

# THE BOLZA CURVE AND SOME ORBIFOLD BALL QUOTIENT SURFACES

VINCENT KOZIARZ, CARLOS RITO, XAVIER ROULLEAU

ABSTRACT. We study Deraux's non-arithmetic orbifold ball quotient surfaces obtained as birational transformations of a quotient  $X$  of a particular Abelian surface  $A$ . Using the fact that  $A$  is the Jacobian of the Bolza genus 2 curve, we identify  $X$  as the weighted projective plane  $\mathbb{P}(1, 3, 8)$ . We compute the equation of the mirror  $M$  of the orbifold ball quotient  $(X, M)$  and by taking the quotient by an involution, we obtain an orbifold ball quotient surface with mirror birational to an interesting configuration of plane curves of degrees 1, 2 and 3. We also exhibit an arrangement of four conics in the plane which provides the above-mentioned ball quotient orbifold surfaces.

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## 1. INTRODUCTION.

Chern numbers of smooth complex surfaces of general type  $X$  satisfy the Bogomolov-Miyaoka-Yau inequality  $c_1^2(X) \leq 3c_2(X)$ . Surfaces for which the equality is reached are ball quotient surfaces: there exists a cocompact torsion-free lattice  $\Gamma$  in the automorphism group  $PU(2, 1)$  of the ball  $B_2$  such that  $X = B_2/\Gamma$ . This description of ball quotient surfaces by uniformisation is of transcendental nature, and in fact among ball-quotient surfaces, very few are constructed geometrically (e.g. by taking cyclic covers of known surfaces or by explicit equations of an embedding in a projective space).

Among lattices in  $PU(2, 1)$ , only 22 commensurability classes are known to be non-arithmetic. The first examples of such lattices were given by Mostow and Deligne-Mostow (see [22] and [10]), and recently Deraux, Parker and Paupert [12, 13] constructed some more, sometimes related to an earlier work of Couwenberg, Heckman and Looijenga [9].

Being rare and difficult to produce, these examples are particularly interesting and one would like a geometric description of them. To do so, Deraux [14] studies the quotient of the Abelian surface  $A = E \times E$ , where  $E$  is the elliptic curve  $E = \mathbb{C}/\mathbb{Z}[i\sqrt{2}]$ , by an order 48 automorphism group isomorphic to  $GL_2(\mathbb{F}_3)$  that we will denote by  $G_{48}$ . The ramification locus of the quotient map  $A \rightarrow A/G_{48}$  is the union of 12 elliptic curves and two orbits of isolated fixed points. The images of these two orbits are singularities of type  $A_2$  and  $\frac{1}{8}(1, 3)$ , respectively.

Then Deraux proves that (on some birational transforms) the 1-dimensional branch locus  $M_{48}$  of the quotient map  $A \rightarrow A/G_{48}$  and the two singularities are the support of four ball-quotient orbifold structures, three of these corresponding to non-arithmetic lattices in  $PU(2, 1)$ . Knowing the branch locus  $M_{48}$  is therefore important for these ball-quotient orbifolds, since it gives an explicit geometric description of the uniformisation maps from the ball to the surface.

Deraux also remarks in [14] that the invariants of  $A/G_{48}$  and its singularities are the same as for the weighted projective plane  $\mathbb{P}(1, 3, 8)$  and, in analogy with cases in [11] and [15] where

weighted projective planes appear in the context of ball-quotient surfaces, he asks whether the two surfaces are isomorphic.

In fact, the quotient  $A/G_{48}$  can also be seen as a quotient  $\mathbb{C}^2/G$  where  $G$  is an affine crystallographic complex reflection group. The Chevalley Theorem asserts that if  $G'$  is a finite reflection group acting on a space  $V$  then the quotient  $V/G'$  is a weighted projective space. Using theta functions, Bernstein and Schwarzman [2] observed that for many examples of affine crystallographic complex reflection groups  $G$  acting on a space  $V$ , the quotient  $V/G$  is also a weighted projective space. Kaneko, Tokunaga and Yoshida [20] worked out some other cases, and it is believed that this analog of the Chevalley Theorem always happens (see [2], [16, p. 17]), although no general method is known (see also the presentation of the problem given by Deraux in [14], where more details can be found).

In this paper we prove that indeed:

**Theorem A.** *The surface  $A/G_{48}$  is isomorphic to  $\mathbb{P}(1, 3, 8)$ .*

We obtain this result by exploiting the fact that  $A$  is the Jacobian of a smooth genus 2 curve  $\theta$ , a curve which was first studied by Bolza [5]. The automorphism group of the curve  $\theta$  induces the action of  $G_{48}$  on the Jacobian  $A$ . The main idea to obtain Theorem A is to understand the image of the curve  $\theta$  in  $A$  by the quotient map  $A \rightarrow A/G_{48}$  and to prove that its strict transform in the minimal resolution is a  $(-1)$ -curve.

We then construct birational transformations of  $\mathbb{P}(1, 3, 8)$  to  $\mathbb{P}^1 \times \mathbb{P}^1$ ,  $\mathbb{P}^2$  and obtain the equations of the images  $M_{\mathbb{P}^1 \times \mathbb{P}^1}$ ,  $M_{\mathbb{P}^2}$  of the branch curve  $M_{48}$  in these surfaces (and also  $M_{48} \subset \mathbb{P}(1, 3, 8)$ ). In particular:

**Theorem B.** *In the projective plane, the mirror  $M_{\mathbb{P}^2}$  is the quartic curve*

$$(x^2 + xy + y^2 - xz - yz)^2 - 8xy(x + y - z)^2 = 0.$$

*This curve has two smooth flex points and singular set  $\mathbf{a}_1 + 2\mathbf{a}_2$  (where an  $\mathbf{a}_k$  singularity has local equation  $y^2 - x^{k+1} = 0$ ). The line  $L_0$  through the two residual points of the flex lines  $F_1, F_2$  contains the node (by flex line we mean the tangent line to a flex point).*

The curve  $M_{\mathbb{P}^2}$  with the two flex lines  $F_1, F_2$  gives rise to the four orbifold ball-quotient surfaces (previously described by Deraux [14]) on suitable birational transformations of the plane. We prove that the configuration of curves described in Theorem B is unique up to projective equivalence.

In [18], Hirzebruch constructed ball quotient surfaces using arrangements of lines and performing Kummer coverings. It is a well-known question whether one can construct other ball quotient surfaces using higher degree curves, the next case being arrangements of conics.

Let  $\varphi$  be the Cremona transformation of the plane centered at the three singularities of  $M_{\mathbb{P}^2}$ . The image by  $\varphi$  of the curves  $M_{\mathbb{P}^2}, F_1, F_2, L_0$  described in Theorem B is a special arrangement of four plane conics. We remark that by performing birational transforms of  $\mathbb{P}^2$  and by taking the images of the 4 conics, one can obtain the orbifold ball-quotients of [14]. To our knowledge that gives the first example of orbifold ball quotients obtained from a configuration of conics (ball quotient orbifolds obtained from a configuration of a conic and three tangent lines are studied in [19] and [28]). However we do not know whether one can obtain ball quotient surfaces by performing Kummer coverings branched at these conics.

When preparing this paper, we observed that the mirror  $M_{\mathbb{P}^1 \times \mathbb{P}^1}$  and one related orbifold ball quotient surface among the four might be invariant by an order 2 automorphism. Using the equation we have obtained for  $M_{\mathbb{P}^2}$ , we prove that this is actually the case: there is an involution  $\sigma$  on  $\mathbb{P}^1 \times \mathbb{P}^1$  with fixed point set a  $(1, 1)$ -curve  $D_i$  such that the quotient surface is  $\mathbb{P}^2$ , moreover the image of  $D_i$  is a conic  $C_o$  and the image of  $M_{\mathbb{P}^2}$  is the unique cuspidal cubic curve  $C_u$ . In the last section we obtain and describe the following result:

**Theorem C.** *There is an orbifold ball-quotient structure on a surface  $W$  birational to  $\mathbb{P}^2$  such that the strict transforms on  $W$  of  $C_o, C_u$  have weights  $2, \infty$  respectively.*

The paper is structured as follows:

In section 2, we recall some results of Deraux on the quotient surface  $A/G_{48}$  and introduce some notation. In section 3, we study properties of the surface  $\mathbb{P}(1, 3, 8)$ . In section 4, we introduce the Bolza curve  $\theta$  and prove that  $A/G_{48}$  is isomorphic to  $\mathbb{P}(1, 3, 8)$ . Section 5 is devoted to the equation of the mirror  $M_{\mathbb{P}^2}$ . Moreover we describe the four conics configuration. Section 6 deals with Theorem C.

Some of the proofs in sections 5 and 6 use the computational algebra system Magma, version V2.24-5. A text file containing only the Magma code that appear below is available as an auxiliary file on arXiv and at [25].

Along this paper we use intersection theory on normal surfaces as defined by Mumford in [23, Section 2].

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## 2. QUOTIENT OF $A$ BY $G_{48}$ AND IMAGE OF THE MIRRORS

**2.1. Properties of  $A/G_{48}$  and image of the mirrors.** In this section, we collect some facts from [14] about the action of the automorphism subgroup  $G_{48}$  on the Abelian surface

$$A := \mathbb{C}^2 / (\mathbb{Z}[i\sqrt{2}])^2.$$

There exists a group  $G_{48}$  of order 48 acting on  $A$  which is isomorphic to  $GL_2(\mathbb{F}_3)$  (see [14, Section 3.1] for generators). The action of  $G_{48}$  on  $A$  has no global fixed points (in particular some elements have a non-trivial translation part).

The group  $G_{48}$  contains 12 order 2 reflections, i.e. their linear parts acting on the tangent space  $T_A \simeq \mathbb{C}^2$  are complex order 2 reflections. The fix point set of a reflection being usually called a mirror, we similarly call the fixed point set of a reflection  $\tau$  of  $G_{48}$  a *mirror*. The mirror of such a  $\tau$  is an elliptic curve on  $A$ . The group  $G_{48}$  acts transitively on the set of the 12 mirrors whose list can be found in [14, Table 1].

We denote by  $M$  the union of the mirrors in  $A$  and by  $M_{48}$  the image of  $M$  in the quotient surface  $A/G_{48}$ . The curve  $M_{48}$  is also called the mirror of  $A/G_{48}$ .

Except the points on  $M$ , there are two orbits of points in  $A$  with non-trivial isotropy, one with isotropy group of order 3 at each point, the other with isotropy group of order 8, see [14, Proposition 4.4]. Correspondingly, the quotient  $A/G_{48}$  has two singular points, which are the images of the two special orbits.

**Proposition 1.** *The surface  $A/G_{48}$  is rational and its singularities are of type  $A_2 + \frac{1}{8}(1, 3)$ . The minimal resolution  $p : X_{48} \rightarrow A/G_{48}$  of the surface  $A/G_{48}$  has invariants  $K_{X_{48}}^2 = 5$  and  $c_2(X_{48}) = 7$ .*

*Proof.* Let us compute the invariants of  $X_{48}$ . Let  $\pi : A \rightarrow A/G_{48}$  be the quotient map. One has

$$(2.1) \quad \mathcal{O}_A = K_A = \pi^* K_{A/G_{48}} + M,$$

moreover, according to [14, §4], each mirror  $M_i$ ,  $i = 1, \dots, 12$ , satisfies  $M_i M = 24$ , therefore  $M^2 = 288$  and

$$(K_{A/G_{48}})^2 = \frac{1}{48} M^2 = 6.$$

We observe that  $M = \pi^* (\frac{1}{2} M_{48})$ , thus by (2.1), one gets  $M_{48} = -2K_{A/G_{48}}$ .

The singularities of the quotient surface  $A/G_{48}$  are computed in [14, Table 2]. Let  $C_1, C_2$  be the two  $(-3)$ -curves above the singularity  $\frac{1}{8}(1, 3)$ ; they are such that  $C_1 C_2 = 1$ . Since the singularity of type  $A_2$  is an  $ADE$  singularity, we obtain:

$$K_{X_{48}} = p^* K_{A/G_{48}} - \frac{1}{2}(C_1 + C_2)$$

and  $(K_{X_{48}})^2 = 5$ .

Let  $\tau$  be a reflection in  $G_{48}$  and let  $G$  be the Klein group of order 4 generated by  $\tau$  and the involution  $[-1]_A \in G_{48}$ . One can check that the quotient surface  $A/G$  is rational. Being dominated by the rational surface  $A/G$ , the surface  $A/G_{48}$  is also rational. Thus the second Chern number is  $c_2(X_{48}) = 7$  by Noether's formula.  $\square$

The mirror  $M_{48}$  (the image of  $M$  by the quotient map) does not contain singularities of  $A/G_{48}$ , moreover:

**Lemma 2.** *The pull-back  $\tilde{M}_{48}$  of the mirror  $M_{48}$  by the resolution map  $p : X_{48} \rightarrow A/G_{48}$  has self-intersection 24. Its singular set is*

$$2\mathfrak{a}_2 + \mathfrak{a}_3 + \mathfrak{a}_5,$$

where  $\mathfrak{a}_k$  denotes a singularity with local equation  $y^2 - x^{k+1} = 0$ .

*Proof.* The singularities of  $\tilde{M}_{48} = p^* M_{48}$  are the same as the singularities of  $M_{48}$  since  $M_{48}$  is in the smooth locus of  $A/G_{48}$ . For the computation of the singularities of  $M_{48}$ , we refer to [14, Table 3], and for the self-intersection of  $\tilde{M}_{48}$  (which is the same as the one of  $M_{48}$ ) to [14, §6.2].  $\square$

### 3. THE WEIGHTED PROJECTIVE SPACE $\mathbb{P}(1, 3, 8)$ .

Since we aim to prove that the quotient surface  $A/G_{48}$  is isomorphic to  $\mathbb{P}(1, 3, 8)$ , one first has to study that weighted projective space: this is the goal of this (technical) section. The reader might at first browse through the main results and notation and proceed to the next section.

**3.1. The surface  $\mathbb{P}(1, 3, 8)$  and its minimal resolution.** The weighted projective space  $\mathbb{P}(1, 3, 8)$  is the quotient of  $\mathbb{P}^2$  by the group  $\mathbb{Z}_3 \times \mathbb{Z}_8$  generated by

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & \zeta \end{pmatrix} \in PGL_3(\mathbb{C}),$$

where  $j^2 + j + 1 = 0$  and  $\zeta$  is a primitive  $8^{\text{th}}$  root of unity. The fixed point set of the order 24 element  $\sigma$  is

$$p_1 = (1 : 0 : 0), p_2 = (0 : 1 : 0), p_3 = (0 : 0 : 1).$$

For  $i, j \in \{1, 2, 3\}$  with  $i \neq j$  let  $L'_{ij}$  be the line through  $p_i$  and  $p_j$ . The fixed point set of an order 3 element (e.g.  $\sigma^8$ ) is  $p_2$  and the line  $L'_{13}$ . The fixed point set of an order 8 element (e.g.  $\sigma^3$ ) and its non-trivial powers is  $p_3$  and the line  $L'_{12}$ . Let  $\pi : \mathbb{P}^2 \rightarrow \mathbb{P}(1, 3, 8)$  be the quotient map:  $\pi$  is ramified with order 3 over  $L'_{13}$  and with order 8 over  $L'_{12}$ . The surface  $\mathbb{P}(1, 3, 8)$  has two singularities, images of  $p_2$  and  $p_3$ , which are respectively a cusp  $A_2$  and a

singularity of type  $\frac{1}{8}(1, 3)$ . We denote by  $p : Z \rightarrow \mathbb{P}(1, 3, 8)$  the minimal desingularization map. The singularity of type  $\frac{1}{8}(1, 3)$  is resolved by two rational curves  $C_1, C_2$  with  $C_1 C_2 = 1$ ,  $C_1^2 = C_2^2 = -3$ , and the singularity  $A_2$  is resolved by two rational curves  $C_3, C_4$  with  $C_3 C_4 = 1$ ,  $C_3^2 = C_4^2 = -2$ , (see e.g. [1, Chapter III]).

**Lemma 3.** *The invariants of the resolution  $Z$  are*

$$K_Z^2 = 5, c_2(Z) = 7, p_q = q = 0.$$

*Proof.* We have:

$$K_{\mathbb{P}^2} \equiv \pi^* K_{\mathbb{P}(1,3,8)} + 2L'_{13} + 7L'_{12},$$

therefore since  $K_{\mathbb{P}^2} \equiv -3L$ , we obtain  $\pi^* K_{\mathbb{P}(1,3,8)} \equiv -12L$  and

$$(K_{\mathbb{P}(1,3,8)})^2 = \frac{(-12L)^2}{24} = 6.$$

We have

$$K_Z \equiv p^* K_{\mathbb{P}(1,3,8)} - \sum_{i=1}^4 a_i C_i$$

where the  $a_i$  are rational numbers. The divisor  $K_Z$  must satisfy the adjunction formula i.e. one must have  $C_i K_Z = -2 - C_i^2$  for  $i \in \{1, 2, 3, 4\}$ . That gives:

$$K_Z = p^* K_{\mathbb{P}(1,3,8)} - \frac{1}{2}(C_1 + C_2)$$

and therefore  $K_Z^2 = 5$ . For the Euler number, one may use the formula in [26, Lemma 3]:

$$e(\mathbb{P}(1, 3, 8)) = \frac{1}{24}(3 + 2(2 - 2) + 7(2 - 2) + 23 \cdot 3) = 3.$$

Thus  $e(Z) = e(\mathbb{P}(1, 3, 8)) - 2 + 3 + 3 = 7$ . Since  $\mathbb{P}(1, 3, 8)$  is dominated by  $\mathbb{P}^2$ , the surface  $Z$  is rational, so that  $q = p_g = 0$ .  $\square$

**3.2. The branch curves in  $\mathbb{P}(1, 3, 8)$  and their pullback in the resolution.** Let  $L_{ij}$  be the image of the line  $L'_{ij}$  on  $\mathbb{P}(1, 3, 8)$  and let  $\bar{L}_{ij}$  be the strict transform of  $L_{ij}$  in  $Z$ .

**Proposition 4.** *We have:*

$$\begin{aligned} \bar{L}_{23}^2 &= -1, \quad \bar{L}_{23} C_1 = \bar{L}_{23} C_3 = 1, \quad \bar{L}_{23} C_2 = \bar{L}_{23} C_4 = 0, \\ \bar{L}_{13}^2 &= 0, \quad \bar{L}_{13} C_2 = 1, \quad \bar{L}_{13} C_1 = \bar{L}_{13} C_3 = \bar{L}_{13} C_4 = 0, \\ \bar{L}_{12}^2 &= 2, \quad \bar{L}_{12} C_4 = 1, \quad \bar{L}_{12} C_1 = \bar{L}_{12} C_2 = \bar{L}_{12} C_3 = 0. \end{aligned}$$

*Proof.* On  $\mathbb{P}(1, 3, 8)$  one has  $L_{23}^2 = \frac{1}{24} L_{23}'^2 = \frac{1}{24}$ . Recall that the resolution map is  $p : Z \rightarrow \mathbb{P}(1, 3, 8)$ . Let  $a_1, \dots, a_4 \in \mathbb{Q}$  such that

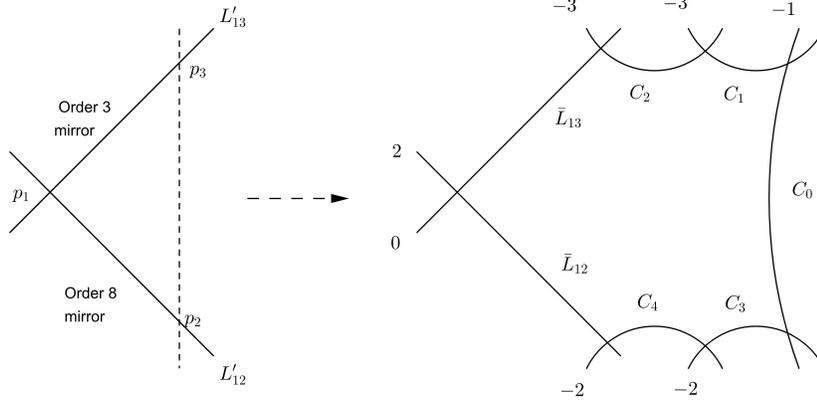
$$\bar{L}_{23} = p^* L_{23} - \sum_{i=1}^4 a_i C_i,$$

then  $C_i p^* L_{23} = 0$  for  $i \in \{1, 2, 3, 4\}$ . Let  $u_i \in \mathbb{N}$  such that  $C_i \bar{L}_{23} = u_i$ . One gets that

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \begin{pmatrix} a_3 \\ a_4 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} u_3 \\ u_4 \end{pmatrix}.$$

We have  $\pi^* K_{\mathbb{P}(1,3,8)} = -12L'_{23}$ , thus

$$K_{\mathbb{P}(1,3,8)} L_{23} = \frac{1}{24} (-12L'_{23} \cdot L'_{23}) = -\frac{1}{2}.$$

FIGURE 3.1. Image of the lines  $L'_{ij}$  in the desingularisation of  $\mathbb{P}(1, 3, 8)$ 

Since  $K_Z = p^*K_{\mathbb{P}(1,3,8)} - \frac{1}{2}(C_1 + C_2)$ , we get

$$\begin{aligned} K_Z \bar{L}_{23} &= (p^*K_{\mathbb{P}(1,3,8)} - \frac{1}{2}(C_1 + C_2)) \left( p^*L - \sum_{i=1}^4 a_i C_i \right) \\ &= -\frac{1}{2} - a_1 - a_2 = -\frac{1}{2}(1 + u_1 + u_2), \end{aligned}$$

which is in  $\mathbb{Z}$ , with  $u_1, u_2 \in \mathbb{N}$ . One computes that

$$\bar{L}_{23}^2 = \frac{1}{24} - \frac{1}{8}(3u_1^2 + 3u_2^2 + 2u_1u_2) - \frac{2}{3}(u_3^2 + u_3u_4 + u_4^2) \in \mathbb{Z}_{\leq 0}.$$

Since  $K_Z \bar{L}_{23} + \bar{L}_{23}^2 = -2$ , the only possibility is

$$\{u_1, u_2\} = \{0, 1\}, \{u_3, u_4\} = \{0, 1\},$$

which gives the intersection numbers with  $\bar{L}_{23}$ .

For the curve  $L_{13}$ , one has  $L_{13}K_{\mathbb{P}(1,3,8)} = -\frac{3}{2}$  and  $L_{13}^2 = \frac{3}{8}$ . Let  $u := \bar{L}_{13}C_1 \in \mathbb{N}$ ,  $v := \bar{L}_{13}C_2 \in \mathbb{N}$ . Then one similarly computes that

$$\bar{L}_{13}K_Z = -\frac{1}{2}(3 + u + v) \leq -\frac{3}{2}$$

and

$$\bar{L}_{13}^2 = \frac{1}{8}(3 - 3u^2 - 3v^2 - 2uv) \leq \frac{3}{8}.$$

Therefore  $\bar{L}_{13}^2 + K_Z \bar{L}_{13} \leq -\frac{9}{8}$  and since  $\bar{L}_{13}^2 + K_Z \bar{L}_{13} \geq -2$ , the only solution is  $\{u, v\} = \{0, 1\}$ , thus  $\bar{L}_{13}^2 = 0$  and  $\bar{L}_{13}K_Z = -2$ .

For the curve  $L_{12}$ , which does not go through the  $\frac{1}{8}(1, 3)$  singularity, one has

$$\bar{L}_{12}K_Z = L_{12}K_{\mathbb{P}(1,3,8)} = -4$$

and  $L_{12}^2 = \frac{8}{3}$ . Let  $w := \bar{L}_{12}C_3$ ,  $t := \bar{L}_{12}C_4$ . Then

$$\bar{L}_{12}^2 = \frac{1}{3}(8 - 2w^2 - 2t^2 - 2wt) \leq \frac{8}{3}.$$

Therefore  $\bar{L}_{12}^2 + K_Z \bar{L}_{12} \leq -\frac{4}{3}$  and the only solution is  $\{w, t\} = \{0, 1\}$ , thus  $\bar{L}_{12}^2 = 2$ .  $\square$

**3.3. From  $\mathbb{P}(1, 3, 8)$  to the Hirzebruch surface  $\mathbb{F}_3$  and back.** By contracting the  $(-1)$ -curve  $C_0 := \bar{L}_{23}$  and then the other  $(-1)$ -curves appearing from the configuration  $C_1, \dots, C_4, \bar{L}$ , one gets a rational surface with

$$K^2 = 2c_2 = 8$$

containing (depending on the choice of the  $(-1)$ -curves we contract) a curve which either is a  $(-2)$ -curve or a  $(-3)$ -curve. Thus that surface is one of the Hirzebruch surfaces  $\mathbb{F}_2$  or  $\mathbb{F}_3$ . Conversely one can reverse the process and obtain the surface  $\mathbb{P}(1, 3, 8)$  by performing a sequence of blow-ups and blow-downs. This process is unique: this follows from the fact that the automorphism group of a Hirzebruch surface  $\mathbb{F}_n$ ,  $n \geq 1$  has two orbits, which are the unique  $(-n)$ -curve and its open complement (see e.g. [4]). In the sequel, only the connection between  $\mathbb{P}(1, 3, 8)$  and  $\mathbb{F}_3$  will be used.

#### 4. THE BOLZA GENUS 2 CURVE IN $A$ AND ITS IMAGE BY THE QUOTIENT MAP

In this section we prove that  $A/G_{48}$  is isomorphic to  $\mathbb{P}(1, 3, 8)$ .

Let us consider the genus 2 curve  $\theta$  whose affine model is

$$(4.1) \quad y^2 = x^5 - x.$$

It was proved by Bolza [5] that the automorphism group of  $\theta$  is  $GL_2(\mathbb{F}_3) \simeq G_{48}$  and  $\theta$  is the unique genus 2 curve with such an automorphism group.

The automorphisms of  $\theta$  are generated by the hyperelliptic involution  $\lambda$  and the lift of the automorphism group  $G$  of  $\mathbb{P}^1$  that preserves the set of 6 branch points  $0, \infty, \pm 1, \pm i$  of the canonical map  $\theta \rightarrow \mathbb{P}^1$  (i.e. the set of points which are fixed by  $\lambda$ ). Note that actually, any map of degree 2 from  $\theta$  to  $\mathbb{P}^1$  is the composition of this map with an automorphism of  $\mathbb{P}^1$ . This is a consequence of the two following facts: on the one hand the 6 ramification points (by the Riemann-Hurwitz formula) of such a map are Weierstrass points, and on the other hand the genus 2 curve  $\theta$  has exactly 6 Weierstrass points.

By the universal property of the Abel-Jacobi map, the group  $GL_2(\mathbb{F}_3)$  acts naturally on the Jacobian variety  $J(\theta)$  of  $\theta$ , the action on  $\theta$  and  $J(\theta)$  being equivariant.

There is only one Abelian surface with an action of  $GL_2(\mathbb{F}_3)$ , which is  $A = E \times E$ , where  $E = \mathbb{C}/\mathbb{Z}[i\sqrt{2}]$  as above (see Fujiki [17] or [3]). We identify  $J(\theta)$  with  $A$ . There are up to conjugation only two possible actions of  $GL_2(\mathbb{F}_3)$  on  $A$  (see [24]):

- a) The action of  $G_{48} \simeq GL_2(\mathbb{F}_3)$  which is described in sub-section 2.1; it has no global fixed points;
- b) The one obtained by forgetting the translation part of that action. That second action globally fixes the 0 point in  $A$ .

Let  $\alpha : \theta \hookrightarrow J(\theta) = A$  be the embedding of  $\theta$  sending the point at infinity of the affine model (4.1) to 0; we identify  $\theta$  with its image.

Note that the morphism  $\theta \times \theta \rightarrow A$ ,  $(x, y) \mapsto [y] - [x] \in \text{Div}_0(\theta) \simeq A$  is onto since  $\theta \times \theta$  and  $A$  are both two-dimensional. Actually, this map has generic degree 2 and contracts the diagonal. Indeed, assume that  $[y] - [x] = [y'] - [x']$  i.e.  $[y] + [x'] - [x] - [y'] = 0 \in \text{Div}_0(\theta)$ . If  $y' = y$  then  $x' = x$  (and conversely) because there is no degree 1 map from  $\theta$  to  $\mathbb{P}^1$ . In the same way,  $y = x$  iff  $y' = x'$ . In the remaining cases, there exists a function of degree 2 from  $\theta$  to  $\mathbb{P}^1$  whose zeroes are  $y$  and  $x'$  and poles are  $x$  and  $y'$ . But by the remark above, we must have  $x' = \lambda(y)$  and  $y' = \lambda(x)$ . Conversely, by the same argument, it is clear that for all  $x$  and  $y$  in  $\theta$ ,  $[\lambda(y)] - [\lambda(x)] = [x] - [y]$ .

This also implies that the points of the type  $[y] - [x]$  with  $x$  and  $y$  being distinct Weierstrass points are exactly the 2-torsion points of  $A$ . Indeed, since there are 6 Weierstrass points on  $\theta$ , we have 15 points of that type in  $A$  satisfying  $[y] - [x] = [\lambda(x)] - [\lambda(y)] = [x] - [y]$  i.e. they are 2-torsion points.

The induced linear action b) is given by  $g([y] - [x]) = [g(y)] - [g(x)]$  for which  $0 \in \text{Div}_0(\theta)$  is a fixed point.

If we fix the base point  $\infty \in \theta$  then for each  $y \in \theta$ ,  $\alpha(x) = [x] - [\infty]$ . The induced action of  $g \in \text{Aut}(\theta)$  on  $A$  is then given by  $g([y] - [x]) = [g(y)] - [g(x)] + [g(\infty)] - [\infty]$ . This is indeed the only action of  $\text{Aut}(\theta)$  on  $A$  commuting with  $\alpha$ .

**Lemma 5.** *The action of  $GL_2(\mathbb{F}_3)$  on  $A$  inducing the action of  $\text{Aut}(\theta)$  on the curve  $\theta \hookrightarrow A$  has no global fixed points.*

*Proof.* The fixed points on  $A$  for the action of the hyperelliptic involution  $\lambda$  are its points of 2-torsion (and 0). Indeed,  $\lambda([y] - [x]) = [\lambda(y)] - [\lambda(x)] \in \text{Div}_0(\theta)$  since  $\infty \in \theta$  is fixed by  $\lambda$  and, as a consequence of the discussion above, if  $[y] - [x] = [\lambda(y)] - [\lambda(x)]$  then either  $y = x$  or  $y = \lambda(x)$  i.e.  $[y] - [x] = [x] - [y]$  and we saw that this implies that  $x$  and  $y$  are Weierstrass points.

But for any pair  $(x, y)$  of distinct Weierstrass points, it is easy to find  $g \in \text{Aut}(\theta)$  (lifting an automorphism of  $\mathbb{P}^1$ ) such that  $g(\infty) = \infty$  but  $[g(y)] - [g(x)] \neq [y] - [x]$ .  $\square$

For  $t \in A$ , let  $\theta_t$  be the curve  $\theta_t = t + \theta$ . The previous result does not depend on the choice of the embedding  $\theta \hookrightarrow A$ : indeed the group of automorphisms acting on  $A$  and preserving  $\theta_t$  is conjugated by the translation  $x \mapsto x + t$  to the group of automorphisms acting on  $A$  and preserving  $\theta$ .

We denote by  $H_{48}$  the order 48 group acting on  $A$  and inducing the automorphism group of the curve  $\theta \hookrightarrow A$  by restriction. As a consequence of Lemma 5, we get:

**Corollary 6.** *There exists an isomorphism between  $H_{48}$  and  $G_{48}$ . That isomorphism is induced by an automorphism  $g$  of the surface  $A$  such that  $H_{48} = gG_{48}g^{-1}$ .*

By [6, Theorem (0.3)], the embedding  $\alpha : \theta \hookrightarrow A$  is such that the torsion points of  $A$  contained in  $\theta$  are 16 torsion points of order 6, 5 torsion points of order 2 and the origin, moreover the  $x$ -coordinates of the 22 torsion points on  $\theta$  satisfy

$$\begin{aligned} x^4 - 4ix^2 - 1 = 0, \quad x^4 + 4ix^2 - 1 = 0 \\ x^5 - x = 0, \quad x = \infty. \end{aligned}$$

**Proposition 7.** (a) *These 22 torsion points of  $\theta$  are not in the mirror of any of the 12 complex reflections of  $H_{48}$ ;*

(b) *Each of these 22 points has a non-trivial stabilizer.*

*Proof.* Let us prove part (a).

The hyperelliptic involution is given by  $(x, y) \rightarrow (x, -y)$ . By [7], the rational map

$$v : (x, y) \mapsto \left( -\frac{x+i}{ix+1}, \sqrt{2} \frac{i-1}{(ix+1)^3} y \right)$$

defines a non-hyperelliptic involution  $v$  on  $\theta$ . The  $x$ -coordinates of the fixed point set of  $v$  are  $x_{\pm} = i(1 \pm \sqrt{2})$ . These coordinates  $x_{\pm}$  are not among the  $x$ -coordinates of the 22 torsion points in  $\theta$ . Let  $\mathbf{v}$  be the automorphism of  $A$  induced by  $v$ . The fixed point set of  $\mathbf{v}$  is a smooth genus 1 curve  $E_v$  (a mirror) and we have just proved that  $E_v$  contains no torsion points of  $\theta$ . By transitivity of the group  $H_{48}$  on its set of 12 non-hyperelliptic involutions, one gets that no mirror contains any of the 22 torsion points.

Let us prove part (b).

The six 2-torsion points are the Weierstrass points of the curve  $\theta$ , they are fixed by the hyperelliptic involution (whose action on  $A$  has only 16 fixed points).

The transformation

$$w : (x, y) \mapsto \left( \frac{(1+i)x - (1+i)}{(1-i)x + (1-i)}, -\frac{1}{((1-i)x + (1-i))^3 y} \right)$$

defines an order 3 automorphism of  $\theta$ , which acts symplectically on  $A$  and one computes that it fixes a torsion point  $p_0 = (x_0, y_0)$  on  $\theta$  with  $x_0$  such that  $x_0^4 + 4ix_0^2 - 1 = 0$ , i.e. it is an order 6 torsion point. This torsion point is an isolated fixed point for each non-trivial element of its stabilizer (since by part (a), it is not on a mirror).

Recall that by [14, Table 2], there are exactly two orbits of points of respective orders 6 and 16 with non-trivial stabilizer under  $G_{48}$  which are isolated fixed points of the non-trivial elements of their stabilizer (by a direct computation one can check that these two orbits are 16 points of order 6 and 6 points of order 2). Since  $H_{48}$  is conjugate to  $G_{48}$ , the 15 other 6-torsion points on  $\theta$  are also isolated fixed points for each non-trivial element of their stabilizer.  $\square$

Since one can change the embedding  $\theta \hookrightarrow A$  by composing with the automorphism  $g$  such that  $H_{48} = gG_{48}g^{-1}$ , let us identify  $H_{48}$  with  $G_{48}$ .

By sub-section 2.1 (or [14]), the images of the 22 torsion points of  $\theta$  on the quotient surface  $A/G_{48}$  give the singularities  $A_2$  and  $\frac{1}{8}(1, 3)$ .

Let  $m$  be the mirror of one of the 12 complex reflections in  $G_{48}$ .

**Lemma 8.** *One has  $\theta \cdot m = 2$ .*

*Proof.* The intersection number  $\theta \cdot m$  is the number of fixed points of the involution  $\iota_m$  with mirror  $m$  restricted to  $\theta$ . Since  $\iota_m$  fixes exactly one holomorphic form, the quotient of  $\theta$  by  $\iota_m$  is an elliptic curve, thus by the Hurwitz formula  $\theta \cdot m = 2$ .  $\square$

Let  $\theta_{48}$  be the image of  $\theta$  in  $A/G_{48}$ . One has:

**Proposition 9.** *The strict transform  $C_0$  of  $\theta_{48}$  by the resolution  $X_{48} \rightarrow A/G_{48}$  is a  $(-1)$ -curve and we have  $\tilde{M}_{48}C_0 = 1$ .*

*Proof.* One has

$$\theta_{48}^2 = \frac{1}{48}\theta^2 = \frac{1}{24}.$$

Let  $\pi : A \rightarrow A/G_{48}$  be the quotient map; it is ramified with order 2 on the union  $M$  of the 12 mirrors. One has  $\pi^*(K_{A/G_{48}} + \frac{1}{2}M_{48}) = K_A = 0$ , thus

$$K_{A/G_{48}}\theta_{48} = -\frac{1}{48}(M\theta) = -\frac{1}{48}12 \cdot 2 = -\frac{1}{2}.$$

The curve  $\theta_{48}$  contains the singularities  $\frac{1}{8}(1, 3)$  and  $A_2$  (image respectively of the 2-torsion points and the 6-torsion points of  $\theta$ ). We are then left with the same combinatorial situation as in the computation of  $\bar{L}_{23}^2$  in Proposition 4, thus we conclude that  $C_0^2 = -1$ .

The two intersection points of  $m$  and  $\theta$  in Lemma 8 are permuted by the hyperelliptic involution of  $\theta$  thus  $M_{48}\theta_{48} = 1$ , which implies  $\tilde{M}_{48}C_0 = 1$ .  $\square$

We obtain:

**Theorem 10.** *The surface  $A/G_{48}$  is isomorphic to  $\mathbb{P}(1, 3, 8)$ .*

*Proof.* Let us denote the resolution map by  $p : X_{48} \rightarrow A/G_{48}$ . Let  $C_1, C_2$  be the resolution curves of the singularity  $\frac{1}{8}(1, 3)$ , and  $C_3, C_4$  be the resolution of  $A_2$ . Let  $a \in A$  be an isolated fixed point of an automorphism  $\tau$  of order 3 or 8. The tangent space  $T_{\theta, a} \subset T_{A, a}$  is stable by the action of  $\tau$ . Since the local setup is the same, we can reason as in Proposition 4 and we obtain that the curve  $C_0$  is such that

$$C_0C_1 = C_0C_3 = 1, \quad C_0C_2 = C_0C_4 = 0.$$

Contracting the curves  $C_0, C_1, C_2$ , one gets a rational surface with a  $(-3)$ -curve and with invariants  $K^2 = 2c_2 = 8$ . This is therefore the Hirzebruch surface  $\mathbb{F}_3$ . From section 3, we know that reversing the contraction process one gets the weighted projective plane  $\mathbb{P}(1, 3, 8)$  (contracting the curves  $C_0, C_1, C_3$ , one would have obtained the Hirzebruch surface  $\mathbb{F}_2$ ).  $\square$

*Remark 11.* Now we identify  $\mathbb{P}(1, 3, 8)$  with  $A/G_{48}$  and we use the notation in section 3. In particular  $Z = X_{48}$  is the minimal resolution of  $\mathbb{P}(1, 3, 8)$ , the curves  $C_1, \dots, C_4$  are exceptional divisors of the resolution map  $Z \rightarrow \mathbb{P}(1, 3, 8)$  and  $C_0 = \bar{L}_{23}$  is a  $(-1)$ -curve in  $Z$ .

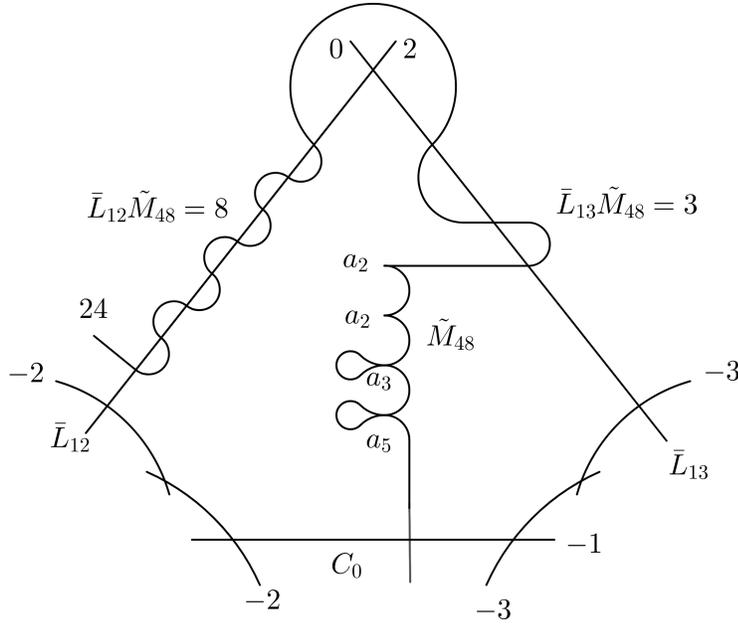
Let us observe that the divisor  $\tilde{F} = C_1 + 3C_0 + 2C_3 + C_4$  satisfies

$$\tilde{F}C_1 = \tilde{F}C_0 = \tilde{F}C_3 = \tilde{F}C_4 = 0,$$

thus  $\tilde{F}^2 = 0$ , moreover  $\tilde{F}C_2 = \bar{L}_{13}\tilde{F} = 1$ ,  $\tilde{F}\bar{L}_{13} = 0$  and  $\bar{L}_{13}^2 = 0$ . This implies that the curves  $\tilde{F}$  and  $\bar{L}_{13}$  are fibers of the same fibration onto  $\mathbb{P}^1$  and  $C_2$  is a section of that fibration.

The curves  $C_0, \dots, C_4$  are exceptional divisors or strict transform of generators of the Néron-Severi group of a minimal rational surface. Thus the Néron-Severi group of the rational surface  $X_{48}$  is generated by these curves. Knowing the intersection of curves  $\bar{L}_{12}, \bar{L}_{13}, \tilde{M}_{48}$  with these curves (see Propositions 4 and 9) it is easy to obtain their classes in the Néron-Severi group, in particular one gets that  $\bar{L}_{12}\tilde{M}_{48} = 8$ ,  $\bar{L}_{13}\tilde{M}_{48} = 3$ .

FIGURE 4.1. Configuration of curves  $\tilde{M}_{48}$ ,  $\bar{L}_{12}$ ,  $\bar{L}_{13}$  etc... in  $X_{48}$  and their intersection numbers



## 5. A MODEL OF THE MIRROR

### 5.1. A birational map from $\mathbb{P}(1, 3, 8)$ to $\mathbb{P}^1 \times \mathbb{P}^1$ ; images of the mirror.

5.1.1. A rational map  $\mathbb{P}(1, 3, 8) \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ . As above, we identify  $\mathbb{P}(1, 3, 8)$  with  $A/G_{48}$ ; we use the notation of sections 3 and 4.

Take a point  $p$  in the Hirzebruch surface  $\mathbb{F}_n$  that is not in the negative section. By blowing-up at  $p$ , and then by blowing-down the strict transform of the fiber through  $p$ , we get the Hirzebruch surface  $\mathbb{F}_{n-1}$ . This process is called an *elementary transformation*.

Recall from sections 3 and 4 that there is a map  $\psi : \mathbb{P}(1, 3, 8) \dashrightarrow \mathbb{F}_3$  that contracts the curves  $C_0, C_3, C_4$  to a smooth point.

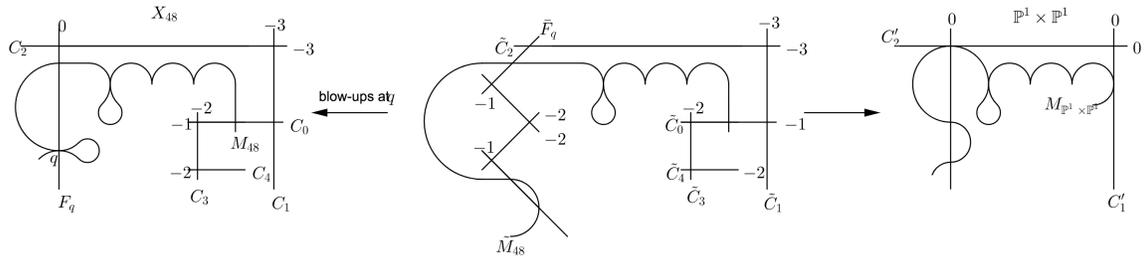
Performing any sequence of three elementary transformations as above, we get a map  $\rho : \mathbb{F}_3 \dashrightarrow \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ . This can be seen as a birational transform that, by blowing-up three times at a point  $q$  not contained in the negative section, takes the fibre  $F_q$  through  $q$  to a chain of curves with self intersections  $(-1), (-2), (-2), (-1)$ , then followed by the contraction of the  $(-1), (-2), (-2)$  chain (which contains the strict transform of  $F_q$ ). For our purpose, we choose the three points to blow-up in a specific way, see subsection 5.1.2.

Consider

$$\phi := \rho \circ \psi : \mathbb{P}(1, 3, 8) \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1.$$

We observe that given any two points  $t, t' \in \mathbb{P}^1 \times \mathbb{P}^1$  not in a common fiber, the map  $\phi$  can be chosen such that the inverse  $\phi^{-1}$  is not defined at  $t, t'$  and  $\phi^{-1}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{P}(1, 3, 8)$ .

FIGURE 5.1. From  $X_{48}$  to  $\mathbb{P}^1 \times \mathbb{P}^1$  and back



5.1.2. *Image of the mirror  $M_{48}$  in  $\mathbb{P}^1 \times \mathbb{P}^1$ .* Let us describe how to choose  $\phi$  such that the image  $M_{\mathbb{P}^1 \times \mathbb{P}^1}$  of the mirror curve  $M_{48}$  is a  $(3, 3)$ -curve with singularities  $\mathfrak{a}_3 + 2\mathfrak{a}_2$  and two special fibers tangent to it with multiplicity 3.

The map  $\mathbb{P}(1, 3, 8) \dashrightarrow \mathbb{F}_3$  factors through a morphism  $\varphi : X_{48} \rightarrow \mathbb{F}_3$ . Consider the point  $t_0 := \varphi(C_0)$ . Since  $M_{48}C_0 = 1$ , then  $\varphi(M_{48})$  is a curve which is smooth at  $t_0$  and its intersection number with the curve  $\varphi(C_1)$  at  $t_0$  is 3. The curve  $C'_1 := \rho \circ \varphi(C_1)$  is a fiber of  $\mathbb{P}^1 \times \mathbb{P}^1$ .

Then we choose  $q$  to be the  $\mathfrak{a}_5$ -singularity of  $M_{48}$ . The fiber  $F_q$  through  $q$  cuts  $M_{48}$  at  $q$  with multiplicity 2 or 3. Suppose that the multiplicity is 3. Then by taking the blow-up at that point and computing the strict transform of the curves  $F_q$  and  $M_{48}$ , one can check that  $F_q M_{48} \geq 4$ . But  $F_q M_{48} = \bar{L}_{13} M_{48} = 3$  by Remark 11. Therefore the fiber  $F_q$  through  $q$  cuts  $M_{48}$  at  $q$  with multiplicity 2, and at another point.

*Remark 12.* An analogous reasoning gives that the fiber through the  $\mathfrak{a}_3$ -singularity has the same property: it is transverse to the tangent of the  $\mathfrak{a}_3$ -singularity.

The three successive blow-ups above  $q$  are chosen such that they resolve the singularity  $\mathfrak{a}_5$ . The three blow-downs we described create a multiplicity 3 tangent point between  $M_{\mathbb{P}^1 \times \mathbb{P}^1}$  (the image of  $M_{48}$  in  $\mathbb{P}^1 \times \mathbb{P}^1$ ) and the curve  $C'_2$  (the image of  $C_2$ ), thus  $C'_2 M_{\mathbb{P}^1 \times \mathbb{P}^1} = 3$ . Moreover  $C'^2_2 = 0, C'_1 C'_2 = 1$  (see figure 5.1).

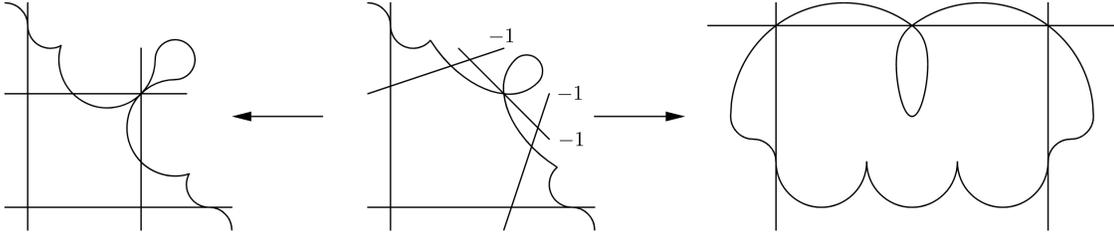
The mirror  $M_{48}$  does not cut the curves  $C_1$  and  $C_2$ . The transforms of these curves in  $\mathbb{P}^1 \times \mathbb{P}^1$  are fibers  $C'_1, C'_2$  such that  $C'_i$  cuts  $M_{\mathbb{P}^1 \times \mathbb{P}^1}$  at one point only, with multiplicity 3.

In particular, the class of  $M_{\mathbb{P}^1 \times \mathbb{P}^1}$  in the Néron-Severi group of  $\mathbb{P}^1 \times \mathbb{P}^1$  is  $3C'_1 + 3C'_2$ . The singularities of  $M_{\mathbb{P}^1 \times \mathbb{P}^1}$  are  $\mathbf{a}_3 + 2\mathbf{a}_2$ .

5.1.3. *From  $\mathbb{P}^1 \times \mathbb{P}^1$  to  $\mathbb{P}^2$  and back.* Let us recall that the blowup of  $\mathbb{P}^1 \times \mathbb{P}^1$  at a point, followed by the blow-down of the strict transform of the two fibers through that point, gives a birational map  $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ .

We choose to blow-up the point at the  $\mathbf{a}_3$ -singularity  $s_0$ , so that the strict transform of  $M_{\mathbb{P}^1 \times \mathbb{P}^1}$  has a node above  $s_0$ . The two fibers  $F_1, F_2$  of  $\mathbb{P}^1 \times \mathbb{P}^1$  passing through  $s_0$  cut  $M_{\mathbb{P}^1 \times \mathbb{P}^1}$  in two other points respectively  $s_1, s_2$  (see Remark 12; the result is preserved through the birational process). The fibers  $F_1, F_2$  are contracted into points in  $\mathbb{P}^2$  by the rational map  $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ , the images of  $s_1, s_2$  by that map are on the image of the exceptional divisor, which is a line  $L_0$  through the node. This implies that the strict transform of  $M_{\mathbb{P}^1 \times \mathbb{P}^1}$  is a plane quartic curve  $M_{\mathbb{P}^2}$ . The process is illustrated in Figure 5.2.

FIGURE 5.2. From  $\mathbb{P}^1 \times \mathbb{P}^1$  to  $\mathbb{P}^2$



The total transform of  $M_{\mathbb{P}^1 \times \mathbb{P}^1}$  in  $\mathbb{P}^2$  is the union of  $2L_0$  with  $M_{\mathbb{P}^2}$ . This quartic  $M_{\mathbb{P}^2}$  has the following properties which follow from its description and the choice of the transformation from  $\mathbb{P}^1 \times \mathbb{P}^1$  to  $\mathbb{P}^2$ :

**Proposition 13.** *The singular set of the quartic curve  $M_{\mathbb{P}^2}$  is  $\mathbf{a}_1 + 2\mathbf{a}_2$ , and the nodal point is contained in the line  $L_0$ . The curve  $M_{\mathbb{P}^2}$  contains two flex points such that each corresponding tangent line meets the quartic at a second point that is contained in the line  $L_0$ .*

5.2. **The yoga between the mirrors  $M_{\mathbb{P}^2}$  and  $M_{48}$ .** Using the previous description the reader can follow the transformations between the surfaces  $\mathbb{P}(1, 3, 8)$  and the plane. The link between Deraux's ball quotient orbifolds described in [14, Theorem 5] and the quartic  $M_{\mathbb{P}^2}$  is as follows:

The singularities  $\mathbf{a}_1 + 2\mathbf{a}_2$  of  $M_{\mathbb{P}^2}$  correspond respectively to singularities  $\mathbf{a}_3 + 2\mathbf{a}_2$  of  $M_{48}$ , so that in order to get the curves  $F, G, H$  in [14, Figure 1] one has to blow-up and contract at these 3 points as it is done in [14]. In order to obtain the curve  $E$  in [14, Figure 1], one has to blow-up the two flexes three times in order to separate  $M_{\mathbb{P}^2}$  and the flex lines. One obtains two chains of  $(-1), (-2), (-2)$  curves. Contracting one of the two  $(-2), (-2)$  chains one gets an  $A_2$ -singularity. The curve  $E$  is the image by the contraction map of the remaining  $(-1)$ -curve of the chain. The resolution of the singularity  $A_2$  on  $\mathbb{P}(1, 3, 8)$  corresponds to the two  $(-2)$ -curves on the other chain of  $(-1), (-2), (-2)$  curves. After taking the blow-up at the residual intersection of the quartic and the flex lines and after separating the flex lines and the mirror  $M_{\mathbb{P}^2}$ , one gets two  $(-3)$ -curves intersecting transversally at one point. In that way the resolution of the singularity  $\frac{1}{8}(1, 3)$  on  $\mathbb{P}(1, 3, 8)$  by two  $(-3)$ -curves corresponds to the two flex lines.

5.3. **A particular quartic curve in  $\mathbb{P}^2$ .** The aim of this sub-section is to prove the following result:

**Theorem 14.** *Up to projective equivalence, there is a unique quartic curve  $Q$  in  $\mathbb{P}^2$  with distinct points  $p_1, \dots, p_7$  such that:*

- (1)  $Q$  has a node at  $p_1$  and ordinary cusps at  $p_2, p_3$ ;
- (2) the points  $p_4, p_5$  are flex points of  $Q$ ;
- (3) the tangent lines to  $Q$  at  $p_4, p_5$  contain  $p_6, p_7$ , respectively;
- (4) the line through  $p_6, p_7$  contains  $p_1$ .

We can assume that

$$p_1 = [0 : 0 : 1], \quad p_2 = [0 : 1 : 1], \quad p_3 = [1 : 0 : 1].$$

Then the equation of  $Q$  is

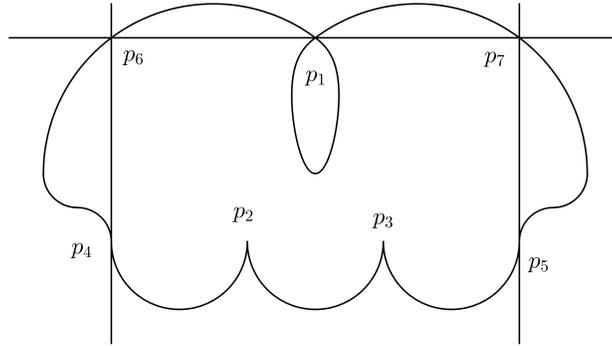
$$(x^2 + xy + y^2 - xz - yz)^2 - 8xy(x + y - z)^2 = 0,$$

and the points  $p_4, p_5$  and  $p_6, p_7$  are, respectively,

$$[\pm 2\sqrt{-2} + 8 : \mp 2\sqrt{-2} + 8 : 25], \quad [\pm 2\sqrt{-2} : \mp 2\sqrt{-2} : 1].$$

**Corollary 15.** *The mirror  $M_{\mathbb{P}^2}$  described on sub-section 5.1.3 satisfies the hypothesis of Theorem 14, thus  $M_{\mathbb{P}^2}$  is projectively equivalent to the quartic  $Q$ .*

FIGURE 5.3. The quartic  $Q$



In order to prove 14, let us first give a criterion for the existence of roots of multiplicity at least 3 on homogeneous quartic polynomials on two variables. We use the computational algebra system Magma; see [25] for a copy-paste ready version of the Magma code.

**Lemma 16.** *The polynomial*

$$P(x, z) = ax^4 + bx^3z + cx^2z^2 + dxz^3 + ez^4$$

*has a root of multiplicity at least 3 if and only if*

$$12ae - 3bd + c^2 = 27ad^2 + 27b^2e - 27bcd + 8c^3 = 0.$$

*Proof.* The computation below is self-explanatory.

```

R<u,v,m,n,a,b,c,d,e>:=PolynomialRing(Rationals(),9);
P<x,z>:=PolynomialRing(R,2);
f:=(u*x+v*z)^3*(m*x+n*z);
s:=Coefficients(f);
I:=ideal<R|a-s[5],b-s[4],c-s[3],d-s[2],e-s[1]>;
EliminationIdeal(I,4);

```

□

Let us now prove Theorem 14:

*Proof.* We have already chosen 3 points  $p_1, p_2, p_3$  in  $\mathbb{P}^2$ . Instead of choosing a fourth point for having a projective base, one can fix two infinitely near points over  $p_2$  and  $p_3$ . Indeed the projective transformations that fix points  $p_1, p_2, p_3$  are of the form

$$\phi : [x : y : z] \mapsto [ax : by : (a-1)x + (b-1)y + z]$$

and these transformations act transitively on the lines through  $p_2$  and  $p_3$ . Thus up to projective equivalence, we can fix the tangent cones (which are double lines) of the curve  $Q$  at the cusps  $p_2, p_3$ . Let us choose for these cones the lines with equations  $y = z$  and  $x = z$ , respectively.

The linear system of quartic curves in  $\mathbb{P}^2$  is 14 dimensional. The imposition of a node and two ordinary cusps (with given tangent cones) corresponds to 13 conditions, thus we get a pencil of curves. We compute that this pencil is generated by the following quartics:

$$(x^2 + xy + y^2 - xz - yz)^2 = 0, \quad xy(x + y - z)^2 = 0.$$

Notice that, at the points  $p_2, p_3$ , the first generator is of multiplicity 2 and the second generator is of multiplicity 3, thus a generic element in the pencil has a cusp singularity at  $p_2, p_3$ .

Let us compute the quartic curves  $Q$  satisfying condition (1) to (4) of Theorem 14. The method is to define a scheme by imposing the vanishing of certain polynomials  $P_i = 0$ , and the non-vanishing of another ones  $D_i \neq 0$ , which is achieved by using an auxiliary parameter  $n$  and imposing  $1 + nD_i = 0$ .

```

K:=Rationals();
R<a,q1,q2,m,d1,d2,n>:=PolynomialRing(K,7);
P<x,y,z>:=ProjectiveSpace(R,2);

```

The defining polynomial of  $Q$ , depending on one parameter:

$$F:=(x^2 + x*y + y^2 - x*z - y*z)^2 + a*x*y*(x + y - z)^2;$$

The points  $p_6, p_7$  are in a line  $y = mx$ , hence they are of the form

```

p6:=[q1,m*q1,1];
p7:=[q2,m*q2,1];

```

and we must have the vanishing of

```

P1:=Evaluate(F,[q1,m*q1,1]);
P2:=Evaluate(F,[q2,m*q2,1]);

```

The defining polynomials of lines through that points are:

```

L1:=-y+d1*x+(m*q1-d1*q1)*z;
L2:=-y+d2*x+(m*q2-d2*q2)*z;

```

We need to impose that these lines are not tangent to  $Q$  at  $p_6, p_7$ , thus the following matrices must be of rank 2.

```

M1:=Matrix([JacobianSequence(F),JacobianSequence(L1)]);
M1:=Evaluate(M1,[q1,m*q1,1]);
M2:=Matrix([JacobianSequence(F),JacobianSequence(L2)]);
M2:=Evaluate(M2,[q2,m*q2,1]);

```

The matrix  $M_i$  is of rank 2 if one of its minors is non-zero. Here we make a choice for these minors, but in order to cover all cases the computations must be repeated for all other choices.

```

D1:=Minors(M1,2)[1];
D2:=Minors(M2,2)[1];

```

Now we intersect the quartic  $Q$  with the lines  $L_1, L_2$  :

```

R1:=Evaluate(F,y,d1*x+(m*q1-d1*q1)*z);
R2:=Evaluate(F,y,d2*x+(m*q2-d2*q2)*z);

```

and we use Lemma 16 to impose that these lines are tangent to  $Q$  at flex points of  $Q$ :

```

c:=Coefficients(R1);
P3:=c[1]*c[5]-1/4*c[2]*c[4]+1/12*c[3]^2;
P4:=c[1]*c[4]^2+c[2]^2*c[5]-c[2]*c[3]*c[4]+8/27*c[3]^3;
c:=Coefficients(R2);
P5:=c[1]*c[5]-1/4*c[2]*c[4]+1/12*c[3]^2;
P6:=c[1]*c[4]^2+c[2]^2*c[5]-c[2]*c[3]*c[4]+8/27*c[3]^3;

```

We note that the lines  $L_1, L_2$  cannot contain the points  $p_2, p_3$  :

```

D3:=Evaluate(L1,[0,1,1]);
D4:=Evaluate(L1,[1,0,1]);
D5:=Evaluate(L2,[0,1,1]);
D6:=Evaluate(L2,[1,0,1]);

```

Also the line  $L_i$  cannot contain the point  $p_1, i = 1, 2$  :

```

D7:=(m-d1)*(m-d2);

```

And it is clear that the following must be non-zero:

```

D8:=a*q1*q2*(q1-q2);

```

Finally we define a scheme with all these conditions.

```

A:=AffineSpace(R);
S:=Scheme(A,[P1,P2,P3,P4,P5,P6,1+n*D1*D2*D3*D4*D5*D6*D7*D8]);

```

We compute (that takes a few hours):

```

PrimeComponents(S);

```

and get the unique solution  $a = -8$ . □

From the equation of the quartic  $Q = M_{\mathbb{P}^2}$ , one can compute a degree 24 equation for the mirror  $M_{48}$ , which is:

$(31072410*r+44060139)*x^{24}+(599304420*r-4660302600)*x^{21}*y+(-106415505000*r+18054913500)*x^{18}$   
 $*y^2+(796474485000*r+3638808225000)*x^{15}*y^3+(-27123660*r-18697014)*x^{16}*z+(34521715125000$   
 $*r-31210968093750)*x^{12}*y^4+(107726220*r+2948918400)*x^{13}*y*z+(-257483985484500*r-$   
 $516632817969000)*x^9*y^5+(42798843000*r-32351244300)*x^{10}*y^2*z+(-1747212737190000*r$   
 $+3228789525752500)*x^6*y^6+(-407331396000*r-935091495000)*x^7*y^3*z+(-655139025450000*r+$   
 $10855982580975000)*x^3*y^7+(7724970*r-2222037)*x^8*z^2+(-3383703150000*r+9052448883750)$   
 $*x^4*y^4*z+(1544666220033750*r+11942493993804375)*y^8+(-102498120*r-465161400)*x^5*y*z^2+$   
 $(-319463676000*r+12613760073000)*x*y^5*z+(-2705586000*r+7086771600)*x^2*y^2*z^2+(-712080*r$   
 $+1186268)*z^3=0$

where  $r = \sqrt{-2}$ .

**5.4. A configuration of four plane conics related to the orbifold ball quotient.** In this subsection we describe the configuration of conics which we announced in the introduction.

Let us consider a conic tangent to two lines of a triangle in  $\mathbb{P}^2$ , and going through two points of the remaining line. Performing a Cremona transformation at the three vertices of the triangle one obtains a quartic curve in  $\mathbb{P}^2$  with singularities  $\mathbf{a}_1 + 2\mathbf{a}_2$ . Conversely, starting with such a quartic, its image by the Cremona transform at the three singularities is a conic with three lines having the above configuration.

Thus we consider the Cremona transform  $\varphi$  at the three singularities of the quartic  $M_{\mathbb{P}^2}$ . Let  $D_1, \dots, D_4$  be respectively the images of  $M_{\mathbb{P}^2}$ , the line  $L_0$  through the node and the two residual points of the flex lines, and the two flex lines. Using Magma, we see that these are 4 conics meeting in 10 points, as follows:

	$q_1$	$q_2$	$q_3$	$q_4$	$q_5$	$q_6$	$q_7$	$q_8$	$q_9$	$q_{10}$
$D_1$	1+	1+	0	0	0	0	1	1	1	1
$D_2$	1	1	1	1	1	0	0	0	1	1
$D_3$	0	1+	1	1	1	1	1	0	0	0
$D_4$	1+	0	1	1	1	1	0	1	0	0

Here two + in the column of  $q_j$  mean that the two curves meet with multiplicity 3 at point  $q_i$ . The other intersections are transverse. We see that the various ball-quotient orbifolds that Deraux described in [14] may be obtained from a configuration of conics by performing birational transformations.

## 6. ONE FURTHER QUOTIENT BY AN INVOLUTION

**6.1. The quotient morphism  $\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^2$ , image of the mirror as the cuspidal cubic.** Consider the plane quartic curve  $Q$  from Theorem 14. Here we show the existence of a birational map

$$\rho: \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$$

and an involution  $\sigma$  on  $\mathbb{P}^1 \times \mathbb{P}^1$  that preserves  $\rho(Q)$  and fixes the diagonal  $D$  of  $\mathbb{P}^1 \times \mathbb{P}^1$  pointwise. Moreover, we have  $(\mathbb{P}^1 \times \mathbb{P}^1)/\sigma = \mathbb{P}^2$ , and the images  $C_u, C_o$  of  $\rho(Q), D$  are curves of degrees 3, 2, respectively. The curve  $C_u$  has a cusp singularity and intersects  $C_o$  at three points, with intersection multiplicities 4, 1, 1. The map  $\rho$  is the inverse of the birational transform  $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$  described in sub-section 5.1.3, whose indeterminacy is at the singularity  $\mathbf{a}_3$  of  $M_{\mathbb{P}^1 \times \mathbb{P}^1}$ .

```

K:=Rationals();
R<r>:=PolynomialRing(K);
K<r>:=ext<K|r^2+2>;
P2<x,y,z>:=ProjectiveSpace(K,2);
Q:=Curve(P2,(x^2+x*y+y^2-x*z-y*z)^2-8*x*y*(x+y-z)^2);
p6:=P2![2*r,-2*r,1];

```

```
p7:=P2![-2*r,2*r,1];
```

We compute the linear system of conics through the cuspidal points  $p_2, p_3$  and take the corresponding map to  $\mathbb{P}^3$ .

```
L:=LinearSystem(LinearSystem(P2,2),[p6,p7]);
```

```
P3<a,b,c,d>:=ProjectiveSpace(K,3);
```

```
rho:=map<P2->P3|Sections(L)>;
```

The image of  $\mathbb{P}^2$  is a quadric surface  $Q_2 (\cong \mathbb{P}^1 \times \mathbb{P}^1)$ .

```
Q2:=rho(P2);Q2;
```

```
C:=rho(Q);C;
```

There is an involution preserving both  $Q_2$  and the curve  $C := \rho(Q)$ .

```
sigma:=map<P3->P3|[d,b,c,a]>;
```

```
C:=rho(Q);C;
```

```
sigma(Q2) eq Q2;
```

```
sigma(C) eq C;
```

We compute the corresponding map to the quotient. The image of  $C$  is a cubic curve, and the image of the diagonal is a conic.

```
psi:=map<P3->P2|[a+d,b,c]>;
```

```
Cu:=psi(C);
```

```
Co:=psi(Scheme(rho(P2),[a-d]));
```

```
Co:=Curve(P2,DefiningEquations(Co));
```

The curve  $C_u$  has a cusp singularity:

```
pts:=SingularPoints(Cu);
```

```
ResolutionGraph(Cu,pts[1]);
```

The intersections of  $C_o$  and  $C_u$  :

```
Degree(ReducedSubscheme(Co meet Cu)) eq 3;
```

```
pt:=Points(Co meet Cu)[1];
```

```
IntersectionNumber(Co,Cu,pt) eq 4;
```

Let  $C'_1, C'_2$  be the fibers that intersect  $M_{\mathbb{P}^1 \times \mathbb{P}^1}$  each at a unique point with multiplicity 3. These fibers are exchanged by the involution  $\sigma$  and are sent to a line  $F_l$  which cuts the cubic curve  $C_u$  at a unique point: this is a flex line. That line  $F_l$  also cuts the conic  $C_o$  at a unique point.

Conversely, let us start from the data of a conic  $C_o$  and a cuspidal cubic  $C_u$  intersecting as above, with the flex line (at the smooth flex point) of the cubic tangent to the conic. One can take the double cover of the plane branched over  $C_o$ , which is  $\mathbb{P}^1 \times \mathbb{P}^1$ . The pull-back of  $C_u$  is then a curve satisfying the properties of Theorem 14, thus the configuration  $(C_o, C_u)$  we described is unique in  $\mathbb{P}^2$ , up to projective automorphisms.

**6.2. An orbifold ball-quotient structure from  $(\mathbb{P}^2, (C_o, C_u))$ .** Let  $C_u \hookrightarrow \mathbb{P}^2$  be the unique plane cuspidal curve and let  $c_1$  be its cuspidal point. Let  $F_l$  be the flex line through the unique smooth flex point  $c_2$  of  $C_u$ . By the previous subsection, one has the following result:

**Proposition 17.** *There exists a unique conic  $C_o \hookrightarrow \mathbb{P}^2$  such that the following holds:*

*i)  $F_l$  is tangent to  $C_o$ ;*

*ii)  $C_o$  cuts  $C_u$  at points  $c_3, c_4, c_5$  ( $\neq c_1, c_2$ ) with intersection multiplicities 4, 1, 1, respectively.*

In this subsection we prove that there is a natural birational transformation  $W \dashrightarrow \mathbb{P}^2$  such that together with the strict transform of the curves  $C_o$  and  $C_u$  one gets an orbifold ball quotient surface. For definitions and results on orbifold theory, we use [8, 11] and [29].

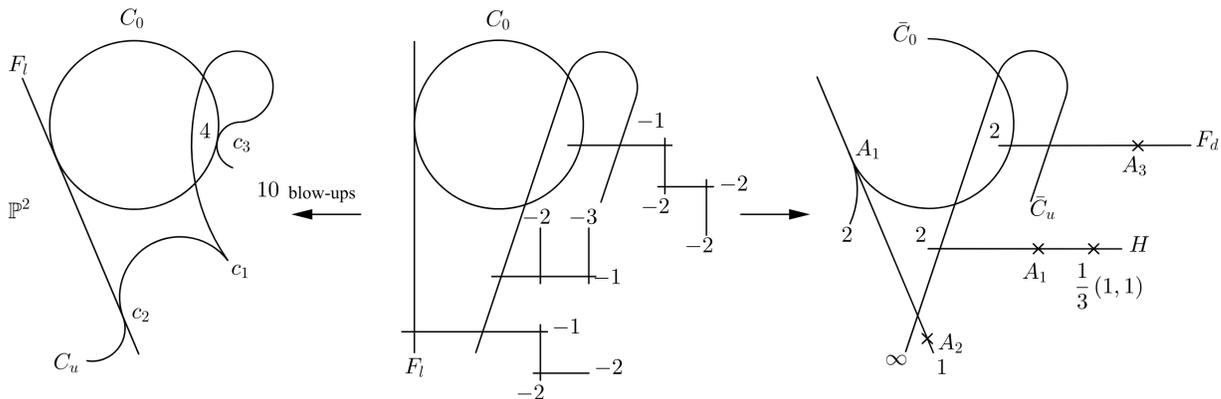
Let us blow-up over points  $c_1, c_2, c_3$  and then contract some divisors as follows (for a pictural description see figure 6.1):

We blow up over  $c_1$  three times, the first blow-up resolves the cusp of  $C_u$  and the exceptional divisor intersects the strict transform of  $C_u$  tangentially, the second blow-up is at that point of tangency and the third blow-up separates the strict transforms of the first exceptional divisor and the curve  $C_u$ . One obtains in that way a chain of  $(-3)$ ,  $(-1)$  and  $(-2)$ -curves. We then contract the  $(-2)$  and  $(-3)$ -curves obtaining in that way singularities  $A_1$  and  $\frac{1}{3}(1,1)$ . The image of the  $(-1)$ -curve by that contraction map is denoted by  $H$ . As an orbifold we put multiplicity 2 on  $H$ .

We blow up over  $c_2$  (the flex point) three times in order that the strict transform of the curves  $F_l$  and  $C_u$  get separated over  $c_2$ . We obtain in that way a chain of  $(-1)$ ,  $(-2)$ ,  $(-2)$ -curves. We then contract the two  $(-2)$ -curves and obtain an  $A_2$ -singularity. The strict transform of the line  $F_l$  is a  $(-2)$ -curve, which we also contract, obtaining in that way an  $A_1$ -singularity. The contracted curve being tangent to  $\tilde{C}_0$ , the image  $\tilde{C}_0$  has a cusp  $a_2$  at the singularity  $A_1$ .

We moreover blow up over  $c_3$  four times, in order that the strict transform of the curves  $C_o$  and  $C_u$  get separated over  $c_3$ . We obtain in that way a chain of  $(-1)$ ,  $(-2)$ ,  $(-2)$ ,  $(-2)$ -curves. We then contract the three  $(-2)$ -curves and obtain an  $A_3$ -singularity. The image of the  $(-1)$ -curve by the contraction map is a curve denoted by  $F_d$ , we give the weight 2 to that curve.

FIGURE 6.1. The plane, the surfaces  $Z$  and  $W$



Let us denote by  $W$  the resulting surface. For a curve  $D$  on  $\mathbb{P}^2$ , we denote by  $\tilde{D}$  its strict transform on  $W$ . Let  $\mathcal{W}$  be the orbifold with same subjacent topological space, with divisorial part:

$$\Delta = \left(1 - \frac{1}{\infty}\right)\tilde{C}_u + \left(1 - \frac{1}{2}\right)(\tilde{C}_o + F_d + H).$$

The singular points of  $W$  are

$$A_1 + A_1 + A_2 + A_3 + \frac{1}{3}(1,1),$$

and they have an isotropy  $\beta$  of order 16, 4, 3, 8, 6 respectively, for  $\mathcal{W}$ . The computation of the isotropy is immediate, except for the first point (that we shall denote by  $r_1$ ), which is also a cusp on the curve  $\tilde{C}_0$  (which has weight 2). Let  $SD_{16}$  be the the semidihedral group of order 16, generated by the matrices

$$g_1 = \begin{pmatrix} 0 & -\zeta \\ -\zeta^3 & 0 \end{pmatrix}, g_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where  $\zeta$  is a primitive 8th root of unity. The order 2 elements  $g_2, g_1^{-1}g_2g_1$  generate an order 8 reflection group  $D_4$ . The quotient of  $\mathbb{C}^2$  by  $SD_{16}$  has a  $A_1$  singularity and one computes that the image of the 4 mirrors of  $D_4$  is a curve with a cusp  $\mathfrak{a}_2$  at the  $A_1$  singularity of  $\mathbb{C}^2/SD_{16}$ . The isotropy group of the point  $r_1$  in the orbifold is therefore the semidihedral group  $SD_{16}$  of order 16. The following proposition is an application of the main result of [21]:

**Proposition 18.** *The Chern numbers of the orbifold  $\mathcal{W} = (W, \Delta)$  satisfy*

$$c_1^2(\mathcal{W}) = 3c_2(\mathcal{W}) = \frac{9}{16},$$

in particular  $\mathcal{W}$  is an orbifold ball quotient.

*Proof.* Let us compute the orbifold second Chern number of  $\mathcal{W}$ . We have (see e.g. [27]):

$$\begin{aligned} c_2(\mathcal{W}) &= e(W) - \left( (1 - \frac{1}{\infty})e(\bar{C}_u \setminus S) + (1 - \frac{1}{2})e(\bar{C}_o \setminus S) + \right. \\ &\quad \left. + (1 - \frac{1}{2})e(F_d \setminus S) + (1 - \frac{1}{2})e(H \setminus S) \right) - \sum_{p \in S} \left( 1 - \frac{1}{\beta(p)} \right), \end{aligned}$$

where  $S$  is the union of the singular points of  $W$  with the singular points of the round-up divisor  $[\Delta]$ , and where moreover  $\beta(p)$  is the isotropy order of the point  $p$ , so that for example for  $p$  on  $\bar{C}_u$ ,  $\beta(p) = \infty$  and the unique point  $p$  in  $F_d$  and  $\bar{C}_o$  has  $\beta(p) = 4$ . Since we have blown-up  $\mathbb{P}^2$  over 10 points and we have contracted 8 rational curves, we get

$$e(W) = 3 + 10 - 8 = 5.$$

We obtain

$$\begin{aligned} c_2(\mathcal{W}) &= 5 - \left( (2 - 4) + \frac{1}{2}(2 - 4) + \frac{1}{2}(2 - 3) + \frac{1}{2}(2 - 3) \right) \\ &\quad - \left( 10 - \frac{1}{16} - \frac{1}{4} - \frac{1}{3} - \frac{1}{8} - \frac{1}{6} - \frac{1}{4} - 4 \cdot \frac{1}{\infty} \right), \end{aligned}$$

thus  $c_2(\mathcal{W}) = \frac{3}{16}$ .

Let us compute  $c_1^2(\mathcal{W})$ . One has

$$c_1^2(\mathcal{W}) = (K_W + \Delta)^2,$$

so that

$$\begin{aligned} c_1^2(\mathcal{W}) &= K_W^2 + 2K_W\bar{C}_u + K_W(\bar{C}_o + F_d + H) + \frac{1}{4}(\bar{C}_o^2 + F_d^2 + H^2) + \bar{C}_u^2 \\ &\quad + \bar{C}_u(\bar{C}_o + F_d + H) + \frac{1}{2}(\bar{C}_oF_d + \bar{C}_oH + F_dH). \end{aligned}$$

Let  $p : Z \rightarrow W$  be the surface above  $W$  which resolves  $W$  and is a blow-up of  $\mathbb{P}^2$ . Since  $Z$  is obtained by 10 blow-ups of  $\mathbb{P}^2$  one has  $K_Z^2 = 9 - 10 = -1$ . Moreover, since all singularities but one are  $ADE$ , one has  $K_Z = p^*K_W - \frac{1}{3}D_1$  where  $D_1$  is the  $(-3)$ -curve on  $Z$  which is contracted to the  $\frac{1}{3}(1, 1)$  singularity on  $W$ . Since  $p^*K_W \cdot D_1 = 0$ , we obtain

$$K_W^2 = -\frac{2}{3}.$$

The curve  $\bar{C}_u$  is a smooth curve of genus 0 on the smooth locus of  $W$ . The blow-up at the  $\mathfrak{a}_2$ -singularity of the cuspidal cubic decreases the self-intersection by 4, the remaining blow-ups decrease the self-intersection by 1. Since one has  $4 + 2 + 3 = 9$  such blow-ups, one gets

$$\bar{C}_u^2 = 3^2 - 4 - 9 = -4,$$

and therefore  $K_W\bar{C}_u = 2$ . Let  $\tilde{D}$  be the strict transform on  $Z$  of a curve  $D$  on  $W$  or  $\mathbb{P}^2$ . We have

$$\tilde{C}_o = p^*\bar{C}_o - aF_1.$$

Since  $\tilde{C}_o F_l = 2$ , then  $a$  is equal to 1. Since moreover  $\tilde{C}_o^2 = 0$ , we get  $0 = (\tilde{C}_o)^2 = \bar{C}_o^2 - 2$ , thus  $\bar{C}_o^2 = 2$ . We have

$$K_W \bar{C}_o = (\tilde{C}_o + F_l) \left( K_W + \frac{1}{3} D_1 \right) = -2.$$

Let  $F_1, F_2, F_3 \subset Z$  be the chain of three  $(-2)$ -curves above the  $A_3$  singularity in  $W$ , so that  $\tilde{F}_d F_1 = 1$ . One computes that

$$\tilde{F}_d = p^* F_d - \frac{1}{4} (3F_1 + 2F_2 + F_3)$$

(it is easy to check that  $\tilde{F}_d F_1 = 1$ ,  $\tilde{F}_d F_2 = \tilde{F}_d F_3 = 0$ ). Then

$$-1 = \tilde{F}_d^2 = F_d^2 - \frac{3}{4}$$

gives  $F_d^2 = -\frac{1}{4}$ . One has

$$K_W F_d = \left( K_Z + \frac{1}{3} D_1 \right) \left( \tilde{F}_d + \frac{1}{4} (3F_1 + 2F_2 + F_3) \right) = -1.$$

Let  $D_1, D_2$  be respectively the  $(-3)$  and  $(-2)$  curves intersecting  $\tilde{H}$ . Since  $\tilde{H} D_1 = \tilde{H} D_2 = 1$ , one has

$$\tilde{H} = p^* H - \frac{1}{3} D_1 - \frac{1}{2} D_2,$$

thus

$$-1 = \tilde{H}^2 = H^2 - \frac{1}{3} - \frac{1}{2}$$

and  $H^2 = -\frac{1}{6}$ . Moreover

$$K_W H = \left( K_Z + \frac{1}{3} D_1 \right) \left( \tilde{H} + \frac{1}{3} D_1 + \frac{1}{2} D_2 \right) = -1 + \frac{1}{3} + \frac{1}{3} - \frac{1}{3} = -\frac{2}{3}.$$

We compute therefore

$$\begin{aligned} c_1^2(\mathcal{W}) &= -\frac{2}{3} + 2 \cdot 2 + \left( -2 - 1 - \frac{2}{3} \right) + \frac{1}{4} \left( 2 - \frac{1}{4} - \frac{1}{6} \right) - 4 \\ &\quad + (2 + 1 + 1) + \frac{1}{2} (1 + 0 + 0) = \frac{9}{16}, \end{aligned}$$

thus  $c_1^2(\mathcal{W}) = 3c_2(\mathcal{W}) = \frac{9}{16}$ . □

*Remark 19.* In [14], Deraux obtains 4 different orbifold ball-quotient structures on surfaces birational to  $A/G_{48}$ . Among these, only the fourth one,  $W'$ , is invariant by the involution  $\sigma$ , the obstruction being the divisor  $E$  in [14] which creates an asymetry, unless it has weight 1. The orbifold  $\mathcal{W}$  we just described can be seen as the quotient of  $W'$  by the involution  $\sigma$ .

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Vincent Koziarz

Univ. Bordeaux, IMB, CNRS, UMR 5251, F-33400 Talence, France

`vincent.koziarz@math.u-bordeaux.fr`

Carlos Rito

*Permanent address:*

Universidade de Trás-os-Montes e Alto Douro, UTAD

Quinta de Prados

5000-801 Vila Real, Portugal

`www.utad.pt`, `crito@utad.pt`

*Temporary address:*

Departamento de Matemática  
Faculdade de Ciências da Universidade do Porto  
Rua do Campo Alegre 687  
4169-007 Porto, Portugal  
[www.fc.up.pt](http://www.fc.up.pt), [crito@fc.up.pt](mailto:crito@fc.up.pt)

Xavier Roulleau  
Aix-Marseille Université, CNRS, Centrale Marseille, I2M UMR 7373,  
13453 Marseille, France  
[Xavier.Roulleau@univ-amu.fr](mailto:Xavier.Roulleau@univ-amu.fr)