Lemma 4.7. *Let $\{f, h, e\} \subset \mathfrak{g}$ be a magical nilpotent and G be a connected Lie group with Lie algebra $\mathfrak{g}$, such that the involution σ_e in (2.6) integrates to G. Let $\mathrm{C} \subset \mathrm{G}$ be the centralizer of $\{f, h, e\}$. Then C centralizes the subalgebra $\mathfrak{g}(e) \subset \mathfrak{g}$ described in Proposition 4.5.*

Proof. In cases (1) and (2) of Theorem 3.1, this is immediate, since C is the center of G in Case (1) and $\mathfrak{g}(e) = \{f, h, e\}$ in Case (2). For cases (3) and (4), note that we have $[\mathfrak{c}, \mathfrak{g}(e)] = 0$ by Proposition 4.5. Thus, we must understand how the group of components $\pi_0(\mathrm{C})$ acts on $\mathfrak{g}(e)$. Note that it suffices to show that C acts trivially when G is simply connected.

For G simply connected and $e \in \mathfrak{g}$ a nilpotent, the fundamental group of the G-orbit $\mathrm{G}\cdot e \subset \mathfrak{g}$ is given by the components of C (see [17, Lemma 6.1.1]),

$$\pi_1(\mathrm{G}\cdot e) = \pi_0(\mathrm{C}).$$

For Case (4) of Theorem 3.1, $\pi_1(\mathrm{G}\cdot e)$ is trivial (see [17, §8.4]). Thus, C is connected and we conclude that C acts trivially $\mathfrak{g}(e)$.

For Case (3), we have $\pi_1(\mathrm{G}\cdot e) = \pi_0(\mathrm{C}) = \mathbb{Z}/2\mathbb{Z}$ [17, §6.1]. The $\mathrm{SO}_N\mathbb{C}$-centralizer of $\{f,h,e\}$ also has two connected components since it is given by $\mathrm{S}(\mathrm{O}_1\mathbb{C}\times\mathrm{O}_{N-2p+1}\mathbb{C}) \cong \mathrm{O}_{N-2p+1}\mathbb{C}$ [17, Theorem 6.1.3]. Thus, it suffices to prove that the $\mathrm{SO}_N\mathbb{C}$-centralizer of $\{f,h,e\}$ also centralizes $\mathfrak{g}(e)$. In this case, we have $\mathfrak{g}(e) \subset \mathfrak{g}$ is $\mathfrak{so}_{2p-1}\mathbb{C} \subset \mathfrak{so}_N\mathbb{C}$, and the $\mathrm{SO}_N\mathbb{C}$-centralizer of $\mathfrak{so}_{2p-1}\mathbb{C}$ is $\mathrm{S}(\mathrm{O}_1\mathbb{C}\times\mathrm{O}_{N-2p+1}\mathbb{C})$. Thus, C centralizes $\mathfrak{g}(e)$. □

4.3. **The Cayley real form.** By Proposition 4.2 the subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ associated to a magical $\mathfrak{sl}_2$-triple has the form

$$\mathfrak{g}_0 = \mathbb{C}^{\mathrm{rk}(\mathfrak{g}(e))} \oplus \mathfrak{g}_{0,ss} = \mathfrak{g}_0 \cap \mathfrak{g}(e) \oplus \mathfrak{g}_{0,ss}.$$

Recall that it has a special real form – the Cayley real form – denoted by $\mathfrak{g}_{\mathcal{C}}^{\mathbb{R}}$ and defined in Definition 2.13.

Proposition 4.8. *Let $\mathfrak{g}_{\mathcal{C}}^{\mathbb{R}} \subset \mathfrak{g}_0$ be the Cayley real form associated a magical $\mathfrak{sl}_2$-triple $\{f,h,e\} \subset \mathfrak{g}$. Then*

$$\mathfrak{g}_{\mathcal{C}}^{\mathbb{R}} \cong \mathbb{R}^{\mathrm{rk}(\mathfrak{g}(e))} \oplus \mathfrak{g}_{0,ss}^{\mathbb{R}},$$

where $\mathfrak{g}_{0,ss}^{\mathbb{R}} \subset \mathfrak{g}_{0,ss}$ is the real form with complexified maximal compact subalgebra $\mathfrak{c} \subset \mathfrak{g}$. Thus,

(1) for Case (1) of Theorem 3.1, $\mathfrak{g}_{\mathcal{C}}^{\mathbb{R}} \cong \mathbb{R}^{\mathrm{rk}\,\mathfrak{g}}$.
(2) for Case (2) of Theorem 3.1,

$\mathfrak{g}$	A_{2n-1}	B_n	C_n	D_n	D_{2n}	E_7
$\mathfrak{g}_{\mathcal{C}}^{\mathbb{R}}$	$\mathbb{R}\oplus\mathfrak{sl}_n\mathbb{C}$	$\mathbb{R}\oplus\mathfrak{so}_{1,2n-2}$	$\mathbb{R}\oplus\mathfrak{sl}_n\mathbb{R}$	$\mathbb{R}\oplus\mathfrak{so}_{1,2n-3}$	$\mathbb{R}\oplus\mathfrak{su}^*_{2n}$	$\mathbb{R}\oplus\mathfrak{e}_6^{-26}$

(3) for Case (3) of Theorem 3.1, $\mathfrak{g}_{\mathcal{C}}^{\mathbb{R}} \cong \mathbb{R}^{p-1}\oplus\mathfrak{so}_{1,N-2p+1}$.
(4) for Case (4) of Theorem 3.1,

$\mathfrak{g}$	E_6	E_7	E_8	F_4
$\mathfrak{g}_{\mathcal{C}}^{\mathbb{R}}$	$\mathbb{R}^2\oplus\mathfrak{sl}_3\mathbb{C}$	$\mathbb{R}^2\oplus\mathfrak{su}^*_6$	$\mathbb{R}^2\oplus\mathfrak{e}_6^{-26}$	$\mathbb{R}^2\oplus\mathfrak{sl}_3\mathbb{R}$

Proof. The Cayley real form is the real form of $\mathfrak{g}_0$ with the property that the complexification of the maximal compact subalgebra is $\mathfrak{c}$. The classification follows from Proposition 4.4. □

Remark 4.9. Note that, in all of the cases, each Z_{2m_j} with $n_{2m_j} = 1$ contributes with an $\mathbb{R}$-factor to $\mathfrak{g}_{\mathcal{C}}^{\mathbb{R}}$. In Case (2), the $\mathbb{R}$-factor of $\mathfrak{g}_{\mathcal{C}}^{\mathbb{R}}$ is given by $\langle h\rangle$, the real span of h, and in Case (3), with p even, an additional $\mathbb{R}$-factor arises from $\mathfrak{g}(e)\cap Z_{2p-2}$.

Let $\mathfrak{g}^{\mathbb{R}} \subset \mathfrak{g}$ be any real form of a complex reductive Lie algebra, with complexified Cartan decomposition $\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{m}$. Recall that the *real rank* of $\mathfrak{g}^{\mathbb{R}}$ is defined to be the maximal dimension of a subalgebra $\mathfrak{a} \subset \mathfrak{m}$ such that the direct sum of $\mathfrak{a}$ with its $\mathfrak{h}$-centralizer is a Cartan subalgebra of $\mathfrak{g}$.

From Propositions 4.1 and 4.8, a simple comparison of the real ranks (see for instance Appendices C.3 and C.4 of [52]) proves the next result.

Proposition 4.10. *Let $\{f,h,e\} \subset \mathfrak{g}$ be a magical $\mathfrak{sl}_2$-triple. Then the real rank of the canonical real form $\mathfrak{g}^{\mathbb{R}}$ equals the real rank of the Cayley real form $\mathfrak{g}_{\mathcal{C}}^{\mathbb{R}}$.*

We will also need the notion of the Cayley group $\mathrm{G}^{\mathbb{R}}_{\mathcal{C}}$.

Definition 4.11. *Let $\{f,h,e\}\subset\mathfrak{g}$ be a magical $\mathfrak{sl}_2$-triple with Cayley real form $\mathfrak{g}^{\mathbb{R}}_{\mathcal{C}} = (\mathbb{R}^+)^{\mathrm{rk}(\mathfrak{g}(e))}\oplus\mathfrak{g}^{\mathbb{R}}_{0,ss}$. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ such that the involution σ_e from (2.6) integrates to an involution $\sigma_e:\mathrm{G}\to\mathrm{G}$. Let $\mathrm{G}^{\mathbb{R}}\subset\mathrm{G}$ be the canonical real form and $\mathrm{C}\subset\mathrm{G}$ be the centralizer of $\{f,h,e\}$. Then the* Cayley group *of $\{f,h,e\}$ and G is the group*

$$\mathrm{G}^{\mathbb{R}}_{\mathcal{C}} = (\mathbb{R}^+)^{\mathrm{rk}(\mathfrak{g}(e))}\times\mathrm{G}^{\mathbb{R}}_{0,ss}\ ,$$

where $\mathrm{G}^{\mathbb{R}}_{0,ss}$ is the real Lie group with Lie algebra $\mathfrak{g}^{\mathbb{R}}_{0,ss}$ and maximal compact $\mathrm{C}\cap\mathrm{G}^{\mathbb{R}}$.

Remark 4.12. In general, the complexification of the maximal compact of the Cayley group $\mathrm{G}^{\mathbb{R}}_{\mathcal{C}}$ is $\mathrm{G}^{\sigma_e}\cap\mathrm{C}=\mathrm{H}\cap\mathrm{C}$. For a principal $\mathfrak{sl}_2$-triple, $\mathrm{C}=\mathrm{Z}(\mathrm{G})\subset\mathrm{G}$ is the center of G and $\mathfrak{g}^{\mathbb{R}}_{0,ss}=0$. Thus $\mathrm{G}^{\mathbb{R}}_{0,ss}=\mathrm{Z}(\mathrm{G}^{\mathbb{R}})$ is the center of $\mathrm{G}^{\mathbb{R}}$. In particular, $\mathrm{C}\neq\mathrm{C}\cap\mathrm{H}$ in general. For example, when $\mathrm{G}=\mathrm{SL}_n\mathbb{C}$ and $\{f,h,e\}$ is a principal $\mathfrak{sl}_2$-triple, $\mathrm{C}=\mathbb{Z}/n\mathbb{Z}$ is the center of $\mathrm{SL}_n\mathbb{C}$ but the center of the canonical real form $\mathrm{SL}_n\mathbb{R}$ is either $\mathbb{Z}/2\mathbb{Z}$ or trivial.

4.4. **Lie theory structure for magical nilpotents in exceptional Lie algebras.** Let $\{f,h,e\}\subset\mathfrak{g}$ be a magical $\mathfrak{sl}_2$-triple from Case (4) of Theorem 3.1. In this section we will study the structure of the magical $\mathfrak{sl}_2$-triple in more detail. The root poset diagrams in §9.2 will be important, so frequently referenced in this discussion. In these diagrams, the labeling of a line connecting a positive root β to a higher positive root γ corresponds to the simple root α_j for which $\gamma=\beta+\alpha_j$. The labeling of every line can be deduced from the labeling of the left most line of each row.

Recall that the $\mathfrak{sl}_2$-module decomposition (2.3) is $\mathfrak{g}=W_0\oplus W_2\oplus W_6\oplus W_{10}$, the $\mathbb{Z}$-grading (2.4) is $\mathfrak{g}=\bigoplus_{j=-5}^{5}\mathfrak{g}_{2j}$ and the subalgebra $\mathfrak{g}(e)\subset\mathfrak{g}$ describe in Proposition 4.5 is $\mathfrak{g}(e)\cong Lie(\mathrm{G}_2)=W_2\oplus W_{10}$. The complexified Cartan decomposition of the involution $\sigma_e:\mathfrak{g}\to\mathfrak{g}$ is $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$, where

$$\mathfrak{h}=\mathfrak{g}_{-8}\oplus\mathfrak{g}_{-4}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_4\oplus\mathfrak{g}_8 \quad\text{and}\quad \mathfrak{m}=\mathfrak{g}_{-10}\oplus\mathfrak{g}_{-6}\oplus\mathfrak{g}_{-2}\oplus\mathfrak{g}_2\oplus\mathfrak{g}_6\oplus\mathfrak{g}_{10}\ . \tag{4.3}$$

Each of the weight spaces $\mathfrak{g}_{2j}$ with $j\neq0$ is a direct sum of root spaces as in (4.1). From the diagrams in §9.2, it is clear that the weight spaces $\mathfrak{g}_{\pm2}$ decompose as a direct sum of two $\mathfrak{g}_0$-representations,

$$\mathfrak{g}_{\pm2}=\mathfrak{g}^b_{\pm2}\oplus\mathfrak{g}_{\pm\tilde{\alpha}}, \tag{4.4}$$

where $\mathfrak{g}_{\tilde{\alpha}}$ is the root space of the simple root $\tilde{\alpha}$ in the diagrams in §9.2, and $\mathfrak{g}^b_{\pm2}$ is the direct sum of root spaces in $\mathfrak{g}_\beta\subset\mathfrak{g}_{\pm2}$ with $\beta\neq\pm\tilde{\alpha}$. We can then decompose $f\in\mathfrak{g}_{-2}$ and $e\in\mathfrak{g}_2$ as

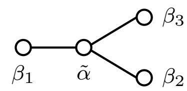
$$f=f_b+\tilde{f}\qquad\text{and}\qquad e=e_b+\tilde{e}, \tag{4.5}$$

where e_b, f_b and $\tilde{e}$, $\tilde{f}$ are the projections of e and f onto $\mathfrak{g}^b_{\pm2}$ and $\mathfrak{g}_{\pm\tilde{\alpha}}$ respectively.

Lemma 4.13. *Each of the terms $\tilde{f}$, f_b, $\tilde{e}$ and e_b in (4.5) is nonzero.*

Proof. The $\mathfrak{sl}_2$-module decomposition $\mathfrak{g}=W_0\oplus W_2\oplus W_6\oplus W_{10}$ implies that $\mathrm{ad}_f:\mathfrak{g}_{10}\to\mathfrak{g}_8$ and $\mathrm{ad}_f:\mathfrak{g}_6\to\mathfrak{g}_4$ are isomorphisms. The map $\mathrm{ad}_f:\mathfrak{g}_{10}\to\mathfrak{g}_8$ is equal to $\mathrm{ad}_{\tilde{f}}$ since $\mathfrak{g}_8$ and $\mathfrak{g}_{10}$ are root spaces which differ by the root $\tilde{\alpha}$. So $\tilde{f}$ cannot be zero. On the other hand, $\mathrm{ad}_f:\mathfrak{g}_6\to\mathfrak{g}_4$ is given by ad_{f_b} since $\mathfrak{g}_4$ and $\mathfrak{g}_6$ are both a direct sum of root spaces $\mathfrak{g}_\beta$, where β has the form $\beta=\sum n_i\alpha_i$ and the coefficient of $\tilde{\alpha}$ is 1. So again $f_b\neq0$. Similar arguments imply $\tilde{e}\neq0$ and $e_b\neq0$. □

Note that $\tilde{\alpha}$ is a red root labeled with a 2 in §9.2. Denote by $\{\beta_1, \beta_2, \beta_3\}$ the other red root spaces which are still labeled with a 2. We claim that $\{\tilde{\alpha}, \beta_1, \beta_2, \beta_3\}$ are a D_4-system with

D_4:


Since there is an action of the symmetric group on three letters on the roots of D_4, the choice of which β_i corresponds to which root space in $\mathfrak{g}_2^b$ is irrelevant.

Lemma 4.14. *The root spaces associated to the red roots in the diagrams of Section 9.2 form a subalgebra isomorphic to $\mathfrak{so}_8\mathbb{C}$.*

Proof. The proof is by direct computation. The positive roots of D_4 are

$$\{\tilde{\alpha}, \beta_1, \beta_2, \beta_3\} \cup \left\{\tilde{\alpha} + \sum_{n_i \in \{0,1\}} n_i\beta_i\right\} \cup \{2\tilde{\alpha}_1 + \beta_1 + \beta_2 + \beta_3\}. \tag{4.6}$$

Using the expression of β_i in terms of the simple roots of $\mathfrak{g}$, one checks that $\{\tilde{\alpha}, \beta_1, \beta_2, \beta_3\}$ satisfy the relations of a D_4 root system and that no other linear combinations of $\{\tilde{\alpha}, \beta_1, \beta_2, \beta_3\}$ define roots in $\mathfrak{g}$. From the diagrams §9.2, it is clear that $\tilde{\alpha} + \beta_i$ is a root, but none of $\beta_i + \beta_j$, $\tilde{\alpha} - \beta_i$ or $\beta_i - \beta_j$ is a root. Any other linear combination of $\tilde{\alpha}, \beta_1, \beta_2, \beta_3$ will have the coefficient of $\tilde{\alpha}$ being nonzero and a coefficient $n_i \geqslant 2$. All such roots in $\mathfrak{g}$ are listed in the tables [52, Appendix C.2] and one checks that the only expressions which are roots of $\mathfrak{g}$ are in (4.6).[2] $\square$

Recall that the coroot h_α associated to a root $\alpha \in \mathfrak{a}^*$ is defined by $h_\alpha = 2\frac{\alpha^*}{\langle \alpha, \alpha \rangle}$ where $\alpha^* \in \mathfrak{a}$ satisfies $\langle \alpha^*, x \rangle = \alpha(x)$ for all $x \in \mathfrak{a}$. Let $\Delta_+ \subset \mathfrak{a}^*$ denote a set of positive roots with simple roots $\{\alpha_1, \ldots, \alpha_{\mathrm{rk}(\mathfrak{g})}\}$, and let $\{f_i, h_{\alpha_i}, e_i\}$ be $\mathfrak{sl}_2$-triples with $e_i \in \mathfrak{g}_{\alpha_i}$ and $f_i \in \mathfrak{g}_{-\alpha_i}$. This data determines a principal $\mathfrak{sl}_2$-triple $\{f, h, e\} \subset \mathfrak{g}$ given by,

$$h = \sum_{\alpha \in \Delta^+} h_\alpha = \sum_{i=1}^{\mathrm{rk}(\mathfrak{g})} r_i h_{\alpha_i}, \qquad e = \sum_{i=1}^{\mathrm{rk}(\mathfrak{g})} \sqrt{r_i} e_i, \qquad f = \sum_{i=1}^{\mathrm{rk}(\mathfrak{g})} \sqrt{r_i} f_i.$$

For the simple roots $\{\tilde{\alpha}, \beta_1, \beta_2, \beta_3\}$ of D_4, the above construction yields

$$\{f, h, e\} = \left\{\sqrt{12} f_{\tilde{\alpha}} + \sqrt{6}\sum_{j=1}^{3} f_{\beta_j}, 12h_{\tilde{\alpha}} + 6\sum_{j=1}^{3} h_{\beta_j}, \sqrt{12} e_{\tilde{\alpha}} + \sqrt{6}\sum_{j=1}^{3} e_{\beta_j}\right\}. \tag{4.7}$$

Lemma 4.15. *A principal $\mathfrak{sl}_2$-triple $\{f, h, e\} \subset \mathfrak{so}_8\mathbb{C} \subset \mathfrak{g}$ in the $\mathfrak{so}_8\mathbb{C}$-subalgebra from Lemma 4.14 is a magical $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ from Case (4) of Theorem 3.1.*

Proof. Consider the simple roots $\{\tilde{\alpha}, \beta_1, \beta_2, \beta_3\}$ of D_4 and the principal $\mathfrak{sl}_2$ given above. We must show that the numbers $\alpha_i(h)$ match the weighted Dynkin diagrams from Case (4) of Theorem 3.1. Let α_i be a simple root of $\mathfrak{g}$ which is not in $\{\beta_1, \beta_2, \beta_3\}$. Then α_i is orthogonal to $\tilde{\alpha}$, and

$$\alpha_i\left(12h_{\tilde{\alpha}} + 6\sum_{j=1}^{3} h_{\beta_j}\right) = \alpha_i\left(6\sum_{j=1}^{3} h_{\beta_j}\right).$$

If α_i is orthogonal to each β_j, then $\alpha_i(\tilde{h}) = 0$. For $\mathfrak{g} = \mathfrak{e}_7$, $\mathfrak{e}_8$ respectively, the simple roots $\{\alpha_4, \alpha_5, \alpha_7\}$, $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ are orthogonal to each β_j. For the remaining simple roots

[2] Note that in the notation of 9.2, $\tilde{\alpha} = \alpha_1, \alpha_8$ for $\mathfrak{g} = \mathfrak{e}_7, \mathfrak{e}_8$ respectively, while in the notation of [52, Appendix C.2], $\tilde{\alpha} = \alpha_7, \alpha_8$ for $\mathfrak{g} = \mathfrak{e}_7, \mathfrak{e}_8$ respectively.

$\alpha_j \notin \{\tilde{\alpha}, \beta_1, \beta_2, \beta_3\}$, there is a unique β_l such that $\alpha_i + \beta_l$ is a root and a unique $\beta_k \neq \beta_l$ such that $-\alpha_i + \beta_k$ is a root. Hence

$$\alpha_i\left(12h_{\tilde{\alpha}} + 6\sum_{j=1}^{3} h_{\beta_j}\right) = 6\alpha_i(h_{\beta_l}) + 6\alpha_i(h_{\beta_k}) = 12\frac{\langle \alpha_i, \beta_l\rangle}{\langle \beta_l, \beta_l\rangle} + 12\frac{\langle \alpha_i, \beta_k\rangle}{\langle \beta_k, \beta_k\rangle}.$$

Since the roots, β_l, β_k, $\alpha_i + \beta_l$ and $\alpha_i - \beta_k$ have the same length we have $\alpha_i(\tilde{h}) = 0$. Finally, if β_1 is the simple root which is also a simple root of $\mathfrak{g}$, then $\tilde{\alpha}(\tilde{h}) = 2$ and $\beta_1(\tilde{h}) = 2$. Thus, the weighted Dynkin diagram of $\{\tilde{f}, \tilde{h}, \tilde{e}\}$ corresponds to a magical $\mathfrak{sl}_2$-triple from Case (4) of Theorem 3.1. □

Lemma 4.16. *Let $\{f, h, e\} \subset \mathfrak{g}$ be a magical $\mathfrak{sl}_2$-triple from Case (4) in Theorem 3.1. Then $\mathfrak{g}_{\tilde{\alpha}} \subset \mathfrak{g}(e)$ is a simple root space for $\mathfrak{g}(e)$ associated to a long root and $\langle e_b\rangle \subset \mathfrak{g}(e)$ is a simple root space for $\mathfrak{g}(e)$ associated to a short root.*

Proof. This follows from the fact that $Lie(\mathrm{G}_2) \subset \mathfrak{so}_8\mathbb{C}$ and a principal $\mathfrak{sl}_2$-triple in $Lie(\mathrm{G}_2)$ is also principal in $\mathfrak{so}_8\mathbb{C}$. Using the decomposition (4.7), we have $e_b = \sqrt{12}(e_{\beta_1} + e_{\beta_2} + e_{\beta_3})$ and $\tilde{e} = \sqrt{6}e_{\tilde{\alpha}}$. A direct computation shows that $\{\tilde{e}, e_b\}$ generate the positive roots of $Lie(\mathrm{G}_2)$ with highest root $[\tilde{e}, [e_b, [e_b, [e_b, \tilde{e}]]]]$. □

Remark 4.17. For Case (4) of Theorem 3.1, this gives a direct proof that the sum $W_2 \oplus W_{10}$ in the $\mathfrak{sl}_2\mathbb{C}$-module decomposition of $\mathfrak{g}$ is a subalgebra isomorphic to $Lie(\mathrm{G}_2)$.

Recall that the canonical real form $\mathfrak{g}^{\mathbb{R}}$ associated to $\{f, h, e\}$ is the quaternionic real form of $\mathfrak{g}$. Thus, $\mathfrak{h} \cong \mathfrak{sl}_2\mathbb{C} \oplus \mathfrak{h}'$, where $\mathfrak{h}'$ is $\mathfrak{sp}_6\mathbb{C}$, $\mathfrak{sl}_6\mathbb{C}$, $\mathfrak{so}_{12}\mathbb{C}$ and $\mathfrak{e}_7$ when $\mathfrak{g}$ is $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$ respectively.

Lemma 4.18. *The decomposition $\mathfrak{h} = \mathfrak{sl}_2\mathbb{C} \oplus \mathfrak{h}'$ is given by*

$$\mathfrak{sl}_2\mathbb{C} = \mathfrak{g}_{-8} \oplus [\mathfrak{g}_{-8}, \mathfrak{g}_8] \oplus \mathfrak{g}_8 \qquad \text{and} \qquad \mathfrak{h}' = \mathfrak{g}_{-4} \oplus [\mathfrak{g}_{-4}, \mathfrak{g}_4] \oplus \mathfrak{g}_4. \tag{4.8}$$

Proof. It is clear that $\mathfrak{h}' = \mathfrak{g}_{-4} \oplus [\mathfrak{g}_{-4}, \mathfrak{g}_4] \oplus \mathfrak{g}_4$ is a subalgebra and $\mathfrak{g}_{-8} \oplus [\mathfrak{g}_{-8}, \mathfrak{g}_8] \oplus \mathfrak{g}_8$ is a subalgebra isomorphic to $\mathfrak{sl}_2\mathbb{C}$. Note that $[\mathfrak{g}_{\pm 8}, \mathfrak{g}_{\pm 4}] = 0$ since there is not a weight-12 summand of the grading. Using the root poset diagrams in §9.2, $\mathfrak{g}_{\pm 8}$ and $\mathfrak{g}_{\pm 4}$ are direct sums of root spaces $\mathfrak{g}_\alpha$ with $\alpha = \pm 1\tilde{\alpha} + \sum_{\alpha_i \neq \tilde{\alpha}} n_i\alpha_i$. This implies $[\mathfrak{g}_8, \mathfrak{g}_{-4}] = 0$ and $[\mathfrak{g}_{-8}, \mathfrak{g}_4] = 0$ since $[\mathfrak{g}_8, \mathfrak{g}_{-4}] \subset \mathfrak{g}_4$ and $[\mathfrak{g}_{-8}, \mathfrak{g}_4] \subset \mathfrak{g}_{-4}$. Now the Jacobi identity implies that $[\mathfrak{h}', \mathfrak{g}_{-8} \oplus [\mathfrak{g}_8, \mathfrak{g}_{-8}] \oplus \mathfrak{g}_8] = 0$. □

Lemma 4.19. *Consider the decomposition $\mathfrak{h} = \mathfrak{sl}_2\mathbb{C} \oplus \mathfrak{h}'$ from (4.8) and the decomposition of $\mathfrak{m}$ from (4.3) and (4.4). Then, $\mathfrak{m}$ decomposes as*

$$\mathfrak{m} = \begin{matrix} \mathfrak{g}_{10} & \oplus & \mathfrak{g}_6 & \oplus & \mathfrak{g}_2^b & \oplus & \mathfrak{g}_{-\tilde{\alpha}} \\ \mathfrak{g}_{+\tilde{\alpha}} & \oplus & \mathfrak{g}_{-2}^b & \oplus & \mathfrak{g}_{-6} & \oplus & \mathfrak{g}_{-10} \end{matrix},$$

where the rows are $\mathfrak{h}'$-invariant and the columns are $\mathfrak{sl}_2\mathbb{C}$-invariant.

Proof. Observe that $\mathfrak{g}_{\pm 4}$, $\mathfrak{g}_{\pm 6}$, $\mathfrak{g}_{\pm 8}$, $\mathfrak{g}_{-\tilde{\alpha}}$ are direct sums root spaces $\mathfrak{g}_\alpha$ with $\alpha = \sum_i n_i\alpha$, where the coefficient of the simple root $\tilde{\alpha}$ is ± 1 and $\mathfrak{g}_{\pm 10}$ is the root space for $\pm\tilde{\alpha} \pm \gamma$ where $\mathfrak{g}_8$ is the root space for the root γ. Thus, the rows are preserved by bracketing with $\mathfrak{g}_{\pm 4}$ and the columns are preserved by bracketing with $\mathfrak{g}_{\pm 8}$. □

Finally, we deduce some bracket relations which will be useful later.

Lemma 4.20. *Let $\{f,h,e\}\subset\mathfrak{g}$ be a magical $\mathfrak{sl}_2$-triple from Case (4) of Theorem 3.1 and let $\mathfrak{g}=W_0\oplus W_2\oplus W_6\oplus W_{10}$ be the $\mathfrak{sl}_2$-module decomposition. Let $f=f_b+\tilde f$ and $e=e_b+\tilde e$ be the decompositions (4.5) and $V_6=\ker(\mathrm{ad}_e|_{W_6})$. Then, for any $\phi\in V_6$,*

$$\mathrm{ad}^3_{f_b}(\tilde f)\neq 0\in\mathfrak{g}_{-8}, \tag{4.9}$$

$$\mathrm{ad}^3_f(\phi)=[f_b,[\tilde f,[f_b,\phi]]]=[[f_b,\tilde f],[f_b,\phi]], \tag{4.10}$$

$$\mathrm{ad}_{f_b+\phi}\tilde f=[f_b,\tilde f], \tag{4.11}$$

$$\mathrm{ad}^3_{f_b+\phi}\tilde f=\mathrm{ad}^3_{f_b}(\tilde f)+3\,\mathrm{ad}^3_f(\phi)+\mathrm{ad}^2_\phi\circ\mathrm{ad}_{f_b}(\tilde f). \tag{4.12}$$

Proof. Equation (4.9) directly from Lemma 4.16 and the bracket relations in $\mathit{Lie}(\mathrm{G}_2)$. For equation (4.10), we have

$$\mathrm{ad}^3_f(\varphi)=[f_b+\tilde f,[f_b+\tilde f,[f_b+\tilde f,\phi]]].$$

Since $\mathrm{ad}_{\tilde f}\mathfrak{g}_6=0\subset\mathfrak{g}_4$, $[\tilde f,\phi]=0$ and

$$\mathrm{ad}^3_f(\phi)=[f_b,[f_b,[f_b,\phi]]]+[\tilde f,[\tilde f,[f_b,\phi]]]+[\tilde f,[f_b,[f_b,\phi]]]+[f_b,f_b,[f_b,\phi]]].$$

We will show that the first three terms are zero. Recall that $\mathfrak{g}_6$ is a direct sum of root spaces $\mathfrak{g}_\alpha$ where the coefficient of $\tilde\alpha$ is 1. Thus,

$$[f_b,[f_b,\phi]]\subset\mathfrak{g}_{\tilde\alpha}\qquad\text{and}[\tilde f,[f_b,\phi]]\subset\mathfrak{g}^b_2.$$

Since $[\mathfrak{g}^b_{\pm2},\mathfrak{g}_{\mp\tilde\alpha}]=0$, the first two terms are zero. For the third term, note that $\mathrm{ad}^2_{f_b}(V_6)\subset\mathfrak{g}_{\tilde\alpha}$ is the projection of $\mathrm{ad}^2_f(V_6)$ onto $\mathfrak{g}_{\tilde\alpha}$. But $\mathfrak{g}_{\tilde\alpha}\subset W_2\oplus W_{10}$ by Lemma 4.16 and $\mathrm{ad}^2_f(V_6)\cap W_2\oplus W_{10}=0$. Hence $\mathrm{ad}^2_{f_b}(\phi)=0$ for $\phi\in V_6$, and

$$\mathrm{ad}^3_f(\phi)=[f_b,[\tilde f,[f_b,\phi]]].$$

The Jacobi identity and $\mathrm{ad}^2_{f_b}(\phi)=0$ imply $[f_b,[\tilde f,[f_b,\phi]]]=[[f_b,\tilde f],[f_b,\phi]]$.

Equation (4.11) follows since $[\mathfrak{g}_{-\tilde\alpha},\mathfrak{g}_6]=0$. For Equation 4.12, we have

$$\mathrm{ad}^3_{f_b+\phi}(\tilde f)=[f_b+\phi,[f_b+\phi,[f_b,\tilde\phi]]]$$

since $[\tilde f,\phi]=0$. Thus,

$$\mathrm{ad}^3_{f_b+\phi}=\mathrm{ad}^3_{f_b}(\tilde f)+[f_b,[\phi,[f_b,\tilde f]]]+[\phi,[f_b,[f_b,\tilde f]]+\mathrm{ad}^2_\phi([f_b,\tilde f]).$$

The middle two terms are in $\mathfrak{g}_0$. Using the Jacobi identity and $[\tilde f,\phi]=0$, we have

$$\begin{aligned}[f_b,[\phi,[f_b,\tilde f]]]+[\phi,[f_b,[f_b,\tilde f]]]&=-[f_b,[\tilde f,[\phi,f_b]]]-[[f_b,\tilde f],[\phi,f_b]]-[f_b,[[f_b,\tilde f],\phi]]\\&=[f_b,[\tilde f,[f_b,\phi]]]+[[f_b,\tilde f],[f_b,\phi]]+[f_b,[\tilde f,[f_b,\phi]]]\\&=3\,\mathrm{ad}^3_f(\phi),\end{aligned}$$

by (4.10). ◻

As a result of the above discussion, we have the following proposition. Recall that a nonzero nilpotent is magical if it belongs to a magical $\mathfrak{sl}_2$-triple.

Proposition 4.21. *The nilpotent $[f_b,\tilde f]\subset\mathfrak{g}_{-4}$ is a magical nilpotent in $\mathfrak{h}'$ of the type of Case (2) of Theorem 3.1 and $[f_b,[f_b,[f_b,\tilde f]]]$ is a magical (i.e. nonzero) nilpotent in $\mathfrak{sl}_2\mathbb{C}$.*

Remark 4.22. Note that $[\mathfrak{g}^b_{-2},\mathfrak{g}^b_2]\subset\mathfrak{g}_0$ is isomorphic to $[\mathfrak{g}_4,\mathfrak{g}_{-4}]$, thus $\{f_b,[e_b,f_b],e_b\}\subset\mathfrak{g}^b_{-2}\oplus[\mathfrak{g}^b_2,\mathfrak{g}^b_{-2}]\oplus\mathfrak{g}^b_2$ is a magical nilpotent from Case (2) of Theorem 3.1.

We also need to understand the group H and its action on $\mathfrak{m}$. Let G and H' be the simply connected groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}'$ respectively. From the description of the Lie algebras $\mathfrak{h}'$ above, H' is $\mathrm{Sp}_6\mathbb{C}$, $\mathrm{SL}_6\mathbb{C}$, $\mathrm{Spin}_{12}\mathbb{C}$, E_7 when $\mathfrak{g}$ is $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$, respectively. The group $\mathrm{H} \subset \mathrm{G}$ is a quotient

$$\mathrm{H} = (\mathrm{H}' \times \mathrm{SL}_2\mathbb{C})/\mathbb{Z}_2,$$

where $\mathbb{Z}_2$ has generator (μ', μ_2) for $\mu' \in \mathrm{H}'$ and $\mu_2 \in \mathrm{SL}_2\mathbb{C}$ the unique order two elements of the center.

As an H-representation, $\mathfrak{m}$ is the tensor product $\mathfrak{m} = V' \otimes V_2$, where V_2 is the standard representation of $\mathrm{SL}_2\mathbb{C}$ and V' is an irreducible H'-representation known as a *minuscule representation*. The decomposition $\mathfrak{h}' = \mathfrak{g}_{-4} \oplus [\mathfrak{g}_{-4}, \mathfrak{g}_4] \oplus \mathfrak{g}_4$ defines a maximal parabolic subgroup $\mathrm{P}' < \mathrm{H}'$ with Lie algebra $[\mathfrak{g}_{-4}, \mathfrak{g}_4] \oplus \mathfrak{g}_4$. In fact, V' is the irreducible representation associated to the Plücker embedding of $\mathrm{H}'/\mathrm{P}' \to \mathbb{P}(V')$, that is, H'/P' is isomorphic to the unique closed H'-orbit in $\mathbb{P}(V')$. For example, when $\mathrm{H}' = \mathrm{SL}_6\mathbb{C}$, $V' = \Lambda^3\mathbb{C}^6$ is the third exterior product of the standard representation of $\mathrm{SL}_6\mathbb{C}$ and $\mathrm{SL}_6\mathbb{C}/\mathrm{P}'$ is the Grassmannian of three planes in $\mathbb{C}^6$. When $\mathfrak{h}' = \mathfrak{e}_7$, then V' is the unique irreducible E_7-representation of dimension 56.

The following result describes the H'-orbit structure of $\mathbb{P}(V')$. We refer the reader to the work of Landsberg–Manivel, specifically [57, §5.3]. For the case $\mathrm{H}' = \mathrm{SL}_6\mathbb{C}$ this orbit structure was described in [23]. For $\mathrm{Sp}_6\mathbb{C}$ and $\mathrm{Spin}_{12}\mathbb{C}$ some aspects of the orbit structure are described in [49], and for E_7 in [44].

Proposition 4.23. *Consider the action of* H' *on* $\mathbb{P}(V')$ *described above. There are four* H'*-orbits,* $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4$*. Moreover, the following facts completely characterize* $\mathcal{O}_1, \mathcal{O}_3, \mathcal{O}_4$

(1) $\mathcal{O}_1$ *is closed and isomorphic to* H'/P'*;*
(2) $\mathcal{O}_3$ *has codimension 1 and* $\overline{\mathcal{O}}_3$ *is the tangent variety of* H'/P'*;*
(3) $p \in \mathcal{O}_3$ *if and only if* p *is contained in a unique tangent line of* H'/P'*;*
(4) $\mathcal{O}_4$ *is open.*

In the decomposition of $\mathfrak{m}$ given by Lemma 4.19, the subspace $\mathfrak{g}_{\tilde{\alpha}} \oplus \mathfrak{g}^b_{-2} \oplus \mathfrak{g}_{-6} \oplus \mathfrak{g}_{-10}$ is H'-invariant and hence isomorphic to the representation V'. The following proposition will be used in the next section.

Proposition 4.24. *Consider the* H'*-invariant subspace of* $\mathfrak{m}$ *given by* $\mathfrak{g}_{\tilde{\alpha}} \oplus \mathfrak{g}^b_{-2} \oplus \mathfrak{g}_{-6} \oplus \mathfrak{g}_{-10}$*.*

(1) The point $(\tilde{e}, 0, 0, 0) \in \mathfrak{g}_{\tilde{\alpha}} \oplus \mathfrak{g}^b_{-2} \oplus \mathfrak{g}_{-6} \oplus \mathfrak{g}_{-10}$ *defines a point in the closed orbit in* $\mathbb{P}(\mathfrak{g}_{\tilde{\alpha}} \oplus \mathfrak{g}^b_{-2} \oplus \mathfrak{g}_{-6} \oplus \mathfrak{g}_{-10})$ *whose stabilizer is the parabolic* P' *with Lie algebra* $[\mathfrak{g}_{-4}, \mathfrak{g}_4] \oplus \mathfrak{g}_4$*.*
(2) For all $\mu \in \mathbb{C}$*, a point* $(\mu\tilde{e}, f_b, 0, 0)$ *defines a point in the codimension one orbit of* $\mathbb{P}(\mathfrak{g}_{\tilde{\alpha}} \oplus \mathfrak{g}^b_{-2} \oplus \mathfrak{g}_{-6} \oplus \mathfrak{g}_{-10})$ *whose stabilizer is contained in the parabolic* P'*.*

Proof. Write a point in $\mathfrak{h}'$ as $(x, y, z) \in \mathfrak{g}_{-4} \oplus [\mathfrak{g}_{-4}, \mathfrak{g}_4] \oplus \mathfrak{g}_4$ and consider $(\tilde{e}, 0, 0, 0) \in V' = \mathfrak{g}_{\tilde{\alpha}} \oplus \mathfrak{g}^b_{-2} \oplus \mathfrak{g}_{-6} \oplus \mathfrak{g}_{-10}$. The bracket is given by

$$[(x, y, z), (\tilde{e}, 0, 0, 0)] = (\lambda(y)\tilde{e}, [x, \tilde{e}], 0, 0),$$

where $\lambda(y) \in \mathbb{C}$ and where $[x, \tilde{e}] \in \mathfrak{g}^b_{-2}$ is zero if and only if $x = 0$. Thus, the H'-stabilizer $[\tilde{e}, 0, 0, 0] \in \mathbb{P}(V')$ is the parabolic subgroup $\mathrm{P}' < \mathrm{H}'$ with Lie algebra $[\mathfrak{g}_{-4}, \mathfrak{g}_4] \oplus \mathfrak{g}_4$.

For the second point, we first analyze the case $\mu = 0$. Note that $\mathrm{ad}_{f_b} : \mathfrak{g}_{-4} \to \mathfrak{g}_{-6}$ is an isomorphism and $\dim(\mathfrak{g}^b_{-2}) = \dim(\mathfrak{g}_{-6})$. Thus,

$$\dim(\mathbb{P}(V')) = \dim(\mathfrak{g}_4 \oplus \mathfrak{g}_{-4}) + 1 = \dim(\mathfrak{h}') - \dim([\mathfrak{g}_4, \mathfrak{g}_{-4}]) + 1.$$

So $[0, f_b, 0, 0] \in \mathbb{P}(V')$ is in the codimension one orbit $\mathcal{O}_3$ if and only if

$$\dim(\{w \in \mathfrak{h}' \mid [w, f_b] = \lambda f_b \text{ for some } \lambda \in \mathbb{C}\}) = \dim([\mathfrak{g}_{-4}, \mathfrak{g}_4]).$$

To show this, write $w = (x, y, z) \in \mathfrak{g}_{-4} \oplus [\mathfrak{g}_{-4}, \mathfrak{g}_4] \oplus \mathfrak{g}_4$. Then the bracket $[w, f_b]$ is given by

$$[(x, y, z), f_b] = ([z, f_b], [y, f_b], [x, f_b], 0) \in V'.$$

Since $\mathrm{ad}_{f_b} : \mathfrak{g}_4 \to \mathfrak{g}_{\tilde{\alpha}}$ is surjective and $\mathrm{ad}_{f_b} : \mathfrak{g}_{-4} \to \mathfrak{g}_{-6}$ is an isomorphism, the space of $(x, 0, z) \in \mathfrak{h}'$ with $\mathrm{ad}_{(x,0,z)} f_b = \lambda f_b$ has dimension $\dim(\mathfrak{g}_4) - 1$.

Recall that $[\tilde{f}, f_b] \in \mathfrak{g}_{-4}$ is a magical nilpotent from Case (2) of Theorem 3.1. For $y \in [\mathfrak{g}_{-4}, \mathfrak{g}_4]$, we decompose $[\mathfrak{g}_{-4}, \mathfrak{g}_4] = [[\tilde{f}, f_b], \mathfrak{g}_4] \oplus \mathfrak{c}$. Then $\mathrm{ad}_{f_b} : [[\tilde{f}, f_b], \mathfrak{g}_4] \to \mathfrak{g}^b_{-2}$ is an isomorphism, so there is a one-dimensional subspace of $[[\tilde{f}, f_b], \mathfrak{g}_4]$ which acts on f_b by scalar multiplication. Since $[\mathfrak{c}, f_b] = 0$, we have

$$\dim(\{w \in \mathfrak{h}' \mid [w, f_b] = \lambda f_b \text{ for some } \lambda \in \mathbb{C}\}) = \dim(\mathfrak{g}_4) - 1 + 1 + \dim(\mathfrak{c}) = \dim([\mathfrak{g}_{-4}, \mathfrak{g}_4]).$$

By the above computation, the Lie algebra of the stabilizer of $[0, f_b, 0, 0] \in \mathbb{P}(V')$ is contained in $[\mathfrak{g}_{-4}, \mathfrak{g}_4] \oplus \mathfrak{g}_4$, so in the Lie algebra of P'. To show that the stabilizer of $[0, f_b, 0, 0]$ is indeed contained in P', we use the description of the codimension 1 orbit $\mathcal{O}_3$ of Proposition 4.23. Namely, there is a unique projective line $\ell \subset \mathbb{P}(V')$ which is tangent to the closed orbit G/P' and passes through $[0, f_b, 0, 0]$. This line is given by

$$\ell(\lambda) = [\tilde{e}, \lambda f_b, 0, 0] \subset \mathbb{P}(V').$$

Since the tangent line is unique, the action of the stabilizer of $[0, f_b, 0, 0]$ on ℓ must fix the intersection of ℓ with the closed orbit, which is given by $[\ell(0)] = [\tilde{e}, 0, 0, 0]$. Since the stabilizer of $\tilde{e}$ is P', we conclude that the stabilizer of $[0, f_b, 0, 0]$ is contained in P'.

Finally, since $\mathrm{ad}_{f_b} : \mathfrak{g}_4 \to \mathfrak{g}_{\tilde{\alpha}}$ is surjective and $[f_{\tilde{\alpha}}, \mathfrak{g}_4] = 0$, for every $\mu \in \mathbb{C}$, there is $x \in \mathfrak{g}_4$ such that $\mathrm{Ad}_{\exp(x)}(0, f_b, 0, 0) = (\mu\tilde{e}, f_b, 0, 0)$. Thus, $[\mu\tilde{e}, f_b, 0, 0]$ and $[0, f_b, 0, 0]$ are in the same H'-orbit. Moreover since the stabilizers of $[\mu\tilde{e}, f_b, 0, 0]$ and $[0, f_b, 0, 0]$ are conjugate via $\exp(x) \in \mathrm{P}'$, we conclude that the stabilizer of $[\mu\tilde{e}, f_b, 0, 0]$ is contained in P', completing the proof. □

5. Higgs bundles and the Cayley map

From now on, X will denote a fixed compact Riemann surface of genus $g \geqslant 2$, with canonical bundle K. All geometric objects we will consider are over X. Let H be a complex reductive Lie group.

5.1. Higgs bundles.

Let $\mathcal{E}_\mathrm{H} \to X$ be a holomorphic principal H-bundle. Given a holomorphic action of H on a space Y, we denote the associated fiber bundle by $\mathcal{E}_\mathrm{H}[Y] = (\mathcal{E}_\mathrm{H} \times Y)/\mathrm{H}$, where $(x, y) \cdot g = (x \cdot g, g^{-1} \cdot y)$. When V is a vector space, $\mathcal{E}_\mathrm{H}[V]$ is a holomorphic vector bundle, and when H acts by group homomorphisms on a complex Lie group G, then $\mathcal{E}_\mathrm{H}[\mathrm{G}]$ is a holomorphic principal G-bundle.

Definition 5.1. *Let* G *be a complex reductive Lie group,* V *be a complex vector space with a holomorphic* G*-action and* L *be a holomorphic line bundle on* X*. An* L-twisted (G, V)-Higgs pair *is a pair* $(\mathcal{E}_\mathrm{G}, \varphi)$ *consisting of a holomorphic* G*-bundle* $\mathcal{E}_\mathrm{G} \to X$ *and a holomorphic section* $\varphi \in H^0(\mathcal{E}_\mathrm{G}[V] \otimes L)$*. The section* φ *is called the* Higgs field.

There is a natural $\mathbb{C}^*$-action on the set of L-twisted (G, V)-Higgs pairs given by

$$\lambda \cdot (\mathcal{E}, \varphi) = (\mathcal{E}, \lambda\varphi). \tag{5.1}$$

Our main objects of interest, Higgs bundles, are a particular class of Higgs pairs.

Definition 5.2. *Let* $\mathrm{G}^{\mathbb{R}} \subset \mathrm{G}$ *be a real form of a complex semisimple Lie group* G*. Let* $\mathrm{H}^{\mathbb{R}} \subset \mathrm{G}^{\mathbb{R}}$ *be a maximal compact subgroup,* $\mathrm{H} \subset \mathrm{G}$ *be its complexification and* $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ *be a complexified Cartan decomposition. An* L-twisted $\mathrm{G}^{\mathbb{R}}$-Higgs bundle *is an* L*-twisted* $(\mathrm{H}, \mathfrak{m})$*-Higgs pair* $(\mathcal{E}_{\mathrm{H}}, \varphi)$*.*

We will denote the set of L-twisted $\mathrm{G}^{\mathbb{R}}$-Higgs bundles by $\mathcal{H}_L(\mathrm{G}^{\mathbb{R}})$. When the twisting line bundle L is the canonical bundle K, we will refer to a K-twisted $\mathrm{G}^{\mathbb{R}}$-Higgs bundle simply as a $\mathrm{G}^{\mathbb{R}}$-*Higgs bundle* and write $\mathcal{H}_K(\mathrm{G}^{\mathbb{R}}) = \mathcal{H}(\mathrm{G}^{\mathbb{R}})$.

Let E_{H} be the smooth underlying bundle of a holomorphic bundle $\mathcal{E}_{\mathrm{H}}$. The gauge group $\mathcal{G}_{\mathrm{H}}$ of smooth bundle automorphisms of E_{H} acts on $\mathcal{H}_L(\mathrm{G}^{\mathbb{R}})$ by pulling back the holomorphic structure and pulling back the Higgs field. In particular, if $(\mathcal{E}_{\mathrm{H}}, \varphi)$ is an L-twisted Higgs bundle then $g \in \mathcal{G}_{\mathrm{H}}(E_{\mathrm{H}})$, then

$$g \cdot \varphi = \mathrm{Ad}_g(\varphi).$$

The automorphism group of an L-twisted Higgs bundle $(\mathcal{E}_{\mathrm{H}}, \varphi)$ is the group of holomorphic gauge transformations g of $\mathcal{E}_{\mathrm{H}}$ such that $\mathrm{Ad}_g(\varphi) = \varphi$.

Example 5.3. Here are some relevant examples of Higgs bundles:

- The complex group G can be regarded as a real form of $\mathrm{G} \times \mathrm{G}$. In this situation $H = G$, $\mathfrak{m} = \mathfrak{g}$, and an L-twisted G-Higgs bundle is thus a pair $(\mathcal{E}_{\mathrm{G}}, \varphi)$, where $\mathcal{E}_{\mathrm{G}}$ is is a holomorphic principal G-bundle and $\varphi \in H^0(\mathcal{E}_{\mathrm{G}}[\mathfrak{g}] \otimes L)$.
- For $\mathrm{G}^{\mathbb{R}} = \mathbb{R}^+$, an L-twisted $\mathrm{G}^{\mathbb{R}}$-Higgs bundle is just a holomorphic section $\varphi \in H^0(L)$.
- For $\mathrm{G}^{\mathbb{R}} = \mathrm{PSL}_2\mathbb{R}$, we have $\mathrm{H} \cong \mathbb{C}^*$ and $\mathfrak{m} = \mathfrak{m}^- \oplus \mathfrak{m}^+ = \langle f \rangle \oplus \langle e \rangle \cong \mathbb{C} \oplus \mathbb{C}$. To be consistent with later notation, we set $\mathrm{H} = \mathrm{T}$ for $\mathrm{G}^{\mathbb{R}} = \mathrm{PSL}_2\mathbb{R}$. The adjoint action of T on $\mathfrak{m}$ is given by

$$\lambda \cdot (f, e) = (\lambda^{-1} f, \lambda e), \tag{5.2}$$

  where $\lambda \in \mathrm{T}$.
- For $\mathrm{G}^{\mathbb{R}} = \mathrm{SL}_2\mathbb{R}$, we have $\mathrm{H} \cong \mathbb{C}^*$, $\mathfrak{m} = \langle f \rangle \oplus \langle e \rangle$ and the action of H is $\lambda \cdot (f, e) = (\lambda^{-2} f, \lambda^2 e)$.

Definition 5.4. *The* uniformizing Higgs bundle *for the compact Riemann surface* X *is the* $\mathrm{PSL}_2\mathbb{R}$*-Higgs bundle* $(\mathcal{E}_{\mathrm{T}}, f)$*, where* $\mathcal{E}_{\mathrm{T}}$ *is the frame bundle of the canonical bundle* $K \to X$ *and* $f \in H^0(\mathcal{E}_{\mathrm{T}}[\langle f \rangle] \otimes K) \cong H^0(\mathcal{O})$ *is a constant nonzero section.*

Remark 5.5. Since $\deg(K) = 2g - 2$ is even, the uniformizing $\mathrm{PSL}_2\mathbb{R}$-Higgs bundle $(\mathcal{E}_{\mathrm{T}}, f)$ lifts to an $\mathrm{SL}_2\mathbb{R}$-Higgs bundle $(\mathcal{E}_{\mathrm{T}'}, f)$, where $\mathcal{E}_{\mathrm{T}'}$ is the frame bundle of one of the 2^{2g} square roots $K^{\frac{1}{2}}$ of the canonical bundle. We will call such a Higgs bundle a *lift of the uniformizing Higgs bundle* of X. Using the standard representation of $\mathrm{SL}_2\mathbb{C}$ on $\mathbb{C}^2$, an $\mathrm{SL}_2\mathbb{C}$-Higgs bundle is a holomorphic rank 2 bundle V with trivial determinant and a holomorphic bundle map $\Phi : V \to V \otimes K$. For a lift of the uniformizing Higgs bundle, we have

$$(V, \Phi) = \left(K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}, \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) : K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}} \to K^{\frac{3}{2}} \oplus K^{\frac{1}{2}}\right).$$

Given two Lie groups $\mathrm{H}_1, \mathrm{H}_2$ and holomorphic principal $\mathrm{H}_1, \mathrm{H}_2$-bundles $\mathcal{E}_{\mathrm{H}_1}$, $\mathcal{E}_{\mathrm{H}_2}$ respectively, the fiber product $\mathcal{E}_{\mathrm{H}_1} \times_X \mathcal{E}_{\mathrm{H}_2}$ is a holomorphic principal $(\mathrm{H}_1 \times \mathrm{H}_2)$-bundle. When $\mathrm{H}_1, \mathrm{H}_2 \subset \mathrm{H}$ are commuting subgroups, the multiplication map $m : \mathrm{H}_1 \times \mathrm{H}_2 \to \mathrm{H}$ is a group homomorphism and $(\mathcal{E}_{\mathrm{H}_1} \times_X \mathcal{E}_{\mathrm{H}_2})[\mathrm{H}]$ is a holomorphic principal H-bundle. This is analogous to twisting a vector bundle by a line bundle. We will use the notation

$$(\mathcal{E}_{\mathrm{H}_1} \star \mathcal{E}_{\mathrm{H}_2})[\mathrm{H}] = (\mathcal{E}_{\mathrm{H}_1} \times_X \mathcal{E}_{\mathrm{H}_2})[\mathrm{H}]. \tag{5.3}$$

5.2. **The Cayley map.** We first describe the global Slodowy slice construction of [16] for an arbitrary even nilpotent $e \in \mathfrak{g}$. When $e \in \mathfrak{g}$ is a magical nilpotent this leads to $\mathrm{G}^{\mathbb{R}}$-Higgs bundles, where $\mathrm{G}^{\mathbb{R}}$ is the canonical real form associated to the corresponding magical $\mathfrak{sl}_2$-triple.

Let $e \in \mathfrak{g}$ be an even nilpotent, $\{f, h, e\} \subset \mathfrak{g}$ be an associated $\mathfrak{sl}_2$-triple and G be a connected Lie group with Lie algebra $\mathfrak{g}$. Let $\mathrm{S} \subset \mathrm{G}$ be the connected subgroup with Lie algebra the $\mathfrak{sl}_2$-subalgebra $\mathfrak{s} = \langle f, h, e\rangle$ and $\mathrm{C} \subset \mathrm{G}$ be the centralizer of $\{f, h, e\}$. When $\mathrm{S} \cong \mathrm{PSL}_2\mathbb{C}$ let $(\mathcal{E}_{\mathrm{T}}, f)$ be the uniformizing Higgs bundle of X, and when $\mathrm{S} \cong \mathrm{SL}_2\mathbb{C}$ let $(\mathcal{E}_{\mathrm{T}}, f)$ be a lift of the uniformizing Higgs bundle of X to $\mathrm{SL}_2\mathbb{R}$. The embedding $\mathrm{T} \hookrightarrow \mathrm{S} \hookrightarrow \mathrm{G}$ defines a holomorphic G-bundle $\mathcal{E}_{\mathrm{G}} = \mathcal{E}_{\mathrm{T}}[\mathrm{G}]$ by extension of structure group.

Given a holomorphic C-bundle $\mathcal{E}_{\mathrm{C}} \to X$, consider the holomorphic G-bundle

$$\mathcal{E}_{\mathrm{G}} = (\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}})[\mathrm{G}]$$

with the notation (5.3). Since C and T preserve the subspaces $\mathfrak{g}_j \cap W_i \subset \mathfrak{g}$ (in particular the highest weight subspaces V_j; cf. (2.5)) and also $\langle f\rangle \subset \mathfrak{g}$, the adjoint bundle $\mathcal{E}_{\mathrm{G}}[\mathfrak{g}]$ decomposes as

$$\mathcal{E}_{\mathrm{G}}[\mathfrak{g}] = (\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}})[\mathfrak{g}] = \bigoplus_{j \in \mathbb{Z}} (\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}})[\mathfrak{g}_j]$$

and $(\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}})[V_j] \subset (\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}})[\mathfrak{g}_j]$ and $(\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}})[\langle f\rangle] \subset (\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}})[\mathfrak{g}_{-2}]$ define holomorphic subbundles. Moreover, since C acts trivially on $\langle f\rangle$,

$$(\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}})[\langle f\rangle] \cong \mathcal{E}_{\mathrm{T}}[\langle f\rangle] \cong K^{-1},$$

by (5.2). So, from a holomorphic C-bundle $\mathcal{E}_{\mathrm{C}}$ and from sections $\phi_j \in H^0((\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}})[V_j] \otimes K)$, we define the G-Higgs bundle

$$(\mathcal{E}_{\mathrm{G}}, \varphi) = ((\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}})[\mathrm{G}], f + \phi_0 + \phi_1 + \cdots + \phi_N). \tag{5.4}$$

Recall that $Z_{2m_j} = W_{2m_j} \cap \mathfrak{g}_0$. We have that $\mathfrak{g}_0 = W_0 \oplus \bigoplus_{j=1}^M Z_{2m_j}$ and, since e is even, $\mathrm{ad}_f^{m_j} : V_{2m_j} \to Z_{2m_j}$ is an isomorphism. Thus, viewing f as a holomorphic section of $(\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}})[\mathfrak{g}] \otimes K$, we have an isomorphism of holomorphic vector bundles

$$\mathrm{ad}_f^{m_j} : (\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}})[V_{2m_j}] \otimes K \xrightarrow{\cong} \mathcal{E}_{\mathrm{C}}[Z_{2m_j}] \otimes K^{m_j+1},$$

where we have used the fact that T acts trivially on Z_{2m_j} to identify $\mathcal{E}_{\mathrm{C}}[Z_{2m_j}] \otimes K^{m_j+1}$ with $(\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}})[Z_{2m_j}] \otimes K^{m_j+1}$.

Let now $\mathcal{B}_e(\mathrm{G})$ denote the set of tuples $((\mathcal{E}_{\mathrm{C}}, \phi_0), \psi_{m_1}, \ldots, \psi_{m_N})$, where $(\mathcal{E}_{\mathrm{C}}, \phi_0)$ is a holomorphic C-Higgs bundle and $\psi_{m_j} \in H^0(\mathcal{E}_{\mathrm{C}}[Z_{2m_j}] \otimes K^{m_j+1})$. By the above discussion, the Higgs bundles of the form (5.4) can be described by the map

$$\begin{array}{lccc} (5.5) & \widehat{\Psi}_e : & \mathcal{B}_e(\mathrm{G}) \longrightarrow \mathcal{H}(\mathrm{G}) \\ & & (\mathcal{E}_{\mathrm{C}}, \phi_0, \psi_{m_1}, \ldots, \psi_{m_N}) \longmapsto ((\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}})[\mathrm{G}], f + \phi_0 + \phi_{m_1} + \cdots + \phi_{m_N}) \end{array}$$

where $\phi_{m_j} \in H^0((\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}})[V_{2m_j}] \otimes K)$ and $\psi_{m_j} = \mathrm{ad}_f^{m_j}(\phi_{m_j})$. We will refer to this map as the *Slodowy map*; see also [16].

Note that the map $\widehat{\Psi}_e$ is equivariant for the action of the C-gauge group $\mathcal{G}_{\mathrm{C}}$. More precisely, if $g \in \mathcal{G}_{\mathrm{C}}$, then $g \star \mathrm{Id}_{\mathrm{T}} \in \mathcal{G}_{\mathrm{G}}$ is a G-gauge transformation of $(\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}})[\mathrm{G}]$, and

$$\widehat{\Psi}_e(g \cdot (\mathcal{E}_{\mathrm{C}}, \psi_{m_1}, \ldots, \psi_{m_N})) = g \star \mathrm{Id}_{\mathrm{T}} \cdot \widehat{\Psi}_e(\mathcal{E}_{\mathrm{C}}, \psi_{m_1}, \ldots, \psi_{m_N}).$$

Lemma 5.6. *Let $e \in \mathfrak{g}$ be a magical nilpotent and $\mathrm{G}^{\mathbb{R}} \subset \mathrm{G}$ be the canonical real form. Then the Higgs bundle $\widehat{\Psi}_e(\mathcal{E}_{\mathrm{C}}, \phi_0, \psi_{m_1}, \ldots, \psi_{m_N})$ from (5.5) is contained in $\mathcal{H}(\mathrm{G}^{\mathbb{R}})$ if and only if $\phi_0 = 0$ and the bundle $\mathcal{E}_{\mathrm{C}}$ reduces to $\mathrm{C} \cap \mathrm{H}$.*

Proof. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be the complexified Cartan decomposition of the real form $\mathfrak{g}^{\mathbb{R}}$, hence given by σ_e. By the definition of a magical nilpotent, $h \in \mathfrak{h}$, $\mathfrak{c} = W_0 \subset \mathfrak{h}$, $V_{2m_j} \subset \mathfrak{m}$ and $f \in \mathfrak{m}$. Thus,

$$(\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}})[\mathrm{G}] \cong (\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}})[\mathrm{H}][\mathrm{G}]$$

if and only if $\mathcal{E}_{\mathrm{C}} \cong \mathcal{E}_{\mathrm{C}\cap\mathrm{H}}[\mathrm{C}]$, and $f + \phi_0 + \phi_{m_1} + \cdots + \phi_{m_N} \in H^0((\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}})[\mathfrak{m}] \otimes K)$ if and only if $\phi_0 = 0$. □

Given a magical $\mathfrak{sl}_2$-triple $\{f, h, e\} \subset \mathfrak{g}$, recall the subalgebra $\mathfrak{g}(e) \subset \mathfrak{g}$ from Proposition 4.5 and the Cayley real form $\mathfrak{g}^{\mathbb{R}}_{\mathcal{C}} = \mathbb{R}^{\mathrm{rk}(\mathfrak{g}(e))} \oplus \mathfrak{g}^{\mathbb{R}}_{0,ss}$ from Proposition 4.8. The Cayley group is defined to be the real Lie group $\mathrm{G}^{\mathbb{R}}_{\mathcal{C}} = (\mathbb{R}^+)^{\mathrm{rk}(\mathfrak{g}(e))} \times \mathrm{G}^{\mathbb{R}}_{0,ss}$, where $\mathrm{G}^{\mathbb{R}}_{0,ss}$ is the real Lie group with Lie algebra $\mathfrak{g}^{\mathbb{R}}_{0,ss}$ and maximal compact $\mathrm{C} \cap \mathrm{G}^{\mathbb{R}}$ (see Definition 4.11). Recall from Proposition 4.3, that the $\mathfrak{sl}_2$-data of a magical $\mathfrak{sl}_2$-triple has at most one $m_j > 0$ with $\dim(Z_{2m_j}) > 1$.

Lemma 5.7. *Let $\{f, h, e\} \subset \mathfrak{g}$ be a magical $\mathfrak{sl}_2$-triple with $\mathfrak{sl}_2$-data $\{m_j\}_{j=1}^M$ and let $\mathfrak{g}(e) \subset \mathfrak{g}$ be the subalgebra from Proposition 4.5. Then there is a natural identification*

$$\{x \in \mathcal{B}_e(\mathrm{G}) \mid \widehat{\Psi}_e(x) \in \mathcal{H}(\mathrm{G}^{\mathbb{R}})\} \longleftrightarrow \mathcal{H}_{K^{m_c+1}}(\mathrm{G}^{\mathbb{R}}_{0,ss}) \times \prod_{j=1}^{\mathrm{rk}(\mathfrak{g}(e))} \mathcal{H}_{K^{l_j+1}}(\mathbb{R}^+).$$

Here m_c is zero in Case (1) of Theorem 3.1 and is the unique positive m_j with $\dim(Z_{2m_j}) > 1$ otherwise. The integers $\{l_j\}$ are the exponents of $\mathfrak{g}(e)$, which are

$$\{l_j\} = \begin{cases} \{m_j\}_{j=1}^M & \text{Cases (1), (2) and (3) with } p\text{-even of Theorem 3.1} \\ \{m_j\}_{j=1}^M \setminus \{p-1\} & \text{Case (3) } p\text{-odd of Theorem 3.1} \\ \{m_j\}_{j=1}^M \setminus \{3\} & \text{Case (4) of Theorem 3.1.} \end{cases}$$

Remark 5.8. Recall that $\mathcal{H}_L(\mathbb{R}^+) \cong H^0(L)$, so

$$\mathcal{H}_{K^{m_c+1}}(\mathrm{G}^{\mathbb{R}}_{0,ss}) \times \prod_{j=1}^{\mathrm{rk}(\mathfrak{g}(e))} \mathcal{H}_{K^{l_j+1}}(\mathbb{R}^+) \cong \mathcal{H}_{K^{m_c+1}}(\mathrm{G}^{\mathbb{R}}_{0,ss}) \times \prod_{j=1}^{\mathrm{rk}(\mathfrak{g}(e))} H^0(K^{l_j+1}).$$

Let $\mathrm{Z}(\mathrm{G}^{\mathbb{R}})$ be the center of $\mathrm{G}^{\mathbb{R}}$. In Case (1) of Theorem 3.1, $\mathcal{H}_{K^{m_c+1}}(\mathrm{G}^{\mathbb{R}}_{0,ss})$ is the finite set of $\mathrm{Z}(\mathrm{G}^{\mathbb{R}})$-bundles on X so the value of m_c is unimportant.

Proof. By Lemma 4.7, C acts trivially on $\mathfrak{g}(e) \cap \mathfrak{g}_0$. When $n_{2m_j} = 1$, we have $Z_{2m_j} \subset \mathfrak{g}(e)$ and thus $\psi_{m_j} \in H^0(\mathcal{E}_{\mathrm{C}}(Z_{2m_j}) \otimes K^{m_j+1}) = H^0(K^{m_j+1})$. This proves Case (1).

From Proposition 4.6, we see that for Case (3) with p odd and Case (4), we have $\mathfrak{g}(e) \cap Z_{2m_c} = \{0\}$ and $\mathfrak{g}_{0,ss} = \mathfrak{c} \oplus Z_{2m_c}$. Thus, $(\mathcal{E}_{\mathrm{C}}, \psi_{m_c})$ is a K^{m_c}-twisted $\mathrm{G}^{\mathbb{R}}_{0,ss}$-Higgs bundle whenever $\mathcal{E}_{\mathrm{C}}$ reduces to $\mathrm{C} \cap \mathrm{H}$. Thus, for Case (3) with p-odd and Case (4), the result follows.

For Case (2) and Case (3) with p-even, we have $Z_{2m_c} \cap \mathfrak{g}(e) \cong \mathbb{C}$, by Propositions 4.5 and 4.6. Hence Z_{2m_c} decomposes C-invariantly as $Z_{2m_c} = \mathbb{C} \oplus \mathfrak{m}_{0,ss}$, where the $\mathbb{C}$-factor is $\mathfrak{g}(e) \cap Z_{2m_c}$ and $\mathfrak{m}_{0,ss} \oplus \mathfrak{c} = \mathfrak{g}^{\mathbb{R}}_{0,ss} \otimes \mathbb{C}$ is complexified Cartan decomposition. Hence

$$\mathcal{E}_{\mathrm{C}}[Z_{2m_c}] \otimes K^{m_c+1} \cong K^{m_c+1} \oplus \mathcal{E}_{\mathrm{C}}[\mathfrak{m}_{0,ss}] \otimes K^{m_c+1.}$$

Thus, $(\mathcal{E}_{\mathrm{C}}, \psi_{m_c}) = (\mathcal{E}_{\mathrm{C}}, q_{m_c+1} \oplus \tilde{\psi}_{m_c})$, where $q_{m_c+1} \in H^0(K^{m_c+1})$ and $(\mathcal{E}_{\mathrm{C}}, \tilde{\psi}_{m_c})$ is a $\mathrm{G}^{\mathbb{R}}_{0,ss}$-Higgs bundle whenever $\mathcal{E}_{\mathrm{C}}$ reduces to $\mathrm{C} \cap \mathrm{H}$. □

To summarize, from a magical $\mathfrak{sl}_2$-triple $\{f,h,e\}\subset\mathfrak{g}$, the Slodowy map (5.5) defines a map

$$\widehat{\Psi}_e : \mathcal{H}_{K^{m_c+1}}(\mathrm{G}^{\mathbb{R}}_{0,ss}) \times \prod_{j=1}^{\mathrm{rk}(\mathfrak{g}(e))} \mathcal{H}_{K^{l_j+1}}(\mathbb{R}^+) \longrightarrow \mathcal{H}(\mathrm{G}^{\mathbb{R}})$$

given by

$$\widehat{\Psi}_e((\mathcal{E}_{\mathrm{C}},\tilde{\psi}_{m_c}),q_{l_1+1},\dots,q_{l_{\mathrm{rk}(\mathfrak{g}(e))}+1}) = \Big(\mathcal{E}_{\mathrm{C}}\star\mathcal{E}_{\mathrm{T}}[\mathrm{H}], f+\tilde{\phi}_{m_c}+\sum_{j=1}^{\mathrm{rk}(\mathfrak{g}(e))} q_{l_j+1}\Big), \tag{5.6}$$

where $\mathrm{G}^{\mathbb{R}}$ is the canonical real form of e, and $\tilde{\phi}_{m_c}=\mathrm{ad}_f^{-m_c}(\tilde{\psi}_{m_c})$. We will refer to the map (5.6) as the *Cayley map* since it generalizes the Cayley correspondence of [7] which concerns Case (2) of Theorem 3.1. In the subsequent sections we will show that the Cayley map actually preserves the polystability conditions, hence descends to a map on moduli spaces, which will be injective, with open and closed image.

Remark 5.9. Note that everything we just described also holds when the line bundle K is replaced by another twisting line bundle $L\to X$. So there is a similarly defined Cayley map, for the L-twisted version, one takes $\mathcal{E}_{\mathrm{T}}$ to be the holomorphic frame bundle of L when $\mathrm{S}\cong\mathrm{PSL}_2\mathbb{C}$ and $\mathcal{E}_{\mathrm{T}}$ the holomorphic frame bundle of a square root of L when $\mathrm{S}\cong\mathrm{SL}_2\mathbb{C}$. In particular, when $\mathrm{S}\cong\mathrm{SL}_2\mathbb{C}$ the degree of L must be even.

5.3. **The Cayley map is injective on gauge orbits.** In this section we prove the Cayley map is injective on gauge orbits. We will use the following lemma.

Lemma 5.10. *Let $\{f,h,e\}\subset\mathfrak{g}$ be an $\mathfrak{sl}_2$-triple and $\mathfrak{g}=\bigoplus_{j\in\mathbb{Z}}\mathfrak{g}_j$ be the associated $\mathbb{Z}$-grading. Let $\mathrm{P}\subset\mathrm{G}$ be the parabolic subgroup with Lie algebra $\mathfrak{p}=\bigoplus_{j\geqslant 0}\mathfrak{g}_j$ and let $x,x'\in V=\ker(\mathrm{ad}_e)$. If an element $g\in\mathrm{P}$ satisfies $\mathrm{Ad}_g(f+x)=f+x'$, then $g\in\mathrm{C}$, with $\mathrm{C}\subset\mathrm{G}$ the centralizer of $\{f,h,e\}$.*

Proof. Since $x\in\mathfrak{p}$, we have $\mathrm{Ad}_g(x)\in\mathfrak{p}$. Thus, $\mathrm{Ad}_g(f+x)=f+x'$ implies $\mathrm{Ad}_g(f)=f$. The intersection of the centralizer of f with P is C. So $g\in\mathrm{C}$. □

Proposition 5.11. *Let $\{f,h,e\}\subset\mathfrak{g}$ be a magical $\mathfrak{sl}_2$-triple and*

$$\widehat{\Psi}_e :\ \mathcal{H}_{K^{m_c+1}}(\mathrm{G}^{\mathbb{R}}_{0,ss}) \times \textstyle\prod_{j=1}^{\mathrm{rk}(\mathfrak{g}(e))} \mathcal{H}_{K^{l_j+1}}(\mathbb{R}^+) \longrightarrow \mathcal{H}(\mathrm{G}^{\mathbb{R}}),$$

be the Cayley map from (5.6). Then two points

$$\widehat{\Psi}_e((\mathcal{E}_{\mathrm{C}\cap\mathrm{H}},\tilde{\psi}_{m_c}),q_{l_1+1},\dots,q_{l_{\mathrm{rk}(\mathfrak{g}(e))}+1}) \quad \text{and} \quad \widehat{\Psi}_e((\mathcal{E}'_{\mathrm{C}\cap\mathrm{H}},\tilde{\psi}'_{m_c}),q'_{l_1+1},\dots,q'_{l_{\mathrm{rk}(\mathfrak{g}(e))}+1})$$

are in the same H-gauge orbit if and only if $(\mathcal{E}_{\mathrm{C}\cap\mathrm{H}},\tilde{\psi}_{m_c})$ and $(\mathcal{E}'_{\mathrm{C}\cap\mathrm{H}},\tilde{\psi}'_{m_c})$ are in the same $\mathrm{C}\cap\mathrm{H}$-gauge orbit and moreover $q_{l_j+1}=q'_{l_j+1}$ for all j.

Proof. We will prove Proposition 5.11 for each case of Theorem 3.1. Note that it suffices to prove the result for the adjoint group G_{Ad}. Indeed, consider a general G and let $\pi:\mathrm{G}\to\mathrm{G}_{\mathrm{Ad}}$ be the covering. An H-gauge transformation $g:\mathcal{E}_{\mathrm{C}}\star\mathcal{E}_{\mathrm{T}}[\mathrm{H}]\to\mathcal{E}'_{\mathrm{C}}\star\mathcal{E}_{\mathrm{T}}[\mathrm{H}]$ induces a gauge transformation between the associated bundles for the adjoint group, and if the induced gauge transformation is valued in $\pi(\mathrm{C}\cap\mathrm{H})$ then g must be valued in $\mathrm{C}\cap\mathrm{H}$. The $\mathrm{C}\cap\mathrm{H}$-gauge group acts trivially on the differentials q_{l_j+1}, so if g is valued in $\mathrm{C}\cap\mathrm{H}$, $q_{l_j+1}=q'_{l_j+1}$ for all j.

Case (1) was proven in [48] using the Hitchin section and moduli spaces. Alternatively, suppose $g : \mathcal{E}_{\mathrm{C}\cap\mathrm{H}} \star \mathcal{E}_{\mathrm{T}}[\mathrm{H}] \to \mathcal{E}'_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}}[\mathrm{H}]$ is a holomorphic gauge transformation such that

$$\mathrm{Ad}_g\left(f + \sum_{j=1}^{\mathrm{rk}(\mathfrak{g})} q_{m_j+1}\right) = f + \sum_{j=1}^{\mathrm{rk}(\mathfrak{g})} q'_{m_j+1}.$$

The Lie algebra bundle decomposes as $\mathcal{E}_{\mathrm{C}\cap\mathrm{H}} \star \mathcal{E}_{\mathrm{T}}[\mathfrak{g}] \otimes K \cong \bigoplus \mathcal{E}_{\mathrm{C}\cap\mathrm{H}} \star \mathcal{E}_{\mathrm{T}}[\mathfrak{g}_j \cap W_{2m_i}] \otimes K$ with each summand $\mathcal{E}_{\mathrm{C}\cap\mathrm{H}} \star \mathcal{E}_{\mathrm{T}}[\mathfrak{g}_j \cap W_{2m_i}] \otimes K \cong K^{j+1}$. Since g is holomorphic, we have

$$\mathrm{Ad}_g\left(\bigoplus_{j\geqslant 0} \mathcal{E}_{\mathrm{C}\cap\mathrm{H}} \star \mathcal{E}_{\mathrm{T}}[\mathfrak{g}_j] \otimes K\right) \subset \bigoplus_{j\geqslant 0} \mathcal{E}_{\mathrm{C}'\cap\mathrm{H}} \star \mathcal{E}_{\mathrm{T}}[\mathfrak{g}_j] \otimes K.$$

Hence g is valued in the intersection of H with the parabolic subgroup associated to $\bigoplus_{j>0} \mathfrak{g}_j$. Thus, g is valued in $\mathrm{C} \cap \mathrm{H}$ by Lemma 5.10.

For Case (2) of Theorem 3.1, the $\mathbb{Z}$-grading is $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2$ with $\mathfrak{h} = \mathfrak{g}_0$. Hence, any gauge transformation $g : \mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}}[\mathrm{H}] \to \mathcal{E}'_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}}[\mathrm{H}]$ is valued in the intersection of H with parabolic subgroup associated to $\mathfrak{g}_0 \oplus \mathfrak{g}_2$. By Lemma 5.10, g is valued in $\mathrm{C} \cap \mathrm{H}$.

For Case (3), Proposition 5.11 was proven in [3, Lemma 4.6] when $\mathrm{G} = \mathrm{SO}_N\mathbb{C}$, i.e., for $\mathrm{G}^{\mathbb{R}} \cong \mathrm{SO}_{p,q}$. As a result, we focus on $\mathrm{G} = \mathrm{PSO}_N\mathbb{C}$. For N-odd, $\mathrm{SO}_N\mathbb{C} = \mathrm{PSO}_N\mathbb{C}$ and we are done. For N-even the centralizer C of the magical $\mathfrak{sl}_2$-triple is $\mathrm{O}_{N-2p+1}\mathbb{C}$ for $\mathrm{G} = \mathrm{SO}_N\mathbb{C}$ and $\mathrm{O}_{N-2p+1}\mathbb{C}/\pm\mathrm{Id}$ for $\mathrm{G} = \mathrm{PSO}_N\mathbb{C}$ (see [17, Theorem 6.1.3]). But N even implies $\mathrm{O}_{N-2p+1}\mathbb{C}/\pm\mathrm{Id} \cong \mathrm{SO}_{N-2p+1}\mathbb{C}$. Since every $\mathrm{SO}_{N-2p+1}\mathbb{C}$-bundle lifts to a $\mathrm{O}_{N-2p+1}\mathbb{C}$-bundle, every $\mathrm{PSO}_N\mathbb{C}$-Higgs bundle in the image of $\hat{\Psi}_e$ lifts to an $\mathrm{SO}_N\mathbb{C}$-Higgs bundle in the image of $\hat{\Psi}_e$.

For Case (4), we use holomorphicity and Proposition 4.24 to apply Lemma 5.10. Recall that the space $\mathfrak{m}$ decomposes as in Lemma 4.19. Write the Higgs field as

$$f + q_2 + \phi_3 + q_6 = \begin{pmatrix} q_6 & \phi_3 & q_2^b & \tilde{f} \\ \tilde{q}_2 & f_b & 0 & 0 \end{pmatrix}, \tag{5.7}$$

where the rows are sections of $\mathcal{E}_{\mathrm{C}}\star\mathcal{E}_{\mathrm{T}}[\mathfrak{g}_{10}\oplus\mathfrak{g}_6\oplus\mathfrak{g}_2^b\oplus\mathfrak{g}_{-\tilde{\alpha}}]\otimes K$ and $\mathcal{E}_{\mathrm{C}}\star\mathcal{E}_{\mathrm{T}}[\mathfrak{g}_{\tilde{\alpha}}\oplus\mathfrak{g}_{-2}^b\oplus\mathfrak{g}_{-6}\oplus\mathfrak{g}_{10}]\otimes K$, respectively. Recall also that $\mathfrak{g} = \mathfrak{sl}_2\mathbb{C} \oplus \mathfrak{h}' = \mathfrak{g}_{-8} \oplus \mathfrak{g}_{-4} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_4 \oplus \mathfrak{g}_8$.

Consider a holomorphic gauge transformation $g : \mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}}[\mathrm{H}] \to \mathcal{E}'_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}}[\mathrm{H}]$. We have $\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}}[\mathfrak{g}_{-8}] \cong K^{-4}$, thus holomorphicity implies

$$\mathrm{Ad}_g(\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}}[\mathfrak{g}_{-4} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_4 \oplus \mathfrak{g}_8]) \subset \mathcal{E}'_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}}[\mathfrak{g}_{-4} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_4 \oplus \mathfrak{g}_8].$$

Hence g is valued in the parabolic of $\mathrm{P} \subset \mathrm{H}$ with Lie algebra $\mathfrak{g}_{-4} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_4 \oplus \mathfrak{g}_8$. The action of P on $\mathfrak{m}$ preserves the top row of (5.7). If preserves the image of $\hat{\Psi}_e$, we have the gauge transformation

$$\mathrm{Ad}_g\begin{pmatrix} q_6 & \phi_3 & q_2^b & \tilde{f} \\ \tilde{q}_2 & f_b & 0 & 0 \end{pmatrix} = \begin{pmatrix} q'_6 & \phi'_3 & (q_2^b)' & \tilde{f} \\ \tilde{q}'_2 & f_b & 0 & 0 \end{pmatrix}.$$

By Proposition 4.24, the gauge transformation g is valued in the parabolic of H with Lie algebra $\mathfrak{g}_0 \oplus \mathfrak{g}_4 \oplus \mathfrak{g}_8$. Thus, Lemma 5.10 implies g is valued in $\mathrm{C} \cap \mathrm{H}$. ☐

We have the following immediate corollary.

Corollary 5.12. *Let* $((\mathcal{E}_{\mathrm{C}\cap\mathrm{H}}, \tilde{\psi}_{m_c}), q_{l_1+1}, \ldots, q_{l_{\mathrm{rk}(\mathfrak{g}(e))}+1})$ *be in the domain of the Cayley map* (5.6)*. Then the automorphism group of* $((\mathcal{E}_{\mathrm{C}\cap\mathrm{H}}, \tilde{\psi}_{m_c}), q_{l_1+1}, \ldots, q_{l_{\mathrm{rk}(\mathfrak{g}(e))}+1})$ *is equal to the automorphism group of* $\widehat{\Psi}_e((\mathcal{E}_{\mathrm{C}\cap\mathrm{H}}, \tilde{\psi}_{m_c}), q_{l_1+1}, \ldots, q_{l_{\mathrm{rk}(\mathfrak{g}(e))}+1})$*.*

6. Moduli spaces of Higgs bundles

6.1. **Stability conditions and moduli spaces.** In this section we introduce the moduli space of L-twisted Higgs bundles, recall some of its features and discuss related objects. For this section we fix a compact Riemann surface X with genus $g \geqslant 2$. We start by recalling the notions of (semi,poly)stability and the moduli spaces for Higgs pairs. See [29] for more details. Let G be a complex reductive Lie group with Lie algebra $\mathfrak{g}$, equipped with a non-degenerate G-invariant $\mathbb{C}$-bilinear pairing $\langle\cdot,\cdot\rangle$. Let $\mathrm{K}^{\mathbb{R}} \subset \mathrm{G}$ be a maximal compact subgroup with Lie algebra $\mathfrak{k}^{\mathbb{R}}$.

An element $s \in i\mathfrak{k}^{\mathbb{R}}$ defines a parabolic subgroup P_s and a Levi subgroup L_s of G

$$\mathrm{P}_s = \{g \in \mathrm{G} \mid e^{ts}ge^{-ts} \text{ is bounded as } t \to \infty\} \subset \mathrm{G},$$
$$\mathrm{L}_s = \{g \in \mathrm{G} \mid e^{ts}ge^{-ts} = s \text{ for all } t\} \subset \mathrm{P}_s.$$

Also, given a holomorphic representation $\rho : \mathrm{G} \to \mathrm{GL}(V)$, we have subspaces

$$V_s = \{v \in V \mid \rho(e^{ts})v \text{ is bounded as } t \to \infty\}, \quad V_s^0 = \{v \in V \mid \rho(e^{ts})v = v \text{ for all } t\} \subset V_s. \tag{6.1}$$

Here, $V_s \subset V$ is P_s-invariant and $V_0^s \subset V_s$ is L_s-invariant. For the adjoint representation $\mathrm{Ad} : \mathrm{G} \to \mathrm{GL}(\mathfrak{g})$, we have $\mathfrak{g}_s^0 \subset \mathfrak{g}_s$ are the Lie algebras $\mathfrak{l}_s \subset \mathfrak{p}_s$ of $\mathrm{L}_s \subset \mathrm{P}_s$. Since, $\langle s, [\mathfrak{p}_s, \mathfrak{p}_s]\rangle = 0$, the element $s \in i\mathfrak{k}^{\mathbb{R}}$ defines the character of $\mathfrak{p}_s$

$$\chi_s := \langle s, -\rangle : \mathfrak{p}_s \to \mathbb{C}.$$

Given a smooth G-bundle E_{G}, we will define the degree of a structure group reduction from G to P_s using Chern-Weil theory and the character χ_s. Let $\mathrm{L}_s^{\mathbb{R}} = \mathrm{K}^{\mathbb{R}} \cap \mathrm{L}_s$ be a maximal compact subgroup of L_s; the inclusion $\mathrm{L}_s^{\mathbb{R}} \subset \mathrm{L}_s$ is a homotopy equivalence. Now suppose $E_{\mathrm{P}_s} \subset E_{\mathrm{G}}$ is a reduction of E_{G} to P_s. There is a further reduction $E_{\mathrm{L}_s^{\mathbb{R}}} \subset E_{\mathrm{P}_s}$ which is unique up to homotopy. Consider a connection A on $E_{\mathrm{L}_s^{\mathbb{R}}}$ with curvature $F_A \in \Omega^2(X, E_{\mathrm{L}_s^{\mathbb{R}}}[\mathfrak{l}_s^{\mathbb{R}}])$. Then $\chi_s(F_A)$ is a 2-form on X with values in $i\mathbb{R}$. Define the degree of the reduction $E_{P_s} \subset \mathcal{E}_{\mathrm{H}}$ to be the real number

$$\deg(E_{P_s}) = \frac{i}{2\pi}\int_X \chi_s(F_A).$$

Let $d\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be the differential of ρ and $\mathfrak{z}^{\mathbb{R}}$ be the center of $\mathfrak{k}^{\mathbb{R}}$. Define

$$\mathfrak{k}_\rho^{\mathbb{R}} = \mathfrak{k}_s^{\mathbb{R}} + \ker(d\rho_{|\mathfrak{z}})^{\perp}.$$

Thus $\mathfrak{k}^{\mathbb{R}} = \mathfrak{k}_\rho^{\mathbb{R}} + \ker(d\rho_{|\mathfrak{z}})$. We are now ready to define α-stability notions, for $\alpha \in i\mathfrak{z}^{\mathbb{R}}$.

Definition 6.1. *Let $\alpha \in i\mathfrak{z}^{\mathbb{R}}$. An L-twisted (G, V)-Higgs pair $(\mathcal{E}_{\mathrm{G}}, \varphi)$ is:*

- α-semistable *if for any $s \in i\mathfrak{k}^{\mathbb{R}}$ and any holomorphic reduction $\mathcal{E}_{\mathrm{P}_s} \subset \mathcal{E}_{\mathrm{G}}$ such that $\varphi \in H^0(\mathcal{E}_{\mathrm{P}_s}[V_s] \otimes L)$, we have* $\deg(\mathcal{E}_{\mathrm{P}_s}) \geqslant \langle\alpha, s\rangle$.
- α-stable *if for any $s \in i\mathfrak{k}_\rho^{\mathbb{R}}$ and any holomorphic reduction $\mathcal{E}_{\mathrm{P}_s} \subset \mathcal{E}_{\mathrm{G}}$ such that $\varphi \in H^0(\mathcal{E}_{\mathrm{P}_s}[V_s] \otimes L)$, we have* $\deg(\mathcal{E}_{\mathrm{P}_s}) > \langle\alpha, s\rangle$.
- α-polystable *if it is α-semistable and whenever $s \in i\mathfrak{k}^{\mathbb{R}}$ and $\mathcal{E}_{\mathrm{P}_s} \subset \mathcal{E}_{\mathrm{G}}$ is a holomorphic reduction with* $\deg(\mathcal{E}_{\mathrm{P}_s}) = \langle\alpha, s\rangle$, *there is a further holomorphic reduction $\mathcal{E}_{\mathrm{L}_s} \subset \mathcal{E}_{\mathrm{P}_s}$ such that $\varphi \in H^0(\mathcal{E}_{\mathrm{L}_s}[V_s^0] \otimes L)$.*

Remark 6.2. In this paper, the case $\alpha \neq 0$ will only appear in very specific situations, therefore we will refer to 0-(semi,poly)stability simply as (semi,poly)stability. It is clear that the (semi,poly)stability of a Higgs pair is preserved by the action of gauge group and the $\mathbb{C}^*$-action from (5.1).

Remark 6.3. Consider an L-twisted G-Higgs bundle $(\mathcal{E}_{\mathrm{G}}, \varphi)$ for a semisimple Lie group G. Using the adjoint representation, we can form the Higgs vector bundle $(\mathcal{E}_{\mathrm{G}}[\mathfrak{g}], \mathrm{ad}_\varphi)$. In this case, 0-polystable of $(\mathcal{E}_{\mathrm{G}}, \varphi)$ is equivalent to the polystability criterion involving degrees of invariant subbundles. Namely, $(\mathcal{E}_{\mathrm{G}}, \varphi)$ is 0-polystable if and only if for any holomorphic subbundle $\mathcal{V} \subset \mathcal{E}_{\mathrm{G}}[\mathfrak{g}]$ with $\mathrm{ad}_\varphi(\mathcal{V}) \subset \mathcal{V} \otimes L$, we have $\deg(\mathcal{V}) \leqslant 0$. This follows from the Hitchin–Kobayashi correspondence (see §6.4).

Remark 6.4. Let $\mathrm{G}_1 \to \mathrm{G}_2$ be a covering and $(\mathcal{E}_{\mathrm{G}_2}, \varphi)$ be a G_2-Higgs bundle which lifts to an G_1-Higgs bundle $(\mathcal{E}_{\mathrm{G}_1}, \varphi)$, i.e. $\mathcal{E}_{\mathrm{G}_1}(\mathrm{G}_2) = \mathcal{E}_{\mathrm{G}_2}$. Then $(\mathcal{E}_{\mathrm{G}_2}, \varphi)$ is polystable if and only if $(\mathcal{E}_{\mathrm{G}_1}, \varphi)$ is polystable. Indeed, any holomorphic parabolic reduction $\mathcal{E}_{\mathrm{P}_s} \subset \mathcal{E}_{\mathrm{G}_1}$ induces a holomorphic parabolic reduction $\mathcal{E}_{\mathrm{P}_s}(\mathrm{G}_2) \subset \mathcal{E}_{\mathrm{G}_2}$ and any holomorphic parabolic reduction $\mathcal{E}_{\mathrm{P}_s} \subset \mathcal{E}_{\mathrm{G}_2}$ lifts to a reduction $\mathcal{E}_{\mathrm{P}'_s} \subset \mathcal{E}_{\mathrm{G}_1}$.

The following result will be useful. For a proof, see [29, §2.10].

Proposition 6.5. *Suppose $(\mathcal{E}_{\mathrm{G}}, \varphi)$ is a strictly polystable L-twisted (G, V)-Higgs pair. Then there exists an $s \in i\mathfrak{k}^{\mathbb{R}}$, a holomorphic reduction $\mathcal{E}_{\mathrm{L}_s} \subset \mathcal{E}_{\mathrm{G}}$ with $\deg(\mathcal{E}_{\mathrm{L}_s}) = 0$ and $\varphi \in H^0(\mathcal{E}_{\mathrm{L}_s}(V_s^0) \otimes L)$ such that $(E_{\mathrm{L}_s}, \varphi)$ is a stable as an L-twisted (L_s, V_s^0)-Higgs pair.*

We will only need to consider the moduli space for L-twisted $\mathrm{G}^{\mathbb{R}}$-Higgs bundles over X, where $\mathrm{G}^{\mathbb{R}}$ is a real form of G. Denote it by $\mathcal{M}_L(\mathrm{G}^{\mathbb{R}})$. We define it as the space of gauge orbits of polystable L-twisted $\mathrm{G}^{\mathbb{R}}$-Higgs bundles

$$\mathcal{M}_L(\mathrm{G}^{\mathbb{R}}) = \mathcal{H}_{\mathrm{L}}^{ps}(\mathrm{G}^{\mathbb{R}})/\mathcal{G}_{\mathrm{H}} \ ,$$

where $\mathcal{H}_{\mathrm{L}}^{ps}(\mathrm{G}^{\mathbb{R}}) \subset \mathcal{H}_{\mathrm{L}}(\mathrm{G}^{\mathbb{R}})$ is the subset of polystable L-twisted $\mathrm{G}^{\mathbb{R}}$-Higgs bundles.

In order to endow $\mathcal{M}_L(\mathrm{G}^{\mathbb{R}})$ with a topology, suitable Sobolev completions must be used in standard fashion; see [25], where a detailed adaptation to Higgs bundles is studied in the case $\mathrm{G} = \mathrm{GL}_n\mathbb{C}$. Then the orbits of the $\mathcal{G}_{\mathrm{H}}$-action on $\mathcal{H}_L(\mathrm{G}^{\mathbb{R}})^{ps}$ are closed in the space of semistable $\mathrm{G}^{\mathbb{R}}$-Higgs bundles, thus the moduli space $\mathcal{M}_L(\mathrm{G}^{\mathbb{R}})$ becomes a Hausdorff topological space. If $\mathcal{H}_L^s(\mathrm{G}^{\mathbb{R}}) \subset \mathcal{H}_L^{ps}(\mathrm{G}^{\mathbb{R}})$ denotes the subset of stable Higgs bundles, then $\mathcal{H}_L^s(\mathrm{G}^{\mathbb{R}})$ is open in $\mathcal{H}_L^{ps}(\mathrm{G}^{\mathbb{R}})$. The stable objects thus define an open subset of $\mathcal{M}_L(\mathrm{G}^{\mathbb{R}})$.

Remark 6.6. A GIT construction of $\mathcal{M}_L(\mathrm{G}^{\mathbb{R}})$ (actually in the more general setting of Higgs pairs) may be found in [63], from which is clear that $\mathcal{M}_L(\mathrm{G}^{\mathbb{R}})$ parameterizes S-equivalence classes of semistable L-twisted $\mathrm{G}^{\mathbb{R}}$-Higgs bundles. This construction generalizes the construction of the moduli space of $\mathrm{G}^{\mathbb{R}}$-Higgs bundles by Ramanathan [62] when $\mathrm{G}^{\mathbb{R}}$ is compact and Simpson [67, 68] when $\mathrm{G}^{\mathbb{R}}$ is complex reductive (see also Nitsure [59] for $\mathrm{G}^{\mathbb{R}} = \mathrm{GL}_n\mathbb{C}$).

6.2. **Local structure of the moduli spaces.** We now recall some deformation theory for Higgs bundles, for more details see [8] and [29]. Fix a holomorphic line bundle L on X and let $(\mathcal{E}_{\mathrm{H}}, \varphi)$ be an L-twisted $\mathrm{G}^{\mathbb{R}}$-Higgs bundle. The double complex of sheaves

$$C^\bullet(\mathcal{E}_{\mathrm{H}}, \varphi) : \ \mathcal{E}_{\mathrm{H}}[\mathfrak{h}] \xrightarrow{\mathrm{ad}_\varphi} \mathcal{E}_{\mathrm{H}}[\mathfrak{m}] \otimes L \tag{6.2}$$

governs infinitesimal deformations of $(\mathcal{E}_{\mathrm{H}}, \varphi)$. Thus, when $(\mathcal{E}_{\mathrm{H}}, \varphi)$ is polystable, (6.2) encodes the local structure of the moduli space $\mathcal{M}_L(\mathrm{G}^{\mathbb{R}})$ near the point defined by $(\mathcal{E}_{\mathrm{H}}, \varphi)$. The complex (6.2) defines a long exact sequence in hypercohomology:

$$\begin{aligned} 0 \longrightarrow \mathbb{H}^0(C^\bullet(\mathcal{E}_{\mathrm{H}}, \varphi)) \longrightarrow H^0(\mathcal{E}_{\mathrm{H}}[\mathfrak{h}]) \xrightarrow{\mathrm{ad}_\varphi} H^0(\mathcal{E}_{\mathrm{H}}[\mathfrak{m}] \otimes L) \longrightarrow \mathbb{H}^1(C^\bullet(\mathcal{E}_{\mathrm{H}}, \varphi)) \\ \longrightarrow H^1(\mathcal{E}_{\mathrm{H}}[\mathfrak{h}]) \xrightarrow{\mathrm{ad}_\varphi} H^1(\mathcal{E}_{\mathrm{H}}[\mathfrak{m}] \otimes L) \longrightarrow \mathbb{H}^2(C^\bullet(\mathcal{E}_{\mathrm{H}}, \varphi)) \longrightarrow 0. \end{aligned} \tag{6.3}$$

We have the following proposition; see [29, Lemma 2.25 and Proposition 3.8].

Proposition 6.7. *If the L-twisted $\mathrm{G}^{\mathbb{R}}$-Higgs bundle $(\mathcal{E}_{\mathrm{H}}, \varphi)$ is polystable, then its automorphism group $\mathrm{Aut}(\mathcal{E}_{\mathrm{H}}, \varphi)$ is a complex reductive group which is identified with a closed subgroup of the automorphisms of the fiber $(\mathcal{E}_{\mathrm{H}}(x), \varphi(x))$ for any $x \in X$. The zeroth hypercohomology group $\mathbb{H}^0(C^\bullet(\mathcal{E}_{\mathrm{H}}, \varphi))$ is the Lie algebra of $\mathrm{Aut}(\mathcal{E}_{\mathrm{H}}, \varphi)$.*

Note that the automorphism group $\mathrm{Aut}(\mathcal{E}_{\mathrm{H}}, \varphi)$ acts on $\mathbb{H}^1(C^\bullet(\mathcal{E}_{\mathrm{H}}, \varphi))$. Using standard slice methods of Kuranishi (see [53, Chapter 7.3] for details for the moduli space of holomorphic bundles), a neighborhood of the isomorphism class of a polystable Higgs bundle $(\mathcal{E}_{\mathrm{H}}, \varphi)$ in $\mathcal{M}_L(\mathrm{G}^{\mathbb{R}})$ is given by

$$\kappa^{-1}(0) /\!\!/ \mathrm{Aut}(\mathcal{E}_{\mathrm{H}}, \varphi)$$

where $\kappa : \mathbb{H}^1(C^\bullet(\mathcal{E}_{\mathrm{H}}, \varphi)) \to \mathbb{H}^2(C^\bullet(\mathcal{E}_{\mathrm{H}}, \varphi))$ is the so called Kuranishi map. When $\mathbb{H}^2(\mathcal{E}_{\mathrm{H}}, \varphi) = 0$, a neighborhood of the isomorphism class of $(\mathcal{E}_{\mathrm{H}}, \varphi)$ in $\mathcal{M}_L(\mathrm{G}^{\mathbb{R}})$ is isomorphic to

$$\mathbb{H}^1(C^\bullet(\mathcal{E}_{\mathrm{H}}, \varphi)) /\!\!/ \mathrm{Aut}(\mathcal{E}_{\mathrm{H}}, \varphi). \tag{6.4}$$

We will use the following result in §7 to prove that for the Higgs bundles considered there, the corresponding $\mathbb{H}^2$ vanishes. Therefore, we have no need to recall the construction of the Kuranishi map.

Proposition 6.8. *Let $\mathrm{G}^{\mathbb{R}} \subset \mathrm{G}$ be a real form of a complex semisimple Lie group G and let L be a holomorphic line bundle with $\deg(L) > 2g-2$. Then for any polystable L-twisted $\mathrm{G}^{\mathbb{R}}$-Higgs bundle $(\mathcal{E}_{\mathrm{H}}, \varphi)$ we have $\mathbb{H}^2(C^\bullet(\mathcal{E}_{\mathrm{H}}, \varphi)) = 0$.*

Proof. It suffices to prove the statement for the L-twisted G-Higgs bundle $(\mathcal{E}_{\mathrm{G}}, \varphi) = (\mathcal{E}_{\mathrm{H}}[\mathrm{G}], \varphi)$ since there is an inclusion $\mathbb{H}^2(C^\bullet(\mathcal{E}_{\mathrm{H}}, \varphi)) \subset \mathbb{H}^2(C^\bullet(\mathcal{E}_{\mathrm{G}}, \varphi))$. Since $(\mathcal{E}_{\mathrm{G}}, \varphi)$ is semistable, any subbundle $\mathcal{V} \subset \mathcal{E}_{\mathrm{G}}[\mathfrak{g}]$ with $\mathrm{ad}_\varphi(\mathcal{V}) = 0$ satisfies $\deg(\mathcal{V}) \leqslant 0$ by Remark 6.3.

Suppose $0 \neq \mathbb{H}^2(C^\bullet(\mathcal{E}_{\mathrm{G}}, \varphi))$. By Serre duality $\mathbb{H}^2(C^\bullet(\mathcal{E}_{\mathrm{G}}, \varphi))$ is isomorphic to $\mathbb{H}^0$ of

$$C^\bullet(\mathcal{E}_{\mathrm{G}}, \varphi)^* \otimes K : \ \mathcal{E}_{\mathrm{G}}[\mathfrak{g}]^* \otimes L^{-1}K \xrightarrow{\mathrm{ad}_\varphi^* \otimes \mathrm{Id}_K} \mathcal{E}_{\mathrm{G}}[\mathfrak{g}]^* \otimes K.$$

The Killing form on $\mathfrak{g}$ identifies $\mathcal{E}_{\mathrm{G}}[\mathfrak{g}]^*$ with $\mathcal{E}_{\mathrm{G}}[\mathfrak{g}]$ and ad_φ^* with $-\mathrm{ad}_\varphi$, so the complex

$$\mathcal{E}_{\mathrm{G}}[\mathfrak{g}] \otimes L^{-1}K \xrightarrow{-\mathrm{ad}_\varphi \otimes \mathrm{Id}_K} \mathcal{E}_{\mathrm{G}}[\mathfrak{g}] \otimes K$$

has nonzero $\mathbb{H}^0$. Thus, there is a nonzero $s \in H^0(\mathcal{E}_{\mathrm{G}}[\mathfrak{g}] \otimes L^{-1}K)$ such that $-\mathrm{ad}_\varphi(s) = 0$. Let $M \subset \mathcal{E}_{\mathrm{G}}[\mathfrak{g}] \otimes L^{-1}K$ be the holomorphic line bundle generated by s, note $\deg(M) \geqslant 0$. However, $M \otimes LK^{-1} \subset \mathcal{E}_{\mathrm{G}}[\mathfrak{g}]$ satisfies $\mathrm{ad}_\varphi(M \otimes LK^{-1}) = 0$. So, by semistability of $(\mathcal{E}_{\mathrm{G}}, \varphi)$,

$$0 \leqslant \deg(M) < \deg(M \otimes LK^{-1}) \leqslant 0.$$

This contradiction implies $\mathbb{H}^2(C^\bullet(\mathcal{E}_{\mathrm{G}}, \varphi)) = 0$. □

6.3. **The Hitchin map.** A fundamental ingredient in the theory of Higgs bundles is the *Hitchin map* [47]. We briefly explain this in the setting of L-twisted $\mathrm{G}^{\mathbb{R}}$-Higgs bundles, for a simple real Lie group $\mathrm{G}^{\mathbb{R}}$; see [34, 47, 22] for more details.

Consider the GIT quotient map $\chi : \mathfrak{m} \to \mathfrak{m} /\!\!/ \mathrm{H}$. Note that χ is $\mathbb{C}^*$-equivariant with respect to the standard scaling action of $\mathbb{C}^*$ on $\mathfrak{m}$ and the action of $\mathbb{C}^*$ on $\mathfrak{m} /\!\!/ \mathrm{H}$ induced by the action of $\mathbb{C}^*$ on the graded ring $\mathbb{C}[\mathfrak{m}]^{\mathrm{H}}$. Namely, if $p \in \mathbb{C}[\mathfrak{m}]^{\mathrm{H}}$ is homogeneous, the $\mathbb{C}^*$-action on $\mathfrak{m} /\!\!/ \mathrm{H}$ is determined by $t \cdot p = t^{\deg(p)} p$. Let $\mathcal{L}$ be the holomorphic $\mathbb{C}^*$-bundle associated to L and consider the rank r vector bundle $\mathcal{L}[\mathfrak{m} /\!\!/ \mathrm{H}]$ associated to $\mathcal{L}$ via the $\mathbb{C}^*$-action on $\mathfrak{m} /\!\!/ \mathrm{H}$. The quotient map $\chi : \mathfrak{m} \to \mathfrak{m} /\!\!/ \mathrm{H}$ defines an H-invariant map $\mathfrak{m} \otimes L \to \mathcal{L}[\mathfrak{m} /\!\!/ \mathrm{H}]$. By H-invariance this defines the Hitchin map:

$$h : \mathcal{M}_L(\mathrm{G}^{\mathbb{R}}) \to \mathcal{B}_L(\mathrm{G}^{\mathbb{R}}) = H^0(\mathcal{L}[\mathfrak{m} /\!\!/ \mathrm{H}]), \quad h(\mathcal{E}_{\mathrm{H}}, \varphi) = \chi(\varphi), \tag{6.5}$$

where the space $\mathcal{B}_L(\mathrm{G}^{\mathbb{R}})$ is called the Hitchin base.

Choosing a homogeneous basis $(\chi_1, \dots, \chi_r)$ of the ring $\mathbb{C}[\mathfrak{m}]^{\mathrm{H}}$ defines and isomorphism of $\mathfrak{m} /\!\!/ \mathrm{H} \xrightarrow{\cong} \mathbb{C}^r$ given by $x \mapsto (\chi_1(x), \dots, \chi_r(x))$. If the degree χ_j is $m'_j + 1$ with $m'_1 < \dots < m'_r$, then the non-negative integers m'_i are the exponents of the real Lie algebra $\mathfrak{g}^{\mathbb{R}}$ (see for example [34, Proposition 4.4]). By definition, they are the exponents of the complex Lie algebra obtained by complexifying the maximal split subalgebra of $\mathfrak{g}^{\mathbb{R}}$ (if $\mathfrak{g}^{\mathbb{R}}$ is complex, these are its exponents appearing in Case (1) of Proposition 4.3).

Any choice of such homogeneous basis $(\chi_1, \dots, \chi_r)$ defines an isomorphism

$$H^0(\mathcal{L}[\mathfrak{m} /\!\!/ \mathrm{H}]) \xrightarrow{\cong} \bigoplus_{j=1}^{r} H^0(L^{m'_j+1}) \ , \ x \mapsto (\chi_1(x), \dots, \chi_r(x)).$$

Using this basis, we obtain the more familiar description of the Hitchin map

$$h : \mathcal{M}_L(\mathrm{G}^{\mathbb{R}}) \to \bigoplus_{j=1}^{r} H^0(L^{m'_j+1}) \ , \ h(\mathcal{E}_{\mathrm{H}}, \varphi) = (\chi_1(\varphi), \dots, \chi_r(\varphi)).$$

For complex Lie groups and $L = K$, the Hitchin map h has many special features, most notably it is an algebraic completely integrable system [47]. The property we will use to prove the Cayley map is closed, and which is true for arbitrary groups and twistings, is that the Hitchin map (6.5) is proper. This follows from [59, Theorem 6.1] for $\mathrm{GL}_n\mathbb{C}$ and from the fact that the moduli space $\mathcal{M}_L(\mathrm{G}^{\mathbb{R}})$ admits a finite (and hence proper) map to $\mathcal{M}_L(\mathrm{GL}_n\mathbb{C})$ for some n in such a way that the Hitchin map of $\mathcal{M}_L(\mathrm{G}^{\mathbb{R}})$ is the restriction of the Hitchin map in $\mathcal{M}_L(\mathrm{GL}_n\mathbb{C})$.

Proposition 6.9. *The Hitchin map $h : \mathcal{M}_L(\mathrm{G}^{\mathbb{R}}) \to \mathcal{B}_L(\mathrm{G}^{\mathbb{R}})$ from (6.5) is proper.*

6.4. **The Hitchin–Kobayashi correspondence.** Finally, we consider an equation for a special metric associated to general L-twisted polystable (G, V)-Higgs pairs. Let G be a complex reductive Lie group and fix a maximal compact subgroup $\mathrm{K}^{\mathbb{R}} \subset \mathrm{G}$ and a $\mathrm{K}^{\mathbb{R}}$-invariant Hermitian inner-product on V so that $d\rho : \mathfrak{k}^{\mathbb{R}} \to \mathfrak{u}(V)$ is the associated unitary representation. Let $(\mathcal{E}_{\mathrm{G}}, \varphi)$ be an L-twisted (G, V)-Higgs pair. Fix a metric h_L on the line bundle L. A metric on $\mathcal{E}_{\mathrm{G}}$ is by definition a reduction of structure group h of $\mathcal{E}_{\mathrm{G}}$ to $\mathrm{K}^{\mathbb{R}}$. Fix a metric h and let $E_h \subset \mathcal{E}_{\mathrm{G}}$ be the associated $\mathrm{K}^{\mathbb{R}}$-bundle. The Hermitian inner-product on V and the metric h_L on L induce a Hermitian metric $h \otimes h_L$ on the bundle $E_h[V] \otimes L$. For $\varphi \in H^0(\mathcal{E}_{\mathrm{G}}[V] \otimes L)$ we can make sense of the following expression:

$$\mu(\varphi) = d\rho^*\Big(-\frac{i}{2}\varphi \otimes \varphi^*_{h \otimes h_L}\Big), \tag{6.6}$$

where we identify $i\varphi \otimes \varphi^{*_{h\otimes h_L}}$ with a section of $E_h(\mathfrak{u}(V))^*$. Hence $\mu(\varphi)$ defines a section of $E_h(\mathfrak{k}^{\mathbb{R}})^*$. Using the nondegenerate pairing, we view $\mu(\varphi)$ as a section of $E_h(\mathfrak{k}^{\mathbb{R}})$.

Remark 6.10. The action of $\mathrm{K}^{\mathbb{R}}$ on V is Hamiltonian and the expression for μ in (6.6) is a bundle version of the moment map for the action.

Now fix a Kähler form ω on X. Given a metric h on $\mathcal{E}_{\mathrm{G}}$ there is a unique connection (the Chern connection) which is compatible with the holomorphic structure and the metric reduction. The *Hitchin–Kobayashi correspondence* states the following.

Theorem 6.11. [29, Theorem 2.24] *An L-twisted (G, V)-Higgs pair $(\mathcal{E}_{\mathrm{G}}, \varphi)$ is α-polystable if and only if there is a metric h on $\mathcal{E}_{\mathrm{G}}$ solving*

$$F_h + \mu(\varphi)\omega = -i\alpha\omega, \tag{6.7}$$

where $F_h \in \Omega^2(E_h[\mathfrak{h}^{\mathbb{R}}])$ denotes the curvature of the Chern connection of h.

Remark 6.12. The existence of solutions h of (6.7) is independent of the choice of h_L. Also, equation (6.7) implies that α depends on the fixed Kähler form ω. If one chooses a different Kähler form ω', then a solution of (6.7) will still be a solution for the corresponding equation with ω', for a different α'. This means that, to check for the existence of solutions of (6.7), we can fix any ω, and always work with it.

When now specialize to the case of Higgs bundles and Higgs pairs arising from $\mathbb{Z}/n\mathbb{Z}$-gradings of $\mathfrak{g}$. Let $\tau : \mathfrak{g} \to \mathfrak{g}$ be the compact real-form associated to $\mathrm{K}^{\mathbb{R}} \subset \mathrm{G}$ and let $\langle\cdot,\cdot\rangle$ be a nondegenerate G-invariant complex bilinear form. The form $\langle x, -\tau(y)\rangle$ is a $\mathrm{K}^{\mathbb{R}}$-invariant positive definite Hermitian inner product on $\mathfrak{g}$. In this case, the moment map $\mu : \mathfrak{g} \to (\mathfrak{k}^{\mathbb{R}})^* \to \mathfrak{k}^{\mathbb{R}}$ is given by $\mu(x) = [x, -\tau(x)]$.

Given a metric h on $\mathcal{E}_{\mathrm{G}}$ and a metric h_L on L, τ defines an involution $\tau_h : E_h(\mathfrak{g}) \otimes L \to E_h(\mathfrak{g}) \otimes L$. Thus, for L-twisted G-Higgs bundles, equation (6.7) is

$$F_h + [\varphi, -\tau_h(\varphi)]\omega = -i\alpha\omega.$$

When $L = K$ we can view the Higgs field as a $(1,0)$-form valued in $E_h(\mathfrak{g})$. In this case, we can use τ and conjugation on 1-forms to define the involution $\tau_h : \Omega^{1,0}(E_h(\mathfrak{g})) \to \Omega^{0,1}(E_h(\mathfrak{g}))$, and solving (6.7) is equivalent to solving

$$F_h + [\varphi, -\tau_h(\varphi)] = -i\alpha. \tag{6.8}$$

Remark 6.13. When $\alpha = 0$, equation (6.8) is usually referred to as the Hitchin equations or the self-duality equations. In this case, the Hitchin–Kobayashi correspondence was proven by Hitchin for $\mathrm{G} = \mathrm{SL}_2\mathbb{C}$ [46] and by Simpson in general [66].

Remark 6.14. The uniformizing $\mathrm{PSL}_2\mathbb{R}$-Higgs bundle $(\mathcal{E}_{\mathrm{T}}, \varphi)$ from Example 5.3 (and any lift of it to $\mathrm{SL}_2\mathbb{R}$) is 0-stable. Since $\mathcal{E}_{\mathrm{T}}$ is the frame bundle of K^{-1}, any metric on $\mathcal{E}_{\mathrm{T}}$ defines a metric on the surface; the metric solving (6.8) has constant curvature [46].

Finally, suppose $\hat{\mathrm{G}} \subset \mathrm{G}$ is a τ-invariant subgroup with maximal compact subgroup $\hat{\mathrm{K}}^{\mathbb{R}} = \hat{\mathrm{G}} \cap \mathrm{K}^{\mathbb{R}}$ and $V \subset \mathfrak{g}$ is a $\hat{\mathrm{G}}$-invariant orthogonal vector subspace of $\mathfrak{g}$. In this case, the moment map equations the action of $\hat{\mathrm{K}}^{\mathbb{R}}$ on V is given orthogonally projecting $[x, -\tau(x)]$ onto the Lie algebra $\hat{\mathfrak{k}}^{\mathbb{R}} \subset \mathfrak{k}^{\mathbb{R}}$. For example, the quiver bundle equations of [2] are an example of this. An important special case of this occurs when the orthogonal projection $\mathfrak{k}^{\mathbb{R}} \to \hat{\mathfrak{k}}^{\mathbb{R}}$ does not loose any information, i.e., when $[V, -\tau(V)] \subset \hat{\mathfrak{k}}^{\mathbb{R}}$. In this case, when $\alpha = 0$ a solution to $(\hat{\mathrm{G}}, V)$-Higgs pair equations also solves the G-Higgs bundle equations. Thus, if $(\mathcal{E}_{\hat{\mathrm{G}}}, \varphi)$ is an 0-polystable L-twisted $(\hat{\mathrm{G}}, V)$-Higgs pair, then the associated G-Higgs bundle $(\mathcal{E}_{\hat{\mathrm{G}}}(\mathrm{G}), \varphi)$ obtained by extending the structure group is polystable as a G-Higgs bundle.

For example, consider a $\mathbb{Z}/n\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} \hat{\mathfrak{g}}_j$, i.e., $[\hat{\mathfrak{g}}_j, \hat{\mathfrak{g}}_k] \subset \hat{\mathfrak{g}}_{j+k \mod n}$. The connected subgroup $\hat{\mathrm{G}}_0 \subset \mathrm{G}$ with Lie algebra $\hat{\mathfrak{g}}_0$ acts on each summand $\hat{\mathfrak{g}}_j$. The compact involution $\tau : \mathfrak{g} \to \mathfrak{g}$ can be chosen so that $\tau(\hat{\mathfrak{g}}_j) = \hat{\mathfrak{g}}_{-j \mod n}$. We have the following proposition, which was observed by Simpson [65].

Proposition 6.15. *Let $(\mathcal{E}_{\hat{\mathrm{G}}_0}, \varphi)$ be a 0-polystable L-twisted $(\hat{\mathrm{G}}_0, \hat{\mathfrak{g}}_1)$-Higgs pair. Then the L-twisted G-Higgs bundle $(\mathcal{E}_{\hat{\mathrm{G}}_0}[\mathrm{G}], \varphi)$ is polystable as a Higgs bundle.*

7. The generalized Cayley correspondence

In this section we prove that the Cayley map Ψ_e from (5.6) descends to an injective map on moduli spaces which is open and closed, thus proving Theorem B from the introduction.

For this section, $\{f, h, e\}$ will be a magical $\mathfrak{sl}_2$-triple, $\mathrm{S} \subset \mathrm{G}$ will the be associated connected subgroup, $\mathrm{C} \subset \mathrm{G}$ will be its centralizer and $\mathrm{G}^{\mathbb{R}} \subset \mathrm{G}$ will be the associated canonical real form. Recall that $\mathrm{H} \subset \mathrm{G}$ is the complexification of the maximal compact $\mathrm{H}^{\mathbb{R}} \subset \mathrm{G}^{\mathbb{R}}$. To simplify notation, we denote $\mathrm{C} \cap \mathrm{H}$ simply by C.

7.1. **Generalized Cayley correspondence and direct consequences.** Recall from (5.6) that the Cayley map is given by

$$\begin{aligned} \widehat{\Psi}_e :\ & \mathcal{H}_{K^{m_c+1}}(\mathrm{G}^{\mathbb{R}}_{0,ss}) \times \textstyle\prod_{j=1}^{\mathrm{rk}(\mathfrak{g}(e))} \mathcal{H}_{K^{l_j+1}}(\mathbb{R}^+) \longrightarrow \mathcal{H}(\mathrm{G}^{\mathbb{R}}), \\ & ((\mathcal{E}_{\mathrm{C}}, \tilde{\psi}_{m_c}), q_{l_1+1}, \ldots, q_{l_{\mathrm{rk}(\mathfrak{g}(e))}+1}) \longmapsto (\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}}[\mathrm{H}], f + \tilde{\phi}_{m_c} + \textstyle\sum_{j=1}^{\mathrm{rk}(\mathfrak{g}(e))} q_{l_j+1}) \end{aligned}$$

where $(\mathcal{E}_{\mathrm{T}}, f)$ is the uniformizing $\mathrm{PSL}_2\mathbb{R}$ (resp. $\mathrm{SL}_2\mathbb{R}$) Higgs bundle if $\mathrm{S} \cong \mathrm{PSL}_2\mathbb{C}$ (resp. $\mathrm{S} \cong \mathrm{SL}_2\mathbb{C}$). There is a natural notion of stability on the domain of the Cayley map since it is a product of Higgs bundle spaces. Moreover, every $q_j \in H^0(K^{j+1}) = \mathcal{H}_{K^{j+1}}(\mathbb{R}^+)$ is polystable. Hence a point

$$((\mathcal{E}_{\mathrm{C}}, \tilde{\psi}_{m_c}), q_{l_1+1}, \ldots, q_{l_{\mathrm{rk}(\mathfrak{g}(e))}+1}) \in \mathcal{H}_{K^{m_c+1}}(\mathrm{G}^{\mathbb{R}}_{0,ss}) \times \prod_{j=1}^{\mathrm{rk}(\mathfrak{g}(e))} \mathcal{H}_{K^{l_j+1}}(\mathbb{R}^+)$$

is polystable if and only if $(\mathcal{E}_{\mathrm{C}}, \tilde{\psi}_{m_c}) \in \mathcal{H}_{K^{m_c+1}}(\mathrm{G}^{\mathbb{R}}_{0,ss})$ is polystable.

Theorem 7.1. *The Cayley map $\hat{\Psi}_e$ descends to an injective map on moduli spaces,*

$$\Psi_e : \mathcal{M}_{K^{m_c+1}}(\mathrm{G}^{\mathbb{R}}_{0,ss}) \times \prod_{j=1}^{\mathrm{rk}(\mathfrak{g}(e))} \mathcal{M}_{K^{l_j+1}}(\mathbb{R}^+) \longrightarrow \mathcal{M}(\mathrm{G}^{\mathbb{R}}). \tag{7.1}$$

which is open and closed.

We also refer to Ψ_e as the *Cayley map*.

Corollary 7.2. *The image of the Cayley map Ψ_e is a union of connected components of $\mathcal{M}(\mathrm{G}^{\mathbb{R}})$ isomorphic to $\mathcal{M}_{K^{m_c+1}}(\mathrm{G}^{\mathbb{R}}_{0,ss}) \times \prod_{j=1}^{\mathrm{rk}(\mathfrak{g}(e))} \mathcal{M}_{K^{l_j+1}}(\mathbb{R}^+)$. Every $\mathrm{G}^{\mathbb{R}}$-Higgs bundle $(\mathcal{E}_{\mathrm{H}}, \varphi)$ in the image of the Cayley map has nowhere vanishing Higgs field φ.*

Definition 7.3. *We refer to the connected components in the image of the Cayley map as the* Cayley components *in $\mathcal{M}(\mathrm{G}^{\mathbb{R}})$.*

Remark 7.4. For Case (2) of Theorem 3.1, the Cayley map generalizes the Cayley correspondence of [7, 30, 36] for Higgs bundles for Hermitian groups of tube type with maximal Toledo invariant. As a result, we refer to the isomorphism defined by the Cayley map as the generalized Cayley correspondence. For Case (1) of Theorem 3.1, the Cayley map recovers the Hitchin section of [48] for split real groups. In fact, for all cases, when the $\mathrm{G}^{\mathbb{R}}_{0,ss}$-Higgs bundle $(\mathcal{E}_{\mathrm{C}}, \tilde{\psi}_{m_c})$ is trivial, the Cayley map recovers the Hitchin section for the split subgroup $\mathrm{G}(e)^{\mathbb{R}} \subset \mathrm{G}^{\mathbb{R}}$ with Lie algebra $\mathfrak{g}(e)^{\mathbb{R}}$. Finally, for $\mathrm{G}^{\mathbb{R}} = \mathrm{SO}_{p,q}$ with $2 \leqslant p \leqslant q$, the Cayley map recovers the connected components of $\mathcal{M}(\mathrm{SO}_{p,q})$ parameterized in [3, 15].

Remark 7.5. When $\mathrm{G}^{\mathbb{R}} \subset \mathrm{G}$ is a split real form with Lie algebra $\mathfrak{sp}_{2n}\mathbb{R}$, $\mathfrak{so}_{n,n+1}$ or the quaternionic real from of $\mathfrak{f}_4$, there are two magical $\mathfrak{sl}_2$-triples, one from Case (1) of Theorem 3.1 and one from Case (2), Case (3) or Case (4), respectively. Note that these are the only cases where the semisimple part $\mathrm{G}^{\mathbb{R}}_{0,ss} \subset \mathrm{G}^{\mathbb{R}}_{\mathcal{C}}$ of the Cayley group is split and contains a unique magical $\mathfrak{sl}_2$-triple. For these groups, the Cayley map for Case (1) of Theorem 3.1 is obtained by iterating the Cayley maps. For example, when $\mathrm{G}^{\mathbb{R}}$ is the quaternionic real from of F_4, we have the following diagram

$$\begin{array}{ccc} H^0(K^2)\oplus H^0(K^6)\oplus H^0(K^8)\oplus H^0(K^{12}) & \xrightarrow{\mathrm{Id}\oplus\Psi_{e,1}^{K^4}} & H^0(K^2)\oplus H^0(K^6)\oplus \mathcal{M}_{K^4}(\mathrm{SL}_3\mathbb{R})\ , \\ & \searrow^{\Psi_{e,1}} \quad \mathcal{M}(\mathrm{G}^{\mathbb{R}}) \quad {}^{\Psi_{e,4}}\swarrow & \end{array}$$

where $\Psi_{e,1}$ is the Cayley map from Case (1) of Theorem 3.1, $\Psi_{e,4}$ is the Cayley map from Case (4) of Theorem 3.1 and $\Psi_{e,1}^{K^4}$ is the K^4-twisted version of the Cayley map from Case (1) of Theorem 3.1 for $\mathrm{SL}_3\mathbb{R}$.

In general the connected components defined by the generalized Cayley correspondence are neither contractible nor smooth. However, in the process of proving Theorem 7.1, we show in Proposition 7.11 that for Higgs bundles in the image of the Cayley map, the second hypercohomology group $\mathbb{H}^2(C^\bullet(\mathcal{E}_{\mathrm{H}},\varphi))$ vanishes. As a result, $\mathbb{H}^1(C^\bullet(\mathcal{E}_{\mathrm{H}},\varphi)) /\!\!/ \mathrm{Aut}(\mathcal{E}_{\mathrm{H}},\varphi)$ is a local model for the moduli space $\mathcal{M}(\mathrm{G}^{\mathbb{R}})$ around $(\mathcal{E}_{\mathrm{H}},\varphi)$. It follows immediately that $\mathcal{M}(\mathrm{G}^{\mathbb{R}})$ is locally irreducible around $(\mathcal{E}_{\mathrm{H}},\varphi)$. Hence, we have the following:

Corollary 7.6. *Every Cayley component in $\mathcal{M}(\mathrm{G}^{\mathbb{R}})$ is locally irreducible and irreducible.*

The proof of Theorem 7.1 is broken into three parts. In §7.2 we prove that the Cayley map is well-defined and injective, we then prove the Cayley map is open in §7.3 and closed in §7.4.

7.2. **The Cayley map descends to moduli spaces.** We first prove the Cayley map descends to an injective map of moduli spaces.

Theorem 7.7. *If $((\mathcal{E}_{\mathrm{C}},\tilde{\psi}_{m_c}),q_{l_1+1},\ldots,q_{l_{\mathrm{rk}(\mathfrak{g}(e))}+1})\in\mathcal{H}_{K^{m_c+1}}(\mathrm{G}^{\mathbb{R}}_{0,ss})\times\prod_{j=1}^{\mathrm{rk}(\mathfrak{g}(e))}\mathcal{H}_{K^{l_j+1}}(\mathbb{R}^+)$ is stable (resp. polystable), then $\widehat{\Psi}_e((\mathcal{E}_{\mathrm{C}},\tilde{\psi}_{m_c}),q_{l_1+1},\ldots,q_{l_{\mathrm{rk}(\mathfrak{g}(e))}+1})$ is a stable (resp. polystable) $\mathrm{G}^{\mathbb{R}}$-Higgs bundle. In particular, the Cayley map (7.1) is well defined.*

Remark 7.8. By Remark 5.9, the Cayley map can be defined for L-twisted Higgs bundles. The proof of Theorem 7.7 given below also applies to this setting when $\deg(L)>0$.

The difficult step in the proof of Theorem 7.7 is proving the following lemma.

Lemma 7.9. *If $(\mathcal{E}_{\mathrm{C}},\tilde{\psi}_{m_c})\in\mathcal{H}_{K^{m_c+1}}(\mathrm{G}^{\mathbb{R}}_{0,ss})$ is stable (resp. polystable), then the $\mathrm{G}^{\mathbb{R}}$-Higgs bundle $\widehat{\Psi}_e((\mathcal{E}_{\mathrm{C}},\tilde{\psi}_{m_c}),0,\ldots,0)$ is a stable (resp. polystable).*

Before proving Lemma 7.9, we will prove Theorem 7.7 assuming Lemma 7.9.

Proof of Theorem 7.7 assuming Lemma 7.9. First note that the map is injective by Proposition 5.11. The idea of the proof that the map is well defined is similar to Hitchin's proof [48] that the image of the Hitchin section consists of stable Higgs bundles. First assume $(\mathcal{E}_{\mathrm{C}},\tilde{\psi}_{m_c})$ is stable. Since stability is preserved by the $\mathbb{C}^*$-action,

$$\widehat{\Psi}_e((\mathcal{E}_{\mathrm{C}},\lambda\tilde{\psi}_{m_c}),0,\ldots,0)=(\mathcal{E}_{\mathrm{C}}\star\mathcal{E}_{\mathrm{T}}[\mathrm{H}],f+\lambda\tilde{\phi}_{m_c})$$

is a stable $\mathrm{G}^{\mathbb{R}}$-Higgs bundle for all $\lambda\in\mathbb{C}^*$ by Lemma 7.9. Since stability is open,

$$\widehat{\Psi}_e((\mathcal{E}_{\mathrm{C}},\tilde{\psi}_{m_c}),t_{l_1+1}q_{l_1+1},\ldots,t_{l_{\mathrm{rk}(\mathfrak{g}(e))}+1}q_{l_{\mathrm{rk}(\mathfrak{g}(e))}+1})=\left(\mathcal{E}_{\mathrm{C}}\star\mathcal{E}_{\mathrm{T}}[\mathrm{H}],f+\tilde{\phi}_{m_c}+\sum_{j=1}^{\mathrm{rk}(\mathfrak{g}(e))}t_{l_j+1}q_{l_j+1}\right)$$

is stable for sufficiently small $t_j \in \mathbb{R}$. Thus, $(\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}}[\mathrm{H}], \lambda^2(f + \tilde{\phi}_{m_c} + \sum_{j=1}^{\mathrm{rk}(\mathfrak{g}(e))} t_{l_j+1}q_{l_j+1}))$ is stable for all $\lambda \in \mathbb{C}^*$.

Let $g_\lambda : \mathcal{E}_{\mathrm{T}} \to \mathcal{E}_{\mathrm{T}}$ be the holomorphic gauge transformation which acts on f by $g_\lambda \cdot f = \lambda^{-2} f$, then $\mathrm{Id}_{\mathcal{E}_{\mathrm{C}}} \star g_\lambda$ acts on $\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}}[\mathfrak{g}_{2j}] \otimes K$ with eigenvalue λ^{2j}. Since stability is also preserved by the gauge group,

$$
\begin{aligned}
(\mathrm{Id}_{\mathcal{E}_{\mathrm{C}}} \star g_\lambda) \cdot & \left(\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}}[\mathrm{H}], \lambda^2 \left(f + \tilde{\phi}_{m_c} + \sum_{j=1}^{\mathrm{rk}(\mathfrak{g}(e))} t_{l_j+1}q_{l_j+1}\right)\right) \\
&= \left(\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}}[\mathrm{H}], f + \lambda^{2m_c+2}\tilde{\phi}_{m_c} + \sum_{j=1}^{\mathrm{rk}(\mathfrak{g}(e))} \lambda^{2l_j+2} t_{l_j+1}q_{l_j+1}\right) \\
&= \widehat{\Psi}_e((\mathcal{E}_{\mathrm{C}}, \lambda^{2m_c+2}\tilde{\psi}_{m_c}), \lambda^{2l_1+2}t_{l_1+1}q_{l_1+1}, \ldots, \lambda^{2l_{\mathrm{rk}(\mathfrak{g}(e))}+2} t_{l_{\mathrm{rk}(\mathfrak{g}(e))}+1} q_{l_{\mathrm{rk}(\mathfrak{g}(e))}+1})
\end{aligned}
$$

is stable for all $\lambda \in \mathbb{C}^*$. Thus, $\Psi_e((\mathcal{E}_{\mathrm{C}}, \tilde{\psi}_{m_c}), q_{l_1+1}, \ldots, q_{l_{\mathrm{rk}(\mathfrak{g}(e))}+1})$ is stable.

If $(\mathcal{E}_{\mathrm{C}}, \tilde{\psi}_{m_c})$ is strictly polystable, then $\widehat{\Psi}_e(\mathcal{E}_{\mathrm{C}}, \tilde{\psi}_{m_c}) = (\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}}[\mathrm{H}], f + \tilde{\phi}_{m_c})$ is a strictly polystable $\mathrm{G}^{\mathbb{R}}$-Higgs bundle by Lemma 7.9. Suppose $s \in i\mathfrak{h}^{\mathbb{R}}$ and $\mathcal{E}_{\mathrm{P}_s} \subset \mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}}[\mathrm{H}]$ is a holomorphic reduction to the parabolic P_s with $\deg(\mathcal{E}_{\mathrm{P}_s}) = 0$ and such that $f + \tilde{\phi}_{m_c} \in H^0(\mathcal{E}_{\mathrm{P}_s}[\mathfrak{m}_s] \otimes K)$. By the definition of polystability there is a further holomorphic reduction $\mathcal{E}_{\mathrm{L}_s} \subset \mathcal{E}_{\mathrm{P}_s}$ such that $f + \tilde{\phi}_{m_c} \in H^0(\mathcal{E}_{L_s}[\mathfrak{m}_s^0] \otimes K)$. We claim that this implies $s \in \mathfrak{c}$. Indeed, write $s = \sum s_{2j}$, where s_{2j} is the projection of s onto the graded piece $\mathfrak{g}_{2j}$ and suppose k is the smallest j with $s_{2j} \neq 0$. If $v \in \mathfrak{g}_{2m_c}$, then $2k-2$-graded piece of $[s, f+v] = [s_{2k}, f]$. Since $\{f, h, e\}$ is magical, $\ker(\mathrm{ad}_f) \cap \mathfrak{h} = \mathfrak{c}$. Thus, $[s_{2k}, f] = 0$ implies $s \in \mathfrak{c}$.

By Proposition 6.5, there is $s \in i\mathfrak{h}^{\mathbb{R}}$ and a holomorphic reduction $\mathcal{E}_{\mathrm{L}_s} \subset \mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}}[\mathrm{H}]$ with $f + \tilde{\phi}_{m_c} \in H^0(\mathcal{E}_{\mathrm{L}_s}[\mathfrak{m}_s^0] \otimes K)$ such that $(\mathcal{E}_{\mathrm{L}_s}, f + \tilde{\phi}_{m_c})$ is a stable $\mathrm{G}_s^{\mathbb{R}}$-Higgs bundles. Here $\mathrm{G}_s^{\mathbb{R}}$ is the real form of the G-centralizer of s associated to the complexified Cartan decomposition $\mathfrak{g}_s^0 = \mathfrak{l}_s \oplus \mathfrak{m}_s^0$. Since, $s \in \mathfrak{c}$ and $[\mathfrak{c}, \mathfrak{g}(e)] = 0$, it follows that $\widehat{\Psi}_e((\mathcal{E}_{\mathrm{C}}, \tilde{\psi}_{m_c}), q_{l_1+1}, \ldots, q_{l_{\mathrm{rk}(\mathfrak{g}(e))}+1})$ is a $\mathrm{G}_s^{\mathbb{R}}$-Higgs bundle. Openness and $\mathbb{C}^*$-invariance of stability implies $\widehat{\Psi}_e((\mathcal{E}_{\mathrm{C}}, \tilde{\psi}_{m_c}), q_{l_1+1}, \ldots, q_{l_{\mathrm{rk}(\mathfrak{g}(e))}+1})$ is a stable $\mathrm{G}_s^{\mathbb{R}}$-Higgs bundle and hence a polystable $\mathrm{G}^{\mathbb{R}}$-Higgs bundle. □

We will prove Lemma 7.9 in each of the four cases of magical nilpotents from Theorem 3.1. The result is immediate for Case (1), it was proven in [7] for Case (2), and for Case (3) the result was proven in [3] for $\mathrm{G} = \mathrm{SO}_N\mathbb{C}$. Our proof in Case (4) relies on the details of the proof of Case (2) so we outline the proof of [7].

Proof of Lemma 7.9 Case (1). For Case (1) of Theorem 3.1, C is the center of $\mathrm{G}^{\mathbb{R}}$ and $\tilde{\phi}_{m_c} = 0$. Thus, $\widehat{\Psi}_e(\mathcal{E}_{\mathrm{C}}, \tilde{\phi}_{m_c}, 0, \ldots, 0) = (\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}}[\mathrm{H}], f)$. This is a polystable Higgs bundle since the solution metric for $(\mathcal{E}_{\mathrm{T}}, f)$ induces a solution to the $\mathrm{G}^{\mathbb{R}}$-Higgs bundle equations. It is stable since a principal nilpotent is not contained in the Levi subalgebra of any proper parabolic subalgebra of $\mathfrak{g}$. □

Proof of Lemma 7.9 Case (3). For Case (3) of Theorem 3.1 with $\mathrm{G} = \mathrm{SO}_N\mathbb{C}$ (and hence $\mathrm{G}^{\mathbb{R}} = \mathrm{SO}_{p,N-p}$), Lemma 7.9 was proven in [3, Lemma 4.5]. Roughly, $m_c + 1 = p$ and there is a $\mathbb{Z}/2p\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus \hat{\mathfrak{g}}_j$ such that $(\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}}[\hat{\mathrm{G}}_0], f + \tilde{\phi}_{p-1})$ is a $(\hat{\mathrm{G}}_0, \hat{\mathfrak{g}}_1)$-Higgs pair. This pair is shown to be polystable and Proposition 6.15 is applied. By Remark 6.4, it suffices to show that every $\mathrm{PSO}_N\mathbb{C}$-Higgs bundle in the image of $\hat{\Psi}_e$ lifts to a $\mathrm{SO}_N\mathbb{C}$-Higgs bundle in the image of $\hat{\Psi}_e$. This was shown in §5.3. □

Proof of Lemma 7.9 Case (2). The proof for Case (2) is the result of Lemmas 5.5, 5.6 and 5.7 of [7]. We outline the argument here in the notation of the current article. In this case, and $m_c = 1$ and $\mathrm{H} = \mathrm{G}_0 \subset \mathrm{G}$ is the centralizer of $h \in \mathfrak{g}$.

Let $(\mathcal{E}_\mathrm{C}, \psi_1)$ be a stable (resp. polystable) K^2-twisted $\mathrm{G}^\mathbb{R}_{0,ss}$-Higgs bundle. By [7, Lemma 5.5], $(\mathcal{E}_\mathrm{C} \star \mathcal{E}_T[\mathrm{H}], \psi_1)$ is an α-stable (resp. α-polystable) K^2-twisted H-Higgs bundle for $\alpha = \frac{h}{2} \in \mathfrak{z}(\mathfrak{h})$. This is proven using equations. Next one proves a finite dimensional GIT result ([7, Lemma 5.6]) for the magical nilpotent $f \in \mathfrak{g}_{-2}$. Namely, if $s \in i\mathfrak{h}$ and $f \in \mathfrak{g}_{-2,s}$, then $\langle h, s\rangle \geqslant 0$ and if equality holds, then $f \in \mathfrak{g}^0_{-2,s}$.

Now consider $\Psi_e(\mathcal{E}_\mathrm{C}, \psi_1) = (\mathcal{E}_\mathrm{C} \star \mathcal{E}_\mathrm{T}[\mathrm{H}], f + \phi_1)$, where $\mathrm{ad}_f(\phi_1) = \psi_1 \in H^0(\mathcal{E}_\mathrm{C} \star \mathcal{E}_\mathrm{T}[\mathfrak{g}_0] \otimes K^2) = H^0(\mathcal{E}_\mathrm{C}[\mathfrak{h}] \otimes K^2)$. Let $s \in i\mathfrak{h}^\mathbb{R}$ and $\mathcal{E}_{\mathrm{P}_s} \subset \mathcal{E}_\mathrm{C} \star \mathcal{E}_\mathrm{T}[\mathrm{H}]$ be a holomorphic reduction such that $f + \phi_1 \in H^0(\mathcal{E}_{\mathrm{P}_s}[\mathfrak{m}_s] \otimes K)$. Since P_s preserves the splitting $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_2$, we have $f \in H^0(\mathcal{E}_{\mathrm{P}_s}[\mathfrak{g}_{-2,s}] \otimes K)$ and $\phi_1 \in H^0(\mathcal{E}_{\mathrm{P}_s}[\mathfrak{g}_{2,s}] \otimes K)$. Hence $\psi_1 = [f, \phi_1] \in H^0(\mathcal{E}_{\mathrm{P}_s}[\mathfrak{h}_s] \otimes K^2)$. We have $\deg(\mathcal{E}_{\mathrm{P}_s}) \geqslant \langle \frac{h}{2}, s\rangle$ by Lemma 5.5 and $\langle \frac{h}{2}, s\rangle \geqslant 0$ by Lemma 5.6. Thus, $\deg(\mathcal{E}_{\mathrm{P}_s}) \geqslant \langle \frac{h}{2}, s\rangle \geqslant 0$.

If $\deg(\mathcal{E}_{\mathrm{P}_s}) = 0$, then $f \in \mathfrak{g}^0_{-2,s}$ and there is a holomorphic reduction $\mathcal{E}_{\mathrm{L}_s} \subset \mathcal{E}_{\mathrm{P}_s}$ such that $\psi_1 = [f, \phi_1] \in H^0(\mathcal{E}_{\mathrm{L}_s}[\mathfrak{h}^0_s] \otimes K^2)$. Note that $[s, \phi_1] = 0$ since $\mathrm{ad}_f : \mathfrak{g}_2 \to \mathfrak{g}_0$ is injective and

$$0 = [s, [f, \phi_1]] = -[\phi_1, [s, f]] - [f, [\phi_1, s]] = [f, [s, \phi_1]].$$

Hence $f + \phi_1 \in H^0(\mathcal{E}_{\mathrm{L}_s}[\mathfrak{m}^0_s] \otimes K)$ and $\widehat{\Psi}_e(\mathcal{E}_\mathrm{C}, \psi_1)$ is a polystable $\mathrm{G}^\mathbb{R}$-Higgs bundle. □

Finally, let $\{f, h, e\} \subset \mathfrak{g}$ be a magical $\mathfrak{sl}_2$-triple from Case (4) of Theorem 3.1. Recall from §4.4 that $m_c = 3$, $\tilde{\phi}_3 = \phi_3$ and the $\mathbb{Z}$-grading is given by $\mathfrak{g} = \bigoplus_{j=-5}^5 \mathfrak{g}_{2j}$. Moreover, $\mathfrak{g}_{-2}$ decomposes $\mathfrak{g}_0$-invariantly as $\mathfrak{g}_{-2} = \mathfrak{g}_{-\tilde{\alpha}} \oplus \mathfrak{g}^b_{-2}$, where $\tilde{\alpha}$ is the simple root in the diagrams in §9. Consider the $\mathbb{Z}/4\mathbb{Z}$-grading given by $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}/4\mathbb{Z}} \hat{\mathfrak{g}}_j$, where

$$\hat{\mathfrak{g}}_0 = \mathfrak{g}_{-8} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_8\ , \qquad \hat{\mathfrak{g}}_1 = \mathfrak{g}_{-10} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_6$$
$$\hat{\mathfrak{g}}_2 = \mathfrak{g}_{-4} \oplus \mathfrak{g}_4, \qquad \hat{\mathfrak{g}}_3 = \mathfrak{g}_{-6} \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_{10}\ .$$

By (4.3), the complexified Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ of the canonical real form satisfies $\mathfrak{h} = \hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_2$ and $\mathfrak{m} = \hat{\mathfrak{g}}_1 \oplus \hat{\mathfrak{g}}_3$. Recall from (4.8) that $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{sl}_2\mathbb{C}$, and note that $\hat{\mathfrak{g}}_0 = \mathfrak{h}'_0 \oplus \mathfrak{sl}_2\mathbb{C}$. Let $\mathrm{G}_0 \subset \hat{\mathrm{G}}_0 \subset \mathrm{G}$ be the connected subgroups with Lie algebras $\mathfrak{g}_0 \subset \hat{\mathfrak{g}}_0$ respectively. The adjoint action of G_0 and $\hat{\mathrm{G}}_0$ preserve the spaces $\mathfrak{g}_j$ and $\hat{\mathfrak{g}}_j$ respectively. Moreover, by Lemma 4.19, $\hat{\mathfrak{g}}_1$ decomposes $\hat{\mathrm{G}}_0$-invariantly as

$$\hat{\mathfrak{g}}_1 = (\mathfrak{g}_{-\tilde{\alpha}} \oplus \mathfrak{g}_{-10}) \oplus (\mathfrak{g}^b_{-2} \oplus \mathfrak{g}_6). \tag{7.2}$$

Consider the K^4-twisted $\mathrm{G}^\mathbb{R}_{0,ss}$-Higgs bundle $(\mathcal{E}_\mathrm{C}, \psi_3)$, and recall

$$\hat{\Psi}_e((\mathcal{E}_\mathrm{C}, \psi_3), 0, 0) = (\mathcal{E}_\mathrm{C} \star \mathcal{E}_\mathrm{T}[\mathrm{H}], f + \phi_3),$$

where $\mathrm{ad}^3_f(\phi_3) = \psi_3$. Since $\mathrm{C} \star \mathrm{T} \subset \mathrm{G}_0$ and $f + \phi_3 \in H^0(\mathcal{E}_\mathrm{C} \star \mathcal{E}_\mathrm{T}[\hat{\mathfrak{g}}_1] \otimes K)$,

$$(\mathcal{E}_{\hat{\mathrm{G}}_0}, \Phi) = (\mathcal{E}_\mathrm{C} \star \mathcal{E}_\mathrm{T}[\hat{\mathrm{G}}_0], f + \phi_3)$$

is a K-twisted $(\hat{\mathrm{G}}_0, \hat{\mathfrak{g}}_1)$-Higgs pair. Using the decomposition (7.2) we write

$$\Phi = (f_b + \phi_3) \oplus (\tilde{f} + 0) \in H^0(\mathcal{E}_{\hat{\mathrm{G}}_0}[\mathfrak{g}^b_{-2} \oplus \mathfrak{g}_6] \otimes K) \oplus H^0(\mathcal{E}_{\hat{\mathrm{G}}_0}[\mathfrak{g}_{-\tilde{\alpha}} \oplus \mathfrak{g}_{-10}] \otimes K).$$

This, implies that $\mathrm{ad}_{f_b + \phi_3}(\tilde{f}) \in H^0(\mathcal{E}_{\hat{\mathrm{G}}_0}[\hat{\mathfrak{g}}_2] \otimes K^2)$. Recall from (4.11), that $\mathrm{ad}_{f_b+\phi_3}(\tilde{f}) = [f_b, \tilde{f}] \in \mathfrak{g}_{-4}$ is a magical nilpotent in $\mathfrak{h}'$ from Case (2) of Theorem 3.1. Since the splitting $\hat{\mathfrak{g}}_2 = \mathfrak{g}_{-4} \oplus \mathfrak{g}_4$ is $\hat{\mathrm{G}}_0$-invariant,

$$\mathrm{ad}_{f_b + \phi_3} \tilde{f} = [f_b, \tilde{f}] \in H^0(\mathcal{E}_{\hat{\mathrm{G}}_0}[\mathfrak{g}_{-4}] \otimes K^2). \tag{7.3}$$

Also, $\theta = \mathrm{ad}^3_{f_b+\phi_3}(\tilde{f}) \in H^0(\mathcal{E}_{\hat{\mathrm{G}}_0}[\hat{\mathfrak{g}}_0] \otimes K^4)$. Thus, $(\mathcal{E}_{\hat{\mathrm{G}}_0}, \theta)$ is a K^4-twisted $\hat{\mathrm{G}}_0$-Higgs bundle. Moreover, the decomposition $\hat{\mathfrak{g}}_0 = \mathfrak{h}_0' \oplus \mathfrak{sl}_2\mathbb{C}$ gives

$$\theta = \theta' \oplus \theta_2 \in H^0(\mathcal{E}_{\hat{\mathrm{G}}_0}[\mathfrak{h}_0'] \otimes K^4) \oplus H^0(\mathcal{E}_{\hat{\mathrm{G}}_0}[\mathfrak{sl}_2\mathbb{C}] \otimes K^4)\ .$$

The bracket relations of Lemma 4.20 now imply

$$\theta' = 3\psi_3 \qquad \text{and} \qquad \theta_2 = \mathrm{ad}^3_{f_b}(\tilde{f}) + \mathrm{ad}^2_{\phi_3} \circ \mathrm{ad}_{f_b}(\tilde{f}). \tag{7.4}$$

In particular, θ_2 is in the K^4-twisted $\mathrm{SL}_2\mathbb{C}$-Hitchin section.

Proof of Lemma 7.9 Case (4). Suppose $(\mathcal{E}_{\mathrm{C}}, \psi_3)$ is a polystable K^4-twisted $\mathrm{G}^{\mathbb{R}}_{0,ss}$-Higgs bundle. To show that $\hat{\Psi}_e((\mathcal{E}_{\mathrm{C}}, \psi_3), 0, 0)$ is a polystable $\mathrm{G}^{\mathbb{R}}$-Higgs bundle, it suffices to show that $(\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}}(\hat{\mathrm{G}}_0), f + \phi_3)$ is a polystable $(\hat{\mathrm{G}}_0, \hat{\mathfrak{g}}_1)$-Higgs pair by Proposition 6.15.

Consider the $(\hat{\mathrm{G}}_0, \hat{\mathfrak{g}}_1)$-Higgs pair $(\mathcal{E}_{\hat{\mathrm{G}}_0}, \Phi) = (\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}}(\hat{\mathrm{G}}_0), f + \phi_3)$. Let $\hat{\mathrm{H}}_0^{\mathbb{R}} \subset \hat{\mathrm{G}}_0$ be a compact real form with Lie algebra $\hat{\mathfrak{h}}_0^{\mathbb{R}}$. Fix $s \in i\hat{\mathfrak{h}}_0^{\mathbb{R}}$ and let $\mathrm{P}_s \subset \hat{\mathrm{G}}_0$ be the corresponding parabolic. Since $\hat{\mathfrak{g}}_0 = \mathfrak{h}_0' \oplus \mathfrak{sl}_2\mathbb{C}$ we can write $s = s' + s_2$, where $s' \in \mathfrak{h}_0'$ and $s_2 \in \mathfrak{sl}_2\mathbb{C}$. Let $\mathcal{E}_{\mathrm{P}_s} \subset \mathcal{E}_{\hat{\mathrm{G}}_0}$ be a holomorphic reduction such that $\Phi \in H^0(\mathcal{E}_{\mathrm{P}_s}[\hat{\mathfrak{g}}_{1,s}] \otimes K)$. Note that the inclusions $\mathrm{P}_s \subset \mathrm{P}_{s'}$ and $\mathrm{P}_s \subset \mathrm{P}_{s_2}$ define holomorphic reductions $\mathcal{E}_{\mathrm{P}_s} \subset \mathcal{E}_{\mathrm{P}_{s'}} \subset \mathcal{E}_{\hat{\mathrm{G}}_0}$ and $\mathcal{E}_{\mathrm{P}_s} \subset \mathcal{E}_{\mathrm{P}_{s_2}} \subset \mathcal{E}_{\hat{\mathrm{G}}_0}$ We are interested in showing

$$\deg(\mathcal{E}_{\mathrm{P}_s}) = \deg(\mathcal{E}_{\mathrm{P}_{s'}}) + \deg(\mathcal{E}_{\mathrm{P}_{s_2}}) \geqslant 0.$$

Since $\hat{\mathrm{G}}_0$ preserves the splitting (7.2), we have

$$f_b + \phi_3 \in H^0(\mathcal{E}_{\mathrm{P}_s}[\hat{\mathfrak{g}}_{1,s}] \otimes K) \qquad \text{and} \qquad \tilde{f} \in H^0(\mathcal{E}_{\mathrm{P}_s}[\hat{\mathfrak{g}}_{1,s}] \otimes K)\ . \tag{7.5}$$

Thus, $\mathrm{ad}^3_{f_b+\phi_3}(\tilde{f}) = \theta \in H^0(\mathcal{E}_{\hat{\mathrm{G}}_0}[\hat{\mathfrak{g}}_{0,s}] \otimes K^4)$, and, using the decomposition (7.4),

$$\theta' = 3\psi_3 \in H^0(\mathcal{E}_{\hat{\mathrm{G}}_0}[\mathfrak{h}'_{0,s'}] \otimes K^4) \qquad \text{and} \qquad \theta_2 \in H^0(\mathcal{E}_{\hat{\mathrm{G}}_0}[\mathfrak{sl}_2\mathbb{C}_{s_2}] \otimes K^4).$$

Since θ_2 is in the K^4-twisted Hitchin section, we have

$$\deg(\mathcal{E}_{\mathrm{P}_{s_2}}) \geqslant 0$$

with equality if and only if $s_2 = 0$.

To show that $\deg(\mathcal{E}_{\mathrm{P}_{s'}}) \geqslant 0$, we use an argument similar to the proof of Case (2) of Theorem 3.1. Write $h = h' + h_2$, where $h' \in \mathfrak{h}_0'$ and $h_2 \in \mathfrak{sl}_2\mathbb{C}$ are both nonzero, and let $\mathrm{T}', \mathrm{T}_2 \subset \mathrm{H}$ be the subgroups generated by $\exp(th')$ and $\exp(th_2)$. The $\hat{\mathrm{G}}_0$-bundle $\mathcal{E}_{\mathrm{T}}(\hat{\mathrm{G}}_0)$ is given by

$$\mathcal{E}_{\mathrm{T}}(\hat{\mathrm{G}}_0) = \mathcal{E}_{\mathrm{T}'} \star \mathcal{E}_{\mathrm{T}_2}(\hat{\mathrm{G}}_0).$$

Fix the Kähler form ω associated to the hyperbolic metric uniformizing the Riemann surface X, so that $F_K = -i\omega$.

Since θ_2 is in the $\mathrm{PSL}_2\mathbb{C}$-Hitchin section, there is a metric h_{T_2} on $\mathcal{E}_{\mathrm{T}_2}$ so that

$$F_{h_{\mathrm{T}_2}} + [\theta_2, -\tau(\theta_2)]\omega = 0.$$

Let $h_{\mathrm{T}'}$ be the uniformizing metric on $\mathcal{E}_{\mathrm{T}'}$ and take $h_{\mathrm{T}} = h_{\mathrm{T}'} \star h_{\mathrm{T}_2}$. Since $(\mathcal{E}_{\mathrm{C}}, 3\psi_3)$ is polystable, there is a metric h_{C} on $\mathcal{E}_{\mathrm{C}}$ such that

$$F_{h_{\mathrm{C}}} + [3\psi_3, -\tau(3\psi)]\omega = 0.$$

Thus, $h_{\mathrm{C}} \star h_{\mathrm{T}_2} \star h_{\mathrm{T}'}$ defines a metric on $\mathcal{E}_{\hat{\mathrm{G}}_0}$ which satisfies

$$F_{h_{\mathrm{C}} \star h_{\mathrm{T}_2} \star h_{\mathrm{T}'}} + [3\psi_3, -\tau(3\psi_3)]\omega + [\theta_2, -\tau(\theta_2)]\omega = F_{h_{\mathrm{T}'}} = -i\lambda\omega h'$$

for some positive constant λ. The exact value of λ is not important. Thus, $(\mathcal{E}_{\hat{\mathrm{G}}_0}, 3\psi_3+\theta_2)$ is an $\alpha=\lambda h'$-polystable K^4-twisted $\hat{\mathrm{G}}_0$-Higgs bundle, and hence

$$\deg(\mathcal{E}_{\mathrm{P}_s}) = \deg(\mathcal{E}_{\mathrm{P}_{s'}}) + \deg(\mathcal{E}_{\mathrm{P}_{s_2}}) \geqslant \deg(\mathcal{E}_{\mathrm{P}'_s}) \geqslant \langle \lambda h', s'\rangle.$$

Note that $\mathrm{ad}_{f_b+\phi_3}(\tilde{f}) = [f_b, \tilde{f}] \in H^0(\mathcal{E}_{\mathrm{P}_{s'}}[\mathfrak{g}_{-4,s'}]\otimes K^2)$ by (7.5) and (7.3). Since $[f_b,\tilde{f}]\subset \mathfrak{g}_{-4}$ is a magical nilpotent in $\mathfrak{h}'$ corresponding to Case (2) of Theorem 3.1, the finite dimensional GIT result [7, Lemma 5.5] applies and gives $\langle \lambda h', s'\rangle \geqslant 0$ with equality if and only if $[f_b,\tilde{f}]\in \mathfrak{g}^0_{-4,s'}$. Thus, $\deg(\mathcal{E}_{\mathrm{P}_s})\geqslant 0$.

So far we have shown that $(\mathcal{E}_{\hat{\mathrm{G}}_0}, \Phi)$ is a semistable $(\hat{\mathrm{G}}_0, \hat{\mathfrak{g}}_1)$-Higgs pair. Suppose $\deg(\mathcal{E}_{\mathrm{P}_s}) = 0$, then $\deg(\mathcal{E}_{\mathrm{P}_{s_2}}) = 0$ and $\deg(\mathcal{E}_{\mathrm{P}_{s'}}) = 0$, and hence $s_2 = 0$. The α-polystable of the K^4-twisted $\hat{\mathrm{G}}_0$-Higgs bundle $(\mathcal{E}_{\hat{\mathrm{G}}_0}, \theta)$ implies there is a holomorphic reduction $\mathcal{E}_{\mathrm{L}_s}\subset \mathcal{E}_{\mathrm{P}_s}$ such that $\theta \in H^0(\mathcal{E}_{\mathrm{L}_s}[\hat{\mathfrak{g}}^0_{0,s}]\otimes K^4)$. In particular, $\psi_3 \in H^0(\mathcal{E}_{\mathrm{L}_s}[\hat{\mathfrak{g}}^0_{0,s}]\otimes K^4)$. Since the splitting $\mathfrak{g}_{-\tilde{\alpha}}\oplus \mathfrak{g}^b_{-2}\oplus \mathfrak{g}_6$ is $\mathfrak{h}'_0$-invariant and $s=s'\in\mathfrak{h}'_0$, we have

$$\tilde{f}\in H^0(\mathcal{E}_{\mathrm{P}_s}[\hat{\mathfrak{g}}_{1,s}]\otimes K)\ , \qquad f_b\in H^0(\mathcal{E}_{\mathrm{P}_s}[\hat{\mathfrak{g}}_{1,s}]\otimes K) \qquad \text{and} \qquad \phi_3\in H^0(\mathcal{E}_{\mathrm{P}_s}[\hat{\mathfrak{g}}_{1,s}]\otimes K).$$

Thus,

$$[f_b,\tilde{f}]\in H^0(\mathcal{E}_{\mathrm{P}_s}[\mathfrak{g}_{-4,s}]\otimes K^2) \qquad \text{and} \qquad [f_b,\phi_3]\in H^0(\mathcal{E}_{\mathrm{P}_s}[\mathfrak{g}_{4,s}]\otimes K^2).$$

We have $0=\deg(\mathcal{E}_{\mathrm{P}_s})\geqslant \langle s, h'\rangle \geqslant 0$, thus the finite dimensional GIT lemma implies $[f_b,\tilde{f}]\in H^0(\mathcal{E}_{\mathrm{L}_s}[\mathfrak{g}^0_{-4,s}]\otimes K^2)$. By (4.12), $\psi_3 = [[f_b,\tilde{f}],[f_b,\phi_3]]$, and hence $[f_b,\phi_3]\in H^0(\mathcal{E}_{\mathrm{L}_s}[\mathfrak{g}^0_{4,s}]\otimes K^2)$. Finally, since $\tilde{f}$, f_b, and ϕ_3 are each in $H^0(\mathcal{E}_{\mathrm{P}_s}[\hat{\mathfrak{g}}_{1,s}]\otimes K)$, we have

$$\tilde{f}\in H^0(\mathcal{E}_{\mathrm{L}_s}[\hat{\mathfrak{g}}^0_{1,s}]\otimes K)\ , \qquad f_b\in H^0(\mathcal{E}_{\mathrm{L}_s}[\hat{\mathfrak{g}}^0_{1,s}]\otimes K) \qquad \text{and} \qquad \phi_3\in H^0(\mathcal{E}_{\mathrm{L}_s}[\hat{\mathfrak{g}}^0_{1,s}]\otimes K).$$

Hence $(\mathcal{E}_{\mathrm{C}}\star\mathcal{E}_{\mathrm{T}}[\hat{\mathrm{G}}_0], f+\phi_3)$ is a 0-polystable K-twisted $(\hat{\mathrm{G}}_0,\hat{\mathfrak{g}}_1)$-Higgs pair. □

7.3. **The Cayley map is open.** We now prove that the Cayley map is open. Recall the deformation complex and description of the local structure of the moduli space from §6.2. By Corollary 3.2, $\ker(\mathrm{ad}_f:\mathfrak{h}\to\mathfrak{m}) = \mathfrak{c}$ and $\mathrm{ad}_f:\mathrm{ad}_f(\mathfrak{m})\to\mathrm{ad}^2_f(\mathfrak{m})$ is an isomorphism. Hence we have $\mathrm{C}\star\mathrm{T}$-invariant splittings

$$\mathfrak{h} = \mathfrak{c}\oplus \mathrm{ad}_f(\mathfrak{m}) \qquad \text{and} \qquad \mathfrak{m} = V_{\mathfrak{m}}\oplus \mathrm{ad}^2_f(\mathfrak{m})\ , \tag{7.6}$$

where $V_{\mathfrak{m}} = \bigoplus_{j>0} V_{2m_j}$ is the set of highest weight spaces in $\mathfrak{m}$.

Recall that the Cayley map (7.1) is defined by

$$\Psi_e(\mathcal{E}_{\mathrm{C}},\psi) = ((\mathcal{E}_{\mathrm{C}}\star\mathcal{E}_{\mathrm{T}})[\mathrm{H}], f+\varphi),$$

where $\psi\in\bigoplus_{j>0} H^0(\mathcal{E}_{\mathrm{C}}[Z_{2m_j}]\otimes K^{m_j+1})$ and $\varphi\in H^0(\mathcal{E}_{\mathrm{C}}\star\mathcal{E}_{\mathrm{T}}[V_{\mathfrak{m}}]\otimes K)$ is determined by ψ using the isomorphisms $\mathrm{ad}^{m_j}_f : (\mathcal{E}_{\mathrm{C}}\star\mathcal{E}_{\mathrm{T}})[V_{2m_j}]\otimes K\to \mathcal{E}_{\mathrm{C}}[Z_{2m_j}]\otimes K^{m_j+1}$. The deformation complex for $(\mathcal{E}_{\mathrm{C}},\psi)$ is

$$C^\bullet_{\mathcal{C}} : \mathcal{E}_{\mathrm{C}}[\mathfrak{c}] \xrightarrow{\mathrm{ad}_\psi} \bigoplus_{j>0}\mathcal{E}_{\mathrm{C}}[Z_{2m_j}]\otimes K^{m_j+1}.$$

On the other hand, since $[\mathfrak{c}, f+V_{\mathfrak{m}}]\subset V_{\mathfrak{m}}$, the deformation complex for $\Psi_e(\mathcal{E}_{\mathrm{C}},\psi)$ is

$$C^\bullet_{\mathcal{H}} : \mathcal{E}_{\mathrm{C}}[\mathfrak{c}]\oplus(\mathcal{E}_{\mathrm{T}}\star\mathcal{E}_{\mathrm{C}})[\mathrm{ad}_f(\mathfrak{m})] \xrightarrow{\begin{pmatrix}\mathrm{ad}_\varphi & \alpha\\ 0 & \beta\end{pmatrix}} (\mathcal{E}_{\mathrm{T}}\star\mathcal{E}_{\mathrm{C}})[V_{\mathfrak{m}}]\otimes K\oplus(\mathcal{E}_{\mathrm{T}}\star\mathcal{E}_{\mathrm{C}})[\mathrm{ad}^2_f(\mathfrak{m})]\otimes K,$$

where we've used the fact that T acts trivially on $\mathfrak{c}$ to identify $(\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}})[\mathfrak{c}] \cong \mathcal{E}_{\mathrm{C}}[\mathfrak{c}]$, and α and β are defined by post composing $\mathrm{ad}_{f+\varphi} : (\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}})[\mathrm{ad}_f] \to (\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}})[\mathfrak{m}] \otimes K$ with the projection onto the $(\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}})[V_{\mathfrak{m}}] \otimes K$ and $(\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}})[\mathrm{ad}_f^2(\mathfrak{m})] \otimes K$, respectively.

The Cayley map induces a short exact sequence of complexes

$$0 \to C^\bullet_{\mathcal{C}} \to C^\bullet_{\mathcal{H}} \to C^\bullet_{\mathcal{H}}/C^\bullet_{\mathcal{C}} \to 0,$$

such that the quotient complex is isomorphic to

$$C^\bullet_{\mathcal{H}}/C^\bullet_{\mathcal{C}} : (\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}})[\mathrm{ad}_f(\mathfrak{m})] \xrightarrow{\beta} (\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}})[\mathrm{ad}_f^2(\mathfrak{m})] \otimes K.$$

Proposition 7.10. *The quotient complex $C^\bullet_{\mathcal{H}}/C^\bullet_{\mathcal{C}}$ has trivial hypercohomology. In particular,*

$$\mathbb{H}^\bullet(C^\bullet_{\mathcal{C}}) \cong \mathbb{H}^\bullet(C^\bullet_{\mathcal{H}}) \ .$$

Proof. It suffices to show that the map $\beta : (\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}})[\mathrm{ad}_f(\mathfrak{m})] \to (\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}})[\mathrm{ad}_f^2(\mathfrak{m})] \otimes K$ is an isomorphism. First, $\mathrm{ad}_f : (\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}})[\mathrm{ad}_f(\mathfrak{m})] \to (\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}})[\mathrm{ad}_f^2(\mathfrak{m})] \otimes K$ induces an isomorphism of holomorphic bundles. Since $v \in V_{\mathfrak{m}} \subset \bigoplus_{j>0} \mathfrak{g}_j$, for any $v \in V_{\mathfrak{m}}$, the composition of $\mathrm{ad}_{f+v} : \mathrm{ad}_f(\mathfrak{m}) \to \mathfrak{m}$ with projection onto $\mathrm{ad}_f^2(\mathfrak{m})$ is injective and hence also defines an isomorphism $\mathrm{ad}_f(\mathfrak{m}) \to \mathrm{ad}_f^2(\mathfrak{m})$. Thus, β is an isomorphism and $C^\bullet_{\mathcal{H}}/C^\bullet_{\mathcal{C}}$ has trivial hypercohomology. □

We can now prove the second hypercohomology of the complexes $C^\bullet_{\mathcal{C}}$ and $C^\bullet_{\mathcal{H}}$ vanishes.

Proposition 7.11. *Suppose $(\mathcal{E}_{\mathrm{C}}, \psi)$ is a polystable object in the domain of Ψ_e. Then*

$$0 = \mathbb{H}^2(C^\bullet_{\mathcal{C}}(\mathcal{E}_{\mathrm{C}}, \psi)) = \mathbb{H}^2(C^\bullet_{\mathcal{H}}(\Psi_e(\mathcal{E}_{\mathrm{C}}, \psi))) = 0.$$

Proof. Since the domain of the Cayley map is identified with a product of moduli spaces of L-twisted Higgs bundles with $\deg(L) > 2g - 2$, Proposition 6.8 implies $\mathbb{H}^2(C^\bullet_{\mathcal{C}}(\mathcal{E}_{\mathrm{C}}, \psi)) = 0$. Now, Proposition 7.10 implies $\mathbb{H}^2(C^\bullet_{\mathcal{H}}(\Psi_e(\mathcal{E}_{\mathrm{C}}, \psi))) = 0$. □

Remark 7.12. Note that isomorphism of hypercohomology groups and vanishing of $\mathbb{H}^2$ in this general context is much cleaner than the one in [3, §4.2] for $\mathrm{G}^{\mathbb{R}} = \mathrm{SO}_{p,q}$, which took several pages. This is a reflect of the power of the magical $\mathfrak{sl}_2$-triple perspective.

We can now prove the Cayley map is open.

Proposition 7.13. *The Cayley map $\Psi_e : \mathcal{M}_{K^{m_c+1}}(\mathrm{G}^{\mathbb{R}}_{0,ss}) \times \prod_{j=1}^{\mathrm{rk}(\mathfrak{g}(e))} H^0(K^{l_j+1}) \to \mathcal{M}(\mathrm{G}^{\mathbb{R}})$ is open. In particular, its image is open in $\mathcal{M}(\mathrm{G}^{\mathbb{R}})$.*

Proof. Let $(\mathcal{E}_{\mathrm{C}}, \psi)$ be a point in the domain of the Ψ_e. By Proposition 6.8, local neighborhoods of $(\mathcal{E}_{\mathrm{C}}, \psi)$ and $\Psi_e(\mathcal{E}_{\mathrm{C}}, \psi)$ are respectively isomorphic to

$$\mathbb{H}^1(C^\bullet_{\mathcal{C}}(\mathcal{E}_{\mathrm{C}}, \psi)) /\!\!/ \mathrm{Aut}(\mathcal{E}_{\mathrm{C}}, \psi) \qquad \text{and} \qquad \mathbb{H}^1(C^\bullet_{\mathcal{H}}(\Psi_e(\mathcal{E}_{\mathrm{C}}, \psi))) /\!\!/ \mathrm{Aut}(\Psi_e(\mathcal{E}_{\mathrm{C}}, \psi)).$$

By Proposition 7.10, Ψ_e induces an isomorphism $\mathbb{H}^1(C^\bullet_{\mathcal{C}}(\mathcal{E}_{\mathrm{C}}, \psi)) \cong \mathbb{H}^1(C^\bullet_{\mathcal{H}}(\Psi_e(\mathcal{E}_{\mathrm{C}}, \psi)))$ which is $\mathrm{Aut}(\mathcal{E}_{\mathrm{C}}, \psi)$-equivariant. By Corollary 5.12 we have $\mathrm{Aut}(\mathcal{E}_{\mathrm{C}}, \psi) = \mathrm{Aut}(\Psi_e(\mathcal{E}_{\mathrm{C}}, \psi))$. Thus, the Cayley map induces a local isomorphism and hence is open. □

7.4. **The Cayley map is closed.** Recall from Remark 2.3, that the Slodowy slice $f + \ker(\mathrm{ad}_e) = f + V \subset \mathfrak{g}$ is a slice for the adjoint action of G. We have an Ad_{H} invariant decomposition $V = \mathfrak{c} \oplus V_{\mathfrak{m}}$, and $f + V_{\mathfrak{m}}$ is a slice through f for the H-action in $\mathfrak{m}$. Moreover, $f + V_{\mathfrak{m}}$ decomposes Ad_{C}-invariantly as

$$f + V_{\mathfrak{m}} = f + \bigoplus_{j=1}^{M} V_{2m_j}, \tag{7.7}$$

where C-acts trivially on every summand except V_{2m_c}. Recall that the Cayley real from $\mathfrak{g}^{\mathbb{R}}_{\mathrm{C}}$ is a real form of $\mathfrak{g}_0$ and has complexified Cartan decomposition $\mathfrak{g}_0 = \mathfrak{c} \oplus Z_{\mathfrak{m}}$, where[3] $Z_{\mathfrak{m}} = \bigoplus_{j=1}^{M} Z_{2m_j}$. There is a C-equivariant isomorphism $\psi_e : Z_{\mathfrak{m}} \to f + V_{\mathfrak{m}}$ induced by the C-equivariant isomorphisms $\mathrm{ad}_f^{m_j} : V_{2m_j} \to Z_{2m_j}$.

Let $\chi : \mathfrak{m} \to \mathfrak{m} /\!\!/ \mathrm{H}$ and $\chi_{\mathcal{C}} : Z_{\mathfrak{m}} \to Z_{\mathfrak{m}} /\!\!/ \mathrm{C}$ be the adjoint quotient maps, and let $\chi_e : f + V_{\mathfrak{m}} \to \mathfrak{m} /\!\!/ \mathrm{H}$ be the restriction of χ to $f + V_{\mathfrak{m}}$. The composition $\chi_e \circ \psi_e : Z_{\mathfrak{m}} \to \mathfrak{m} /\!\!/ \mathrm{H}$ defines a map

$$\gamma_e : Z_{\mathfrak{m}} /\!\!/ \mathrm{C} \to \mathfrak{m} /\!\!/ \mathrm{H} \tag{7.8}$$

such that

$$\gamma_e \circ \chi_{\mathcal{C}} = \chi_e \circ \psi_e. \tag{7.9}$$

Recall that, by choosing a homogeneous basis of invariant polynomials, $\mathfrak{m} /\!\!/ \mathrm{H}$ and $Z_{\mathfrak{m}} /\!\!/ \mathrm{C}$ are identified with affine spaces of dimension the real rank of $\mathfrak{g}^{\mathbb{R}}$ and $\mathfrak{g}^{\mathbb{R}}_{\mathcal{C}}$ respectively. Thus, by Proposition 4.10, $\mathfrak{m} /\!\!/ \mathrm{H}$ and $Z_{\mathfrak{m}} /\!\!/ \mathrm{C}$ have the same dimension.

Proposition 7.14. *Let $\{f, h, e\} \subset \mathfrak{g}$ be a magical $\mathfrak{sl}_2$-triple. Then $\chi_e : f + V_{\mathfrak{m}} \to \mathfrak{m} /\!\!/ \mathrm{H}$ and $\gamma_e : Z_{\mathfrak{m}} /\!\!/ \mathrm{C} \to \mathfrak{m} /\!\!/ \mathrm{H}$ are flat and surjective, thus faithfully flat. Moreover, γ_e has finite fibers.*

Proof. By [54, Theorem 9], every fiber of the surjective morphism $\chi : \mathfrak{m} \to \mathfrak{m} /\!\!/ \mathrm{H}$ has pure dimension equal to $\dim(\mathfrak{m}) - \dim(\mathfrak{m} /\!\!/ \mathrm{H})$. Since both $\mathfrak{m}$ and $\mathfrak{m} /\!\!/ \mathrm{H}$ are affine spaces, the so called "miracle flatness theorem" implies that χ_e is flat; see, for example [45, Exercise III.10.9] or [26, pg. 158].

On the other hand, the orbit map $\mu : \mathrm{H} \times S_{\mathfrak{m}} \to \mathfrak{m}$ is smooth, and hence flat since $f + V_{\mathfrak{m}}$ is a slice for the H-action on $\mathfrak{m}$. Thus, $\chi_e \circ \mu : \mathrm{H} \times (f + V_{\mathfrak{m}}) \to \mathfrak{m} /\!\!/ \mathrm{H}$ is also flat. However, this morphism factors through $f + V_{\mathfrak{m}}$, so that we have a commutative diagram

$$\begin{array}{ccccc} \mathrm{H} \times S_{\mathfrak{m}} & \xrightarrow{\mu} & \mathfrak{m} & \xrightarrow{\chi} & \mathfrak{m} /\!\!/ \mathrm{H} \\ & \searrow^{\mathrm{pr}_2} & \uparrow & \nearrow_{\chi_e} & \\ & & S_{\mathfrak{m}} & & \end{array}$$

where pr_2 is the canonical projection. Since both $\chi \circ \mu$ and pr_2 are flat, the morphism $\chi_e : S_{\mathfrak{m}} \to \mathfrak{m} /\!\!/ \mathrm{H}$ is flat by [38, Corollary 2.2.11].

As in [69, §7.4], to show that χ_e is surjective, we show that it is equivariant with respect to a $\mathbb{C}^*$-action with positive weights. Choose a basis $(p_1, \ldots, p_r)$ of H-invariant polynomials on $\mathfrak{m}$ which are homogeneous of degree $m_1', \cdots m_r'$. This identifies $\mathfrak{m} /\!\!/ \mathrm{H}$ with $\mathbb{C}^r$, via $[y] \mapsto (p_1(y), \ldots, p_r(y))$ for $y \in \mathfrak{m}$. We have

$$\chi_e(t^2 y) = (t^{2m_1'} p_1(y), \ldots, t^{2m_r'} p_r(y)). \tag{7.10}$$

[3] Note that $Z_{\mathfrak{m}}$ is not a subset of $\mathfrak{m}$.

Now consider the $\mathbb{C}^*$-action on $f + V_{\mathfrak{m}}$ by

$$t \cdot \left(f + \sum_{j=1}^{M} v_{2m_j} \right) = f + \sum_{j=1}^{M} t^{2+2m_j} v_{2m_j}, \tag{7.11}$$

where $v_{2m_j} \in V_{2m_j}$. There is an element $g \in \mathrm{T} \subset \mathrm{H}$ so that

$$\mathrm{Ad}_g \left(t^2 f + \sum_{j=1}^{M} t^2 v_{2m_j} \right) = f + \sum_{j=1}^{M} t^{2m_j+2} v_{2m_j} = t \cdot \left(f + \sum_{j=1}^{M} v_{2m_j} \right).$$

Since the polynomials p_j are H-invariant, the map $\chi_e : f + V_{\mathfrak{m}} \to \mathfrak{m} /\!\!/ \mathrm{H}$ is equivariant with respect to the $\mathbb{C}^*$-actions (7.11) and (7.10).

Now, flatness implies $\chi_e : f + V_{\mathfrak{m}} \to \mathfrak{m} /\!\!/ \mathrm{H}$ is open, so its image is an open set $U \subset \mathfrak{m} /\!\!/ \mathrm{H}$ containing $0 = \chi_e(f)$. By $\mathbb{C}^*$-equivariance, it follows that U must be $\mathbb{C}^*$-invariant. Since the weights of the $\mathbb{C}^*$-action are positive, we conclude that $U = \mathfrak{m} /\!\!/ \mathrm{H}$, and thus $\chi_e^{\mathfrak{m}}$ is surjective.

For the map γ_e, surjectivity follows immediately from surjectivity of χ_e. To prove flatness we use a similar argument as above. The argument for flatness of $\chi : \mathfrak{m} \to \mathfrak{m} /\!\!/ \mathrm{H}$ also applies to the $\chi_{\mathcal{C}} : Z_{\mathfrak{m}} \to Z_{\mathfrak{m}} /\!\!/ \mathrm{C}$, thus $\chi_{\mathcal{C}}$ is flat. Hence, both $\chi_e \circ \psi_e = \gamma_e \circ \chi_{\mathcal{C}}$ and $\chi_{\mathcal{C}}$ are flat. Therefore, again by [38, Corollary 2.2.11], γ_e is flat as well. Finally, a faithfully flat morphism between affine spaces of the same dimension has finite fibers. So γ_e has finite fibers since $\dim(Z_{\mathfrak{m}} /\!\!/ \mathrm{C}) = \dim(\mathfrak{m} /\!\!/ \mathrm{H})$. □

Remark 7.15. Note that the proof that $\chi_e : f + V_{\mathfrak{m}} \to \mathfrak{m} /\!\!/ \mathrm{H}$ is flat and surjective holds for general normal $\mathfrak{sl}_2$-triples $\{f, h, e\} \subset \mathfrak{h} \oplus \mathfrak{m}$.

The global version of the above picture is given by taking the Hitchin maps from §6.3 on the domain and target of the Cayley map Ψ_e defined in (7.1). Let $\mathcal{K}$ be the holomorphic frame bundle of K. The Hitchin base on the domain is

$$\mathcal{B}_{\mathrm{C}} = \mathcal{B}_{K^{m_c+1}}(\mathrm{G}^{\mathbb{R}}_{0,ss}) \times \prod_{j=1}^{\mathrm{rk}(\mathfrak{g}(e))} H^0(K^{l_j+1}) \cong \bigoplus_{j>0} H^0(\mathcal{K}^{m_j+1}[Z_{2m_j} /\!\!/ \mathrm{C}]),$$

while the Hitchin base for $\mathcal{M}(\mathrm{G}^{\mathbb{R}})$ is $\mathcal{B}(\mathrm{G}^{\mathbb{R}}) = H^0(\mathcal{K}[\mathfrak{m} /\!\!/ \mathrm{H}])$. Let $h_{\mathcal{C}}$ and h be the respective Hitchin maps. From the previous discussion, we conclude that the Cayley map Ψ_e is compatible with the Hitchin maps $h_{\mathcal{C}}$ and h in the sense of the next proposition.

Proposition 7.16. *There is a commutative diagram*

$$\begin{array}{ccc} \mathcal{M}_{K^{m_c+1}}(\mathrm{G}^{\mathbb{R}}_{0,ss}) \times \prod_{j=1}^{\mathrm{rk}(\mathfrak{g}(e))} H^0(K^{l_j+1}) & \xrightarrow{\Psi_e} & \mathcal{M}(\mathrm{G}^{\mathbb{R}}) \\ \downarrow{\scriptstyle h_{\mathcal{C}}} & & \downarrow{\scriptstyle h} \\ \mathcal{B}_{\mathcal{C}} & \xrightarrow{\Gamma_e} & \mathcal{B}(\mathrm{G}^{\mathbb{R}}) \end{array} \tag{7.12}$$

where Γ_e is a proper map.

Proof. By Proposition 7.14, the map $\gamma_e : Z_{\mathfrak{m}} /\!\!/ \mathrm{C} \to \mathfrak{m} /\!\!/ \mathrm{H}$ defines a proper map

$$\Gamma_e : \mathcal{B}_{\mathcal{C}} \longrightarrow \mathcal{B}(\mathrm{G}^{\mathbb{R}}),$$

and the commutativity of the diagram follows from (7.9). □

Remark 7.17. We expect that the map Γ_e is an isomorphism, but, for our purposes, being proper is sufficient.

We now complete the proof of Theorem 7.1 by showing the Cayley map is closed.

Proposition 7.18. *The image* $\mathrm{Im}(\Psi_e)$ *of the Cayley map* Ψ_e *is closed in* $\mathcal{M}(\mathrm{G}^{\mathbb{R}})$.

Proof. Consider a sequence $x_n = \Psi_e(y_n)$ that diverges in $\mathrm{Im}(\Psi_e)$. In particular y_n diverges in the domain of the Cayley map. Since the maps $h_{\mathcal{C}}$ and Γ_e in the diagram (7.12) are proper, we conclude that $h_{\mathcal{C}}(y_n)$ diverges in $\mathcal{B}_{\mathcal{C}}$ and $\Gamma_e(h_{\mathcal{C}}(y_n))$ diverges in the Hitchin base $\mathcal{B}(\mathrm{G}^{\mathbb{R}})$ of $\mathcal{M}(\mathrm{G}^{\mathbb{R}})$. Since the diagram (7.12) commutes and the Hitchin map h is proper, we conclude that x_n diverges in $\mathcal{M}(\mathrm{G}^{\mathbb{R}})$. Hence the image of Ψ_e is closed in $\mathcal{M}(\mathrm{G}^{\mathbb{R}})$. □

7.5. **Remarks on local minima of energy and components.** The connected components of the moduli spaces of $\mathrm{G}^{\mathbb{R}}$-Higgs bundles have been subject to an extensive study through the last three decades (see for example, [46, 48, 36, 10, 11, 60, 31, 15]). Most of the works dealt with $\mathrm{G}^{\mathbb{R}}$ in a case-by-case basis, and the main tool, pioneered by Hitchin [46, 48], to detect and count such components was the *Hitchin function* defined by taking the L^2-norm of the Higgs field. Namely, the L^2-norm of the Higgs field with respect to the metric solving the Hitchin equations (6.8) defines a proper function on the moduli space

$$F : \mathcal{M}(\mathrm{G}^{\mathbb{R}}) \to \mathbb{R} \ : \ (\mathcal{E}_{\mathrm{H}}, \varphi) \mapsto \int_X ||\varphi||^2. \tag{7.13}$$

Since proper maps attain their minimum on every closed set we have an inequality

$$|\pi_0(\mathcal{M}(\mathrm{G}^{\mathbb{R}}))| \leqslant |\pi_0(\text{local min of } F)|.$$

Remark 7.19. The strategy is then to classify local minimum of F and show that each component of the local minimum define a component of $\mathcal{M}(\mathrm{G}^{\mathbb{R}})$. There is an obvious global minimum which occurs when the Higgs field φ is identically zero. The component count of the global minimum is then given by the component count of the moduli space of polystable H-bundles. By [62], the number of such components is determined by the number of different topological types of H-bundles.

We briefly recall the local minimum criterion for stable Higgs bundles whose second hypercohomology $\mathbb{H}^2$ vanishes; see for example the Appendix of [3] for details. The local minima of F are in particular fixed points of the $\mathbb{C}^*$-action on $\mathcal{M}(\mathrm{G}^{\mathbb{R}})$. If $(\mathcal{E}_{\mathrm{H}}, \varphi)$ is a stable $\mathbb{C}^*$-fixed point with $\varphi \neq 0$, then there is a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{h}_j \oplus \mathfrak{m}_j$ and a holomorphic H_0-bundle $\mathcal{E}_{\mathrm{H}_0}$, where $\mathrm{H}_0 \subset \mathrm{H}$ is the connected with Lie algebra $\mathfrak{h}_0$, such that

$$\mathcal{E}_{\mathrm{H}_0}[\mathrm{H}] \cong \mathcal{E}_{\mathrm{H}} \qquad \text{and} \qquad \varphi \in H^0(\mathcal{E}_{\mathrm{H}_0}[\mathfrak{m}_{-1}] \otimes K).$$

As a result, for all j, the Higgs field φ defines a map

$$\mathrm{ad}_\varphi : \mathcal{E}_{\mathrm{H}_0}[\mathfrak{h}_j] \to \mathcal{E}_{\mathrm{H}_0}[\mathfrak{m}_{j-1}] \otimes K. \tag{7.14}$$

It turns out that if the stable $\mathrm{G}^{\mathbb{R}}$-Higgs bundle $(\mathcal{E}_{\mathrm{H}}, \varphi)$ is such that $\mathbb{H}^2(C^\bullet(\mathcal{E}_{\mathrm{H}}, \varphi)) = 0$, then it is a local minimum of F if and only if $\mathrm{ad}_\varphi : \mathcal{E}_{\mathrm{H}_0}[\mathfrak{h}_j] \xrightarrow{\cong} \mathcal{E}_{\mathrm{H}_0}[\mathfrak{m}_{j-1}] \otimes K$ is an isomorphism for all $j < 0$; see [12, §3.4] and [10, Remark 4.16].

Recall from Corollary 3.2 that if $\{f, h, e\} \subset \mathfrak{g}$ is a magical $\mathfrak{sl}_2$-triple, then $\mathrm{ad}_f : \mathfrak{h}_j \to \mathfrak{m}_{j-1}$ is injective for all $j < 0$. This implies the $\mathrm{G}^{\mathbb{R}}$-Higgs bundle $\Psi_e(\mathcal{E}_{\mathrm{C}}) = (\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}}[\mathrm{H}], f)$ defines a local minimum of the Hitchin function.

Proposition 7.20. *Let* $\{f, h, e\} \subset \mathfrak{g}$ *be a magical* $\mathfrak{sl}_2$*-triple and* $\mathrm{C} \subset \mathrm{H}$ *be its* H*-centralizer. Then the* $\mathrm{G}^{\mathbb{R}}$*-Higgs bundle* $(\mathcal{E}_{\mathrm{C}} \star \mathcal{E}_{\mathrm{T}}[\mathrm{H}], f)$ *is a local minimum of the Hitchin function* F.

Since the image of the Cayley map Ψ_e is a union of connected components of the moduli space $\mathcal{M}(\mathrm{G}^{\mathbb{R}})$, it is natural to ask how many components are those. Of course that number equals the number of connected components of the moduli space $\mathcal{M}_{K^{m_c+1}}(\mathrm{G}^{\mathbb{R}}_{0,ss})$. This question has been studied whenever $\mathrm{G}^{\mathbb{R}}$ is one of the classical groups corresponding to Cases (1), (2) and (3) of Theorem 3.1.

The classification of local minima of the Hitchin function (7.13) also applies to L-twisted Higgs bundles when $\deg(L) > 2g-2$, the only difference being that a metric on L-must be fixed to make sense of the L^2-norm. Moreover, all the results of [3, Appendix 1] hold for L-twisted $\mathrm{G}^{\mathbb{R}}$-Higgs bundles. To count the components in the image of the Cayley map, one first classifies the stable local minima of the L-twisted Hitchin function $F_L : \mathcal{M}_L(\mathrm{G}^{\mathbb{R}}) \to \mathbb{R}$ and then the polystable local minima. As in the K-twisted case, the crucial computation to detect the stable local minima among the $\mathbb{C}^*$-fixed points is [12, Lemma 3.11] (see also [10, Remark 4.16]). These results can be easily adapted the L-twisted. Consider the L-twisted version of (7.14),

$$\mathrm{ad}_\varphi : \mathcal{E}_{\mathrm{H}_0}[\mathfrak{h}_j] \to \mathcal{E}_{\mathrm{H}_0}[\mathfrak{m}_{j-1}] \otimes L. \tag{7.15}$$

Proposition 7.21. *If $\deg(L) > 2g-2$, a stable L-twisted $\mathrm{G}^{\mathbb{R}}$-Higgs bundle $(\mathcal{E}_{\mathrm{H}}, \varphi)$ with $\varphi \neq 0$ is a local minimum of the Hitchin function F_L if and only if (7.15) is an isomorphism for every $j < 0$.*

Recall from Proposition 6.5 that a strictly polystable $\mathrm{G}^{\mathbb{R}}$-Higgs bundle admits a Jordan–Hölder reduction to a stable $\hat{\mathrm{G}}^{\mathbb{R}}$-Higgs bundle, for a subgroup $\hat{\mathrm{G}}^{\mathbb{R}} < \mathrm{G}^{\mathbb{R}}$. Such subgroup $\hat{\mathrm{G}}^{\mathbb{R}}$ is independent of the twisting line bundle [29, §2.10]. So the identification of strictly polystable local minima of F_L is done by identifying stable local minima for F_L in $\mathcal{M}(\hat{\mathrm{G}}^{\mathbb{R}})$ and then checking if such minima still define local minima in $\mathcal{M}(\mathrm{G}^{\mathbb{R}})$. Using Proposition 7.21 and the minima classification in the literature, we arrive at the following count of Cayley components, i.e. of connected components in the image of the Cayley map, for Case (4) of Theorem 3.1.

Proposition 7.22. *Let G be a complex simple Lie group of type F_4, E_6, or E_7 and $\mathrm{G}^{\mathbb{R}} \subset \mathrm{G}$ be the quaternionic real form. Let Ψ_e be the Cayley map from Theorem 7.1. Then,*

- $|\pi_0(\mathrm{Im}(\Psi_e))| = 3$ *for G of type F_4;*
- $|\pi_0(\mathrm{Im}(\Psi_e))| = 1$ *for G the simply connected group of type E_6;*
- $|\pi_0(\mathrm{Im}(\Psi_e))| = 3$ *for G the adjoint group of type E_6;*
- $|\pi_0(\mathrm{Im}(\Psi_e))| = 1$ *for G the simply connected group of type E_7;*
- $|\pi_0(\mathrm{Im}(\Psi_e))| = 2$ *for G the adjoint group of type E_7.*

Proof. Suppose $\mathrm{G}^{\mathbb{R}}$ is a quaternionic real form of the simply connected group of type F_4, E_6, E_7 or E_8. By Proposition 4.8, the semisimple part $\mathrm{G}^{\mathbb{R}}_{0,ss}$ of the Cayley group $\mathrm{G}^{\mathbb{R}}_{\mathcal{C}}$ is $\mathrm{SL}_3\mathbb{R}$, $\mathrm{SL}_3\mathbb{C}$ and SU^*_6 and E_6^{-26}, respectively. For F_4 and E_8, the adjoint group is simply connected but for E_6 and E_7 the adjoint groups have centers $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$, respectively, and $\mathrm{G}^{\mathbb{R}}_{0,ss}$ is $\mathrm{PSL}_3\mathbb{C}$ and PSU^*_6, respectively. The number of connected components of the image of the Cayley map Ψ_e is equal to the number of connected components of the moduli space $\mathcal{M}_{K^4}(\mathrm{G}^{\mathbb{R}}_{0,ss})$.

For $\mathrm{G}^{\mathbb{R}}_{0,ss} = \mathrm{SL}_3\mathbb{R}$, the number of connected components of $\mathcal{M}_K(\mathrm{SL}_3\mathbb{R})$ is 3. This was computed in [48] by showing the only nonzero local minima of the Hitchin function arises from Case (1) of Theorem 3.1. These methods can easily be adapted to the K^4-twisted situation and no extra local minima arise. Thus, $|\pi_0(\mathrm{Im}(\Psi_e))| = 3$ for G of type F_4. Similarly, when $\mathrm{G}^{\mathbb{R}}_{0,ss}$ is $\mathrm{SL}_3\mathbb{C}$ or $\mathrm{PSL}_3\mathbb{C}$, there are no nonzero local minima of the Hitchin function by [31] and the number of components is 1 or 3, respectively. These methods also generalize directly to the K^4-twisted situation, and give the desired component count. Finally, for $\mathrm{G}^{\mathbb{R}}_{0,ss} = \mathrm{SU}^*_6$ it is shown in [32, Proposition 4.6] that there are no nonzero local minima of the Hitchin function. This computation also applies to $\mathrm{G}^{\mathbb{R}}_{0,ss} = \mathrm{PSU}^*_6$. These techniques also generalize immediately to the K^4-twisted case and give the desired component counts. □

Remark 7.23. When G has type E_8, we expect the image of the Cayley map Ψ_e to be connected since the maximal compact of the Cayley group has type F_4 which is simply connected, and hence has only one topological type. In general, it is expected that the Cayley map is the only source of connected components of the moduli space of $\mathrm{G}^{\mathbb{R}}$-Higgs bundles which are not labeled by topological invariants of $\mathrm{G}^{\mathbb{R}}$-bundles. This has been proven for the real groups $\mathrm{SL}_n\mathbb{R}$ [48, 35], $\mathrm{U}_{p,q}$ [10, 9], $\mathrm{PGL}_n\mathbb{R}$ [60], SU^*_{2n} [32], $\mathrm{SO}_{p,q}$ with $p = 1$ or $2 < p \leqslant q$ [3], $\mathrm{SO}_{2,3}$ [37, 28] and $\mathrm{Sp}_{2p,2q}$ [33]. Moreover, when there is a Cayley map for these groups, the number of connected components in the image of the Cayley map is counted.

8. Positive surface group representations

In this section we deduce properties of the surface group representations associated Higgs bundles in the image of the Cayley map via the nonabelian Hodge correspondence.

For this section G is a complex simple Lie group and $\mathrm{G}^{\mathbb{R}} \subset \mathrm{G}$ is a real form. We fix a maximal compact subgroup $\mathrm{H}^{\mathbb{R}} \subset \mathrm{G}^{\mathbb{R}}$ with complexification H, and consider the Cartan decomposition $\mathfrak{g}^{\mathbb{R}} = \mathfrak{h}^{\mathbb{R}} \oplus \mathfrak{m}^{\mathbb{R}}$, and its complexification $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.

8.1. **Surface group representations.** Let Σ be a compact smooth oriented surface, without boundary, and $\pi_1\Sigma$ be its fundamental group. Consider the space $\mathrm{Hom}(\pi_1\Sigma, \mathrm{G}^{\mathbb{R}})$ of all representations of $\pi_1\Sigma \to \mathrm{G}^{\mathbb{R}}$. The group $\mathrm{G}^{\mathbb{R}}$ acts on $\mathrm{Hom}(\pi_1\Sigma, \mathrm{G}^{\mathbb{R}})$ by conjugation. Recall that a representation $\pi_1\Sigma \to \mathrm{G}^{\mathbb{R}}$ is called *reductive* if its composition with the adjoint representation of $\mathrm{G}^{\mathbb{R}}$ in $\mathfrak{g}^{\mathbb{R}}$ decomposes as a direct sum of irreducible representations. Let $\mathrm{Hom}^+(\pi_1\Sigma, \mathrm{G}^{\mathbb{R}})$ be the $\mathrm{G}^{\mathbb{R}}$-invariant the subspace consisting of reductive representations.

Definition 8.1. *The* $\mathrm{G}^{\mathbb{R}}$-character variety $\mathcal{X}(\Sigma, \mathrm{G}^{\mathbb{R}})$ *of* $\pi_1\Sigma$ *is defined as the orbit space*

$$\mathcal{X}(\Sigma, \mathrm{G}^{\mathbb{R}}) = \mathrm{Hom}^+(\pi_1\Sigma, \mathrm{G}^{\mathbb{R}})/\mathrm{G}^{\mathbb{R}}.$$

Example 8.2. Let $\mathrm{S}^{\mathbb{R}}$ be $\mathrm{PSL}_2\mathbb{R}$ or $\mathrm{SL}_2\mathbb{R}$. The space of *Fuchsian representations* $\mathrm{Fuch}(\Sigma, \mathrm{S}^{\mathbb{R}}) \subset \mathcal{X}(\Sigma, \mathrm{S}^{\mathbb{R}})$ is defined to be the subset of conjugacy classes of *faithful* representations $\rho_{\mathrm{Fuch}} : \pi_1\Sigma \to \mathrm{S}^{\mathbb{R}}$ with *discrete image*. The space $\mathrm{Fuch}(\Sigma, \mathrm{PSL}_2\mathbb{R})$ defines two connected components of $\mathcal{X}(\Sigma, \mathrm{PSL}_2\mathbb{R})$ [35] and is in one-to-one correspondence with the Teichmüller space of isotopy classes of marked Riemann surface structures on the surface Σ with either of its orientations. Every Fuchsian representation $\rho \in \mathrm{Fuch}(\Sigma, \mathrm{PSL}_2\mathbb{R})$ lifts to a representation $\tilde{\rho}_{\mathrm{Fuch}} \in \mathrm{Fuch}(\Sigma, \mathrm{SL}_2\mathbb{R})$. There are 2^{2g} such lifts and each lift lies in a distinct connected component of $\mathcal{X}(\Sigma, \mathrm{SL}_2\mathbb{R})$.

If $\mathfrak{g}^{\mathbb{R}}$ is the Lie algebra of $\mathrm{G}^{\mathbb{R}}$ and $e \in \mathfrak{g}^{\mathbb{R}}$ is a nonzero nilpotent, the inclusion of the associated $\mathfrak{sl}_2\mathbb{R}$-subalgebra in $\mathfrak{g}^{\mathbb{R}}$ induces an embedding $\iota_e : \mathrm{S}^{\mathbb{R}} \to \mathrm{G}^{\mathbb{R}}$, which in turn defines a map on character varieties

$$\iota_e : \mathrm{Fuch}(\Sigma, \mathrm{S}^{\mathbb{R}}) \to \mathcal{X}(\Sigma, \mathrm{G}^{\mathbb{R}}). \tag{8.1}$$

Such maps define ways to deform the Teichmüller space of Σ inside the character variety $\mathcal{X}(\Sigma, \mathrm{G}^{\mathbb{R}})$. We will call the set $\iota_e(\mathrm{Fuch}(\Sigma, \mathrm{S}^{\mathbb{R}}))$ the *Fuchsian locus*.

The following theorem links the $\mathrm{G}^{\mathbb{R}}$-character variety and the $\mathrm{G}^{\mathbb{R}}$-Higgs bundle moduli space and is known as the *nonabelian Hodge correspondence*. It was proven by Hitchin [46], Donaldson [24], Corlette [18] and Simpson [66] in various generalities (see also [29]).

Theorem 8.3. *Let Σ be a closed oriented surface of genus $g \geqslant 2$ and $\mathrm{G}^{\mathbb{R}}$ be a real semisimple Lie group. For each Riemann surface structure X on Σ, there is a homeomorphism between the moduli space $\mathcal{M}(X, \mathrm{G}^{\mathbb{R}})$ of $\mathrm{G}^{\mathbb{R}}$-Higgs bundles on X and the $\mathrm{G}^{\mathbb{R}}$-character variety $\mathcal{X}(\Sigma, \mathrm{G}^{\mathbb{R}})$.*

One direction of the nonabelian Hodge correspondence is given by considering solutions to the Hitchin equations (6.8). Namely, given a polystable $\mathrm{G}^{\mathbb{R}}$-Higgs bundle $(\mathcal{E}_{\mathrm{H}}, \varphi)$, there is a metric h on $\mathcal{E}_{\mathrm{H}}$ such that $F_h + [\varphi, -\tau_h(\varphi)] = 0$, where F_h is the curvature of the Chern connection A_h associated to h. If $E_h \subset \mathcal{E}_{\mathrm{H}}$ is the associated $\mathrm{H}^{\mathbb{R}}$-bundle, then the connection $D = A_h + \varphi - \tau(\varphi)$ defines a flat connection on the smooth $\mathrm{G}^{\mathbb{R}}$-bundle $E_h[\mathrm{G}^{\mathbb{R}}]$. The flat connection D defines the associated reductive representation $\rho : \pi_1\Sigma \to \mathrm{G}^{\mathbb{R}}$.

For the other direction, let $\rho : \pi_1\Sigma \to \mathrm{G}^{\mathbb{R}}$ be a reductive representation and consider the associated $\mathrm{G}^{\mathbb{R}}$-bundle with flat connection D_ρ,

$$E_\rho = \widetilde{\Sigma} \times_\rho \mathrm{G}^{\mathbb{R}},$$

where $\widetilde{\Sigma}$ is the universal cover of Σ. Each metric h on E_ρ defines a decomposition of the flat connection $D_\rho = A_h + \Psi$, where A_h preserves the metric. Fixing a Riemann surface structure X on Σ allows us to decompose A_h and Ψ into $(1,0)$ and $(0,1)$-parts. If $E_h \subset E_\rho$ is the $\mathrm{H}^{\mathbb{R}}$-bundle associated to h, then the $(0,1)$-part of A_h defines a holomorphic structure on the H-bundle $E_h[\mathrm{H}]$ and the $(1,0)$-part of Ψ defines a section of $E_h[\mathfrak{m}] \otimes K$. By Corlette's Theorem [18], there is a metric h on E_ρ (the *harmonic metric*) which defines a polystable $\mathrm{G}^{\mathbb{R}}$-Higgs bundle

$$(\mathcal{E}_{\mathrm{H}}, \varphi) = ((E_h[\mathrm{H}], A_h^{0,1}), \Psi^{1,0}).$$

Note that for complex groups G we have H = G and the underlying smooth bundle of $\mathcal{E}_{\mathrm{G}}$ is $E_\rho = E_h[\mathrm{G}]$.

Definition 8.4. *Let* G *be a complex reductive Lie group, $\mathrm{G}^{\mathbb{R}} \subset \mathrm{G}$ be a real form, and $\hat{\mathrm{G}}^{\mathbb{R}} \subset \mathrm{G}^{\mathbb{R}}$ be a reductive subgroup. Let $\hat{\mathrm{H}} \subset \mathrm{H} \subset \mathrm{G}$ be the complexifications of maximal compact subgroups of $\hat{\mathrm{G}}^{\mathbb{R}} \subset \mathrm{G}^{\mathbb{R}}$ and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $\hat{\mathfrak{g}} = \hat{\mathfrak{h}} \oplus \hat{\mathfrak{m}}$ be associated complexified Cartan decompositions with $\hat{\mathfrak{m}} \subset \mathfrak{m}$.*

- *A representation $\rho : \pi_1\Sigma \to \mathrm{G}^{\mathbb{R}}$* factors *through $\hat{\mathrm{G}}^{\mathbb{R}}$ if $\rho = \iota \circ \hat{\rho}$, where $\hat{\rho} : \pi_1\Sigma \to \hat{\mathrm{G}}^{\mathbb{R}}$ and $\iota : \hat{\mathrm{G}}^{\mathbb{R}} \to \mathrm{G}^{\mathbb{R}}$ is the inclusion.*
- *A $\mathrm{G}^{\mathbb{R}}$-Higgs bundle $(\mathcal{E}_{\mathrm{H}}, \varphi)$* reduces *to a $\hat{\mathrm{G}}^{\mathbb{R}}$-Higgs bundle $(\mathcal{E}_{\hat{\mathrm{H}}}, \varphi)$ if there is a holomorphic $\hat{\mathrm{H}}$-subbundle $\mathcal{E}_{\hat{\mathrm{H}}} \subset \mathcal{E}_{\mathrm{H}}$ such that $\varphi \in H^0(\mathcal{E}_{\hat{\mathrm{H}}}[\hat{\mathfrak{m}}] \otimes K)$.*

The following is an immediate consequence of the nonabelian Hodge correspondence.

Proposition 8.5. *A reductive representation $\rho : \pi_1\Sigma \to \mathrm{G}^{\mathbb{R}}$ factors through a reductive subgroup $\hat{\mathrm{G}}^{\mathbb{R}} \subset \mathrm{G}^{\mathbb{R}}$ if and only if the associated $\mathrm{G}^{\mathbb{R}}$-Higgs bundle $(\mathcal{E}_{\mathrm{H}}, \varphi)$ reduces to a $\hat{\mathrm{G}}^{\mathbb{R}}$-Higgs bundle. In particular, ρ factors though a compact subgroup if and only if the Higgs field φ is identically zero.*

The *centralizer of a representation* $\rho : \pi_1\Sigma \to \mathrm{G}^{\mathbb{R}}$ is the reductive subgroup

$$Z_{\mathrm{G}^{\mathbb{R}}}(\rho) = \{g \in \mathrm{G}^{\mathbb{R}} \mid g \cdot \rho(\gamma) \cdot g^{-1} = \rho(\gamma) \text{ for all } \gamma \in \pi_1\Sigma\}.$$

The centralizer of the centralizer $Z_{\mathrm{G}^{\mathbb{R}}}(Z_{\mathrm{G}^{\mathbb{R}}}(\rho)) \subset \mathrm{G}^{\mathbb{R}}$ is reductive, and by construction, ρ factors through $Z_{\mathrm{G}^{\mathbb{R}}}(Z_{\mathrm{G}^{\mathbb{R}}}(\rho))$.

Proposition 8.6. *Let* G *be a complex reductive Lie group and $\rho : \pi_1\Sigma \to \mathrm{G}$ be a reductive representation. Then the centralizer $Z_{\mathrm{G}}(\rho)$ of ρ is naturally a subgroup of the automorphism group of the associated* G*-Higgs bundle.*

Proof. Set $\hat{\mathrm{G}} = Z_{\mathrm{G}}(Z_{\mathrm{G}}(\rho))$ and write $\rho = \iota \circ \hat{\rho}$, where $\hat{\rho} : \pi_1\Sigma \to \hat{\mathrm{G}}$. The flat bundle E_ρ is given by $E_{\hat{\rho}}[\mathrm{G}]$. Thus, the associated G-Higgs bundle $(\mathcal{E}_{\mathrm{G}}, \varphi)$ reduces to a $\hat{\mathrm{G}}$-Higgs bundle

$$(\mathcal{E}_{\mathrm{G}}, \varphi) = (\mathcal{E}_{\hat{\mathrm{G}}}[\mathrm{G}], \varphi).$$

Any element $g \in Z_{\mathrm{G}}(\rho)$ defines a constant gauge transformation g of the flat bundle $E_\rho = E_{\hat{\rho}}[\mathrm{G}]$. Since G and $\hat{\mathrm{G}}$ are complex, this defines a gauge transformation of resulting G-Higgs bundle. But the constant gauge transformation g acts trivially on $(\mathcal{E}_{\hat{\mathrm{G}}}[\mathrm{G}], \varphi)$ and hence defines an element of $\mathrm{Aut}(\mathcal{E}_{\hat{\mathrm{G}}}[\mathrm{G}], \varphi)$. □

Proposition 8.7. *Let $(\mathcal{E}_{\mathrm{H}}, \varphi)$ be a $\mathrm{G}^{\mathbb{R}}$-Higgs bundle and $(\mathcal{E}_{\mathrm{H}}[\mathrm{G}], \varphi)$ be the G-Higgs bundle obtained by extension of structure group. If the second hypercohomology group $\mathbb{H}^2(C^\bullet(\mathcal{E}_{\mathrm{H}}, \varphi))$ from (6.3) vanishes, then we have an isomorphism*

$$\mathbb{H}^0(C^\bullet(\mathcal{E}_{\mathrm{H}}, \varphi)) \cong \mathbb{H}^0(C^\bullet(\mathcal{E}_{\mathrm{H}}[\mathrm{G}], \varphi)).$$

In particular, we have an isomorphism of the Lie algebras $\mathrm{aut}(\mathcal{E}_{\mathrm{H}}, \varphi) \cong \mathrm{aut}(\mathcal{E}_{\mathrm{H}}[\mathrm{G}], \varphi)$.

Proof. Serre duality for the complex $C^\bullet(\mathcal{E}_{\mathrm{H}}[\mathrm{G}], \varphi)$ yields an isomorphism

$$\mathbb{H}^0(C^\bullet(\mathcal{E}_{\mathrm{H}}[\mathrm{G}], \varphi)) \cong \mathbb{H}^0(C^\bullet(\mathcal{E}_{\mathrm{H}}, \varphi)) \oplus \mathbb{H}^2(C^\bullet(\mathcal{E}_{\mathrm{H}}, \varphi))^*;$$

see [29, Corollary 3.16]. So $\mathbb{H}^2(C^\bullet(\mathcal{E}_{\mathrm{H}}, \varphi)) = 0$ implies $\mathbb{H}^0(C^\bullet(\mathcal{E}_{\mathrm{H}}[\mathrm{G}], \varphi)) \cong \mathbb{H}^0(C^\bullet(\mathcal{E}_{\mathrm{H}}, \varphi))$. □

We are now set up to prove Theorem A from the introduction.

Theorem 8.8. *Let G be a complex simple Lie group with Lie algebra $\mathfrak{g}$ and $\{f, h, e\} \subset \mathfrak{g}$ be a magical $\mathfrak{sl}_2$-triple with canonical real form $\mathrm{G}^{\mathbb{R}} \subset \mathrm{G}$. Let Σ be a closed orientable surface of genus $g \geqslant 2$ and $\mathcal{X}(\Sigma, \mathrm{G}^{\mathbb{R}})$ be the $\mathrm{G}^{\mathbb{R}}$-character variety. Then, there exists a nonempty open and closed subset*

$$\mathcal{P}_e(\Sigma, \mathrm{G}^{\mathbb{R}}) \subset \mathcal{X}(\Sigma, \mathrm{G}^{\mathbb{R}}),$$

such that every $\rho \in \mathcal{P}_e(\Sigma, \mathrm{G}^{\mathbb{R}})$ has a compact centralizer and does not factor through a compact subgroup. Moreover, the components $\mathcal{P}_e(\Sigma, \mathrm{G}^{\mathbb{R}})$ contain the Fuchsian locus defined by $\{f, h, e\}$

$$\iota_e(\mathrm{Fuch}(\Sigma, \mathrm{S}^{\mathbb{R}})) \subset \mathcal{P}_e(\Sigma, \mathrm{G}^{\mathbb{R}}),$$

where $\iota_e : \mathrm{S}^{\mathbb{R}} \to \mathrm{G}^{\mathbb{R}}$ is the subgroup associated to the $\mathfrak{sl}_2\mathbb{R}$-subalgebra defined by $\{f, h, e\}$.

Remark 8.9. The components $\mathcal{P}_e(\Sigma, \mathrm{G}^{\mathbb{R}}) \subset \mathcal{X}(\Sigma, \mathrm{G}^{\mathbb{R}})$ are obtained by applying the nonabelian Hodge correspondence to the components defined by the Cayley map Ψ_e from Theorem 7.1. For the magical $\mathfrak{sl}_2$-triples from Case (1) of Theorem 3.1, the components $\mathcal{P}_e(\Sigma, \mathrm{G}^{\mathbb{R}})$ are the spaces of Hitchin representations and the above theorem was proven by Hitchin in [48]. For the magical triples from Case (2) of Theorem 3.1, the components $\mathcal{P}_e(\Sigma, \mathrm{G}^{\mathbb{R}})$ are the spaces of maximal representations and most aspects of the above theorem were proven in [7]. For Case (3), the statement was proven in [3].

Since the center of a proper parabolic $\mathrm{P}^{\mathbb{R}} < \mathrm{G}^{\mathbb{R}}$ is not compact, the following is immediate.

Corollary 8.10. *If ρ is any representation in $\mathcal{P}_e(\Sigma, \mathrm{G}^{\mathbb{R}})$, then there is no proper parabolic subgroup $\mathrm{P}^{\mathbb{R}} < \mathrm{G}^{\mathbb{R}}$ such that ρ factors through $\mathrm{P}^{\mathbb{R}}$.*

Proof of Theorem 8.8. By Theorem 7.1, the image of the Cayley map Ψ_e defines nonempty connected components of the moduli space $\mathcal{M}(\mathrm{G}^{\mathbb{R}})$. Applying the nonabelian Hodge correspondence to these components defines an nonempty, open and close subset $\mathcal{P}_e(\Sigma, \mathrm{G}^{\mathbb{R}})$ of the $\mathrm{G}^{\mathbb{R}}$-character variety $\mathcal{X}(\Sigma, \mathrm{G}^{\mathbb{R}})$. Since the Higgs field in the image of the Cayley map is never zero, the associated representations never factor through compact subgroups.

By construction of the Cayley map, when $\mathcal{E}_{\mathrm{C}}$ is the trivial C-bundle and all sections $\tilde{\psi}_{m_c}$ and q_{l_j+1} are zero, the resulting Higgs bundle reduces to the uniformizing $\mathrm{S}^{\mathbb{R}}$-Higgs bundle for the Riemann surface X. Applying the nonabelian Hodge correspondence to this point defines a point in the Fuchsian locus $\iota_e(\mathrm{Fuch}(\Sigma, \mathrm{S}^{\mathbb{R}}))$. Actually, $\iota_e(\mathrm{Fuch}(\Sigma, \mathrm{S}^{\mathbb{R}}))$ corresponds, under the nonabelian Hodge correspondence, to

$$\Psi_e\left(\left\{((\mathcal{E}_{\mathrm{C}}, 0), q_2, 0, \ldots, 0) \,|\, \mathcal{E}_{\mathrm{C}} \text{ trivial, } q_2 \in H^0(K^2)\right\}\right).$$

Thus, $\mathcal{P}_e(\Sigma, \mathrm{G}^{\mathbb{R}})$ contains the Fuchsian locus defined by the magical $\mathfrak{sl}_2$-triple.

Finally we show that the centralizer is compact. Let $\rho : \pi_1\Sigma \to \mathrm{G}^{\mathbb{R}}$ be a representation in such a component and let $Z_{\mathrm{G}^{\mathbb{R}}}(\rho) \subset \mathrm{G}^{\mathbb{R}}$ be its centralizer. Consider the induced complex representation $\rho : \pi_1\Sigma \to \mathrm{G}^{\mathbb{R}} \subset \mathrm{G}$, we have

$$Z_{\mathrm{G}^{\mathbb{R}}}(\rho) = Z_{\mathrm{G}}(\rho) \cap \mathrm{G}^{\mathbb{R}}.$$

It suffices to show that the Lie algebra $\mathfrak{z}_{\mathfrak{g}^{\mathbb{R}}}(\rho) \subset \mathfrak{g}^{\mathbb{R}}$ is contained in $\mathfrak{h}^{\mathbb{R}}$. By Proposition 6.7 and Proposition 5.11, the automorphism group $\mathrm{Aut}(\mathcal{E}_{\mathrm{H}}, \varphi)$ is identified with a closed subgroup of C, and hence $\mathrm{aut}(\mathcal{E}_{\mathrm{C}}, \varphi) \subset \mathfrak{c}$. Thus,

$$\mathfrak{z}_{\mathrm{G}}(\rho) \subset \mathrm{aut}(\mathcal{E}_{\mathrm{H}}[\mathrm{G}], \varphi) = \mathrm{aut}(\mathcal{E}_{\mathrm{H}}, \varphi) \subset \mathfrak{c}.$$

Since $\mathfrak{g}^{\mathbb{R}} \cap \mathfrak{c} = \mathfrak{c}^{\mathbb{R}} \subset \mathfrak{h}^{\mathbb{R}}$, we conclude that the centralizer $Z_{\mathrm{G}^{\mathbb{R}}}(\rho)$ of ρ is compact. □

Points in the domain of the Cayley map (7.1) are given by

$$((\mathcal{E}_{\mathrm{C}}, \tilde{\psi}_{m_c}), q_{l_1+1}, \ldots, q_{l_{\mathrm{rk}(\mathfrak{g}(e))}+1}) \in \mathcal{M}_{K^{m_c+1}}(\mathrm{G}^{\mathbb{R}}_{0,ss}) \times \prod_{j=1}^{\mathrm{rk}(\mathfrak{g}(e))} H^0(K^{l_j+1}).$$

When $\tilde{\psi}_{m_c} = 0$, the associated Higgs bundle reduces to a $\mathrm{G}(e)^{\mathbb{R}} * \mathrm{C}^{\mathbb{R}}$-Higgs bundle, where $\mathrm{G}(e)^{\mathbb{R}} \subset \mathrm{G}^{\mathbb{R}}$ is the connected group with Lie algebra $\mathfrak{g}(e)^{\mathbb{R}}$ and $\mathrm{C}^{\mathbb{R}}$ is the compact real form of C. Moreover, by construction of the Cayley map the Higgs field of the associated Higgs bundle is in the image of the Cayley map for the magical $\mathfrak{sl}_2$-triple in $\mathfrak{g}(e)$ from Case (1) of Theorem 3.1. Hence, the associated representations $\rho : \pi_1\Sigma \to \mathrm{G}^{\mathbb{R}}$ are of the form $\rho = \rho_{Hit} * \rho_{\mathrm{C}^{\mathbb{R}}}$, where $\rho_{Hit} : \pi_1\Sigma \to \mathrm{G}(e)^{\mathbb{R}}$ is a Hitchin representation into $\mathrm{G}(e)^{\mathbb{R}}$ and $\rho_{\mathrm{C}^{\mathbb{R}}} : \pi_1\Sigma \to \mathrm{C}^{\mathbb{R}}$ is any representations into the compact group $\mathrm{C}^{\mathbb{R}}$. In particular, we have the following proposition.

Proposition 8.11. *Each of the sets $\mathcal{P}_e(\Sigma, \mathrm{G}^{\mathbb{R}})$ contains all representations of the form*

$$\rho = \rho_{Hit} * \rho_{\mathrm{C}^{\mathbb{R}}} : \pi_1\Sigma \to \mathrm{G}(e)^{\mathbb{R}} * \mathrm{C}^{\mathbb{R}} \subset \mathrm{G}^{\mathbb{R}},$$

where $\rho_{Hit} : \pi_1\Sigma \to \mathrm{G}(e)^{\mathbb{R}}$ is any $\mathrm{G}(e)^{\mathbb{R}}$-Hitchin representation and $\rho_{\mathrm{C}^{\mathbb{R}}} : \pi_1\Sigma \to \mathrm{C}^{\mathbb{R}}$ is any $\mathrm{C}^{\mathbb{R}}$-representation.

8.2. **Positive Anosov representations.** Anosov representations were introduced by Labourie in [56] and have many interesting geometric and dynamic properties, generalizing convex cocompact representations into rank one Lie groups. Important examples of Anosov representations include Fuchsian representations, quasi-Fuchsian representations, Hitchin representations into split real groups and maximal representations into Lie groups of Hermitian type. We will briefly recall the important points for our applications and refer the reader to [56, 42, 39, 51] for more details.

Let $\mathrm{G}^{\mathbb{R}}$ be a real semisimple Lie group, $\mathrm{P}^{\mathbb{R}} \subset \mathrm{G}^{\mathbb{R}}$ be a proper parabolic subgroup and $\mathrm{L}^{\mathbb{R}} \subset \mathrm{G}^{\mathbb{R}}$ be a Levi subgroup of $\mathrm{P}^{\mathbb{R}}$. If $\mathrm{P}^{\mathbb{R}}_{opp}$ is the opposite parabolic of $\mathrm{P}^{\mathbb{R}}$, then $\mathrm{L}^{\mathbb{R}} = \mathrm{P}^{\mathbb{R}} \cap \mathrm{P}^{\mathbb{R}}_{opp}$ and the homogeneous space $\mathrm{G}^{\mathbb{R}}/\mathrm{L}^{\mathbb{R}}$ is realized as the unique open $\mathrm{G}^{\mathbb{R}}$-orbit in $\mathrm{G}^{\mathbb{R}}/\mathrm{P}^{\mathbb{R}} \times \mathrm{G}^{\mathbb{R}}/\mathrm{P}^{\mathbb{R}}_{opp}$. The pairs of elements $(x, y) \in \mathrm{G}^{\mathbb{R}}/\mathrm{P}^{\mathbb{R}} \times \mathrm{G}^{\mathbb{R}}/\mathrm{P}^{\mathbb{R}}_{opp}$ which lie in this open orbit are called *transverse*.

Definition 8.12. *Let Σ be a closed orientable surface of genus $g \geqslant 2$. Let $\partial_\infty \pi_1 \Sigma$ be the Gromov boundary of the fundamental group $\pi_1\Sigma$. Topologically $\partial_\infty \pi_1 \Sigma \cong \mathbb{RP}^1$. A representation $\rho : \pi_1\Sigma \to \mathrm{G}^{\mathbb{R}}$* is $\mathrm{P}^{\mathbb{R}}$-Anosov *if there exists a unique continuous boundary map $\xi_\rho : \partial_\infty \pi_1 \Sigma \to \mathrm{G}^{\mathbb{R}}/\mathrm{P}^{\mathbb{R}}$ which satisfies the following properties:*

- *Equivariance: $\xi(\gamma \cdot x) = \rho(\gamma)\cdot \xi(x)$ for all $\gamma \in \pi_1\Sigma$ and all $x \in \partial_\infty \pi_1 S$.*
- *Transversality: for all distinct $x, y \in \partial_\infty \pi_1 S$ the generalized flags $\xi(x)$ and $\xi(y)$ are transverse.*
- *Dynamics preserving: see* [56, 42, 39, 51] *for the precise notion.*

The map ξ_ρ will be called the $\mathrm{P}^{\mathbb{R}}$-Anosov boundary curve*.*

An important property of Anosov representations is that they are stable, that is, they define an *open* set of the character variety [56]. However, in general, the set of Anosov representations is *not closed*. For example, the set of Anosov representations in the $\mathrm{PSL}_2\mathbb{C}$-character variety is the open set of quasi-Fuchsian representations, which is not closed. On the other hand, the set of Hitchin representations in split real groups and the set of maximal representations in Hermitian Lie groups do define sets of Anosov representations which are both open and closed in the character variety. For both of these cases, the representations satisfy an additional positivity property [56, 27, 14]. These notions have been unified into the notion of Θ-positive Anosov representations by Guichard–Wienhard [43], which we now briefly recall.

Let $\mathrm{P}^{\mathbb{R}} \subset \mathrm{G}^{\mathbb{R}}$ be a parabolic subgroup, $\mathrm{L}^{\mathbb{R}} \subset \mathrm{P}^{\mathbb{R}}$ be a Levi subgroup and $\mathrm{U}^{\mathbb{R}} < \mathrm{P}^{\mathbb{R}}$ be the unipotent radical. The Lie algebra $\mathfrak{p}^{\mathbb{R}}$ decomposes $\mathrm{Ad}_{\mathrm{L}^{\mathbb{R}}}$-invariantly as $\mathfrak{p}^{\mathbb{R}} = \mathfrak{l}^{\mathbb{R}} \oplus \mathfrak{u}^{\mathbb{R}}$. Moreover, the nilpotent Lie algebra $\mathfrak{u}^{\mathbb{R}}$ decomposes into irreducible $\mathrm{L}^{\mathbb{R}}$-representations

$$\mathfrak{u}^{\mathbb{R}} = \bigoplus \mathfrak{g}_\beta.$$

The parabolic subgroup $\mathrm{P}^{\mathbb{R}}$ is determined by fixing a restricted root system Δ of a maximal $\mathbb{R}$-split torus of $\mathrm{G}^{\mathbb{R}}$ and then choosing a subset $\Theta \subset \Delta$ of simple roots. The each simple root $\beta_j \in \Theta$, there is a corresponding irreducible $\mathrm{L}^{\mathbb{R}}$-representation $\mathfrak{u}_{\beta_j}$.

Definition 8.13. [43, Definition 4.2] *A pair $(\mathrm{G}^{\mathbb{R}}, \mathrm{P}^{\mathbb{R}}_\Theta)$ admits* a Θ-positive structure *if, for all $\beta_j \in \Theta$, the $\mathrm{L}^{\mathbb{R}}_\Theta$-representation space $\mathfrak{u}_{\beta_j}$ has an $(\mathrm{L}^{\mathbb{R}}_\Theta)_0$-invariant acute convex cone $c^\Theta_{\beta_j}$, where $(\mathrm{L}^{\mathbb{R}}_\Theta)_0$ denotes the identity component of $\mathrm{L}^{\mathbb{R}}_\Theta$.*

The set of pairs $(\mathrm{G}^{\mathbb{R}}, \mathrm{P}^{\mathbb{R}}_\Theta)$ which admit a positive structure were classified in [43, Theorem 4.3], and we now relate this classification with the classification of magical $\mathfrak{sl}_2$-triples given in Theorem 3.1. Fix a magical $\mathfrak{sl}_2$-triple $\{f, h, e\} \subset \mathfrak{g}$ and let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be the complexified Cartan decomposition defined by the involution σ_e from (2.6). Fix an involution $\tau_e : \mathfrak{g} \to \mathfrak{g}$ which commutes with σ_e. Recall that τ_e defines the canonical real form $\mathfrak{g}^{\mathbb{R}}$ associated to the magical $\mathfrak{sl}_2$-triple. Recall also from §2.4 that $\{f, h, e\}$ is a normal $\mathfrak{sl}_2$-triple and its Cayley transform $\gamma^{-1}(\{f, h, e\}) = \{\hat{f}, \hat{h}, \hat{e}\}$ is a Cayley triple (see (2.12)) which is a magical $\mathfrak{sl}_2\mathbb{R}$-triple of $\mathfrak{g}^{\mathbb{R}}$. In particular, the nonzero nilpotent $\hat{e}$ belongs to $\mathfrak{g}^{\mathbb{R}}$ and hence it determines a parabolic subgroup $\mathrm{P}^{\mathbb{R}}_{\hat{e}} \subset \mathrm{G}^{\mathbb{R}}$ of the canonical real form.

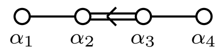
Theorem 8.14. *A pair $(\mathrm{G}^{\mathbb{R}}, \mathrm{P}^{\mathbb{R}}_\Theta)$ admits a Θ-positive structure if and only if there is a magical $\mathfrak{sl}_2\mathbb{R}$-triple $\{\hat{f}, \hat{h}, \hat{e}\} \subset \mathfrak{g}^{\mathbb{R}}$ such that $(\mathrm{G}^{\mathbb{R}}, \mathrm{P}^{\mathbb{R}}_\Theta) = (\mathrm{G}^{\mathbb{R}}, \mathrm{P}^{\mathbb{R}}_{\hat{e}})$. In particular, there are four such families*

(1) $\mathrm{G}^{\mathbb{R}}$-split and $\mathrm{P}^{\mathbb{R}}_\Theta$ is the Borel subgroup.
(2) $\mathrm{G}^{\mathbb{R}}$ is a Hermitian group of tube type and $\mathrm{P}^{\mathbb{R}}_\Theta$ is the maximal parabolic associated the Shilov boundary.

(3) $\mathrm{G}^{\mathbb{R}}$ *is locally isomorphic to* $\mathrm{SO}_{p,q}$ *and* $\mathrm{P}^{\mathbb{R}}_{\Theta}$ *stabilizes an isotropic flag of the form*

$$\mathbb{R} \subset \mathbb{R}^2 \subset \cdots \subset \mathbb{R}^{p-1} \subset \mathbb{R}^{q+1} \subset \cdots \subset \mathbb{R}^{p+q-1} \subset \mathbb{R}^{p+q}.$$

(4) $\mathrm{G}^{\mathbb{R}}$ *is a quaternionic real form of* E_6*,* E_7*,* E_8 *or* F_4*, so that its restricted root system is that of* F_4*, and* $\Theta = \{\alpha_1, \alpha_2\}$*, where*

F_4 :


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Proof. By [43, Theorem 4.3], the set of pairs $(\mathrm{G}^{\mathbb{R}}, \mathrm{P}_{\Theta})$ which admit a Θ-positive structure are given by the above list. The correspondence with magical $\mathfrak{sl}_2$-triples follows from Theorem 3.1 and Proposition 4.1. □

Remark 8.15. The cones in Cases (1) and (2) are the only relevant cones. For Case (1), we have $\mathfrak{u}_{\beta_j} \cong \mathbb{R}$ for all j and the cone is $\mathbb{R}^+ \subset \mathbb{R}$. For Case (2), the cones are related to the causal structure on the Shilov boundary (see [50]). For example, when $\mathrm{G}^{\mathbb{R}} = \mathrm{SU}_{n,n}$, the positive cone is the set of positive definite $(n \times n)$-matrices inside the set of all $(n \times n)$-matrices and for $\mathrm{G}^{\mathbb{R}} = \mathrm{SO}_{2,N-2}$ the cone is the light cone in $\mathbb{R}^{1,N-3}$. For Case (3), there are $p-2$ cones isomorphic to $\mathbb{R}^+ \subset \mathbb{R}$ and one isomorphic to the cone in Case (2) for $\mathrm{SO}_{2,q-p+1}$. For Case (4), there are two invariant cones. One is $\mathbb{R}^+ \subset \mathbb{R}$ corresponding to the simple root space $\tilde{\alpha}$ from (4.4), and the other is isomorphic to the cone in Case (2) for $\mathrm{Sp}_6\mathbb{R}$, $\mathrm{SU}_{3,3}$, SO^*_{12} and E_7^{-25}, for $\mathrm{G}^{\mathbb{R}}$ given by the quaternionic real form of F_4, E_6, E_7 and E_8 respectively.

For pairs $(\mathrm{G}^{\mathbb{R}}, \mathrm{P}^{\mathbb{R}}_{\Theta})$ which admit a Θ-positive structure, there is a distinguished semigroup $\mathrm{U}^{\mathbb{R}}_{\Theta,+} \subset \mathrm{U}^{\mathbb{R}}_{\Theta}$ of the unipotent radical [43, Theorem 4.5], which allows one to define a notion of positively ordered triples in $\mathrm{G}^{\mathbb{R}}/\mathrm{P}^{\mathbb{R}}_{\Theta}$ as follows. Since the group $\mathrm{G}^{\mathbb{R}}$ acts transitively on the space of transverse points in $\mathrm{G}^{\mathbb{R}}/\mathrm{P}^{\mathbb{R}}$, any two points $x, y \in \mathrm{G}^{\mathbb{R}}/\mathrm{P}^{\mathbb{R}}_{\Theta}$ can be mapped to the points (x_+, x_-) associated to $\mathrm{P}^{\mathbb{R}}_{\Theta}$ and $\mathrm{P}^{\mathbb{R}}_{\Theta,opp}$ respectively.

Definition 8.16. [43, Definition 4.6] *Let* $x_+, x_- \in \mathrm{G}^{\mathbb{R}}/\mathrm{P}^{\mathbb{R}}_{\Theta}$ *be the points associated to* $\mathrm{P}^{\mathbb{R}}_{\Theta}$ *and* $\mathrm{P}^{\mathbb{R}}_{\Theta,opp}$ *respectively. A point* x_0 *which is transverse to* x_+ *is the image of* x_- *under a unique element* $u_0 \in \mathrm{U}^{\mathbb{R}}_{\Theta}$*. The triple* (x_+, x_0, x_-) *is* positive *if* $u_0 \in \mathrm{U}^{\mathbb{R}}_{\Theta,+}$*.*

With respect to the orientation on $\partial_\infty\Gamma$, we say that a triple of pairwise distinct points (a, b, c) is a *positive triple* if the points appear in this order.

Definition 8.17. [43, Definition 5.3] *Suppose the pair* $(\mathrm{G}^{\mathbb{R}}, \mathrm{P}^{\mathbb{R}}_{\Theta})$ *admits a* Θ*-positive structure. Then a* $\mathrm{P}^{\mathbb{R}}_{\Theta}$*-Anosov representation* $\rho : \pi_1\Sigma \to \mathrm{G}^{\mathbb{R}}$ *is called* Θ-positive *if the Anosov boundary curve* $\xi : \partial_\infty\pi_1\Sigma \to \mathrm{G}^{\mathbb{R}}/\mathrm{P}^{\mathbb{R}}_{\Theta}$ *sends positively ordered triples in* $\partial_\infty\pi_1 S$ *to positive triples in* $\mathrm{G}^{\mathbb{R}}/\mathrm{P}^{\mathbb{R}}_{\Theta}$*.*

Conjecture 8.18. [41, 43] *If* $(\mathrm{G}, \mathrm{P}^{\mathbb{R}}_{\Theta})$ *admits a* Θ*-positive structure, then the set* Θ*-positive Anosov representations is an open and closed subset of* $\mathcal{X}(\Sigma, \mathrm{G}^{\mathbb{R}})$*.*

This conjecture aims to characterize connected components of the character variety consisting entirely of discrete and faithful representations as precisely those arising from positive Anosov representations. Such components are now commonly referred to as *higher Teichmüller spaces* (cf. [43]).

The construction of the positive semigroup $\mathrm{U}^{\mathbb{R}}_{\Theta,+} \subset \mathrm{U}^{\mathbb{R}}_{\Theta}$ is defined by exponentiating certain combinations of elements in the cones $c_{\beta_j} \subset \mathfrak{u}_{\beta_j}$. Namely, there is a certain Weyl group $\mathcal{W}_\Theta$, and if $w_\Theta = \sigma_{i_1} \cdots \sigma_{i_l}$ is an expression for the longest word in $\mathcal{W}_\Theta$, it defines the map

$$F_{\sigma_{i_1}\cdots\sigma_{i_l}} : c^0_{\beta_{i_1}} \times \cdots \times c^0_{\beta_{i_l}} \longrightarrow \mathrm{U}^{\mathbb{R}}_{\Theta} \ ; \ F_{\sigma_{i_1}\cdots\sigma_{i_l}}(v_{i_1}, \ldots, v_{i_l}) = \exp(v_{i_1}) \cdots \exp(v_{i_l}),$$

where $c^0_{\beta_{i_j}}$ is the interior of $c_{\beta_{i_l}}$. By [43, Theorem 4.5], the semigroup $\mathrm{U}^{\mathbb{R}}_{\Theta,+} \subset \mathrm{U}^{\mathbb{R}}_{\Theta}$ is given by

$$\mathrm{U}^{\mathbb{R}}_{\Theta,+} = F_{\sigma_{i_1}\cdots\sigma_{i_l}}(c^0_{\beta_{i_1}} \times \cdots \times c^0_{\beta_{i_l}}).$$

Recall from Proposition 4.5, that if $\{f,h,e\} \subset \mathfrak{g}$ is a magical $\mathfrak{sl}_2$-triple and $\mathfrak{c} \subset \mathfrak{g}$ is its centralizer, then we denoted the semisimple part of the centralizer of $\mathfrak{c}$ by $\mathfrak{g}(e) \subset \mathfrak{g}$. For magical triples we showed that $\mathfrak{g}(e)$ is simple and $\{f,h,e\} \subset \mathfrak{g}(e)$ is a principal $\mathfrak{sl}_2$-triple in $\mathfrak{g}(e)$. The next result relates the Weyl group $\mathcal{W}_\Theta$ with the Weyl group of $\mathfrak{g}(e)$, for each one of positive families from Theorem 8.14.

Proposition 8.19. *Let $\{f,h,e\} \subset \mathfrak{g}$ be a magical $\mathfrak{sl}_2$-triple with canonical real from $\mathrm{G}^{\mathbb{R}}$ and let $\mathfrak{g}(e) \subset \mathfrak{g}$ be the semisimple part of the centralizer of the centralizer of $\{f,h,e\}$. Then the relevant Weyl group $\mathcal{W}_\Theta$ used to define the semigroup $\mathrm{U}^{\mathbb{R}}_{\Theta,+}$ is the Weyl group of $\mathfrak{g}(e)$. In particular,*

(1) For Case (1) of Theorem 8.14, $\mathfrak{g}(e) = \mathfrak{g}$ and $\mathcal{W}_\Theta$ is the Weyl group of $\mathfrak{g}$.
(2) For Case (2) of Theorem 8.14, $\mathfrak{g}(e) = \{f,h,e\}$ and $\mathcal{W}_\Theta$ is the Weyl group of $\mathfrak{sl}_2\mathbb{C}$.
(3) For Case (3) of Theorem 8.14, $\mathfrak{g}(e) \cong \mathfrak{so}_{2p-1}\mathbb{C}$ and $\mathcal{W}_\Theta$ is the Weyl group of $\mathfrak{so}_{2p-1}\mathbb{C}$.
(4) For Case (4) of Theorem 8.14, $\mathfrak{g}(e) \cong Lie(\mathrm{G}_2)$ and $\mathcal{W}_\Theta$ is the Weyl group of $Lie(\mathrm{G}_2)$.

Recall that the canonical real form $\tau_e : \mathfrak{g} \to \mathfrak{g}$ associated to a magical $\mathfrak{sl}_2$-triple $\{f,h,e\}$ preserves the subalgebra $\mathfrak{g}(e) \oplus \mathfrak{c}$ and the fixed point set defines a subalgebra

$$\mathfrak{g}(e)^{\mathbb{R}} \oplus \mathfrak{c}^{\mathbb{R}} \subset \mathfrak{g}^{\mathbb{R}}.$$

Here $\mathfrak{g}(e)^{\mathbb{R}}$ is the split real form of $\mathfrak{g}(e)$ and contains the Cayley transform $\{\hat{f},\hat{h},\hat{e}\}$ of $\{f,h,e\}$ and $\mathfrak{c}^{\mathbb{R}}$ is the compact real form of $\mathfrak{c}$. This defines an embedding of the connected subgroup with Lie algebra $\mathfrak{g}(e)^{\mathbb{R}}$

$$\iota : \mathrm{G}(e)^{\mathbb{R}} \to \mathrm{G}^{\mathbb{R}}.$$

Moreover, the intersection of the parabolic $\mathrm{P}_\Theta = \mathrm{P}_{\hat{e}} \subset \mathrm{G}^{\mathbb{R}}$ defined by $\hat{e}$ with $\mathrm{G}(e)^{\mathbb{R}}$ is the Borel subgroup $\mathrm{B}^{\mathbb{R}}_e$ of $\mathrm{G}(e)^{\mathbb{R}}$. As a result, there are two important semigroups appearing: the semigroup $\mathrm{U}^{\mathbb{R}}_{\Theta,+} \subset \mathrm{U}^{\mathbb{R}}_{\Theta}$ coming from Θ-positivity for $\{f,h,e\} \subset \mathfrak{g}$, and the semigroup $\mathrm{U}^{\mathbb{R}}_{e,+} \subset \mathrm{U}^{\mathbb{R}}_{e} \subset \mathrm{B}^{\mathbb{R}}_e$ coming from Θ-positivity of $\{f,h,e\} \subset \mathfrak{g}(e)$ from Case (1) of Theorem 8.14.

Proposition 8.20. *Let $\{f,h,e\} \subset \mathfrak{g}$ be a magical $\mathfrak{sl}_2$-triple with canonical real form $\mathfrak{g}^{\mathbb{R}}$ and consider the parabolic $\mathrm{P}_\Theta \subset \mathrm{G}$ and the Borel subgroup $\mathrm{B}^{\mathbb{R}}_e \subset \mathrm{G}(e)^{\mathbb{R}}$. Then the inclusion $\iota : \mathrm{B}^{\mathbb{R}}_e \to \mathrm{P}^{\mathbb{R}}_\Theta$ induces an inclusion of the positive semigroups*

$$\iota : \mathrm{U}^{\mathbb{R}}_{e,+} \to \mathrm{U}^{\mathbb{R}}_{\Theta,+}.$$

Proof. For Case (1) of Theorem 8.14, there is nothing to prove since $\mathfrak{g}(e) = \mathfrak{g}$. For Case (2) of Theorem 8.14, $\mathfrak{g}(e) = \{f,h,e\}$ and the semigroup is just the exponential of the positive cone $c^0_{\beta_1}$. In this case the Cayley transform $\hat{e}$ of e is contained in the cone, and hence $\exp(t\hat{e})$ is contained in $c^0_{\beta_1}$ for $t > 0$. For Case (3) of Theorem 8.14 the statement was proven in [15] for $\mathrm{G}^{\mathbb{R}} = \mathrm{SO}_{p,p+1}$ and the proof for $\mathrm{SO}_{p,q}$ is identical; see [3, §7.2].

Finally we focus on Case (4) of Theorem 8.14. Note that there are two simple roots $\alpha_3, \alpha_4 \notin \Theta$, and the $\mathrm{L}^{\mathbb{R}}_\Theta$-invariant decomposition $\mathfrak{u}^{\mathbb{R}}_{\alpha_3} \oplus \mathfrak{u}^{\mathbb{R}}_{\alpha_4}$ is a real version of the decomposition $\mathfrak{g}_2 = \mathfrak{g}^b_2 \oplus \mathfrak{g}_{\tilde{\alpha}}$ in (4.4). Recall from Remark 8.15, that the two cones $c_{\alpha_3} \subset \mathfrak{u}_{\alpha_3}$ and $c_{\alpha_4} \subset \mathfrak{u}_{\alpha_4}$ are described as follows: $c_{\alpha_4} \subset \mathfrak{u}_{\alpha_4}$ is $\mathbb{R}^+ \subset \mathbb{R}$ and $c_{\alpha_3} \subset \mathfrak{u}_{\alpha_3}$ is the cone from Case (2) for the Lie algebras $\mathfrak{sp}_6\mathbb{R}$, $\mathfrak{su}(3,3)$, $\mathfrak{so}^*_{12}$ and $\mathfrak{e}^{-25}_7$, with $\mathfrak{g}^{\mathbb{R}}$ equal to the quaternionic real forms of $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$ and $\mathfrak{e}_8$ respectively.

We claim that the Cayley transform $\hat{e}$ of the magical nilpotent e is contained in $c^0_{\alpha_3} \times c^0_{\alpha_4}$. First note, that the projections $\hat{e}_{\alpha_3}$ and $\hat{e}_{\alpha_4}$ of $\hat{e}$ onto each factor $\mathfrak{u}^{\mathbb{R}}_{\alpha_3} \oplus \mathfrak{u}^{\mathbb{R}}_{\alpha_4}$ are nonzero since the parabolic $\mathfrak{p}^{\mathbb{R}}_{\Theta} = \mathfrak{p}^{\mathbb{R}}_{\hat{e}}$ is determined by $\hat{e}$. Since the projection onto $\hat{e}$ onto $\mathfrak{u}^{\mathbb{R}}_{\alpha_4}$ is nonzero, we conclude that it is in the cone $c^0_{\alpha_4} \subset \mathfrak{u}^{\mathbb{R}}_{\alpha_4}$. Recall from Remark 4.22, that $\{f_b, [f_b, e_b], e_b\}$ is a magical $\mathfrak{sl}_2$-triple from Case (2). Since the Cayley transform of e_b is contained in the cone from Case (2), the projection of $\hat{e}$ onto $\mathfrak{u}_{\alpha_3}$ is contained in the cone $c^0_{\alpha_3} \subset \mathfrak{u}_{\alpha_3}$. Now, the Weyl group $\mathcal{W}_\Theta$ is the Weyl group of $\mathfrak{g}(e)^{\mathbb{R}}$, thus that of $Lie(\mathrm{G}_2)$, and $\mathfrak{g}(e)^{\mathbb{R}}$ is the split real form of G_2. Moreover, the projections $\hat{e}_{\alpha_3}$ and $\hat{e}_{\alpha_4}$ generate the nilpotent part of the Borel subalgebra $\mathfrak{b}^{\mathbb{R}}_e \subset \mathfrak{g}(e)^{\mathbb{R}}$. Hence, the inclusion $\iota : \mathrm{B}^{\mathbb{R}}_e \to \mathrm{P}^{\mathbb{R}}_\Theta$ induces an inclusion $\iota : \mathrm{U}^{\mathbb{R}}_{e,+} \to \mathrm{U}^{\mathbb{R}}_{\Theta,+}$. ☐

As in [15, Theorem 7.13], we can now prove that, for a magical $\mathfrak{sl}_2$-triple $\{f, h, e\} \subset \mathfrak{g}$ with canonical real form $\mathrm{G}^{\mathbb{R}}$, the set of representations in $\mathcal{P}_e(\Sigma, \mathrm{G}^{\mathbb{R}})$ described by Proposition 8.11 are Θ-positive Anosov representations. Using openness of Θ-positive Anosov representations, we conclude from this that the union of connected components $\mathcal{P}_e(\Sigma, \mathrm{G}^{\mathbb{R}})$ contains an open set of Θ-positive Anosov representations.

Theorem 8.21. *Let* G *be a simple complex Lie group and with Lie algebra* $\mathfrak{g}$. *Let* $\{f, h, e\} \subset \mathfrak{g}$ *be a magical* $\mathfrak{sl}_2$*-triple with canonical real form* $\mathrm{G}^{\mathbb{R}} \subset \mathrm{G}$. *Then the set of representations* $\rho_{Hit} * \rho_{\mathrm{C}^{\mathbb{R}}}$ *from Proposition 8.11 are* Θ*-positive Anosov representations. In particular, each of the sets* $\mathcal{P}_e(\Sigma, \mathrm{G}^{\mathbb{R}}) \subset \mathcal{X}(\Sigma, \mathrm{G}^{\mathbb{R}})$ *from Theorem 8.8 contains a nonempty open set of* Θ*-positive Anosov representations.*

Proof. Consider a $\mathrm{G}(e)^{\mathbb{R}}$-Hitchin representation $\rho_{Hit} : \pi_1\Sigma \to \mathrm{G}(e)^{\mathbb{R}}$. Since ρ_{Hit} is a Θ-positive Anosov representation for Case (1) of Theorem 8.14, the Anosov boundary curve

$$\xi_{\rho_{Hit}} : \partial_\infty \pi_1\Sigma \to \mathrm{G}(e)^{\mathbb{R}}/\mathrm{B}^{\mathbb{R}}_e$$

sends positive triples in $\partial_\infty\pi_1\Sigma$ to positive triples of transverse points in $\mathrm{G}(e)^{\mathbb{R}}/\mathrm{B}^{\mathbb{R}}_e$. The inclusion $\iota : \mathrm{G}(e)^{\mathbb{R}} \to \mathrm{G}^{\mathbb{R}}$ induces a representation $\iota \circ \rho_{Hit}$ and an Anosov boundary curve

$$\iota \circ \xi_{\rho_{Hit}} : \partial_\infty\pi_1\Sigma \to \mathrm{G}(e)^{\mathbb{R}}/\mathrm{B}^{\mathbb{R}}_e \hookrightarrow \mathrm{G}^{\mathbb{R}}/\mathrm{P}^{\mathbb{R}}_\Theta.$$

By Proposition 8.20, $\iota\circ\xi_{\rho_{Hit}}$ also sends positive triples in $\partial_\infty\pi_1\Sigma$ to positive triples of transverse points in $\mathrm{G}(e)^{\mathbb{R}}/\mathrm{B}^{\mathbb{R}}_e$, and hence $\iota \circ \rho_{Hit}$ is a Θ-positive Anosov representation.

The centralizer of $\iota \circ \rho_{Hit}$ is $\mathrm{C}^{\mathbb{R}}$, so compact. Since multiplication by an element in the compact part of the centralizer does not change the boundary curve and does not affect the Anosov property, the boundary curve $\iota \circ \xi_{\rho_{Hit}}$ is also the Anosov boundary curve for the representation $\rho = (\iota \circ \rho_{Hit}) * \rho_{\mathrm{C}^{\mathbb{R}}}$, where $\rho_{\mathrm{C}^{\mathbb{R}}} : \pi_1\Sigma \to \mathrm{C}^{\mathbb{R}}$ is any representation into the compact group $\mathrm{C}^{\mathbb{R}}$. Therefore, all representations from Proposition 8.11 are Θ-positive Anosov representations. Since the set of Θ-positive Anosov representations is open, each of the spaces $\mathcal{P}_e(\Sigma, \mathrm{G}^{\mathbb{R}})$ contain an open set of Θ-positive Anosov representations. ☐

Remark 8.22. By Corollary 8.10, none of the representations in $\mathcal{P}_e(\Sigma, \mathrm{G}^{\mathbb{R}})$ factors through a proper parabolic subgroup of $\mathrm{G}^{\mathbb{R}}$. This fact should be important in proving that in fact every connected component of $\mathcal{P}_e(\Sigma, \mathrm{G}^{\mathbb{R}})$ which contains the Θ-positive Anosov representations described in Theorem 8.21 consists entirely of Θ-positive Anosov representations. There are known examples of components in $\mathcal{P}_e(\Sigma, \mathrm{G}^{\mathbb{R}})$ which do not contain the locus described in Theorem 8.21, namely for the group $\mathrm{SO}_{p,p+1}$ [15]. However, each of these components lie in a component of $\mathcal{P}_e(\Sigma, \mathrm{SO}_{p,p+2})$ which does contain representations in the locus of Theorem 8.21. In fact, one expects that all Θ-positive Anosov representations do not factor through proper parabolic subgroups. This gives further evidence that the space of Θ-positive Anosov representations is exactly described by the space $\mathcal{P}_e(\Sigma, \mathrm{G}^{\mathbb{R}})$, and thus that the higher Teichmüller spaces coincide precisely with the spaces $\mathcal{P}_e(\Sigma, \mathrm{G}^{\mathbb{R}})$.

9. Diagrams and Tables

9.1. Tables.

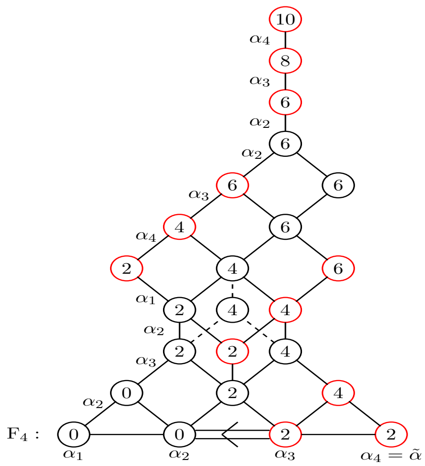
Table 1. Table of magical triples for inner real forms of exceptional Lie algebras

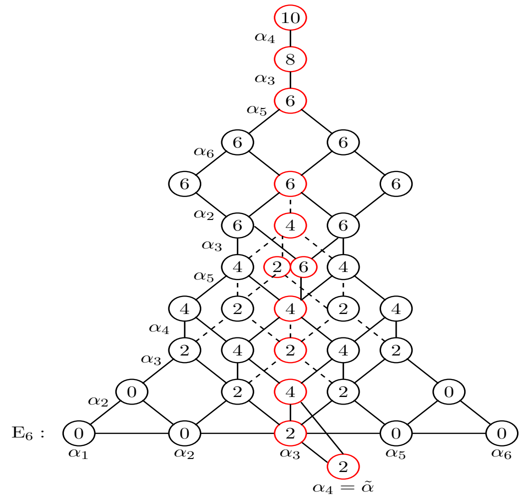
real form	table in [20]	row(s)	Columns 4 & 5	$\mathfrak{c}^{\mathbb{R}}$	weighted Dynkin diagram
$\mathfrak{g}_2^2$	VI	5	0	0	Theorem 3.1 Case (1)
$\mathfrak{f}_4^4$	VII	19	3	$\mathfrak{so}_3$	Theorem 3.1 Case (1)
$\mathfrak{f}_4^4$	VII	26	0	0	Theorem 3.1 Case (1)
$\mathfrak{f}_4^{-20}$	VIII	—	—	—	—
$\mathfrak{e}_6^2$	IX	23	8	$\mathfrak{su}_3$	Theorem 3.1 Case (4)
$\mathfrak{e}_6^{-14}$	X	—	—	—	—
$\mathfrak{e}_7^7$	XI	$93, 94$	0	0	Theorem 3.1 Case (1)
$\mathfrak{e}_7^{-5}$	XII	22	21	$\mathfrak{sp}_6$	Theorem 3.1 Case (4)
$\mathfrak{e}_7^{-25}$	XIII	$6, 7$	52	$\mathfrak{f}_4^{-52}$	Theorem 3.1 Case (2)
$\mathfrak{e}_8^8$	XIV	115	0	0	Theorem 3.1 Case (1)
$\mathfrak{e}_8^{-24}$	XV	21	52	$\mathfrak{f}_4^{-52}$	Theorem 3.1 Case (4)

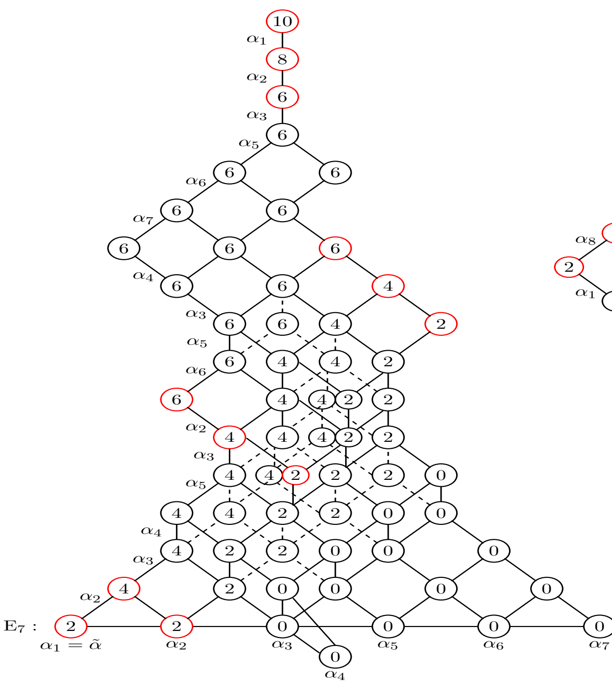
Table 2. Table of noncompact real forms of classical simple Lie algebras

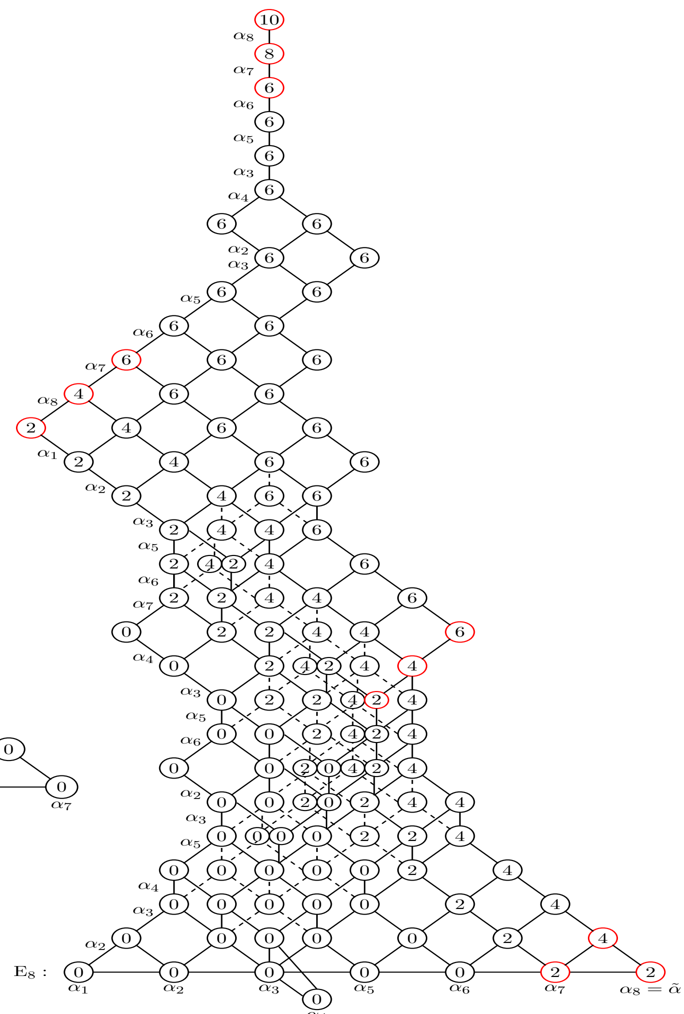
$\mathfrak{g}$	$\mathfrak{g}_{\mathbb{R}}$	Description	$\dim \mathfrak{m} - \dim \mathfrak{h}$
$\mathfrak{sl}_n\mathbb{C}$	$\mathfrak{sl}_n\mathbb{R}$	traceless $(n \times n)$ $\mathbb{R}$-matrices	$n-1$
$\mathfrak{sl}_{p+q}\mathbb{C}$	$\mathfrak{su}_{p,q}$	traceless $(p+q) \times (p+q)$ $\mathbb{C}$-matrices which are skew-adjoint w.r.t. a nondegenerate signature (p,q) Hermitian form	$1-(q-p)^2$
$\mathfrak{sl}_{2m}\mathbb{C}$	$\mathfrak{su}^*_{2m}$	$m \times m$ $\mathbb{H}$-matrices with purely imaginary trace	$-2m-1$
$\mathfrak{so}_{p+q}\mathbb{C}$	$\mathfrak{so}_{p,q}$	$(p+q) \times (p+q)$ $\mathbb{R}$-matrices which are skew-adjoint w.r.t. a nondegenerate signature (p,q) symmetric form	$\frac{1}{2}(p+q-(q-p)^2)$
$\mathfrak{so}_{2m}\mathbb{C}$	$\mathfrak{so}^*_{2m}$	$(m \times m)$ $\mathbb{H}$-matrices which are skew-adjoint w.r.t. a nondegenerate skew-Hermitian form	$-m$
$\mathfrak{sp}_{2m}\mathbb{C}$	$\mathfrak{sp}_{2m}(\mathbb{R})$	$(2m \times 2m)$ $\mathbb{R}$-matrices which are skew-adjoint w.r.t. a nondegenerate skew-symmetric form	m
$\mathfrak{sp}_{2p+2q}\mathbb{C}$	$\mathfrak{sp}_{2p,2q}$	$(m \times m)$ $\mathbb{H}$-matrices which are skew-adjoint w.r.t. a nondegenerate signature (p,q) Hermitian form	$-2(p-q)^2-p-q$

9.2. Weighted root poset for magical nilpotents in E_6, E_7, E_8 and F_4.









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