ON EULER'S ROTATION THEOREM

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It is well known that a rigid motion of the Euclidean plane can be written as the composition of at most three reflections. It is perhaps not so widely known that a similar result holds for Euclidean space in any number of dimensions.

The purpose of the present article is, firstly, to present a natural proof of this result in dimension 3 by explicitly constructing a suitable sequence of reflections [1] and, secondly, to show how a careful analysis of this construction provides a quick and pleasant geometric path to Euler's rotation theorem, and to the complete classification of rigid motions of space, whether orientation preserving or not. Finally, we present an example where we use the general scheme of our proofs to classify the composition of two explicitly given orientation preserving isometries.

We believe that our presentation will highlight the elementary nature of the results and hope that readers, perhaps especially those more familiar with the usual linear algebra approach, will appreciate the simplicity and geometric flavour of the arguments.

In view of the topic of the article any list of references is bound to be inadequate, so we provide just two: our article [1] which deals with the case of the plane, and the article [2] which gives a thorough discussion of Euler's theorem and, among several proofs, includes Euler's original one and a modern one using linear algebra.

Let $\pi = \mathbf{plane}\, ABC$ be the plane through three noncollinear points, A, B and C. We write σ^{ABC} or σ^{π} for the reflection in π and if $A \neq B$ we write $\operatorname{bis}\overline{AB}$ for the plane through the midpoint of $\overline{AB}$ perpendicular to this line, which is formed by the points at equal distance of A and B. In particular, $\sigma^{\operatorname{bis}\overline{AB}}(A) = B$ and vice-versa.

Theorem 1. *Given points A, A', B, B', C and C' in space such that A, B and C are noncollinear and $|AB| = |A'B'|$, $|AC| = |A'C'|$ and $|BC| = |B'C'|$, there exist exactly two rigid motions sending A to A', B to B' and C to C'. The first of these can be written as the composition of three reflections. The second one is obtained composing the first one with the reflection in the plane $A'B'C'$.*

Proof. First, there exists no third motion with the stated properties, since if the images of four non-collinear points are the same for two given isometries, then the images by the two isometries of any point

[1] We opted to state and prove this theorem in three dimensions, but note that this proof can be easily adapted to fit any number of dimensions.

are equal (because the distances to any four such points determine any point), and hence the isometries are the same.

Our proof now proceeds by first constructing a rigid motion i, written as the composition of three reflections and satisfying $i(A) = A'$, $i(B) = B'$ and $i(C) = C'$. The second motion will then be $j := \sigma^{A'B'C'} \circ i \neq i$. In the generic situation, when $A \neq A'$, we define $\alpha = \text{bis}\,\overline{AA'}$, so that $A' = \sigma^\alpha(A)$. Next, supposing that $B^* := \sigma^\alpha(B)$ is distinct from B', we define $\beta = \text{bis}\,\overline{B^*B'}$. Since

$$|A'B^*| = |\sigma^\alpha(A)\ \sigma^\alpha(B)| = |AB| = |A'B'|,$$

$\sigma^\beta(A') = A'$. Finally, we define $C^* = \sigma^\beta \circ \sigma^\alpha(C)$, $\gamma = \text{bis}\,\overline{C^*C'}$, and $i = \sigma^\gamma \circ \sigma^\beta \circ \sigma^\alpha$ here supposing $C^* \neq C'$). Again,

$$|A'C^*| = |\sigma^\beta \circ \sigma^\alpha(A)\ \sigma^\beta \circ \sigma^\alpha(C)| = |AC| = |A'C'|$$

and similarly

$$|B'C^*| = |\sigma^\beta \circ \sigma^\alpha(B)\ \sigma^\beta \circ \sigma^\alpha(C)| = |BC| = |B'C'|.$$

Hence, $i(A) = A'$, $i(B) = B'$ and $i(C) = C'$.

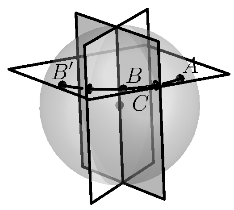
Now, if $A = A'$ then we may instead define $\alpha = \textbf{plane}\,ABC$, if $B^* = B'$ then we define $\beta = \textbf{plane}\,A'B'C$ if $C \notin \overleftrightarrow{A'B'}$ and $\beta = \textbf{plane}\,ABC$ otherwise, and if $C^* = C'$ then $\gamma = \textbf{plane}\,A'B'C'$. In all these cases [1], one still has $i(A) = A'$, $i(B) = B'$ and $i(C) = C'$. Note that if $A = A'$, $B = B'$ and $C = C'$ then $i = \sigma^{ABC}$. □

Theorem 2. *Let* $\mathfrak{m}$ *be a rigid motion in space with a fixed point,* C*, and suppose* $\mathfrak{m}$ *is not the identity.*

- (Euler) *If* $\mathfrak{m}$ *is an orientation preserving isometry then* $\mathfrak{m}$ *is a rotation about a line through* C*.*
- *If* $\mathfrak{m}$ *does not preserve orientations then* $\mathfrak{m}$ *is either an inversion in* C*, a reflection in a plane through* C*, or a* rotary reflection*, a reflection in a plane* π *through* C *followed by a rotation about a line through* C *perpendicular to* π*.*

Proof. Let A be such that $B = \mathfrak{m}(A) \neq A$. Since $|AC| = |BC|$, if for every point A the points A, B and C are collinear then for every point A, C is the midpoint of $\overline{AB}$. Hence, $\mathfrak{m}$ is the inversion in C.

Now, suppose that A, B and C are not collinear and let $B' = \mathfrak{m}(B)$. Then, $|BB'| = |AB| \neq 0$, and $|AC| = |BC| = |B'C|$. This means either that A, B and B' are not collinear or that $B' = A$. In any case, we may apply Theorem 1 to these points.



[1]In each of the cases of coincidence, if we were to instead simply omit the respective reflection from our sequence, we would still obtain a motion with the desired properties. The reason why we do not do so, is that having the specific sequence of reflections will be essential in our proof of Theorem 2.

Let us construct i and j as defined in the proof of Theorem 1. Since $\alpha = \operatorname{bis}\overline{AB}$ and $B^* = \sigma^\alpha(B) = A$, if $B' = A$ then $B' = B^*$ and $\beta = \mathbf{plane}\,ABC$, whereas if $B' \neq A$ then $B' \neq B^*$ and $\beta = \operatorname{bis}\overline{B^*B}$.

Now, $i = \sigma^\gamma \circ \sigma^\beta \circ \sigma^\alpha$ and $j = \sigma^{A'B'C} \circ i = \mathfrak{h}$ since $\mathbf{plane}\,A'B'C = \gamma$. Hence, j is a rotation about a line.

In both cases, $\mathfrak{h} = \sigma^\beta \circ \sigma^\alpha$ is a rotation about a line: if $B' = A$, it is a half turn about the line ℓ connecting C with the midpoint of $\overline{AB}$ and if $B' \neq A$ it is a rotation about the line $k = \operatorname{bis}\overline{AB} \cap \operatorname{bis}\overline{BB'} \ni C$. In both cases, $C^* = \mathfrak{h}(C) = C$ and $\gamma = \mathbf{plane}\,B'BC$. Note that since A, B and C are not collinear then $\overleftrightarrow{AB} \not\parallel \overleftrightarrow{BB'}$ and hence $\alpha \not\parallel \beta$.

As for i, if $B' = A$ then $\gamma = \beta$ and so $i = \sigma^\alpha$ is a reflection in $\operatorname{bis}\overline{AB}$, and if $B' \neq A$ then i is a rotation about k followed by a reflection in $\mathbf{plane}\,B'BC \perp k$. □

Corollary 3. *Let $\mathfrak{m}$ be a rigid motion in space different from the identity.*

- (Mozzi-Chasles) *If $\mathfrak{m}$ is an orientation preserving isometry then $\mathfrak{m}$ is a "screw displacement", that is, a rotation about a line ℓ followed (or preceded) by a (possibly trivial) translation in the direction of ℓ.*
- *If $\mathfrak{m}$ is not orientation preserving then $\mathfrak{m}$ is either an inversion, or a reflection in a plane, or a rotary reflection, or a* glide plane operation, *a reflection in a plane π followed (or preceded) by a translation in π.*

Proof. Let us fix a point A and let $\mathfrak{l}(X) = \mathfrak{m}(X) - u$, where $u = \mathfrak{m}(A) - A$. Then $\mathfrak{l}(A) = A$ and, by Theorem 2, $\mathfrak{l}$ fixes a plane π. Let $u = n + v$, where $n \perp \pi$ and $v \perp n$ is a direction of π.

- If $\mathfrak{m}$ is an orientation preserving isometry then $\mathfrak{l}$ is a rotation about a line through A, and $\mathfrak{p} = \mathfrak{l} + v$ is a rotation about a parallel line. Hence, $\mathfrak{m} = \mathfrak{p} + n$ is a screw displacement.
- If $\mathfrak{l}$ is an inversion in A and $B' = \mathfrak{l}(B)$ for a point B (and so $B' - A = -(B - A)$), then $(B' + u) - (A + u/2) = -\big(B - (A + u/2)\big)$ and hence $\mathfrak{m}$ is an inversion in $A + u/2$.

  If $\mathfrak{l}$ is a reflection in a plane through A, then $\mathfrak{l}+n$ is a reflection in a parallel plane and $(\mathfrak{l} + n) + v$ is a glide plane operation.

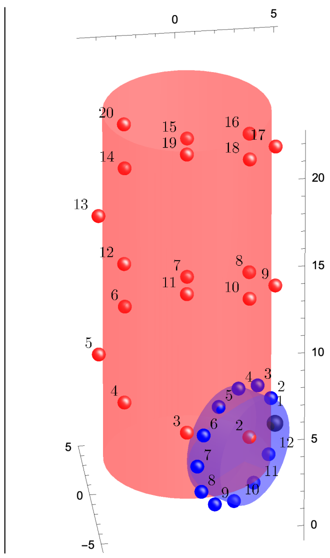
  Finally, if $\mathfrak{l}$ is a reflection σ^π in a plane π through A, followed by a rotation ρ^ℓ about a line ℓ through A perpendicular to π, then $\mathfrak{m} = (\rho^\ell + v) \circ (\sigma^\pi + n)$, a reflection in a plane parallel to π, followed by a rotation about a line parallel to ℓ. □

Example. We consider an example, based on calculations produced by *Mathematica* that follow the procedures described above.

Let f be a rotation of $\pi/6$ radians about a line ℓ through $(1, 0, 0)$ with the direction of $u = (1, -1, 0)$, followed by a translation by $v = (1, 1, 1)$.

Since $u \perp v$, it is still a rotation of $\pi/6$ about a horizontal line that one can find by solving $f(x,y,z) = (x,y,z)$.

In the figure on the right we represent with small blue spheres the iterates $f^i(x)$ for a given point x marked 1, and draw a transparent cylinder about ℓ. The same representation takes place, now in red, for a rotation g of $\pi/4$ radians about the line connecting $P = (0,0,1)$ with the origin O, followed by a translation by $w = (0,0,1)$. Note that w has the direction of the axis, and so g is a screw displacement. We want to study $h = g \circ f$. Let $k(x,y,z) = h(x,y,z) - h(0,0,0)$. Since $k(0,0,0) = (0,0,0)$, by Euler's Theorem 2, k is a rotation about a line.



Let us use the arguments of the proof of Theorem 2 to study k, so that afterwards we may describe h.

A fixed point of k is $C = (0,0,0)$. Let us take for instance $A = (1,2,-2)$; then $B = \left(\frac{\sqrt{6}}{2} + \sqrt{2}, \frac{\sqrt{6}}{2}, 1 - \sqrt{3}\right)$ and

$$B' = \left(\frac{1}{4}(7 - \sqrt{2} + 2\sqrt{3} + \sqrt{6}), \frac{1}{4}(-1 - \sqrt{2} - 2\sqrt{3} + \sqrt{6}), \frac{1}{4}(-6 + 2\sqrt{3} + \sqrt{6})\right)$$

Let $\pi =$ **plane** ABC be the fixed plane of k. Note that $k = \sigma^\beta \circ \sigma^\alpha$ where α and β are the perpendicular bisectors of $\overline{AB}$ and $\overline{BB'}$, respectively.

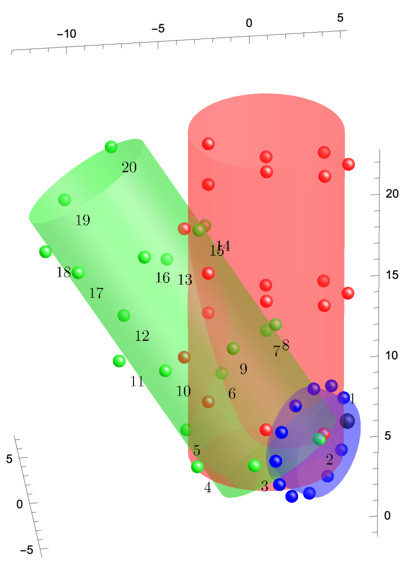
The axis of k is the intersection of both planes and has the direction of $n = \left(-1 - \sqrt{2}, 1, 2 + \sqrt{3}\right) \perp \pi$. The component of $p = h(0,0,0) = h(x,y,z) - k(x,y,z)$ in the direction of n is

$$m = \frac{p \cdot n}{n \cdot n}\, n \approx (-0.539178, 0.223335, 0.833496)$$

and

$$h(x,y,z) \approx k(x,y,z) + m + (1.25261, 0.490099, 0.678976)$$

Since the last vector is perpendicular to n, this is a rotation about an axis parallel to the axis of k and with the same angle. We now evaluate this angle from elements of k.



A normal vector to the perpendicular bisector of $\overline{AB}$ is

$$\left(-\frac{-2+2\sqrt{2}+\sqrt{6}}{2(\sqrt{3}-3)},-\frac{\sqrt{6}-4}{2(\sqrt{3}-3)},1\right)\approx(1.29261,-0.611424,1)$$

and a normal vector to the perpendicular bisector of $\overline{BB'}$ is

$$\left(\frac{7-5\sqrt{2}+2\sqrt{3}-\sqrt{6}}{-10+6\sqrt{3}+\sqrt{6}},\frac{-1-\sqrt{2}-2\sqrt{3}-\sqrt{6}}{-10+6\sqrt{3}+\sqrt{6}},1\right)\approx(0.332024,-2.93047,1)$$

Thus, the angle of the two normals (which is half the angle of rotation) is

$$\theta=\cos^{-1}\left(\frac{1}{8}\left(-4+2\sqrt{2}+2\sqrt{3}+\sqrt{6}\right)\right)\approx 0.936324\,\text{rad}\,.$$

and the rotation is about the line

$$\left\{\left(x,-\frac{(2-11\sqrt{2}+3\sqrt{3}+2\sqrt{6})x}{-20-9\sqrt{2}+7\sqrt{3}+5\sqrt{6}},-\frac{(-15+27\sqrt{2}-11\sqrt{3}+5\sqrt{6})x}{(\sqrt{3}-3)(-20-9\sqrt{2}+7\sqrt{3}+5\sqrt{6})}\right)\middle|\,x\in\mathbb{R}\right\}$$

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References

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[2] Palais, B, Palais, R, Rodi, S. A Disorienting Look at Euler's Theorem on the Axis of a Rotation, *Amer Math Monthly* **116** (2009) 892–909. URL: `https://doi.org/10.4169/000298909X477014`

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