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EXTREMAL BEHAVIOUR OF CHAOTIC DYNAMICS

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Abstract

In this work, we study the existence of limiting laws for dynamically defined stochastic processes. This type of stochastic processes are constructed by evaluating a given observable function along the orbits of a dynamical system. We consider observable functions maximized at uncountable sets that present some fractal structure, such as the ternary Cantor set or the Cantor dust. Using such observables, we establish the existence of extreme value laws for processes created using one dimensional and two dimensional piecewise uniformly expanding maps.

By making use of tools from fractal geometry, for example box dimension or Digraph Iterated Function Systems, we establish a link between the existence of clustering in the limiting law and the compatibility between the dynamics and the limiting set of the observable function. This compatibility is translated by the difference between the box dimension of the maximal set and the box dimension of its iterates. In the examples considered throughout this work, we were able to show that when exists full compatibility between the dynamics and the maximal set, in the sense that the set is preserved by the map, then the high recurrence of the maximal set to itself leads to the appearance of clusters of exceedances resulting in a Extremal Index strictly smaller than 1. On other hand, when the box dimension of the maximal set is higher than that of its iterates then there exists a negligible recurrence effect of the limiting set to itself resulting in low clustering that leads to an Extremal Index equal to 1.

To finish, we present a numerical study that intends to illustrate the usage of the Extremal Index as an indicator of the compatibility between the dynamics and the fractal structure of a set. We tested several different dynamics, such as uniformly and non-uniformly expanding maps or even irrational rotations. By using the ternary Cantor set as a limiting set, we were able to obtain numerical values for the Extremal Index of the correspondent stochastic process. The Extremal Index revealed itself as a good indicator of the compatibility between the dynamics and the geometrical structure of the limiting set. In particular, the simulations performed allowed the numerical validation of some of the theoretical results proved in this work.

Resumo

Neste trabalho, foi estudada a existência de leis limite para processos estocásticos definidos dinamicamente. Este tipo de processos estocásticos é construído avaliando um observável ao longo das orbitas do sistema. Foram considerados observáveis maximizados em conjuntos não contáveis que apresentam uma estrutura fractal, como o conjunto ternário de cantor ou a poeira de Cantor. Usando este tipo de observáveis foi estabelecida a existência de leis dos valores extremos para processos estocásticos criados usando mapas uniformemente expansores unidimensionais e bidimensionais.

Usando ferramentas de geometria fractal, como por exemplo dimensão fractal ou “Digraph Iterated Function Systems” foi estabelecida uma ligação entre a intensidade de aglomeração de observações que excedem um certo patamar (“clustering”) e a compatibilidade entre a dinâmica e o conjunto maximizante. Esta compatibilidade é traduzida pela diferença entre a dimensão fractal do conjunto limitante e dos seus iterados. Nos problemas considerados no decorrer deste trabalho foi possível demostrar que quando existe compatibilidade completa entre a dinâmica e o conjunto limitante, no sentido em que o conjunto é totalmente preservado pela dinâmica, então a alta recorrência do conjunto maximizante para ele próprio resulta numa grande intensidade de aglomeração de excedências resultando num Índice Extremal estritamente inferior a 1. Por outro lado, quando a dimensão fractal do conjunto maximizante é superior á dos seus iterados isto resulta numa menor intensidade de aglomeração de excedências o que leva a um Índice Extremal igual a 1.

Para finalizar, é apresentado um estudo numérico que pretende ilustrar o uso do Índice Extremal como indicador da compatibilidade entre a dinâmica e um dado conjunto. Foram testados vários exemplos, como dinâmicas uniformemente e não uniformemente expansoras ou rotações irracionais. Usando o conjunto ternário de Cantor como conjunto limitante foram obtidos valores numéricos para o Índice Extremal do respetivo processo estocástico. O Índice Extremal revelou-se um bom indicador da compatibilidade entre a dinâmica e a estrutura geométrica do conjunto maximizante. Em particular, foi possível validar numericamente alguns dos resultados teóricos obtidos.

Table of contents

List of figures

Chapter 1

Introduction

The study of extreme events has been a relevant field of investigation for a long time. An extreme event or rare event is usually understood as an event that has a small probability of happening. Due to this aspect, the study of such occurrences is tied to abnormal situations that pose serious hazard or cause high stress in the human population.
We come across situations like these in many fields of science and technology. As a title of example, we can consider geophysical extremes such as earthquakes, who do not only cause a huge number of fatalities but also cause serious economic damage. Another example, linked this time to climate dynamics, are heat waves or hurricanes that, again, pose a situation of serious risk to the affected communities.
Due to the impact of such events, being able to predict their regularity is of the utmost importance. This is one of the primary goals of the theory of extreme events and makes it a field of transdisciplinary research by including tools from mathematics, engineering, finances or geosciences.
It is in the overlapping between mathematics and the theory of extreme values that we can find the main subject of this thesis: the study of rare events for dynamically defined stochastic processes. A dynamically defined stochastic process is a process created by evaluating a given observable φ through the orbits of a system. In this context, a rare event is just an area of the phase space with small measure. In recent years, this field of research has seen great development. We refer the book [30] and the review paper [40] for a broad view of the field.

In this work, we will perform the study of rare events for stochastic processes arising from dynamical systems by analyzing the distribution of the maximum of the process. For that purpose, we consider that the observable φ is maximized in a region of the phase space denoted by $\mathscr{M}$. The limiting laws are achieved by considering a series of thresholds increasing to the maximum value of φ.
In the literature about this topic, the maximal set is usually considered as one single point. There exists, however, a trend to consider more sophisticated maximal sets. In [23] and [3], it was considered maximal sets containing a finite set of points and in [4] $\mathscr{M}$ was chosen to be a countable set. There are even papers, such as [7], [25] or [16], that considers a one dimensional submanifold, such as the diagonal of product spaces, as a maximal set of an observable.

In here, we will follow up on this trend and consider maximal sets with a more irregular geometric structure, such as Cantor sets. The motivation for such choice of maximal sets is the paper [31], where the authors consider the situation of fractal landscapes, with $\mathscr{M}$ taken as the ternary Cantor set. They performed a numerical study which sustained the conjecture that the same distributional limits observed when $\mathscr{M}$ was a singular point should apply for such complex maximal sets.

The use of maximal sets with such complex fractal geometric structure can be justified by the possibility of real-life applications. If one considers cases such as mine swiping, the movement of air masses, road traffic, network communications, structural safety or stock market, we easily realize that the sensitive regions of the phase space that worth study possess a complex structure.
To better illustrate the possible applications of considering more intricate maximal sets, we present a few examples from the meteorology area. The models used in weather forecast often include strange attractors. This means that the sensitive regions of the phase space that have interest may very well have a fractal geometric structure.
To be more concrete, in the paper [17] the anomalies for the precipitation frequency data have a complex geometric structure that is compatible with that of a fractal set. In a more recent paper [15], the authors observe that when greenhouse gases are enhanced, the attractor acquires some fractal structure. To finish, we mention the paper [29], where the basins of attraction of the two metastable states (Warm and Snow Ball) have a fractal structure.
All the scenarios stated above could benefit from a fully developed theory of extremes based on fractal maximal sets.

Throughout this thesis, we consider as maximal sets, various examples of low dimensional fractal sets. We take sets like the ternary Cantor set or the Cantor dust and combine them with simple dynamics such as piecewise uniformly expanding maps. This way, we present a framework that, in spite of being simple, can still capture the fractal complexity of sensitive regions. In fact, we were able to demonstrate rigorously some of the results conjectured in [31].
Furthermore, we were able to relate the appearance of clusters of exceedances to the fractal structure of the iterates of the maximal set. The existence of clusters of exceedances, *i.e* a large number of exceedances happening in a short period of time, plays a big role in the limiting laws achieved for a stochastic process. The existence of such clusters determines a parameter usually referred to as the Extremal Index (EI). This parameter ranges from 0 to 1 and essentially measures the intensity of the clusters. The Extremal Index is higher when the intensity of clustering is smaller.
It was discovered in previous works, such as [23], [3] or [4], that the recurrence of the maximal set to itself by the map T is the key to determine the intensity level of the clusters of exceedances and therefore to determine the Extremal Index. For example, when $\mathscr{M}$ is composed of one single point the periodicity of such point determines the Extremal Index. If a point is periodic, then there exists a high tendency for the exceedances to agglomerate resulting in an Extremal Index strictly smaller than 1. On other hand, if a point is not periodic then there exists a low tendency for the exceedances to agglomerate leading to an Extremal Index of 1.
In this case, however, due to complexity of the maximal sets it is necessary a finer analyses of the sets $T^{-j}(\mathscr{M}) \cap \mathscr{M}$ to determine the Extremal Index. Through the usage of concepts from fractal

geometry, we were able to identify how the nature of $T^{-j}(\mathscr{M}) \cap \mathscr{M}$ contributes for the agglomeration of exceedances.
This clustering mechanism is linked to the compatibility between the dynamics T and the geometric structure of the limiting set $\mathscr{M}$. For example, when $T = 3x \mod 1$, then the map preserves the ternary Cantor set. Using such set as the maximal set, we obtain that $\mathscr{M}$ plays the role of a periodic point, *i.e* $T^{-j}(\mathscr{M}) = \mathscr{M}$. This leads to clustering resulting in a low Extremal Index. If, however, the box dimension of $T^{-j}(\mathscr{M}) \cap \mathscr{M}$ is lower than the box dimension of $\mathscr{M}$, this implies low compatibility between the dynamics and the maximal set resulting in a Extremal Index of 1 and in the absence of clustering.

The link identified between the Extremal Index and the compatibility of a map with a maximal set opens a new possibility for the usage of the EI. One can think of using the Extremal Index as an indicator of the compatibility between a map and a given set to indirectly express how relevant is $T^{-j}(\mathscr{M}) \cap \mathscr{M}$ when compared with $\mathscr{M}$ itself. To illustrate this possibility, we present a numerical study where we test several dynamics. Using these dynamics in conjunction with the ternary Cantor set, we created stochastic processes and obtain an estimate, using the formula presented by Hsing in [24], for the correspondent EI. In our simulations, the EI has accurately detected the relevance of $T^{-j}(\mathscr{M}) \cap \mathscr{M}$ when compared with $\mathscr{M}$. In particular, we were able to numerically confirm some of the theoretical statements made in this work.

The contents of this thesis are largely based in the article “Rare Events for Cantor Target Sets” submitted to the scientific magazine “Communications in Mathematical Physics” and already available as a pre-print on arXiv (see [35]).
This thesis will follow the structure described below:

- In Chapter 2, we present an introduction to the general laws and concepts used in the classical theory of extreme events. At the same time, we introduce some of the framework necessary to study the limiting behaviour of dynamically defined stochastic processes.

- In Chapter 3, we establish sufficient conditions that allow to prove the existence of extreme value laws. We provide results that guarantee the existence of limiting laws for stochastic processes built using unidimensional and two dimensional maps with sufficiently fast decay of correlations.

- In Chapters 4 and 5, using the ternary Cantor set and the Cantor dust as limiting sets of observables, we rigorously prove the existence of cylinder extreme value laws for the correspondent stochastic processes. This results are achieved using unidimensional and two dimensional piecewise uniformly expanding maps. During the exposition, we identify a mechanism that links the geometrical structure of the limiting set and the dynamics which is responsible for the value of the Extremal Index.

- In Chapter 6, we provide a numerical study that intents to demonstrate that the Extremal Index can be a good indicator of the compatibility between the dynamics and the geometrical structure of a given set. In particular, this study numerically validates some of the results presented in Chapters 4 and 5.

- In the Appendix A, we state some preliminary definitions and results necessary for the theoretical results presented in this work.

Chapter 2

Laws of Extreme Events

The purpose of the theory of extreme events is to analyze the occurrence of events that have a small probability to happen. This small probability needs a quantification which requires a probabilistic framework. The starting point of the Extreme Value Theory (EVT) is then a probability space $\mathscr{X}$ associated with a σ-algebra β and a probability measure μ. This probability measure measures the likelihood that an event, $A \in \mathscr{B}$, has to occur. With this setup, the quantification of the word small becomes more clear. We say that A is a rare event if $\mu(A) \leq c$, where c is chosen to be small. This choice of c is still rather vague and depends upon the context of the events that we are studying. The analyses of rare events draw its motivation from our necessity to understand unwanted scenarios or high risk incidents, such as, climate incidents or financial crisis. The objective is, somehow, to assert how likely is for such situations to repeat itself in the future. Due to this time concern, we consider that $\mathscr{X}$ is the space of realization of a collection of random variables, $X_0, X_1, X_2, \ldots$, that can represent any quantity that is relevant for the considered scenario. Such a collection of random variables is called a stochastic process and is denoted by $(X_n)_{n \in \mathbb{N}}$.
To a random variable, X_i, we associate a distribution function, F, that is defined as

$$F(x) = \mu(X_i \leq x),$$

for all x in $\mathbb{R}$.
Similarly, for all $x \in \mathbb{R}$, the joint distribution function, F_j, of a finite collection of random variables, $X_{i_1}, X_{i_2}, X_{i_3}, \ldots, X_{i_j}$ is defined as

$$F_{i_1, i_2, \ldots, i_j}(x) = \mu(X_{i_1} \leq x, X_{i_2} \leq x, X_{i_3} \leq x, \ldots, X_{i_j} \leq x).$$

We deal only with stochastic processes that are stationary. This means that the joint distribution of any finite collection of random variables, $X_{i_1}, X_{i_2}, X_{i_3}, \ldots, X_{i_j}$, that belong to the stochastic process is the same as the joint distribution function of $X_{i_1+t}, X_{i_2+t}, X_{i_3+t}, \ldots, X_{i_j+t}$, for every time displacement t.
Under this setup, rare events become tied to abnormal observations in a collection of random variables of the stochastic process. This abnormality can be expressed into very large or very small values taken

by the random variables. Typically, we consider only very large values. So, a rare event corresponds to an exceedance of a high threshold u by a random variable X_i, that is

$$U(u) := \{X_i > u\}.$$

To further justify the word rare, this threshold u is chosen to be close of the right endpoint of the distribution function of X_i, *i.e*

$$u_F = \sup\{x : F(x) < 1\}.$$

As stated before, we are interested in the occurrence of exceedances of a high threshold. For that purpose, we consider the collection of random variables $X_1, X_2, X_3, \ldots, X_n$ and define

$$M_n = \max\{X_1, X_2, X_3, \ldots, X_n\}. \tag{2.0.1}$$

Note that, $(M_n)_{n\in\mathbb{N}}$ is itself a stochastic process. The knowledge of M_n allows us to determine whether an exceedance of a threshold u has occurred upon the first n observations. It is only necessary to observe if $\{M_n \leq u\}$ has occurred or not. Hence, it is natural to ask if we can find a distributional limit for M_n. This is the primary concern of EVT.

Definition 2.0.1. We have an Extreme Value Law (EVL) for M_n if it exists a non-degenerated distribution function $H : \mathbb{R} \to [0,1]$, with $H(0) = 0$ and for every $\tau > 0$ and all $n \in \mathbb{N}$, there exists a sequence of thresholds $u_n = u_n(\tau)$ such that

$$n\mu(X_0 > u_n) \to \tau \text{ as } n \to \infty, \tag{2.0.2}$$

and for which the following holds:

$$\mu(M_n \leq u_n) \to (1 - H)(\tau) \text{ as } n \to \infty, \tag{2.0.3}$$

for all continuity points of $H(\tau)$.

By non-degenerated distribution function, we mean that there is no $x_0 \in \mathbb{R}$ such that $H(x_0) = 1$ and $H(x) = 0$, for all $x > x_0$.
In the definition above, the limiting law for M_n is found using a normalizing sequence $(u_n)_{n\in\mathbb{N}}$ satisfying (2.0.2). This normalizing sequence has its roots in the case where the stochastic process X_n is composed by independent identically distributed (i.i.d) random variables. In this case, the distribution function is the same for all random variables. If F represents such a distribution function, then using (2.0.2),

$$\mu(M_n \leq u_n) = (1 - \mu(X_0 > u_n))^n \sim \left(1 - \frac{\tau}{n}\right)^n \to e^{-\tau} \text{ as } n \to \infty$$

and, in this case, $H(\tau) = 1 - e^{-\tau}$ is the exponential distribution function.
In this setting, there exists a more heuristic view of condition (2.0.2). It expresses that, the mean number of exceedances approaches τ as the time goes to infinity.

The sequence of thresholds u_n is usually taken in the form

$$u_n = \frac{y}{a_n} + b_n$$

where $y \in \mathbb{R}$ and $a_n > 0$, for all $n \in \mathbb{N}$. Consequently, the distributional limit of M_n is written in the form

$$\mu(a_n(M_n - b_n) \leq y).$$

It is for the i.i.d setting that appears the first main theorem of EVT. It is usually referred to as the *Extremal Types Theorem* and is due to Gnedenko [22].

Theorem 2.0.2. *Let $X_0, X_1, \ldots$ be a sequence of i.i.d random variables and assume that there exists linear normalizing sequences $(a_n)_n$ and $(b_n)_n$, with $a_n > 0$ for all $n \in \mathbb{N}$, such that*

$$\mu(a_n(M_n - b_n) \leq y)) \to G(y), \tag{2.0.4}$$

where $G(y)$ is non-degenerate.
Then

$$G(y) = e^{-\tau(y)}$$

where, under linear normalization, $\tau(y)$ is one of the next three types:

- $\tau_1(y) = e^{-y}$ *for* $y \in \mathbb{R}$
- $\tau_2(y) = y^{-\beta}$ *for* $y, \beta > 0$
- $\tau_3(y) = (-y)^{\gamma}$ *for* $y \leq 0$ *and* $\gamma > 0$.

The three types mentioned in the Extremal Types Theorem are usually called Gumbel or Type 1 if $\tau(y) = \tau_1(y)$, Fréchet or Type 2 if $\tau(y) = \tau_2(y)$ and Weibull or Type 3 if $\tau(y) = \tau_3(y)$. This theorem is quite remarkable since it limits the search for limiting laws to only three types of non-degenerated distribution functions.
Next, we present an example of how to compute an EVL for a stochastic process constituted by i.i.d random variables.

Example 2.0.3. Let $X_0, X_1, \ldots$ to be i.i.d random variables with exponential distribution of parameter 1, that is,

$$F(x) = 1 - e^{-x} \text{ for } x > 0.$$

For any $\tau > 0$, consider u_n such that $n(1 - F(u_n)) = \tau$.
This implies that,

$$u_n = \log(n) - \log(\tau).$$

Hence,

$$\mu(M_n - \log(n) \leq -\log(\tau)) \to e^{-\tau}.$$

Putting $\tau = e^{-y}$, we get

$$\mu(M_n - \log(n) \leq y)) \to e^{-e^{-y}}.$$

Therefore, in this case, M_n follows a Gumbel EVL with $a_n = 1$ and $b_n = \log(n)$.

With an i.i.d setting the computation of an EVL is very straightforward once we have access to the distribution function, F, of the random variables. In fact, it is the behaviour of the tail of F that determines the type of limiting law. More precisely, if $\bar{F} = (1-F)$ then is the speed at which $\bar{F}(u)$ approaches 0 when $u \to u_F$ that determines the type of EVL. We can state necessary and sufficient conditions on the tail of F that determine such type:

- $G(y)$ is Gumbel if and only if there exists a strict positive $h : \mathbb{R} \to \mathbb{R}$ such that, for all $y \in \mathbb{R}$,

$$\lim_{s \to u_F} \frac{\bar{F}(s + yh(s))}{\bar{F}(s)} = e^{-y}.$$

- $G(y)$ is Fréchet if and only if $u_F = +\infty$ and there exists $\beta > 0$ such that, for all $y > 0$,

$$\lim_{s \to u_F} \frac{\bar{F}(sy)}{\bar{F}(s)} = y^{-\beta}.$$

- $G(y)$ is Weibull if and only if $u_F < +\infty$ and there exists $\gamma > 0$ such that, for all $y > 0$,

$$\lim_{s \to 0} \frac{\bar{F}(u_F - sy)}{\bar{F}(u_F - s)} = y^{-\beta}.$$

2.1 Extreme Value Laws and Stationarity

What was written above is a very concise resume of EVT in the i.i.d. setting. After such accomplishments, the focus was changed to the study of stationary dependent stochastic processes. This work was started by Loynes in [28] and further developed by Leadbetter in [26]. It was Leadbetter who proposed a sort of mixing conditions on the stochastic process that guarantee the existence of the same distributional limits as in the i.i.d case.
These conditions depend upon the thresholds u_n and are called $D(u_n)$ and $D'(u_n)$.

Condition ($D(u_n)$). We say that condition $D(u_n)$ holds for the sequence $X_0, X_1, \ldots$ if for any integers $i_1 < \ldots < i_p$ and $j_1 < \ldots < j_k$ for which $j_1 - i_p > m$ and any large $n \in \mathbb{N}$,

$$|F_{i_1,\ldots,i_p,j_1,\ldots,j_k}(u_n) - F_{i_1,\ldots,i_p}(u_n)F_{j_1,\ldots,j_k}(u_n)| \leq \alpha(n,t) \tag{2.1.1}$$

uniformly for every $p, k \in \mathbb{N}$, where $\alpha(n, t_n) \to 0$ as $n \to$ for some sequence $t_n = o(n)$.

The form of condition $D(u_n)$ resembles the condition of independence between samples of random variables. This happens since condition $D(u_n)$ is imposing that any two blocks of random variables,

$X_{i_1},\dots,X_{i_p}$ and $X_{j_1},\dots,X_{j_k}$, that are separated by a sufficiently big time gap, are in some sort independent.

It is in this context of breaking the stochastic process X_n into blocks of random variables, that appears condition $D'(u_n)$.

Let $(k_n)_{n\in\mathbb{N}}$ be a sequence satisfying

$$k_n \to \infty, \qquad \lim_{n\to\infty} k_n\alpha(n,t) = 0 \qquad \text{and} \qquad k_n t_n = o(w_n). \tag{2.1.2}$$

Condition $(D'(u_n))$**.** We say that condition $D'(u_n)$ holds for the sequence $X_0, X_1, \dots$ if it exists $(k_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ satisfying (2.1.2) such that,

$$\limsup_{n\to\infty} n \sum_{j=1}^{\lfloor n/k_n \rfloor} \mu(X_0 > u_n, X_j > u_n) = 0. \tag{2.1.3}$$

Condition $D'(u_n)$ essentially breaks the stochastic process into k_n blocks of $\lfloor n/k_n \rfloor$ random variables and limits the number of exceedances in each of the blocks. The dependence of the process X_n may lead to the occurrence of several exceedances in a short period of time. These bursts of extreme observations happening in a short time gap are usually referred to as clustering. Since $D'(u_n)$ limits the number of exceedances in each of the blocks, this condition is seen as an anti-clustering condition.

When coupled together, these two conditions, $D(u_n)$ and $D'(u_n)$ sustain the following result.

Theorem 2.1.1 ([27], Theorem 1.2)**.** *Consider a stationary stochastic process* $(X_n)_{n\in\mathbb{N}}$ *and let* $(u_n)_{n\in\mathbb{N}}$ *be a sequence of thresholds such that*

$$n\mu(X_0 > u_n) \to \tau \text{ as } n \to \infty,$$

for some $\tau > 0$.

Assume that conditions $D(u_n)$ *and* $D'(u_n)$ *hold. Then,*

$$\lim_{n\to\infty} \mu(M_n \le u_n) = \mathrm{e}^{-\tau}.$$

Depending upon the dependence of τ on the thresholds u_n, the theorem above guarantees that the type of limiting laws achieved for the stationary case are the same as in the i.i.d case.

The theorem depends upon two conditions, hence, we can ask what happens if condition $D(u_n)$ holds and condition $D'(u_n)$ does not hold. In this case, we cannot assert the existence of a limiting law for M_n. However, if such limit exists, we can say something about the type of law expected.

Theorem 2.1.2 ([27], Theorem 2.2)**.** *Consider a stationary stochastic process* $(X_n)_{n\in\mathbb{N}}$ *and let* $(u_n)_{n\in\mathbb{N}}$ *be a sequence of thresholds such that* (2.0.2) *holds for some* $\tau > 0$.

Assume that condition $D(u_n)$ *holds for every* τ*. If the limit* (2.0.3) *exists, then exists a parameter* $0 \le \theta \le 1$ *such that*

$$(1-H)(\tau) = e^{-\theta\tau}.$$

We stated before that condition $D'(u_n)$ prevented the observation of clusters of exceedances. From Theorem 2.1.2, we can deduce that clustering does not affect the type of limiting law one can expect. It leads, however, to the appearance of a parameter θ. This parameter is called the Extremal Index (EI) associated with the EVL.
Dependent stochastic processes may have a tendency to have memory. This implies that whenever an exceedance occurs, this memory effect can result in the appearance of a large number of extreme observations in a reduced time gap.
This Extremal Index is then a sort of measure of how much memory a stochastic process has.
If $\theta = 1$ the process is low dependent and as practically no memory. The result is a low tendency to observe clusters of exceedances. On other hand, if θ is close to zero, then the process carries a lot of memory and we should expect exceedances to agglomerate.
It is possible to assert the existence of a distributional limit for heavy dependent stochastic processes. It is only necessary to introduce a condition to replace $D'(u_n)$. Here, we present a condition presented in [8] and denoted by $D^{(k)}(u_n)$.

Condition ($D^{(k)}(u_n)$)**.** We say that condition $D^{(k)}(u_n)$ holds for the sequence $X_0, X_1, \ldots$ if it exists $(k_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ satisfying (2.1.2) such that,

$$\lim_{n\to\infty} n\mu\left(X_0 > u_n \geq M_{1,k-1}, M_{k,\lfloor n/k_n \rfloor -1} > u_n\right) = 0,$$

where $M_{i,j} := +\infty$ for $i > j$ and $M_{i,j} := \max\{X_i, \ldots, X_j\}$ for $i \leq j$.

We point out that, when $k = 1$ condition $D^{(k)}(u_n)$ is equivalent to condition $D'(u_n)$.
Together with $D(u_n)$, condition $D^{(k)}(u_n)$ permits to prove the existence of an EVL given by

$$(1-H)(\tau) = e^{-\theta\tau},$$

for a stationary stochastic process.
The computation of the EI is done using O'Brien's formula presented in [37], that is

$$\theta = \lim_{n\to\infty} \theta_n = \lim_{n\to\infty} \frac{\mu\left(X_0 > u_n, X_1 \leq u_n, \ldots, X_{k-1} \leq u_n\right)}{\mu\left(X_0 > u_n\right)}. \tag{2.1.4}$$

The next result summarizes such findings.

Theorem 2.1.3 ([8])**.** *Consider $(X_n)_{n\in\mathbb{N}}$ to be a stationary stochastic process and let $(u_n)_{n\in\mathbb{N}}$ be a sequence of thresholds satisfying* (2.0.2)*. Assume that $D^{(k)}(u_n)$ holds and that* $\liminf_{n\to\infty} \mu\left(M_n \leq u_n\right) > 0$*. If, for each positive k, $D^{(k)}(u_n)$ holds, then*

$$\lim_{n\to\infty} \left(\mu\left(M_n \leq u_n\right) - e^{-\theta_n\tau}\right) = 0.$$

Moreover, if the limit in formula (2.1.4) *exists, then*

$$\lim_{n\to\infty} \mu\left(M_n \leq u_n\right) = e^{-\theta\tau}.$$

2.2 Extreme Value Laws and Dynamical Systems

The theory exposed so far does not make any assumption on how the information about the phenomena we are analyzing is obtained. The starting point is simply a stochastic process.
Mathematical sciences are, however, able to provide a wide variety of mathematical models capable of capture the time evolution of many natural, social or even engineering phenomena. One of the mathematical research fields that is more capable of providing such models is called the field of *dynamical systems*.
A dynamical system is composed by one phase space, $\mathscr{X}$, where each point represents a state of the system. The time evolution of the system is then represented by a map $T : \mathscr{X} \to \mathscr{X}$. We are interested in the probabilistic study of extremes. Therefore, we endow $\mathscr{X}$ with a measure-theoretical structure, that is, there exists a $\sigma-$ algebra $\mathscr{B}$ and a probability measure μ associated to $\mathscr{X}$. We refer to this structure as a *probability space* and denote it by $(\mathscr{X}, \mathscr{B}, \mu)$.
The dynamics T is always a measurable map with respect to the measure μ. Moreover, we will require μ to be an invariant measure for the map T.

Definition 2.2.1. Let $(\mathscr{X}, \mathscr{B}, \mu)$ be a probabilistic space. We say that μ is an invariant measure for a measurable map $T : \mathscr{X} \to \mathscr{X}$ if $\forall A \in \mathscr{B}$,

$$\mu(T^{-1}(A)) = \mu(A).$$

In this work, we always consider time-discrete models. We start with a random point, $x \in \mathscr{X}$ and analyze the orbit of such point, that is, $x, T(x), T^2(x), \dots, T^n(x)$, where

$$T^n(x) = \underbrace{T \circ T \circ T \dots \circ T(x)}_{n \text{ times}}.$$

In the last section, we labeled a rare event as an event that has low probability to occur. In the framework of dynamical systems, this can be translated by the orbit of a point entering a region A of the phase space such that $\mu(A)$ is small.
Define the *first hitting time* $r_A : \mathscr{X} \to \mathbb{N} \cup \{+\infty\}$ as

$$r_A := \inf\{n \in \mathbb{N} : T^n(x) \in A\}.$$

We say that a system is ergodic with respect to a probability measure μ if for all $A \in \mathscr{B}$, where $T^{-1}(A) = A$, then either $\mu(A) = 0$ or $\mu(A) = 1$.
Under the ergodic assumption, there exists a theorem by Kac that states that the mean return time to A is equal to $1/\mu(A)$. This implies that, in average, r_A should go to infinity as $\mu(A) \to 0$. This way the study of distributional limits for r_A is an approach to the study of rare events in the context of dynamical systems. This approach is usually called *Hitting Time Statistics*.
One can, however, build upon the classical Extreme Value Theory to present another approach.
We consider an observable function $\varphi : \mathscr{X} \to \mathbb{R} \cup \{+\infty\}$ and evaluate the value of φ along the orbit of

a point $x \in \mathscr{X}$. This way, we construct a stochastic process, $(X_n)_{n\in\mathbb{N}}$, such that

$$X_n = \varphi \circ T^n. \tag{2.2.1}$$

With this framework, having an exceedance of a threshold u_n is equivalent to say that the orbit of x has entered the region of the phase space determined by

$$U(u_n) = \{X_0 > u_n\}.$$

There is a connection between Hitting Time Statistics and this extremes based approach. The first hint to such connection comes from noting that, if X_n shows no exceedances up to time n then $r_{U(u_n)}$ must be larger than n, *i.e*

$$T^{-1}(\{M_n \leq u_n\}) = \left\{r_{U(u_n)} > u_n\right\}.$$

A link has, in fact, been established for the case where exists an exact correspondence between the points where φ exceeds a high threshold and the sets used for the study of Hitting Time Statistics. We will not go deep into such connection, but more information on the topic can be found in [19] and [30].

The dynamical systems considered for this analyses based on the extremes must exhibit a behaviour that is hard to be understood. Chaotic dynamical systems provide excellent models of such behaviour. These systems possess high sensibility to initial conditions and exhibit limits for the deterministic prediction of the behaviour of its orbits. We can take as an example the doubling map. This map is a uniformly expanding map defined on the circle $\mathscr{S}^1$ such that, to each point x, it associates the point $2x$ mod 1. Figure 2.1 shows a graphical representation of this system, where $\mathscr{S}^1$ is identified with the segment $[0,1]$.

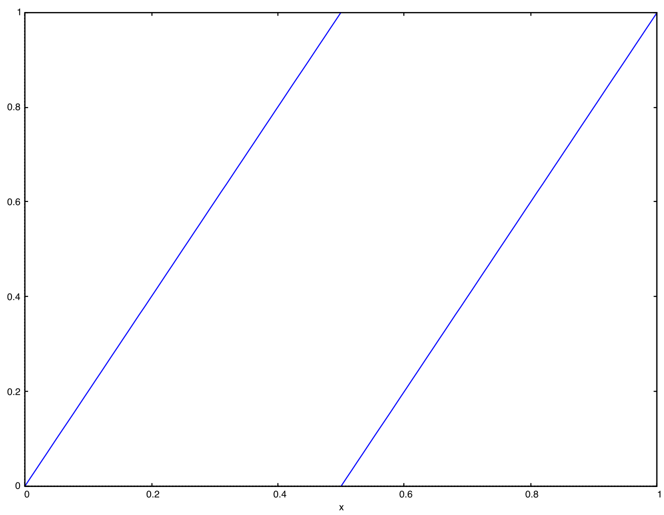

Fig. 2.1 Graphical representation of the doubling map.

The orbits of this system can be fully categorized depending upon its initial condition. If the initial

condition is a rational number then we have a periodic orbit, however, if the initial condition is an irrational number the resulting orbit is non-periodic and eventually reaches every subset of $[0,1]$ that is detected by the one dimensional Lebesgue measure.
It is this erratic behaviour that makes chaotic dynamical systems suitable for a probabilistic analysis.

The invariance of the probability measure, with respect to the map T assures the stationarity of the stochastic process $(X_n)_{n\in\mathbb{N}}$ given by (2.2.1). One could think about applying conditions $D(u_n)$ and $D'(u_n)$, in association with O'Brien's formula in (2.1.4) to obtain a distributional limit for the process of partial maxima M_n. However, this approach is not adequate for stochastic processes arising from dynamical systems.
To be able to prove condition $D(u_n)$ of Leadbetter in the dynamical systems setting one must rely on the mixing properties of the systems. But condition $D(u_n)$ requires a bound that it is independent of the number of random variables considered in the blocks. It is this requirement for a uniform bound that makes this condition very hard to prove using the knowledge about mixing rates of the dynamical systems.
Several attempts were made to find milder conditions that can be verified in this context. This revision of the theory started with the work of Collet in [9]. In [18] Freitas and Freitas propose a condition called $D_2(u_n)$. This condition was much weaker than the original condition $D(u_n)$ of Leadbetter. Most importantly, it was constructed to follow from the decay of correlations of the underlying dynamical system.
The pursue for more weak mixing conditions continued and in [34] and [23] the authors were able to obtain limiting laws for processes based on Benedicks-Carleson quadratic maps and other non-uniformly expanding dynamical systems.
Nevertheless, all of these achievements were made in the absence of clustering.
The presence of clustering in dynamically generated stochastic processes is linked to the periodicity of the points that constitute the set of maximal points of the observable φ. This link appeared in [49] and new conditions were devised to prove limiting laws with $\theta < 1$. These conditions were denoted by $Д_q(u_n)$ and $Д'_q(u_n)$. The use of such conditions also allowed the choice of more general maximal sets, but always of finite nature. In the next chapter, we will present, in detail, the last achievements. Following [4], we present conditions $Д_{q_n}(u_n)$ and $Д'_{q_n}(u_n)$. These conditions allow proving the existence of extreme value laws for a large set of dynamically generated stochastic processes. In particular, as we will see in this work, they allow the possibility of maximizing the observable function in more intricate and geometrically rich sets of infinite nature, such as, one dimensional and two dimensional fractal sets.

2.3 Observables and Maximal sets

There are two main ingredients to generate stochastic processes from dynamical systems. One is the dynamics itself, where the mixing rates of the dynamical system, in the form of decay of correlations,

play a big role in the dependence conditions necessary to achieve a distributional limit. The other ingredient is the observable function, in particular, the set where it is maximized.
For a first approach to this matter one can consider observables whose maximal set, $\mathscr{M}$, consists of one point of the phase space, *i.e.*, $\mathscr{M} = \{\zeta\}$, where, $\zeta \in \mathscr{X}$. To that purpose let $\varphi : \mathscr{X} \to \mathbb{R} \cup \{+\infty\}$ be an observable, such that

$$\varphi(x) = g(\mathrm{dist}(x, \zeta)). \tag{2.3.1}$$

We assume that, the function $g : [0, +\infty] \to \mathbb{R} \cup \{+\infty\}$ achieves its global maximum at 0, possibly $+\infty$, and is a strictly decreasing bijection on a neighbourhood of 0. Note that, for an observable of this form the set $U(u_n)$ corresponds to a topological ball centered at the point ζ.
Under sufficiently fast decay of correlations of the map in some specific spaces, in [2] a dichotomy was achieved:

- $\zeta \in \mathscr{X}$ is a non periodic point and the EI associated with the EVL is 1.
- $\zeta \in \mathscr{X}$ is a periodic point and the EI associated with the EVL is lower than 1.

This result starts to show the importance of the maximal set to determine the level of clustering in the limiting law. The recurrence to itself of the maximal set, in this case determined by the periodicity of the point ζ, is the responsible for the appearance of clusters of exceedances.
The next step was to consider larger maximal sets. In [23] and [3], it was considered a maximal set formed by a finite group of points.
The observable considered was a version of the one presented in (2.3.1) adapted to the context of multiple maximal points. More precisely, denoting by $\zeta_1, \ldots, \zeta_k$ the points of $\mathscr{M}$, the observable φ could by written in the form,

$$\varphi(x) = g_i(\mathrm{dist}(x, \zeta_i)), \tag{2.3.2}$$

for a family of functions g_i, with $i \in \{1, \ldots, k\}$, satisfying similar conditions to the ones presented for the function g.
In the first article, it was considered maximal points chosen independently as typical points for the invariant probability measure. The result was uncorrelated maximal points in the sense that an orbit of a point in $\mathscr{M}$ does not contain any other point in the maximal set. This assumption on $\mathscr{M}$ led to low recurrence of the maximal set to itself and to an Extremal Index of 1.
On other hand, in [3] it was studied the effect of correlated maxima in the maximal set. It was considered that a point in $\mathscr{M}$ contains in its orbit the other points of the maximal set. The result was a clustering effect created by this correlation even without considering periodic points in the composition of the maximal set. Consequently, an Extremal Index lower than 1 was achieved.
Similar results were proved for the case where $\mathscr{M}$ is composed by a countable number of points. In [4], the maximal set was considered to be the closure of the orbit of a point in the phase space. The type of observable used was the one given by (2.3.2) considering, however, a countable family of functions g_i.
It was discovered that a fast recurrence of the maximal set to itself results in EI smaller than 1 and a slow recurrence of the maximal set to itself results in a negligible clustering effect leading, in the limit, to an EI equal to 1.

The common ground in these results, and ultimately responsibly for the clustering effect, is the orbit of the maximal set as a whole. Denoting the dynamics by T, this recurrence of the maximal set can be captured by the study of the sets $T^{-j}(\mathscr{M})\cap\mathscr{M}$. In the case where $\mathscr{M}$ is reduced to a single periodic point, we know that, due to this periodicity, $T^{-j}(\mathscr{M})$ will collide with $\mathscr{M}$ creating multiple exceedances in a short gap of time.
For a maximal set composed of uncorrelated maxima, the fact that the orbits of the points in $\mathscr{M}$ do not hit $\mathscr{M}$ leads to an absence of clustering. If the maximal set is composed by a finite number of correlated points, then $T^{-j}(\mathscr{M})\cap\mathscr{M}$ is a non-empty set and bursts of exceedances are expected.
The infinite nature of a countable maximal set in [4] adds something new to this mechanism. To create cluster we not only need $T^{-j}(\mathscr{M})\cap\mathscr{M}$ to be non empty but also that the maximal set reoccurs relatively fast.

In this work, we present an extension to the previous results. We will consider an observable that is maximized in a fractal set. The prototype set chosen to demonstrate the results is the ternary Cantor set, although some of the results are proven for more general Cantor sets such as attractors of Iterated Functions Systems.
This work draws its motivation from the study published in [31]. It was conjectured that the same limiting laws that apply to the case where $\mathscr{M}$ is a single point or a set of points should apply to this more complicated sets. Using the ternary Cantor set, the authors were able to numerically verify the existence of the distributional limits. We present here rigorous proofs of the existence of such limiting laws for uniformly expanding maps.
The existence of clustering was not detected in any of the experiments made in [31]. However, in line of the observations made for the cases where $\mathscr{M}$ is single point or a set of points, we were able to show the existence of clusters depending upon the choice of dynamics.
We identified the mechanism leading to the appearance of clusters and, again, the recurrence of the maximal set played a crucial role. Choosing the ternary Cantor set as the maximal set of an observable, we selected a map that preserves the entire set. The choice was the map $3x \mod 1$. For this map the ternary Cantor set acts as fixed point leading to a very big overlap between $T^{-j}(\mathscr{M})$ and $\mathscr{M}$, which resulted in the appearance of an EI smaller than 1.
Using concepts from fractal geometry, we were capable of translating to a number the overlap between $T^{-j}(\mathscr{M})$ and $\mathscr{M}$. This was done by evaluating the box dimension of the set $T^{-j}(\mathscr{M})\cap\mathscr{M}$. The relevance of the recurrence of the maximal set to itself was then assessed by comparing the box dimension of the sets $\mathscr{M}$ and $T^{-j}(\mathscr{M})\cap\mathscr{M}$. If the dynamics T is compatible with $\mathscr{M}$, in the sense that $T^{-j}(\mathscr{M})\cap\mathscr{M}$ and $\mathscr{M}$ have the same box dimension, we were able to show the existence of clustering due to the big overlap between $\mathscr{M}$ and its own orbit.
However, if $T^{-j}(\mathscr{M})\cap\mathscr{M}$ has a smaller box dimension than $\mathscr{M}$, then we prove the existence of distributional limits with EI equal to 1.
These results follow the trend initiated with the case of a countable number of points in the maximal set, where an EI equal to one was achieved even if $T^{-j}(\mathscr{M})\cap\mathscr{M}$ is not an empty set. In this case, the recurrence velocity of the maximal set was a factor to have in consideration for the appearance of

clusters. If $\mathscr{M}$ is a Cantor set, for example the ternary Cantor set, its distribution along the segment $[0,1]$ makes very easy for the set $T^{-j}(\mathscr{M}) \cap \mathscr{M}$ to be non-empty. What dictates the appearance or not of clusters of exceedances is how big is the overlap between $T^{-j}(\mathscr{M})$ and $\mathscr{M}$ in terms of their box dimension.

Building upon the one dimensional results achieved for one dimensional fractal sets, we were able to extend the theory for two dimensional fractal maximal sets.
We considered two dimensional fractal sets constructed using the direct product between one dimensional sets. This product structure allowed for a calculation of the EI based upon a decomposition of $(X_n)_{n\in\mathbb{N}}$ into processes generated by one dimensional maps and observables.
Under some compatibility conditions between the observable used to generate X_n and the observables used to construct its one dimensional decomposition, we were able to assert the existence of limiting laws independently of the level of clustering.

Chapter 3

Existence of Limiting Laws

The main purpose of this chapter is to provide sufficient conditions that allow us to prove the existence of limiting laws for dynamically generated stochastic processes.
In this first part of the chapter, we rigorously establish the notation and concepts used throughout this work.

Start by considering a discrete dynamical system $(\mathscr{X},\mathscr{B},T,\mu)$, where $\mathscr{X}$ is a compact set, $\mathscr{B}$ is the respective Borel sigma algebra, $T:\mathscr{X}\to\mathscr{X}$ is a measurable map and μ is an invariant measure with respect to T.
Considering an observable function $\varphi:\mathscr{X}\to\mathbb{R}^+\cup\{+\infty\}$, we define a stochastic process, $(X_n)_{n\in\mathbb{N}}$, as in (2.2.1), *i.e*

$$X_n(x)=\varphi\circ T^n(x).$$

The process $(X_n)_{n\in\mathbb{N}}$, constructed as above, is stationary due to the invariance of the measure μ.
As seen in the last chapter, the extremal behaviour of the process is linked to the recurrence properties of the set of global maxima of φ, hence, we assume that there exists $Z=\max_{x\in\mathscr{X}}\varphi(x)$, where we allow $Z=+\infty$.
Following the notation already introduced, we define the set of maximal points of φ as

$$\mathscr{M}=\{x\in\mathscr{X}:\ \varphi(x)=Z\},$$

and denote by ζ a generic point of this set.
From the stochastic process $(X_n)_{n\in\mathbb{N}}$, we define the process of partial maxima $(M_n)_{n\in\mathbb{N}}$ whose limiting distribution we want to analyse:

$$M_n=\max\{X_0,\dots,X_{n-1}\}. \tag{3.0.1}$$

Given a sequence of thresholds $(u_n)_{n\in\mathbb{N}}$, our objective is to achieve a non-degenerated distribution law, H, such that

$$\mu(M_n\le u_n)\to(1-H)(\tau)\ \text{as}\ n\to\infty.$$

In the definition of Extreme Value Law given by 2.0.1, this thresholds u_n are dependent on τ. This dependence is translated by relation (2.0.2), *i.e*

$$n\mu(X_0 > u_n) \to \tau \text{ as } n \to \infty.$$

When the observables are based on topological balls around the points that form the maximal set of the observable, such as in (2.3.1) and (2.3.2), it is assumed that $n\mu(X_0 > u_n)$ is sufficiently smooth to allow the existence of thresholds u_n such that the limit in (2.0.2) exists. However, this is not always the case. In [20], it was considered observables based on cylinders, such as

$$\varphi(x) = g(\mu(Z_n(\zeta))), \tag{3.0.2}$$

where n is maximal, such that, $x \in Z_n(\zeta)$ and g is a function satisfying certain regularity conditions. Using the *tent map*, $T : [0,1] \to [0,1]$ given by

$$T(x) = 1 - |2x - 1|$$

and for this choice of observable, the quantity $n\mu(X_0 > u_n)$ varies too much, hence, it does not exists a sequence of thresholds u_n for which the limit in (2.0.2) is verified.
This issue was addressed in [20] and the authors introduced the concept of *cylinder* EVL. In this definition, especially fitted for cylinders based observables, the thresholds u_n are taken such that,

$$w_n\mu(X_0 > u_n) \to \tau \text{ as } n \to \infty. \tag{3.0.3}$$

In this normalization, we take a subsequence w_n of the time n. Such modification makes more likely to exist a sequence of thresholds u_n such that the limit in (3.0.3) holds, even if $n\mu(X_0 > u_n)$ varies wildly. For this type of limiting law, instead of built the dependence of $n\mu(X_0 > u_n)$ on τ in the thresholds, we built the dependence on τ into the time scale. This way for all $n \in \mathbb{N}$, we can consider

$$w_n = w_n(\tau) = \lfloor \tau(\mu(X_0 > u_n))^{-1} \rfloor. \tag{3.0.4}$$

We now state the definition of cylinder EVL.

Definition 3.0.1. Let $(X_n)_{n\in\mathbb{N}}$ and $(M_n)_{n\in\mathbb{N}}$ be the stochastic processes defined as in (2.2.1) and (3.0.1). Consider $(w_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$ such that,

$$w_n\mu(X_0 > u_n) \to \tau \text{ as } n \to \infty. \tag{3.0.5}$$

We say that a cylinder EVL for the stochastic process $(M_n)_{n\in\mathbb{N}}$ exists if

$$\mu(M_{w_n} \leq u_n) \to 1 - H(\tau) \tag{3.0.6}$$

for some non-degenerate H.

We note that, the definition of a cylinder EVL is weaker than the regular definition of EVL given by 2.0.1. We are only require convergence for certain subsequences of the time. It is, however, this weakness that allows a larger applicability assuring, nevertheless, the existence of a non-degenerated distribution function.
When maximizing observables in Cantor sets, we will be considering observables based upon the algorithmic construction of these sets, which is essentially a discrete process. The result is observables that are not smooth and it is not possible to guarantee the existence of thresholds u_n as in (2.0.2). For that reason, all the limiting laws we achieve are of the form given by (3.0.6).
Through the remaining of this work, we will be referring to a cylinder EVL simply as EVL.

3.1 Conditions $Д_{q_n}(u_n, w_n)$ and $Д'_{q_n}(u_n, w_n)$

In this section, we present two conditions, $Д_{q_n}(u_n, w_n)$ and $Д'_{q_n}(u_n, w_n)$, acting on the dependence structure of the stochastic process that, in conjunction, allow to prove the existence the distributional limits stated in Definition 3.0.1.

Let $(u_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ be as in (3.0.5) and consider a sequence $(q_n)_{n\in\mathbb{N}}$, such that

$$\lim_{n\to\infty} q_n = \infty \qquad \text{and} \qquad \lim_{n\to\infty} \frac{q_n}{w_n} = 0. \tag{3.1.1}$$

Denote the i-th preimage by the map T by T^{-i} and fix $u \in \mathbb{R}$ and $q \in \mathbb{N}$. We define the following events,

$$\begin{aligned} U(u) &:= \{X_0 > u\}, \\ \mathscr{A}_q(u) &:= U(u) \cap \bigcap_{i=1}^{q} T^{-i}(U(u)^c) = \{X_0 > u, X_1 \le u, \ldots, X_q \le u\}. \end{aligned} \tag{3.1.2}$$

The event $U(u)$ corresponds to the occurrence of an exceedance and the event $\mathscr{A}_q(u)$ corresponds to the occurrence of an exceedance which terminates a cluster of exceedances, *i.e.*, if $T^{-j}(\mathscr{A}_q(u))$ occurs, then the next exceedance after the one observed at time j must belong to a new and different cluster of exceedances. In particular, q can be thought as the maximal waiting time between two exceedances within the same cluster.

Let $B \in \mathscr{B}$ be an event and for $s, \ell \in \mathbb{N}$, define,

$$\mathcal{W}_{s,\ell}(B) = \bigcap_{i=s}^{s+\ell-1} T^{-i}(B^c).$$

Observe that, under the notation above $\mathcal{W}_{0,n}(U(u_n)) = \{M_n \le u_n\}$.
For each $n \in \mathbb{N}$, set $U_n := U(u_n)$ and $\mathscr{A}_{q_n,n} := \mathscr{A}_{q_n}(U(u_n))$. O'Brien's formula to compute the EI, stated

in (2.1.4) has a natural reformulation using the setting above. Just consider

$$\theta_n := \frac{\mu\left(\mathscr{A}_{q_n,n}\right)}{\mu(U_n)}. \tag{3.1.3}$$

and the Extremal Index is given by

$$\theta = \lim_{n\to\infty} \theta_n.$$

Using the notation established, we now state conditions $Д_{q_n}(u_n, w_n)$ and $Д'_{q_n}(u_n, w_n)$. This conditions are a variation, adapted to cylinders and non-smooth observables, of conditions $Д_{q_n}(u_n)$ and $Д'_{q_n}(u_n)$ that appear in [4]. Such conditions were primarily devised to deal with countable maximal sets where a point $\zeta \in \mathscr{M}$ has the possibility of having arbitrarily large periods. This was accomplished by introducing a sequence $(q_n)_{n\in\mathbb{N}}$, satisfying (3.1.1), to replace the static factor $q \in \mathbb{N}$ that appears in the conditions $Д_q(u_n)$ and $Д'_q(u_n)$. Conditions $Д_q(u_n)$ and $Д'_q(u_n)$ appear in [2], [3] and [20], where they are used to prove the existence of limiting laws in the case of finite maximal sets.
The uncountable structure of Cantor sets makes the possibility of a point to have arbitrarily large periods a very likely scenario. This justifies our use of conditions based on a sequence of integers q_n instead of the more usual conditions $Д_q(u_n)$ and $Д'_q(u_n)$ based on a fixed q.

Condition ($Д_{q_n}(u_n, w_n)$)**.** We say that condition $Д_{q_n}(u_n, w_n)$ holds for the stochastic process $(X_n)_{n\in\mathbb{N}}$ if for every $\ell, t, n \in \mathbb{N}$

$$\left|\mu\left(\mathscr{A}_{q_n,n} \cap \mathcal{W}_{t,\ell}\left(\mathscr{A}_{q_n,n}\right)\right) - \mu\left(\mathscr{A}_{q_n,n}\right)\mu\left(\mathcal{W}_{0,\ell}\left(\mathscr{A}_{q_n,n}\right)\right)\right| \le \gamma(n,t), \tag{3.1.4}$$

where $\gamma(n,t)$ is decreasing in t for each n and there exists a sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n = o(w_n)$ and $w_n\gamma(n,t_n) \to 0$ when $n \to \infty$.

Consider the sequence $(t_n)_{n\in\mathbb{N}}$ given by condition $Д_{q_n}(u_n, w_n)$ and let $(k_n)_{n\in\mathbb{N}}$ be another sequence of integers such that

$$k_n \to \infty \quad \text{and} \quad k_n t_n = o(w_n). \tag{3.1.5}$$

Condition ($Д'_{q_n}(u_n, w_n)$)**.** We say that condition $Д'_{q_n}(u_n, w_n)$ holds for the sequence $(X_n)_{n\in\mathbb{N}}$ if there exists a sequence $(k_n)_{n\in\mathbb{N}}$ satisfying (3.1.5) such that

$$\lim_{n\to\infty} w_n \sum_{j=q_n+1}^{\lfloor w_n/k_n\rfloor - 1} \mu\left(\mathscr{A}_{q_n,n} \cap T^{-j}\left(\mathscr{A}_{q_n,n}\right)\right) = 0. \tag{3.1.6}$$

Condition $Д_{q_n}(u_n, w_n)$ establishes a sort of asymptotic independence between the occurrence of the event $\mathscr{A}_{q_n,n}$ and the absence of occurrences of such an event in the time gap $t - l$. That is, if after a cluster, we do not observe any exceedance for q_n time steps, then the observation of the next exceedance is almost an independent even. This type of reasoning is similar to that expressed by condition $D(u_n)$ stated in (2.1.1). As in $D(u_n)$, condition $Д_{q_n}(u_n, w_n)$ requires an asymptotic independence on blocks of random variables that are well spaced in time. In this case, this blocks are represented by the set $\mathscr{A}_{q_n,n}$. There exists, however, a crucial difference. This new condition $Д_{q_n}(u_n, w_n)$ imposes a bound only on

q_n random variables. On the other hand, condition $D(u_n)$ imposes a bound that is independent of the number of random variables considered in the blocks. It is precisely this aspect that allows $Д_{q_n}(u_n, w_n)$ to follow from the decay of correlations of the maps involved.
Condition $Д'_{q_n}(u_n, w_n)$ is pursuing the same objective as condition $D'(u_n)$ stated in (2.1.3). It is imposing that clusters of exceedances are spaced through time to prevent its concentration.

Coupling conditions $Д_{q_n}(u_n, w_n)$ and $Д'_{q_n}(u_n, w_n)$ provides a general result that allows us to establish the existence of a limiting extreme value law.

Theorem 3.1.1. *Let* $(X_n)_{n\in\mathbb{N}}$ *be a stochastic process constructed as in* (2.2.1). *Consider the sequences* $(u_n)_{n\in\mathbb{N}}$ *and* $(w_n)_{n\in\mathbb{N}}$ *satisfying* (3.0.5) *for some* $\tau \geq 0$. *Assume that conditions* $Д_{q_n}(u_n, w_n)$ *and* $Д'_{q_n}(u_n, w_n)$ *hold for some* $q_n \in \mathbb{N}_0$ *satisfying* (3.1.1). *Moreover, assume that the sequence* $(\theta_n)_{n\in\mathbb{N}}$ *defined in* (3.1.3) *converges to some* $0 \leq \theta \leq 1$, i.e. $\theta = \lim_{n\to\infty} \theta_n$.
Then,

$$\lim_{n\to+\infty} \mu(M_{w_n} \leq u_n) = e^{-\theta\tau}.$$

The proof of this theorem follows from an easy adjustment of the proof of [30, Corollary 4.1.7].

3.2 Application to One Dimensional Systems

The main advantage of the conditions presented in the previous section, when compared with the usual ones from the classical Extreme Value Theory, is that they were constructed to follow easily for system with nice decay of correlations.

Definition 3.2.1 (Decay of correlations)**.** Let $\mathscr{C}_1, \mathscr{C}_2$ denote Banach spaces of real-valued measurable functions defined on $\mathscr{X}$. We define the *correlation* of non-zero functions $\phi \in \mathscr{C}_1$ and $\psi \in \mathscr{C}_2$ with respect to a measure μ as

$$\operatorname{Cor}_\mu(\phi, \psi, n) := \frac{1}{\|\phi\|_{\mathscr{C}_1}\|\psi\|_{\mathscr{C}_2}} \left| \int \phi\,(\psi \circ T^n)\,\mathrm{d}\mu - \int \phi\,\mathrm{d}\mu \int \psi\,\mathrm{d}\mu \right|.$$

We say that the dynamical sytem $(\mathscr{X}, \mathscr{B}, T, \mu)$ has *decay of correlations*, with respect to the measure μ, for observables in $\mathscr{C}_1$ *against* observables in $\mathscr{C}_2$ if there exists a rate function $\rho : \mathbb{N} \to \mathbb{R}$, with

$$\lim_{n\to\infty} \rho(n) = 0,$$

such that, for every $\phi \in \mathscr{C}_1$ and every $\psi \in \mathscr{C}_2$, we have

$$\operatorname{Cor}_\mu(\phi, \psi, n) \leq \rho(n).$$

The definition of decay of correlations presumes that the observables used belong to some Banach space. For that purpose, we will define the Banach space of functions of Bounded Variation, starting with the concept of *variation* of an observable.

Definition 3.2.2. Given an observable $\psi : I \to \mathbb{R} \cup \{+\infty\}$ defined on an interval I, the *variation* of ψ is denoted by

$$\mathrm{Var}(\psi) := \sup \left\{ \sum_{i=0}^{n-1} |\psi(x_{i+1}) - \psi(x_i)| \right\},$$

where the supremum is taken over all finite ordered sequences $(x_i)_{i=0}^{n} \subset I$.

Using the norm $\|\psi\|_{BV} = \sup|\psi| + \mathrm{Var}(\psi)$, the space of functions of Bounded Variation,

$$BV := \{\psi : I \to \mathbb{R} : \|\psi\|_{BV} < \infty\},$$

is a Banach space.
The one dimensional systems that we will work with have decay of correlations of functions of Bounded Variation against observables in $L^1(\mu)$. It is our objective to see that conditions $Д_{q_n}(u_n, w_n)$ and $Д'_{q_n}(u_n, w_n)$ follow from the above mentioned type of decay of correlations of the underlying dynamical system.

Theorem 3.2.3. *Let $(\mathscr{X}, \mathscr{B}, T, \mu)$ be a dynamical system and consider an observable φ achieving a global maximum on a set $\mathscr{M}$. Let $(X_n)_{n\in\mathbb{N}}$ be the stochastic process constructed as in* (2.2.1) *and consider $(u_n)_{n\in\mathbb{N}}$, $(w_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ as sequences such that* (3.0.5) *and* (3.1.1) *hold. If the system has decay of correlations of observables in $\mathscr{C}_1$ against observables in $L^1(\mu)$ and if*

1. $\lim_{n\to\infty} \|\mathbf{1}_{\mathscr{A}_{q_n,n}}\|_{\mathscr{C}_1} w_n \rho(t_n) = 0$ *or* $\lim_{n\to\infty} w_n \left(\|\mathbf{1}_{U_n}\|_{\mathscr{C}_1} \rho(t_n) + 2\mu(U_n \setminus \mathscr{A}_{q_n,n})\right) = 0$*, for some sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n = o(w_n)$*
2. $\lim_{n\to\infty} \|\mathbf{1}_{U_n}\|_{\mathscr{C}_1} \sum_{j=q_n}^{\infty} \rho(j) = 0$

and if the sequence $(\theta_n)_{n\in\mathbb{N}}$ defined in (**??**) *converges to some $0 \le \theta \le 1$, then conditions $Д_{q_n}(u_n, w_n)$ and $Д'_{q_n}(u_n, w_n)$ are satisfied and*

$$\lim_{n\to\infty} \mu(M_{w_n} \le u_n) = \mathrm{e}^{-\theta\tau}.$$

Remark 3.2.4. We remark that, under the assumption of summable decay of correlations against L^1, hypothesis (1) implies $Д_{q_n}(u_n, w_n)$, while hypothesis (2) implies $Д'_{q_n}(u_n, w_n)$.

Proof. By Theorem 3.1.1, we only need to check that the stochastic process $(X_n)_{n\in\mathbb{N}}$ satisfies conditions $Д_{q_n}(u_n, w_n)$ and $Д'_{q_n}(u_n, w_n)$.
Consider $\phi = \mathbf{1}_{\mathscr{A}_{q_n,n}}$ and $\psi = \mathbf{1}_{\mathscr{W}_{t,\ell}(\mathscr{A}_{q_n,n})}$ in Definition 3.2.1. Then, there exists $C > 0$, such that, for

any positive numbers ℓ and t, we have

$$\begin{aligned}|\mu(\mathscr{A}_{q_n,n}\cap \mathcal{W}_{t,\ell}(\mathscr{A}_{q_n,n}))-\mu(\mathscr{A}_{q_n,n})\mu(\mathcal{W}_{0,\ell}(\mathscr{A}_{q_n,n}))|\\ &=\left|\int_{\mathscr{X}}\mathbf{1}_{\mathscr{A}_{q_n,n}}\cdot(\mathbf{1}_{\mathcal{W}_{0,\ell}(\mathscr{A}_{q_n,n})}\circ T^t)d\mu-\int_{\mathscr{X}}\mathbf{1}_{\mathscr{A}_{q_n,n}}d\mu\int_{\mathscr{X}}\mathbf{1}_{\mathcal{W}_{0,\ell}(\mathscr{A}_{q_n,n})}d\mu\right|\\ &\leq C\|\mathbf{1}_{\mathscr{A}_{q_n,n}}\|_{\mathscr{C}_1}\rho(t).\end{aligned}$$

Hence, if there exists a sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n=o(w_n)$ and $\lim_{n\to\infty}\|\mathbf{1}_{\mathscr{A}_{q_n,n}}\|_{\mathscr{C}_1}w_n\rho_{t_n}=0$ condition $Д_{q_n}(u_n,w_n)$ follows.
To verify the alternate version of hypothesis (1), consider $\phi=\mathbf{1}_{(U_n\setminus\mathscr{A}_{q_n,n})\cup\mathscr{A}_{q_n,n}}$ and $\psi=\mathbf{1}_{\mathcal{W}_{t,\ell}(\mathscr{A}_{q_n,n})}$ in Definition 3.2.1. Again, there exists a $C>0$, such that, for any positive numbers ℓ and t,

$$\begin{aligned}|\mu\left((U_n\setminus\mathscr{A}_{q_n,n})\cup\mathscr{A}_{q_n,n})\cap\mathcal{W}_{t,\ell}(\mathscr{A}_{q_n,n})\right)-\mu\left((U_n\setminus\mathscr{A}_{q_n,n})\cup\mathscr{A}_{q_n,n}\right)\mu(\mathcal{W}_{0,\ell}(\mathscr{A}_{q_n,n}))|\\ \leq C\|\mathbf{1}_{U_n}\|_{\mathscr{C}_1}\rho(t).\end{aligned}\tag{3.2.1}$$

Moreover, since $U_n\setminus\mathscr{A}_{q_n,n}$ and $\mathscr{A}_{q_n,n}$ are disjoint, note that

$$\begin{aligned}&|\mu((U_n\setminus\mathscr{A}_{q_n,n})\cup\mathscr{A}_{q_n,n})\cap\mathcal{W}_{t,\ell}(\mathscr{A}_{q_n,n}))-\mu((U_n\setminus\mathscr{A}_{q_n,n})\cup\mathscr{A}_{q_n,n})\mu(\mathcal{W}_{0,\ell}(\mathscr{A}_{q_n,n}))|\\ =&|\mu(((U_n\setminus\mathscr{A}_{q_n,n})\cap\mathcal{W}_{t,\ell}(\mathscr{A}_{q_n,n}))\cup(\mathscr{A}_{q_n,n}\cap\mathcal{W}_{t,\ell}(\mathscr{A}_{q_n,n}))-\mu((U_n\setminus\mathscr{A}_{q_n,n})\cup\mathscr{A}_{q_n,n})\mu(\mathcal{W}_{0,\ell}(\mathscr{A}_{q_n,n}))|\\ =&|\mu((U_n\setminus\mathscr{A}_{q_n,n})\cap\mathcal{W}_{t,\ell}(\mathscr{A}_{q_n,n}))-\mu(U_n\setminus\mathscr{A}_{q_n,n})\mu(\mathcal{W}_{0,\ell}(\mathscr{A}_{q_n,n}))\\ &\qquad+\mu(\mathscr{A}_{q_n,n}\cap\mathcal{W}_{t,\ell}(\mathscr{A}_{q_n,n}))-\mu(\mathscr{A}_{q_n,n})\mu(\mathcal{W}_{0,\ell}(\mathscr{A}_{q_n,n}))|.\end{aligned}$$

Let $A:=\mu((U_n\setminus\mathscr{A}_{q_n,n})\cap\mathcal{W}_{t,\ell}(\mathscr{A}_{q_n,n}))-\mu(U_n\setminus\mathscr{A}_{q_n,n})\mu(\mathcal{W}_{0,\ell}(\mathscr{A}_{q_n,n}))$ and $B:=\mu(\mathscr{A}_{q_n,n}\cap\mathcal{W}_{t,\ell}(\mathscr{A}_{q_n,n}))-\mu(\mathscr{A}_{q_n,n})\mu(\mathcal{W}_{0,\ell}(\mathscr{A}_{q_n,n}))$, then using (3.2.1), we obtain that

$$|A+B|+|A|\leq C\|\mathbf{1}_{U_n}\|_{\mathscr{C}_1}\rho(t)+|A|.$$

But, using the triangular inequality,

$$|A|\leq 2\mu(U_n\setminus\mathscr{A}_{q_n,n})$$

and, therefore

$$|A+B|+|A|\leq C\|\mathbf{1}_{U_n}\|_{\mathscr{C}_1}\rho(t)+2\mu(U_n\setminus\mathscr{A}_{q_n,n}).$$

Using again the triangular inequality, we achieve that

$$|\mu(\mathscr{A}_{q_n,n}\cap\mathcal{W}_{t,\ell}(\mathscr{A}_{q_n,n}))-\mu(\mathscr{A}_{q_n,n})\mu(\mathcal{W}_{0,\ell}(\mathscr{A}_{q_n,n}))|\leq C\|\mathbf{1}_{U_n}\|_{\mathscr{C}_1}\rho(t)+2\mu(U_n\setminus\mathscr{A}_{q_n,n}).$$

Consequently, condition $Д_{q_n}(u_n,w_n)$ follows if there exists a sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n=o(w_n)$ and

$$\lim_{n\to\infty}w_n\left(\|\mathbf{1}_{U_n}\|_{\mathscr{C}_1}\rho(t_n)+2\mu(U_n\setminus\mathscr{A}_{q_n,n})\right)=0.$$

To prove that condition $Д'_{q_n}(u_n, w_n)$ holds, start by observing that

$$w_n \sum_{j=q_n+1}^{\lfloor w_n/k_n \rfloor} \mu\left(\mathscr{A}_{q_n,n} \cap T^{-j}(\mathscr{A}_{q_n,n})\right) \leq w_n \sum_{j=q_n+1}^{\lfloor w_n/k_n \rfloor} \mu\left(U_n \cap T^{-j}(U_n)\right)$$

Then, we take $\phi = \psi = \mathbf{1}_{U_n}$ in Definition 3.2.1, to obtain that

$$\mu\left(U_n \cap T^{-j}(U_n)\right) = \int_{\mathscr{X}} \phi \cdot (\phi \circ T^j) d\mu \leq (\mu(U_n))^2 + \|\mathbf{1}_{U_n}\|_{\mathscr{C}_1} \mu(U_n) \rho(j).$$

Let t_n be as above and take $(k_n)_{n\in\mathbb{N}}$ as in (3.1.5).
Recalling that $\lim_{n\to\infty} w_n \mu(U_n) = \tau$, it follows that

$$\begin{aligned} w_n \sum_{j=q_n+1}^{\lfloor w_n/k_n \rfloor} \mu\left(U_n \cap T^{-j}(U_n)\right) &\leq w_n \left\lfloor \tfrac{w_n}{k_n} \right\rfloor \mu(U_n)^2 + w_n \|\mathbf{1}_{U_n}\|_{\mathscr{C}_1} \mu(U_n) \sum_{j=q_n+1}^{\lfloor w_n/k_n \rfloor} \rho_j \\ &\leq \frac{w_n^2 \mu(U_n)^2}{k_n} + w_n \|\mathbf{1}_{U_n}\|_{\mathscr{C}_1} \mu(U_n) \sum_{j=q_n}^{\infty} \rho(j) \\ &\leq \frac{\tau^2}{k_n} + \tau \|\mathbf{1}_{U_n}\|_{\mathscr{C}_1} \sum_{j=q_n}^{\infty} \rho(j) \xrightarrow[n\to\infty]{} 0, \end{aligned}$$

by choice of k_n and hypothesis (2). □

3.3 Application to Two Dimensional Systems

Similarly to the last section, the objective is to achieve a set of sufficient conditions that guarantee that $Д_{q_n}(u_n, w_n)$ and $Д'_{q_n}(u_n, w_n)$ hold for a class of systems defined in a two dimensional space with some sort of decay of correlations against L^1. When working with maps defined in a two dimensional space, we will assume that they have decay of correlations for quasi-Hölder functions against observables in $L^1(\mu)$.
We start by defining the space of quasi-Hölder functions.

Definition 3.3.1. Given an observable $\psi : I \to \mathbb{R}^n$ and a Borel set $Z \subseteq \mathbb{R}^n$, we define the *oscillation* of $\psi \in L^1(\mu)$ over Z as

$$\operatorname{osc}(\psi, Z) := \operatorname*{ess\,sup}_Z \psi - \operatorname*{ess\,inf}_Z \psi.$$

It is possible to verify that $x \mapsto \operatorname{osc}(\psi, B_\varepsilon(x))$ is a measurable function (see [39, Proposition 3.1]). Consider real numbers $0 < \alpha \leq 1$ and $\varepsilon_0 > 0$. The α-seminorm of ψ is defined as

$$|\psi|_\alpha = \sup_{0<\varepsilon\leq\varepsilon_0} \varepsilon^{-\alpha} \int_{\mathbb{R}^n} \operatorname{osc}(\psi, B_\varepsilon(x)) d\mu.$$

The space of functions with bounded α-seminorm is denoted by

$$V_\alpha = \left\{ \psi \in L^1(\mu) : |\psi|_\alpha < \infty \right\}.$$

If we endow V_α with the norm,

$$\|\cdot\|_\alpha = \|\cdot\|_{L^1(\mu)} + |\cdot|_\alpha,$$

then, it becomes a Banach space, called the space of quasi-Hölder functions.

We will be considering two dimensional dynamical systems that are constructed as the product of unidimensional maps.
Consider the dynamical systems $(\mathscr{X}, \mathscr{B}, T_1, \mu)$ and $(\mathscr{X}, \mathscr{B}, T_2, \mu)$. From these maps, we define a product map $T : \mathscr{X}^2 \to \mathscr{X}^2$, whose invariant measure is $\mu \times \mu$, as

$$T(x_1, x_2) = (T_1(x_1), T_2(x_2)). \tag{3.3.1}$$

Choosing an observable $\psi : \mathscr{X}^2 \to \mathbb{R}$, we will see that conditions $Д_{q_n}(u_n, w_n)$ and $Д'_{q_n}(u_n, w_n)$ hold for the stochastic process $X_n = \psi \circ T^n$ and follow from the decay of correlations mentioned above. Moreover, we will show that is possible to prove that condition $Д'_{q_n}(u_n, w_n)$ holds using only the decay of correlations of the maps T_1 and T_2.
For that purpose, let φ_1 and φ_2 be two observables achieving a global maximum in the sets $\mathscr{M}_1$ and $\mathscr{M}_2$, respectively. Define the stochastic processes

$$X_n^1 = \varphi_1 \circ T_1^n(x) \qquad \text{and} \qquad X_n^2 = \varphi_2 \circ T_2^n(x).$$

Let $(u_n)_{n \in \mathbb{N}}$ be a sequence of thresholds and consider the sets

$$U_n^{T_1} = \{x \in \mathscr{X} : \varphi_1(x) > u_n\} \qquad \text{and} \qquad U_n^{T_2} = \{x \in \mathscr{X} : \varphi_2(x) > u_n\}$$

associated with X_1 and X_2, respectively.
Assume that the observable ψ achieves a global maximum on the set $\mathscr{M}_1 \times \mathscr{M}_2$, such that, the set $U_n = \{x \in \mathscr{X}^2 : \psi(x) > u_n\}$ can be written as

$$U_n = U_n^{T_1} \times U_n^{T_2}. \tag{3.3.2}$$

Denoting $\mu \times \mu$ by μ^2 and using the setting presented above and under the hypothesis of decay of correlations against L^1 of the maps involved, the next Theorem states sufficient conditions for $Д_{q_n}(u_n, w_n)$ and $Д'_{q_n}(u_n, w_n)$ to hold.

Theorem 3.3.2. *Let T be a dynamical system defined as in* (3.3.1) *and consider an observable ψ, achieving a global maximum on a set $\mathscr{M}_1 \times \mathscr{M}_2$. Let $(X_n)_{n \in \mathbb{N}}$ be the stochastic process constructed as in* (2.2.1) *and consider sequences $(u_n)_{n \in \mathbb{N}}$, $(w_n)_{n \in \mathbb{N}}$ and $(q_n)_{n \in \mathbb{N}}$ such that* (3.0.5), (3.1.1) *and* (3.3.2) *hold. Assume that T has decay of correlations of functions in $\mathscr{C}_1$ against observables in $L^1(\mu^2)$ and*

that T_1 and T_2 have decay of correlations of functions in $\mathscr{C}_2$ against observables in $L^1(\mu)$.
If,

1. $\lim_{n\to\infty} \|\mathbf{1}_{\mathscr{A}_{q_n,n}}\|_{\mathscr{C}_1} w_n\rho(t_n) = 0$ *or* $\lim_{n\to\infty} w_n \left(\|\mathbf{1}_{U_n}\|_{\mathscr{C}_1}\rho(t_n) + 2\mu^2(U_n \setminus \mathscr{A}_{q_n,n})\right) = 0$*, for some sequence* $(t_n)_{n\in\mathbb{N}}$ *such that* $t_n = o(w_n)$

2. $\lim_{n\to\infty} \left\|\mathbf{1}_{U_n^{T_1}}\right\|_{\mathscr{C}_2} \mu\left(U_n^{T_2}\right) \sum_{j=q_n}^{\infty} \rho_j^1 = 0$

3. $\lim_{n\to\infty} \left\|\mathbf{1}_{U_n^{T_2}}\right\|_{\mathscr{C}_2} \mu\left(U_n^{T_1}\right) \sum_{j=q_n}^{\infty} \rho_j^2 = 0$

4. $\lim_{n\to\infty} \left\|\mathbf{1}_{U_n^{T_1}}\right\|_{\mathscr{C}_2} \left\|\mathbf{1}_{U_n^{T_2}}\right\|_{\mathscr{C}_2} \sum_{j=q_n}^{\infty} \rho_j^1\rho_j^2 = 0$

then conditions $Д_{q_n}(u_n,w_n)$ *and* $Д'_{q_n}(u_n,w_n)$ *are satisfied. Furthermore, if the sequence* $(\theta_n)_{n\in\mathbb{N}}$ *defined in* (3.1.3) *converges to some* $0 \le \theta \le 1$ *then*

$$\lim_{n\to\infty} \mu^2(M_{w_n} \le u_n) = \mathrm{e}^{-\theta\tau}.$$

Remark 3.3.3. As in Theorem 3.2.3, we remark that, under the assumption of summable decay of correlations against L^1, hypothesis (1) implies $Д_{q_n}(u_n,w_n)$, while hypothesis (2) implies $Д'_{q_n}(u_n,w_n)$.

Proof. The dynamical system T has decay of correlations, with respect to the measure μ^2, for functions in $\mathscr{C}_1$ against observables in $L^1(\mu^2)$. Denote the correspondent rate function by ρ.
Similarly, the maps T_1 and T_2 have decay of correlations, with respect to the measure μ, for functions in $\mathscr{C}_2$ against observables in $L^1(\mu)$. Let ρ^1 and ρ^2 denote the respective rate functions.
By Theorem 3.1.1, we only need to check that the stochastic process $(X_n)_{n\in\mathbb{N}}$ satisfies conditions $Д_{q_n}(u_n,w_n)$ and $Д'_{q_n}(u_n,w_n)$.
As in the proof of Theorem 3.2.3, consider $\phi = \mathbf{1}_{\mathscr{A}_{q_n,n}}$ and $\psi = \mathbf{1}_{\mathcal{W}_{t,\ell}(\mathscr{A}_{q_n,n})}$ in Definition 3.2.1. Then, there exists $C>0$, such that, for any positive numbers ℓ and t, we have

$$\begin{aligned}|\mu^2(\mathscr{A}_{q_n,n}\cap \mathcal{W}_{t,\ell}(\mathscr{A}_{q_n,n})) - \mu^2(\mathscr{A}_{q_n,n})\mu^2(\mathcal{W}_{0,\ell}(\mathscr{A}_{q_n,n}))| \\ = \left|\int_{\mathscr{X}^2} \mathbf{1}_{\mathscr{A}_{q_n,n}}\cdot(\mathbf{1}_{\mathcal{W}_{0,\ell}(\mathscr{A}_{q_n,n})}\circ T^t)d\mu^2 - \int_{\mathscr{X}^2}\mathbf{1}_{\mathscr{A}_{q_n,n}}d\mu^2\int_{\mathscr{X}^2}\mathbf{1}_{\mathcal{W}_{0,\ell}(\mathscr{A}_{q_n,n})}d\mu^2\right| \\ \le C\|\mathbf{1}_{\mathscr{A}_{q_n,n}}\|_{\mathscr{C}_1}\rho(t).\end{aligned}$$

If there exists a sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n = o(w_n)$ and $\lim_{n\to\infty}\|\mathbf{1}_{\mathscr{A}_{q_n,n}}\|_{\mathscr{C}_1} w_n\rho_{t_n} = 0$, which is the content of the hypothesis (1), then condition $Д_{q_n}(u_n,w_n)$ follows.
Again, as in the proof of Theorem 3.2.3, to verify the alternate version of hypothesis (1), consider $\phi = \mathbf{1}_{(U_n\setminus\mathscr{A}_{q_n,n})\cup\mathscr{A}_{q_n,n}}$ and $\psi = \mathbf{1}_{\mathcal{W}_{t,\ell}(\mathscr{A}_{q_n,n})}$ in Definition 3.2.1. Then, there exists a $C>0$, such that, for

any positive numbers ℓ and t,

$$|\mu^2((U_n \setminus \mathscr{A}_{q_n,n}) \cup \mathscr{A}_{q_n,n}) \cap \mathcal{W}_{t,\ell}(\mathscr{A}_{q_n,n})) - \mu^2((U_n \setminus \mathscr{A}_{q_n,n}) \cup \mathscr{A}_{q_n,n})\mu^2(\mathcal{W}_{0,\ell}(\mathscr{A}_{q_n,n}))| \\ \leq C\|\mathbf{1}_{U_n}\|_{\mathscr{C}_1}\rho(t). \tag{3.3.3}$$

Since $U_n \setminus \mathscr{A}_{q_n,n}$ and $\mathscr{A}_{q_n,n}$ are disjoint, we have that

$$\begin{aligned} &|\mu^2((U_n \setminus \mathscr{A}_{q_n,n}) \cup \mathscr{A}_{q_n,n}) \cap \mathcal{W}_{t,\ell}(\mathscr{A}_{q_n,n})) - \mu^2((U_n \setminus \mathscr{A}_{q_n,n}) \cup \mathscr{A}_{q_n,n})\mu^2(\mathcal{W}_{0,\ell}(\mathscr{A}_{q_n,n}))| \\ =&|\mu^2((U_n \setminus \mathscr{A}_{q_n,n}) \cap \mathcal{W}_{t,\ell}(\mathscr{A}_{q_n,n})) \cup (\mathscr{A}_{q_n,n} \cap \mathcal{W}_{t,\ell}(\mathscr{A}_{q_n,n})) - \mu^2((U_n \setminus \mathscr{A}_{q_n,n}) \cup \mathscr{A}_{q_n,n})\mu^2(\mathcal{W}_{0,\ell}(\mathscr{A}_{q_n,n}))| \\ =&|\mu^2((U_n \setminus \mathscr{A}_{q_n,n}) \cap \mathcal{W}_{t,\ell}(\mathscr{A}_{q_n,n})) - \mu^2(U_n \setminus \mathscr{A}_{q_n,n})\mu^2(\mathcal{W}_{0,\ell}(\mathscr{A}_{q_n,n})) \\ &\qquad + \mu^2(\mathscr{A}_{q_n,n} \cap \mathcal{W}_{t,\ell}(\mathscr{A}_{q_n,n})) - \mu^2(\mathscr{A}_{q_n,n})\mu^2(\mathcal{W}_{0,\ell}(\mathscr{A}_{q_n,n}))|. \end{aligned}$$

Let $A := \mu^2((U_n \setminus \mathscr{A}_{q_n,n}) \cap \mathcal{W}_{t,\ell}(\mathscr{A}_{q_n,n})) - \mu^2(U_n \setminus \mathscr{A}_{q_n,n})\mu^2(\mathcal{W}_{0,\ell}(\mathscr{A}_{q_n,n}))$ and $B := \mu^2(\mathscr{A}_{q_n,n} \cap \mathcal{W}_{t,\ell}(\mathscr{A}_{q_n,n})) - \mu^2(\mathscr{A}_{q_n,n})\mu^2(\mathcal{W}_{0,\ell}(\mathscr{A}_{q_n,n}))$, then using (3.3.3), we obtain that

$$|A+B| + |A| \leq C\|\mathbf{1}_{U_n}\|_{\mathscr{C}_1}\rho(t) + |A|.$$

Using the triangular inequality,

$$|A| \leq 2\mu^2(U_n \setminus \mathscr{A}_{q_n,n})$$

and

$$|A+B| + |A| \leq C\|\mathbf{1}_{U_n}\|_{\mathscr{C}_1}\rho(t) + 2\mu^2(U_n \setminus \mathscr{A}_{q_n,n}).$$

Using again the triangular inequality, we finally achieve that

$$|\mu^2(\mathscr{A}_{q_n,n} \cap \mathcal{W}_{t,\ell}(\mathscr{A}_{q_n,n})) - \mu^2(\mathscr{A}_{q_n,n})\mu^2(\mathcal{W}_{0,\ell}(\mathscr{A}_{q_n,n}))| \leq C\|\mathbf{1}_{U_n}\|_{\mathscr{C}_1}\rho(t) + 2\mu^2(U_n \setminus \mathscr{A}_{q_n,n}).$$

Therefore, condition $Д_{q_n}(u_n, w_n)$ follows if there exists a sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n = o(w_n)$ and

$$\lim_{n\to\infty} w_n \left(\|\mathbf{1}_{U_n}\|_{\mathscr{C}_1}\rho(t_n) + 2\mu^2(U_n \setminus \mathscr{A}_{q_n,n})\right) = 0.$$

To prove condition $Д'_{q_n}(u_n, w_n)$, we start by noting that, due to (3.3.2) and since $\mathscr{A}_{q_n,n} \subseteq U_n$, we have

$$\begin{aligned} \mu^2\left(\mathscr{A}_{q_n,n} \cap T^{-j}(\mathscr{A}_{q_n,n})\right) &\leq \mu^2\left((U_n^{T_1} \times U_n^{T_2}) \cap T^{-j}(U_n^{T_1} \times U_n^{T_2})\right) \\ &= \mu^2\left((U_n^{T_1} \times U_n^{T_2}) \cap (T_1^{-j}(U_n^{T_1}) \times T_2^{-j}(U_n^{T_2}))\right) \\ &= \mu\left(U_n^{T_1} \cap T_1^{-j}(U_n^{T_1})\right)\mu\left(U_n^{T_2} \cap T_2^{-j}(U_n^{T_2})\right). \end{aligned}$$

The last inequality allows us to write

$$w_n \sum_{j=q_n+1}^{\lfloor w_n/k_n \rfloor} \mu^2\left(\mathscr{A}_{q_n,n} \cap T^{-j}(\mathscr{A}_{q_n,n})\right) \leq w_n \sum_{j=q_n+1}^{\lfloor w_n/k_n \rfloor} \mu\left(U_n^{T_1} \cap T^{-j}(U_n^{T_1})\right)\mu\left(U_n^{T_2} \cap T^{-j}(U_n^{T_2})\right). \tag{3.3.4}$$

Take $\phi = \psi = \mathbf{1}_{U_n^{T_1}}$ in Definition 3.2.1, to obtain that

$$\mu\left(U_n^{T_1} \cap T_1^{-j}(U_n^{T_1})\right) = \int_{\mathscr{X}} \phi \cdot (\phi \circ T_1^j) d\mu \leq \left(\mu(U_n^{T_1})\right)^2 + \left\|\mathbf{1}_{U_n^{T_1}}\right\|_{\mathscr{C}_2} \mu\left(U_n^{T_1}\right) \rho^1(j). \tag{3.3.5}$$

Likewise, choosing $\phi = \psi = \mathbf{1}_{U_n^{T_2}}$ in Definition 3.2.1, we obtain that

$$\mu\left(U_n^{T_2} \cap T_2^{-j}(U_n^{T_2})\right) = \int_{\mathscr{X}} \phi \cdot (\phi \circ T_2^j) d\mu \leq \left(\mu(U_n^{T_2})\right)^2 + \left\|\mathbf{1}_{U_n^{T_2}}\right\|_{\mathscr{C}_2} \mu\left(U_n^{T_2}\right) \rho^2(j). \tag{3.3.6}$$

Take $(k_n)_{n\in\mathbb{N}}$ as in (3.1.5) and consider t_n as above. Recalling that $\lim_{n\to\infty} w_n \mu(U_n)$ and combining (3.3.4), (3.3.5) and (3.3.6), we can state that

$$\begin{aligned}
&w_n \sum_{j=q_n+1}^{\lfloor w_n/k_n \rfloor} \left(\mu(U_n^{T_1})^2 + \left\|\mathbf{1}_{U_n^{T_1}}\right\|_{\mathscr{C}_2} \mu(U_n^{T_1})\rho_j^1\right)\left(\mu(U_n^{T_2})^2 + \left\|\mathbf{1}_{U_n^{T_2}}\right\|_{\mathscr{C}_2} \mu(U_n^{T_2})\rho_j^2\right) \\
&\leq \frac{\tau^2}{k_n} + \tau \left\|\mathbf{1}_{U_n^{T_2}}\right\|_{\mathscr{C}_2} \mu\left(U_n^{T_1}\right) \sum_{j=q_n}^{\infty} \rho_j^2 + \tau \left\|\mathbf{1}_{U_n^{T_1}}\right\|_{\mathscr{C}_2} \mu\left(U_n^{T_2}\right) \sum_{j=q_n}^{\infty} \rho_j^1 + \tau \left\|\mathbf{1}_{U_n^{T_1}}\right\|_{\mathscr{C}_2} \left\|\mathbf{1}_{U_n^{T_2}}\right\|_{\mathscr{C}_2} \sum_{j=q_n}^{\infty} \rho_j^1 \rho_j^2.
\end{aligned}$$

Hence, condition $Д'_{q_n}(u_n, w_n)$ holds if we can verify the following conditions,

$$\begin{aligned}
&\lim_{n\to\infty} \left\|\mathbf{1}_{U_n^{T_1}}\right\|_{\mathscr{C}_2} \mu\left(U_n^{T_2}\right) \sum_{j=q_n}^{\infty} \rho_j^1 = 0 \\
&\lim_{n\to\infty} \left\|\mathbf{1}_{U_n^{T_2}}\right\|_{\mathscr{C}_2} \mu\left(U_n^{T_1}\right) \sum_{j=q_n}^{\infty} \rho_j^2 = 0 \\
&\lim_{n\to\infty} \left\|\mathbf{1}_{U_n^{T_1}}\right\|_{\mathscr{C}_2} \left\|\mathbf{1}_{U_n^{T_2}}\right\|_{\mathscr{C}_2} \sum_{j=q_n}^{\infty} \rho_j^1 \rho_j^2 = 0.
\end{aligned} \tag{3.3.7}$$

□

Remark 3.3.4. When the maps T_1 and T_2 denote the same dynamics, that is $T_1 = T_2$, the conditions in (3.3.7) can be further simplified. Under this assumption, condition $Д'_{q_n}(u_n, w_n)$ holds if

$$\lim_{n\to\infty} \left\|\mathbf{1}_{U_n^{T_1}}\right\|_{\mathscr{C}_2} \mu(U_n^{T_1}) \sum_{j=q_n}^{\infty} \rho_j^1 = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \left\|\mathbf{1}_{U_n^{T_1}}\right\|_{\mathscr{C}_2}^2 \sum_{j=q_n}^{\infty} \rho_j^1 \rho_j^2 = 0. \tag{3.3.8}$$

Assuming that relation (3.3.2) holds, one can determine a lower bound for the level of clustering associated with X_n by means of the level of clustering appearing in the processes X_n^1 and X_n^2.
For that purpose, let $\mathscr{A}_{q_n,n}^{T_1}$ denote the set $\mathscr{A}_{q_n}(U_n^{T_1})$ and $\mathscr{A}_{q_n,n}^{T_2}$ denote the set $\mathscr{A}_{q_n}(U_n^{T_2})$.
Assume that exists $(q_n)_{n\in\mathbb{N}}$ such that the following limits exist:

$$\theta_1 := \lim_{n\to\infty} \frac{\mu(\mathscr{A}_{q_n,n}^{T_1})}{\mu(U_n^{T_1})} \qquad \text{and} \qquad \theta_2 := \lim_{n\to\infty} \frac{\mu(\mathscr{A}_{q_n,n}^{T_2})}{\mu(U_n^{T_2})}. \tag{3.3.9}$$

The product structure of the maximal set $\mathscr{M}_1 \times \mathscr{M}_2$ permits us to find a partial decomposition of $\mathscr{A}_{q_n,n}$ using $\mathscr{A}^{T_1}_{q_n,n}$ and $\mathscr{A}^{T_2}_{q_n,n}$. Such fact will allow us to prove the following result.

Theorem 3.3.5. *Let T be a dynamical system defined as in* (3.3.1) *and consider an observable ψ, achieving a global maximum on a set $\mathscr{M}_1 \times \mathscr{M}_2$. Let $(X_n)_{n\in\mathbb{N}}$ be the stochastic process constructed as in* (2.2.1) *and consider sequences $(u_n)_{n\in\mathbb{N}}$, $(w_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ such that* (3.0.5), (3.1.1), (3.3.2) *and* (3.3.9) *hold.*
Assume, moreover, that conditions $Д_{q_n}(u_n,w_n)$ and $Д'_{q_n}(u_n,w_n)$ hold and that the Extremal Index θ exists. Then,

$$\lim_{n\to\infty} \mu^2(M_{w_n} \leq u_n) = \mathrm{e}^{-\theta\tau}, \tag{3.3.10}$$

where,

$$\theta > \theta_1 + \theta_2 - \theta_1\theta_2. \tag{3.3.11}$$

Proof. Since conditions $Д_{q_n}(u_n,w_n)$ and $Д'_{q_n}(u_n,w_n)$ hold, to conclude the proof of the Theorem it is only necessary to obtain a lower bound for θ.
Let $x = (a,b)$ be a point in U_n and assume that $T_1^j(a) \in (U_n^{T_1})^c$ or $T_2^j(b) \in (U_n^{T_2})^c$ for all $j \leq q_n$. This implies that $x \in T^{-j}(U_n^c)$, for all $j \leq q_n$ and consequently $x \in \mathscr{A}_{q_n,n}$. Hence, we can state that

$$\left(\mathscr{A}^{T_1}_{q_n,n} \times U_n^{T_2}\right) \cup \left(U_n^{T_1} \times \mathscr{A}^{T_2}_{q_n,n}\right) \subseteq \mathscr{A}_{q_n,n}. \tag{3.3.12}$$

But, the union of sets described above is not disjoint. The elements of the set $\mathscr{A}^{T_1}_{q_n,n} \times \mathscr{A}^{T_2}_{q_n,n}$ are being counted twice in (3.3.12). This implies that, we can write

$$\mu^2(\mathscr{A}_{q_n,n}) \geq \mu(\mathscr{A}^{T_1}_{q_n,n})\mu(U_n^{T_2}) + \mu(\mathscr{A}^{T_2}_{q_n,n})\mu(U_n^{T_1}) - \mu(\mathscr{A}^{T_1}_{q_n,n})\mu(\mathscr{A}^{T_2}_{q_n,n}).$$

Using O'Brien's formula, we obtain a lower bound for θ,

$$\begin{aligned}
\lim_{n\to\infty} \theta_n &= \lim_{n\to\infty} \frac{\mu^2(\mathscr{A}_{q_n,n})}{\mu^2(U_n)} \\
&\geq \lim_{n\to\infty} \frac{\mu(\mathscr{A}^{T_1}_{q_n,n})\mu(U_n^{T_2}) + \mu(\mathscr{A}^{T_2}_{q_n,n})\mu(U_n^{T_1}) - \mu(\mathscr{A}^{T_1}_{q_n,n})\mu(\mathscr{A}^{T_2}_{q_n,n})}{\mu(U_n^{T_1})\mu(U_n^{T_2})} \\
&\geq \theta_1 + \theta_2 - \theta_1\theta_2.
\end{aligned}$$

□

The closed formula achieved in Theorem 3.3.5 for the lower bound of the EI supports some useful conclusions. Not only allows us to prove the existence of the Extremal Index, but also allows us to show, as illustrated by Figure 3.1, that θ will be 1 if and only if θ_1 or θ_2 is also 1. This implies that the absence of clustering in either X_n^1 or X_n^2 will imply an absence of clustering in the process X_n.
One can take this reasoning one step further and state that the level of clustering appearing in X_n is always smaller than the level of clustering appearing in X_n^1 or X_n^2. This conclusion can be reached by

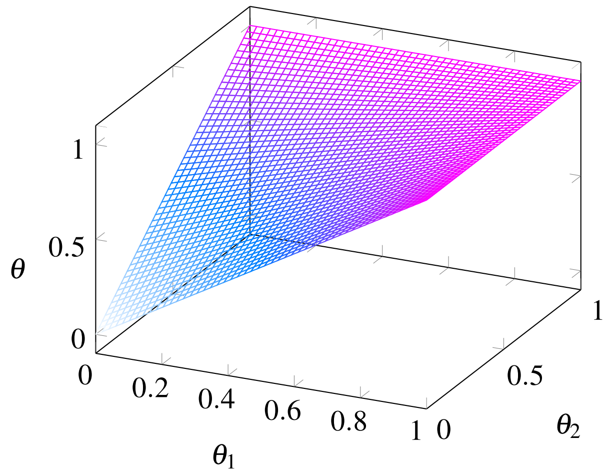

Fig. 3.1 The relation between θ, θ_1 and θ_2.

noting that, for $\theta_1, \theta_2 \in]0,1[$, we have

$$\theta_1 < \theta_1 + \theta_2 - \theta_1\theta_2 \qquad \text{and} \qquad \theta_2 < \theta_1 + \theta_2 - \theta_1\theta_2.$$

This smoothing effect of the clustering can be linked to the product structure present in $\mathscr{M}_1 \times \mathscr{M}_2$. Due to this nature of the maximal set, if $T_1^{-j}(\mathscr{M}_1) \cap \mathscr{M}_1$ or $T_2^{-j}(\mathscr{M}_2) \cap \mathscr{M}_2$ is negligible, this is enough to guarantee that $T^{-j}(\mathscr{M}_1 \times \mathscr{M}_2) \cap (\mathscr{M}_1 \times \mathscr{M}_2)$ has low relevance when compared with $\mathscr{M}_1 \times \mathscr{M}_2$ resulting in a low level of clustering appearing in X_n.
This fact will be used to prove Theorem 5.3.1 and Theorem 5.3.2 of Chapter 5.

Chapter 4

Existence of Clustering with Fractal Maximal Sets

Our primary goal is to achieve limiting laws for dynamically generated stochastic processes whose observable is maximized in a set with fractal properties. We will use, as a prototype, the ternary Cantor set that we shall denote by $\mathscr{C}$.
To construct $\mathscr{C}$, we start by removing the middle third of the interval $\mathscr{C}_0 := [0,1]$ and define in this way the first approximation $\mathscr{C}_1$. Then, we start an iterative process where we build $\mathscr{C}_n$ by removing the middle third of each connected component of $\mathscr{C}_{n-1}$, as represented in Figure 4.1. Repeating this process indefinitely, we obtain the set $\mathscr{C} = \cap_{n\geq 1}\mathscr{C}_n$.

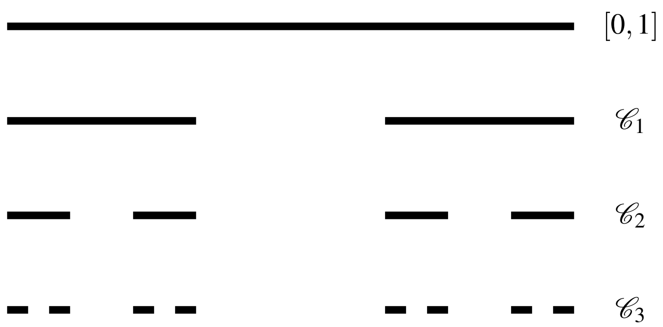

Fig. 4.1 The construction of the ternary Cantor set.

The ternary Cantor set $\mathscr{C}$ has features that are commonly linked to a possible definition of a fractal set. It is a simple set that, as we saw above, can be defined recursively leading to self-similarity. In fact, if we could zoom in, for example, on the first third of the set, we would see a set identical to the original set $\mathscr{C}$. It is also an example of a more general type of sets designated by *attractors* of *Iterated Function Systems* (IFS), which are usually characterized as fractal sets.
In addition to these aspects, it is an uncountable set of Lebesgue measure zero which makes it a good

candidate to extend the previous results concerning extreme value laws on more general maximal sets.

Through the use of the algorithm described above to generate $\mathscr{C}$, we define the observable to be the Cantor ladder function. This observable was also used in [31] to model a fractal landscape.
For each $n \in \mathbb{N}$, let $B_n := \mathscr{C}_{n-1} \setminus \mathscr{C}_n$ so that $B_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$, $B_2 = \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right), \ldots$, *i.e.*, the sets B_n correspond to the gaps of the Cantor set formed at the n-th step of its construction.
Now, the Cantor ladder function is taken to be

$$\varphi(x) = \begin{cases} n, & \text{if } x \in B_n,\ n = 1, 2, 3\ldots \\ \infty, & \text{otherwise.} \end{cases} \tag{4.0.1}$$

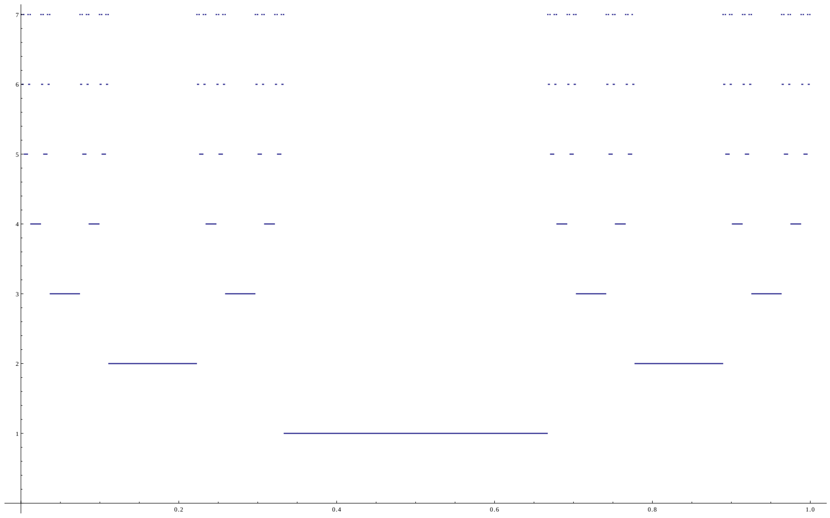

Fig. 4.2 The Cantor ladder function.

Note that, if $x \in \mathscr{C}$ then $x \notin B_n$ for all $n \in \mathbb{N}$, which implies that $\varphi(x) = \infty$. If $x \notin \mathscr{C}$ then $x \in B_n$ for some $n \in \mathbb{N}$. Therefore, in this case, we have that the set of maximal points $\mathscr{M}$ of φ is exactly $\mathscr{C}$.
We will be considering dynamical systems given by:

$$\begin{aligned} T\colon\ [0,1] &\longrightarrow [0,1] \\ x &\longmapsto m \cdot x \mod 1, \end{aligned} \tag{4.0.2}$$

where $m \in \mathbb{N}$.
These are full branched uniformly expanding maps, which preserve Lebesgue measure (that we shall denote by Leb) and have exponential decay of correlations of BV observables against L^1(Leb), accordingly with [6, Corollary 8.3.1].

In [31, Section 3], the authors considered the same observable φ defined in (4.0.1) and the dynamics generated by an asymmetric tent map. This map is also a full branched uniformly hyperbolic map

and the authors conjectured the existence of a limiting extreme value law with an EI equal to 1. This conjecture was supported by the numerical simulations performed.
We prove, in Chapter 5, that when the dynamics considered is not compatible with the self-similar structure of the maximal set (which happens here when $m \neq 3^k$ for all $k \in \mathbb{N}$) then indeed the conjectured extreme limiting behaviour applies. The numerical simulations performed in [31, Section 3] also showed that, interestingly, the same limiting laws seem to apply when the dynamics is replaced by that of irrational rotations. The fact that these ergodic maps are not mixing and yet the agreement was still good, lead the authors of [31] to conjecture that the role of the fast decay of correlations in assuring the validity of conditions such as Д_{q_n} and $\text{Д}'_{q_n}$ was played, in this situation, by the complexity of the observable function.
In all numerical studies performed in [31] with observables maximized on fractal sets, with strictly positive Hausdorff dimension, the observed EI was always 1. In this chapter, however, not only we provide examples where the EI is strictly less than 1, as we explain how the EI is related with the compatibility between the dynamics and the fractal structure of the maximal set.

Theorem 4.0.1. *Let* $(X_n)_{n\in\mathbb{N}}$ *be the stochastic process constructed as in* (2.2.1) *for a dynamical system* T *defined in* (4.0.2)*, with* $m = 3^k$ *for some* $k \in \mathbb{N}$. *Consider a sequence of thresholds* $(u_n)_{n\in\mathbb{N}}$ *such that* $u_n = n + k - 1$ *and a sequence of times* $(w_n)_{n\in\mathbb{N}}$ *such that* $w_n = \left\lfloor \tau\,(3/2)^{n+k-1} \right\rfloor$.
Then, condition (3.0.5) *holds and, moreover*

$$\lim_{n\to\infty} Leb(M_{w_n} \leq n) = \mathrm{e}^{-\left(1-\frac{2^k}{3^k}\right)\tau}.$$

In fact, in Section 4.1, we prove Theorem 4.0.1 as a corollary of Theorem 4.1.2 which applies to more general Cantor sets, namely attractors of IFS. In the context of both Theorems 4.0.1 and 4.1.2, the compatibility between the dynamics and the maximal set becomes obvious when we observe that $T(\mathscr{M}) = \mathscr{M}$. This means that T preserves the structure of the Cantor set, which play the role of a periodic point in the context of when $\mathscr{M}$ is reduced to a single point.
The proof of the existence of the limiting laws will follow more or less the same strategy used in [21] and generalised later in [3, 4]. This strategy basically exploits the periodicity of the maximal set in order to be able to compute the EI from the O'Brien's formula (3.1). Then, it proceeds to the verification of conditions $\text{Д}_{q_n}(u_n, w_n)$ and $\text{Д}'_{q_n}(u_n, w_n)$ that were designed to be easily checked from the decay of correlations of the systems considered.

4.1 Compatibility and Clustering

When the dynamics is compatible with the self-similar structure of the maximal set, we observe the appearance of clustering and a limiting law with a non-trivial EI. In this context, we consider more general fractal sets. Namely, we will consider Cantor sets generated by an IFS satisfying some regularity conditions. These Cantor sets can also be identified as survivor sets for some conveniently chosen dynamical systems, which will also provide a common ground to assess the compatibility of the self-similarity structure with the original dynamics. We will start by providing a description of these more general dynamically defined Cantor sets. Then, we establish the existence of a limiting law with a non-trivial EI when the dynamics is compatible with the system generating the Cantor set and, finally, we apply it to the usual ternary Cantor set.

4.1.1 Dynamically Defined Cantor Sets

We start with a description of a class of Iterated Function Systems.
Consider a finite family of contractions, $\mathfrak{F} = \{f_1, f_2, \dots, f_s\}$, where each f_i is a C^1 diffeomorphism on $[0,1]$ and satisfies

$$|f_i(x) - f_i(y)| \leq \lambda_i |x - y|,$$

for some ratio λ_i.
Let $J_i = f_i([0,1])$ and assume that $J_i \cap J_j = \emptyset$, for all $i, j \in \mathbb{N}$. The family $\mathfrak{F}$ defines an Iterated Function System satisfying certain regularity conditions that assert the existence of a unique attractor Λ, *i.e*, a unique compact set that satisfies the equation $\Lambda = \cup_{i=1}^{s} f_i(\Lambda)$. In this case, both the Hausdorff and box dimension of the attractor (see Definitions A.1.1 and A.1.2) are equal to d, where $\sum_{i=1}^{s} \lambda_i^d = 1$.
We call this set a dynamically generated Cantor set in the sense that it can be identified as the survivor set of a dynamical system $G : \mathbb{R} \to \mathbb{R}$ defined as

$$G(x) = \begin{cases} f_i^{-1}(x), & \text{if } x \in J_i \\ 2, & \text{otherwise.} \end{cases}$$

Using the function $G(x)$, Λ can be described as the set of points whose orbit never leaves the interval $[0,1]$, that is

$$\Lambda = \{x \in [0,1] : \; G^n(x) \in [0,1], \text{ for all } n \in \mathbb{N}\}.$$

We can define the n-th approximation to Λ, denoted by Λ_n, by letting $\Lambda_0 = [0,1]$ and setting for all $n \in \mathbb{N}$

$$\Lambda_n = G^{-1}(\Lambda_{n-1}) = \{x \in [0,1] : G^l(x) \in [0,1], \text{ for all } l = 1, \dots, n\}.$$

Under this notation, note that $\Lambda = \cap_{n \geq 0} \Lambda_n$. For more details on IFS, we refer to [13, Chapter 9].

4.1.2 Limiting Laws and Dynamically Defined Cantor Sets

We adapt the definition of the observable function $\varphi : [0,1] \to \mathbb{R}$ of (4.0.1) so that, in this case, the maximal set is Λ.
We set,

$$\varphi_\Lambda(x) = \begin{cases} n, & \text{if } x \in \Lambda_n \setminus \Lambda_{n+1},\; n = 1,2,3\dots \\ \infty, & \text{if } x \in \Lambda. \end{cases} \tag{4.1.1}$$

Note that, when $\Lambda = \mathscr{C}$ the observable φ of (4.0.1) follows directly from φ_Λ.
Now, we define a dynamics $\bar{T}$ that is compatible with the set Λ. This map is constructed using the

function $G(x)$.
Let I denote a connected component of $[0,1]\setminus\cup_{i=1}^{s}J_i$ and set $g_I(x)$ to be a linear function that maps I onto $[0,1]$. With this notation, we define $F:[0,1]\to[0,1]$ as

$$F(x)=\begin{cases} G(x), & \text{if } x\in\cup_{i=1}^{s}J_i\\ g_I(x), & \text{if } x\in I.\end{cases}$$

The function F is a piecewise uniformly expanding map and, henceforth, admits an absolutely continuous invariant measure μ. Also, accordingly to [6, Corollary 8.3.1] this maps have decay of correlations of BV observables against $L^1(\mu)$ as stated in Definition 3.2.1.
The dynamics $\bar{T}$ is constructed by setting $\bar{T}=F^k$, for some $k\in\mathbb{N}$. Now, we prove a result that formalizes the compatibility of $\bar{T}$ with Λ.

Lemma 4.1.1. *If $j\le n/k$, then, $\bar{T}^{-j}(\Lambda_n)\cap\Lambda_n=\Lambda_{n+kj}$.*

Proof. Let $x\in\bar{T}^{-j}(\Lambda_n)\cap\Lambda_n$, we start by establish that $\bar{T}^{-j}(\Lambda_n)\cap\Lambda_n\subseteq\Lambda_{n+kj}$.
Since $x\in\Lambda_n$ then $G^l(x)\in[0,1]$ for all $l=1,\dots,n$. Moreover, if $x\in\bar{T}^{-j}(\Lambda_n)$ and as $j\le n/k$ then $G^{kj}(x)\in\Lambda_n$ and $G^{kj}(x)\in[0,1]$ for all integers j not bigger than n/k.
Hence, $G^l(G^{kj}(x))\in[0,1]$ for $l=1,\dots,n$ and in particular, $G^{n+kj}(x)\in[0,1]$. It follows that $G^l(x)\in[0,1]$ for all $l=1,\dots,n+kj$ and $x\in\Lambda_{n+kj}$.
To prove the remaining inclusion, consider $x\in\Lambda_{n+kj}$. Since $\Lambda_{n+kj}\subseteq\Lambda_n$ then $x\in\Lambda_n$. Furthermore, as $x\in\Lambda_{n+kj}$ then $G^l(x)\in[0,1]$ for all $l=1,\dots,n+kj$. In particular, we have that $G^{n+kj}(x)=G^n(G^{kj}(x))\in[0,1]$. Hence, $G^{kj}(x)=\bar{T}^j(x)\in\Lambda_n$ and $x\in\bar{T}^{-j}(\Lambda_n)$. □

Lemma 4.1.1, in particular, implies that $T^{-j}(\Lambda)=\Lambda$ and, therefore, the system preserves the whole maximal set. It is this recurrence of the maximal set to itself by effect of the dynamics, which we call compatibility, that results in clusters of exceedances. The next theorem formalizes this idea.

Theorem 4.1.2. *Let $(X_n)_{n\in\mathbb{N}}$ be the stochastic process constructed as in (2.2.1) for the dynamical system $\bar{T}=F^k$, for some $k\in\mathbb{N}$, where F and the observable φ are as defined just above. Consider a sequence of thresholds $(u_n)_{n\in\mathbb{N}}$ and a sequence of times $(w_n)_{n\in\mathbb{N}}$ such that $w_n=\left\lfloor\tau(\mu(\Lambda_{\lfloor u_n\rfloor}))^{-1}\right\rfloor$.*

Assume that there exists a sequence $(t_n)_{n\in\mathbb{N}}$, with $t_n=o(w_n)$, and a sequence $(q_n)_{n\in\mathbb{N}}$, with $1\le q_n\le u_n/k$, satisfying (3.1.1).
Assume, moreover, that

$$\lim_{n\to\infty}\left\|\mathbf{1}_{\mathscr{A}_{q_n,n}}\right\|_{BV}w_n r^{t_n}=0$$
$$\lim_{n\to\infty}\left\|\mathbf{1}_{U_n}\right\|_{BV}r^{q_n}=0$$

and that there exists $0\le\theta\le1$, such that

$$\theta=\lim_{n\to\infty}\frac{\mu(\Lambda_{\lfloor u_n\rfloor}\setminus\Lambda_{\lfloor u_n\rfloor+k})}{\mu(\Lambda_{\lfloor u_n\rfloor})}.$$

Then,

$$\lim_{n\to\infty}\mu(M_{w_n}\leq n)=\mathrm{e}^{-\theta\tau}.$$

Proof. Start by noting that for any sequence of thresholds u_n, we have $U_n=\Lambda_{\lfloor u_n\rfloor}$ and then the definition of w_n makes condition (3.0.5) satisfied.

Using Lemma 4.1.1, we can easily characterize the sets $\mathscr{A}_{q_n,n}$. Observe that $\Lambda_n^c\subset\Lambda_{n+1}^c$ for all $n\in\mathbb{N}$. Hence, for n sufficiently large, such that $1\leq q_n\leq u_n/k$ and recalling the definition of $\mathscr{A}_{q_n,n}$, we obtain:

$$\mathscr{A}_{q_n,n}=\bigcap_{i=1}^{q_n}T^{-i}(\Lambda^c_{\lfloor u_n\rfloor})\cap\Lambda_{\lfloor u_n\rfloor}=\Lambda^c_{\lfloor u_n\rfloor+k}\cap\Lambda_{\lfloor u_n\rfloor}=\Lambda_{\lfloor u_n\rfloor}\setminus\Lambda_{\lfloor u_n\rfloor+k}.$$

The fact that F has decay of correlations of BV against L^1, together with the assumptions

$$\lim_{n\to\infty}\left\|\mathbf{1}_{\mathscr{A}_{q_n,n}}\right\|_{BV}w_n r^{t_n}=0\qquad\text{and}\qquad\lim_{n\to\infty}\|\mathbf{1}_{U_n}\|_{BV}r^{q_n}=0,$$

guarantee that conditions (1) and (2) from Theorem 3.2.3 hold.
Moreover, the assumption on θ, gives that

$$\theta_n=\frac{\mu(\Lambda_{\lfloor u_n\rfloor}\setminus\Lambda_{\lfloor u_n\rfloor+k})}{\mu(\Lambda_{\lfloor u_n\rfloor})}\xrightarrow[n\to\infty]{}\theta,$$

as required.
Consequently, the result follows from direct application of Theorem 3.2.3. □

4.1.3 Application to the Ternary Cantor Set

The objective of this subsection is to apply Theorem 4.1.2 to the ternary Cantor set and prove the limiting law stated in Theorem 4.0.1.

Proof of Theorem 4.0.1. We start by checking the hypothesis of Theorem 4.1.2 and then verify the formula provided for the Extremal Index θ. In this case, the IFS is given by $f_1(x)=1/3x$ and $f_2(x)=1/3x+2/3$, the map F is given by $F(x)=3x \mod 1$, the fractal set Λ is $\mathscr{C}$ and $\Lambda_n=\mathscr{C}_n$.
The invariant measure μ is the Lebesgue measure, denoted by Leb, and the rate of decay of correlations expressed in Definition 3.2.1 is such that $r=1/3$.
We set $u_n=n+k-1$, $q_n=\lfloor(n+k-1)/k\rfloor$ and observe that $U_n=\mathscr{C}_{n+k-1}$ and $\mathscr{A}_{q_n,n}=\mathscr{C}_{n+k-1}\setminus\mathscr{C}_{n+2k-1}$. Since $\mathscr{C}_n\subset\mathscr{C}_{n-1}$, for all $n\in\mathbb{N}$, and $\mu(\mathscr{C}_n)=(\frac{2}{3})^n$, then $w_n=\left\lfloor\tau(3/2)^{n+k-1}\right\rfloor$, $q_n=o(w_n)$ and we obtain

$$\mathrm{Leb}(\mathscr{A}_{q_n,n})=\left(\frac{2}{3}\right)^{n+k-1}-\left(\frac{2}{3}\right)^{n+2k-1}=\left(1-\frac{2^k}{3^k}\right)\left(\frac{2}{3}\right)^{n+k-1}$$

and, moreover,

$$\|\mathbf{1}_{U_n}\|_{BV}\leq 2^{n+k+1},\qquad\|\mathbf{1}_{\mathscr{A}_{q_n,n}}\|_{BV}\leq 2^{n+2k}+1\leq 2^{n+2k+1}.$$

Let $t_n = n^2$ and note that clearly $t_n = o(w_n)$. Since $r = 1/3$, then

$$\lim_{n\to\infty} \left\|\mathbf{1}_{\mathscr{A}_{q_n,n}}\right\|_{BV} w_n r^{t_n} \leq \lim_{n\to\infty} \left\lfloor \tau (3/2)^{n+k-1} \right\rfloor 2^{n+2k+1} r^{n^2}$$
$$\leq 2 \lim_{n\to\infty} 2^{k-1} \tau 3^{n+k-1} \left(1/3^k\right)^{n^2} + 2^{n+2k} \left(1/3^k\right)^{n^2} = 0.$$

Furthermore, there exists some constant $C' > 0$ such that

$$\lim_{n\to\infty} \|\mathbf{1}_{U_n}\|_{BV} r^{q_n} = \lim_{n\to\infty} 2^{n+k+1} r^{q_n} \leq \lim_{n\to\infty} 2^{n+k+1} (1/3^k)^{n/k+1-1/k} \leq C' \lim_{n\to\infty} (2/3)^n = 0.$$

Finally, we use O'Brien's formula to compute the EI:

$$\lim_{n\to\infty} \theta_n = \lim_{n\to\infty} \frac{\operatorname{Leb}(\mathscr{A}_{q_n,n})}{\operatorname{Leb}(U_n)} = \lim_{n\to\infty} \frac{\left(1-\frac{2^k}{3^k}\right)\left(\frac{2}{3}\right)^{n+k-1}}{\left(\frac{2}{3}\right)^{n+k-1}} = \left(1 - \frac{2^k}{3^k}\right) =: \theta.$$

As a consequence of Theorem 4.1.2, we obtain $\lim_{n\to\infty} \operatorname{Leb}(M_{w_n} \leq n) = \mathrm{e}^{-\left(1-\frac{2^k}{3^k}\right)\tau}$. □

4.2 Clustering and Two Dimensional Uniformly Expanding Maps

In the last section, we achieved limiting laws for dynamically generated stochastic processes built upon unidimensional uniformly expanding maps. The presence of clusters of exceedances was linked to the compatibility between the maps and the maximal set of the observable function. The objective, here, is to expand these results to stochastic processes whose underlying dynamics is defined in a two dimensional space.

Our starting point is the Cantor set $\mathscr{C}$ used in the last section. Using this set, we can define a new fractal set given by $\mathfrak{C} := \mathscr{C} \times \mathscr{C}$. This set, usually called Cantor dust, is a self-similar set contained in the two dimensional space $[0,1] \times [0,1]$ and will be the prototype chosen to illustrate the results.

The Cantor dust can be seen as the final product of an algorithmic construction similar to the one presented for $\mathscr{C}$ and represented in Figure 4.3. One can define the n-th approximation to $\mathfrak{C}$, denoted by $\mathfrak{C}_n$, as the product $\mathscr{C}_n \times \mathscr{C}_n$. The set $\mathfrak{C}$ can then be described as $\cap_{n\geq 1} \mathfrak{C}_n$.

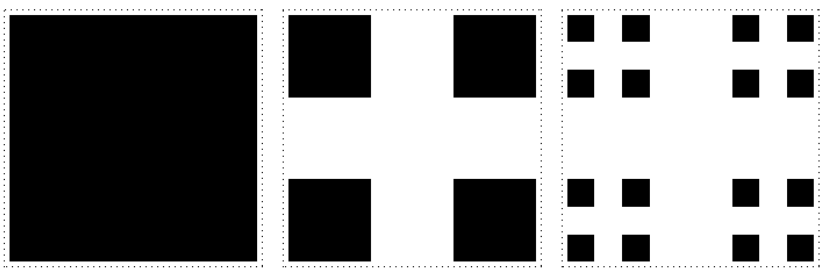

Fig. 4.3 Representation of the algorithmic construction leading to the Cantor dust.

Using the Cantor ladder function in (4.0.1), we construct an observable, $\psi : [0,1]^2 \to \mathbb{R}$, whose maximal set $\mathscr{M}$ is exactly $\mathfrak{C}$ in the following way:

$$\psi(x,y) = \begin{cases} n, & \text{if } \min(\varphi(x), \varphi(y)) = n \\ \infty, & \text{otherwise.} \end{cases} \tag{4.2.1}$$

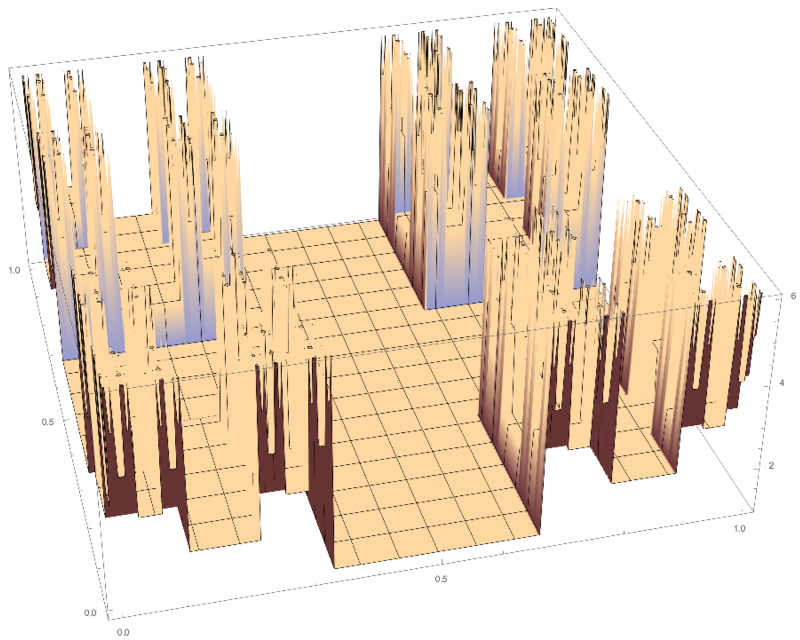

Fig. 4.4 The observable ψ.

The two dimensional dynamical systems that we will consider are given by,

$$\begin{aligned} T\colon\ [0,1]^2 &\longrightarrow [0,1]^2 \\ (x,y) &\mapsto (m_1 \cdot x \mod 1, m_2 \cdot y \mod 1), \end{aligned} \tag{4.2.2}$$

where $m_1, m_2 \in \mathbb{N}$.
Again, we will use Leb to denote the Lebesgue measure in $\mathbb{R}$. The systems in (4.2.2) preserve the product measure Leb $\times$ Leb, that we will denote by Leb^2 and belong to a larger class of maps defined by Saussol in [39], designated by multidimensional piecewise expanding systems. Moreover, assuming that the system is sufficiently expanding and following the setting presented by Saussol, it is possible to prove that these systems have decay of correlations for quasi-Hölder observables against $L^1(\text{Leb}^2)$ for some $0 < \alpha \leq 1$.
We will rely on the results achieved in Section 3.3 of Chapter 3 to prove the existence of the distributional limits, so, accordingly with (4.2.2), the natural choice for the maps T_1 and T_2 are the uniformly expanding maps $m_1 \cdot x \mod 1$ and $m_2 \cdot y \mod 1$, respectively.
For this type of maps, in conjunction with the observable φ whose maximal set is $\mathscr{C}$, it was proved in

Section 4.1.3 the existence of a limiting extreme value law for the associated stochastic process. We were able to link the value of the Extremal Index to the compatibility between the maximal set and the dynamics. It was discovered that, when for example $m_1 = 3^k$, for some $k \in \mathbb{N}$, then $T_1(\mathscr{C}) = \mathscr{C}$ and the Cantor set is playing the role of a periodic point resulting in clustering.
In the same sprit, we will demonstrate that if $m_1 = 3^{k_1}$ and $m_2 = 3^{k_2}$ for some k_1, k_2 in $\mathbb{N}$, then, there exists full compatibility between $\mathfrak{C}$ and $T^{-j}(\mathfrak{C})$. This will lead to the following theorem.

Theorem 4.2.1. *Consider* $(X_n)_{n\in\mathbb{N}}$ *to be the stochastic process constructed as in* (2.2.1) *for the dynamical system* T *defined in* (4.2.2) *with* $m_1 = 3^{k_1}$ *and* $m_2 = 3^{k_2}$, *for some* k_1, k_2 *in* $\mathbb{N}$ *satisfying*

$$1 + \frac{min\{k_1, k_2\}}{max\{k_1, k_2\}} > \log_3(4).$$

Consider a sequence of thresholds $(u_n)_{n\in\mathbb{N}}$ *such that* $u_n = n$ *and a sequence of times* $(w_n)_{n\in\mathbb{N}}$, *such that* $w_n = \left\lfloor \tau\,(3/2)^{2n} \right\rfloor$.
Then, condition (3.0.5) *holds and*

$$\lim_{n\to\infty} Leb^2(M_{w_n} \le n) = \mathrm{e}^{-\left(1-\frac{2^{k_1+k_2}}{3^{k_1+k_2}}\right)\tau}.$$

The strategy followed to prove this statement is identical to the one adopted to prove Theorem 4.0.1. We will prove a Theorem analogous to Theorem 4.1.2 that will guarantee the existence of limiting laws, with a non-trivial EI, for stochastic processes where $\mathscr{M}$ is the direct product of two dynamically generated Cantor Sets, Λ_1 and Λ_2. This is done by defining a dynamics, adapted from $\bar{T}$, fully compatible with $\Lambda_1 \times \Lambda_2$.
Theorem 4.2.1 will then follow as a direct application of such result when $\Lambda_1 = \Lambda_2 = \mathscr{C}$.

4.2.1 Clustering and Product Structure

Start by considering the observable φ_Λ stated in (4.1.1), *i.e*

$$\varphi_\Lambda(x) = \begin{cases} n, & \text{if } x \in \Lambda_n \setminus \Lambda_{n+1},\ n = 1,2,3\ldots \\ \infty, & \text{otherwise.} \end{cases}$$

Using this observable and the dynamics $\bar{T}$, we constructed in Section 4.1.1, a stochastic process $X_n = \varphi_\Lambda \circ \bar{T}^n$. In Theorem 4.1.2, it was established sufficient conditions for the existence of a limiting law, with a non-trivial EI, for X_n using the compatibility between the dynamics and the set Λ.
In what follows, we consider two dynamically generated sets, Λ_1 and Λ_2, with associated compatible maps denoted, respectively, by $T_1 = F_1^{k_1}$ and $T_2 = F_2^{k_2}$, for $k_1, k_2 \in \mathbb{N}$.
The direct product of Λ_1 and Λ_2 represented by Σ, is a fractal set contained in $[0,1]^2$. The n-th approximation to Σ is defined as $\Sigma_n = \Lambda_{1,n} \times \Lambda_{2,n}$, where $\Lambda_{1,n}$ and $\Lambda_{2,n}$ represent the n-th approximation to, respectively, Λ_1 and Λ_2.

Consider the map $T : [0,1]^2 \to [0,1]^2$, given by

$$T(x,y) = (T_1(x), T_2(y)). \tag{4.2.3}$$

Due to the product structure present in Σ, we can use Lemma 4.1.1 to establish the compatibility between T and Σ.
Let $k = \max\{k_1, k_2\}$, we claim that, if $j \le n/k$, then

$$T^{-j}(\Sigma_n) \cap \Sigma_n = \Lambda_{1,n+k_1 j} \times \Lambda_{2,n+k_2 j}. \tag{4.2.4}$$

Using the properties of the direct product, we write that,

$$T^{-j}(\Sigma_n) \cap \Sigma_n = \left(T_1^{-j}(\Lambda_{1,n}) \cap \Lambda_{1,n}\right) \times \left(T_2^{-j}(\Lambda_{2,n}) \cap \Lambda_{2,n}\right). \tag{4.2.5}$$

Applying Lemma 4.1.1 to 4.2.5 we obtain that

$$T^{-j}(\Sigma_n) \cap \Sigma_n = \Lambda_{1,n+k_1 j} \times \Lambda_{2,n+k_2 j}$$

and the claim follows.
We remark that this result implies that $T^{-j}(\Sigma) = \Sigma$ and clustering is to be expected when Σ is defined to be the maximal set of the considered observable.
To construct an observable whose maximal set is Σ, we use the function φ_Λ.
The observable that we will be considering depends upon the chosen sets Λ_1 and Λ_2 and it is defined as

$$\hat{\psi}(x,y) = \begin{cases} n, & \text{if } \min(\varphi_{\Lambda_1}(x), \varphi_{\Lambda_2}(y)) = n \\ \infty, & \text{otherwise.} \end{cases} \tag{4.2.6}$$

The maximal set of this observable is, precisely, the set Σ. Moreover, the function ψ follows directly from $\hat{\psi}$ when $\Lambda_1 = \Lambda_2$ is the ternary Cantor set $\mathscr{C}$.
The map T is a multidimensional piecewise expanding map as defined by Saussol in [39]. Consequently, it has decay of correlations for quasi-Hölder observables against L^1, with respect to the invariant measure $\mu \times \mu$ denoted by μ^2. This allows us to prove the following Theorem.

Theorem 4.2.2. *Consider $(X_n)_{n\in\mathbb{N}}$ to be the stochastic process constructed as in* (2.2.1) *for the dynamical system T in* (4.2.3) *and the observable $\hat{\psi}$ in* (4.2.6)*. Consider a sequence of thresholds $(u_n)_{n\in\mathbb{N}}$ and a sequence $(w_n)_{n\in\mathbb{N}}$ such that,*

$$w_n = \left\lfloor \tau \left[\mu(\Lambda_{1,\lfloor u_n \rfloor})\mu(\Lambda_{2,\lfloor u_n \rfloor})\right]^{-1} \right\rfloor.$$

Let $k = \max\{k_1, k_2\}$ and consider a sequence $(q_n)_{n\in\mathbb{N}}$ as in (3.1.1) *satisfying $1 \le q_n \le u_n/k$.*
Assume that exists $(t_n)_{n\in\mathbb{N}}$, where $t_n = o(w_n)$, such that the following conditions hold:

1. $\lim_{n\to\infty} \|\mathbf{1}_{\mathscr{A}_{q_n,n}}\|_\alpha w_n \rho(t_n) = 0$ *for some* $0 < \alpha \le 1$

2. $\lim_{n\to\infty} \left\| \mathbf{1}_{U_n^{T_1}} \right\|_{BV} \mu\left(U_n^{T_2}\right) \sum_{j=q_n}^{\infty} \rho_j^1 = 0$

3. $\lim_{n\to\infty} \left\| \mathbf{1}_{U_n^{T_2}} \right\|_{BV} \mu\left(U_n^{T_1}\right) \sum_{j=q_n}^{\infty} \rho_j^2 = 0$

4. $\lim_{n\to\infty} \left\| \mathbf{1}_{U_n^{T_1}} \right\|_{BV} \left\| \mathbf{1}_{U_n^{T_2}} \right\|_{BV} \sum_{j=q_n}^{\infty} \rho_j^1 \rho_j^2 = 0.$

Furthermore, assume that exists $0 \le \theta \le 1$ *such that*

$$\theta = \lim_{n\to\infty} \frac{\mu^2(\Sigma_{\lfloor u_n \rfloor} \setminus (\Lambda_{1,\lfloor u_n \rfloor + k_1} \times \Lambda_{2,\lfloor u_n \rfloor + k_2}))}{\mu^2(\Lambda_{1,\lfloor u_n \rfloor} \times \Lambda_{2,\lfloor u_n \rfloor})}.$$

Then,

$$\lim_{n\to\infty} \mu^2(M_{w_n} \le n) = \mathrm{e}^{-\theta\tau}.$$

Proof. Start by considering a sequence of thresholds $(u_n)_{n\in\mathbb{N}}$.
Due to the definition of φ_Λ,

$$U_n^{T_1} = \{x \in [0,1] : \hat{\varphi}_{\Lambda_1}(x) > u_n\} = \Lambda_{1,\lfloor u_n \rfloor}$$

and

$$U_n^{T_2} = \{y \in [0,1] : \hat{\varphi}_{\Lambda_2}(y) > u_n\} = \Lambda_{2,\lfloor u_n \rfloor}.$$

By construction of $\hat{\psi}$, the point $(x,y) \in [0,1]^2$ satisfies the equation $\hat{\psi}(x,y) > u_n$, if and only if, $\varphi_{\Lambda_1}(x) > u_n$ and $\varphi_{\Lambda_2}(y) > u_n$.
Hence,

$$U_n = \left\{(x,y) \in [0,1]^2 : \hat{\psi}(x,y) > u_n\right\} = \Lambda_{1,\lfloor u_n \rfloor} \times \Lambda_{2,\lfloor u_n \rfloor} = U_n^{T_1} \times U_n^{T_2}.$$

Moreover, condition (3.0.5) is verified since,

$$w_n \mu^2(U_n) = \left\lfloor \tau(\mu(\Lambda_{1,\lfloor u_n \rfloor})\mu(\Lambda_{2,\lfloor u_n \rfloor}))^{-1} \right\rfloor \mu(\Lambda_{1,\lfloor u_n \rfloor})\mu(\Lambda_{2,\lfloor u_n \rfloor}) \xrightarrow[n\to\infty]{} \tau.$$

The maps T_1 and T_2 have decay of correlations for observables in BV against $L^1(\mu)$. Moreover, the dynamics T has decay of correlations for quasi-Hölder observables against $L^1(\mu^2)$. Denote the rate functions of T_1, T_2 and T by ρ^1, ρ^2 and ρ, respectively.
Using Theorem 3.3.2, we obtain that conditions (1) thru (4) guarantee that conditions $Д_{q_n}(u_n, w_n)$ and $Д'_{q_n}(u_n, w_n)$ hold.
Noting that, $\Lambda_{1,i+i^*} \times \Lambda_{2,j+j^*} \subseteq \Lambda_{1,i} \times \Lambda_{2,j}$, for all $i, i^*, j, j^* \in \mathbb{N}$ and using relation (4.2.4), we can establish that

$$\mathscr{A}_{q_n,n} = \Sigma_{\lfloor u_n \rfloor} \setminus \left(\Lambda_{1,\lfloor u_n \rfloor + k_1} \times \Lambda_{2,\lfloor u_n \rfloor + k_2}\right).$$

for all q_n satisfying $1 \le q_n \le u_n/k$.
The value of the EI follows from O'Brien's formula. □

4.2.2 Application to the Cantor Dust

As pointed out before, we now use Theorem 4.2.2 to prove the limiting law stated in Theorem 4.2.1.

Proof of Theorem 4.2.1. The set $\mathscr{C}$ is dynamically generated by the IFS $f_1 = x/3$ and $f_2 = x/3 + 2/3$. Theorem 4.2.1 follows by making $\Lambda_1 = \Lambda_2 = \mathscr{C}$ and applying Theorem 4.2.2.

The map considered is $T = (T_1(x), T_2(y))$, where $T_1(x) = 3^{k_1}x \mod 1$ and $T_2(x) = 3^{k_2}x \mod 1$, for $k_1, k_2 \in \mathbb{N}$ satisfying

$$1 + \frac{\min\{k_1,k_2\}}{\max\{k_1,k_2\}} > \log_3(4).$$

Observe that, the invariant measure associated with T is Leb^2 and the invariant measure associated with T_1 and T_2 is Leb.
Set $u_n = n$, $w_n = \lfloor \tau(2/3)^{-2n} \rfloor$ and $q_n = \lfloor n/k \rfloor$, where $k = \max\{k_1,k_2\}$ and observe that,

$$U_n^{T_1} = U_n^{T_2} = \mathscr{C}_n.$$

Consequently,

$$\mu(U_n^{T_1}) = \mu(U_n^{T_2}) = \left(\frac{2}{3}\right)^n \qquad \text{and} \qquad \left\|\mathbf{1}_{U_n^{T_1}}\right\|_{BV} = \left\|\mathbf{1}_{U_n^{T_2}}\right\|_{BV} \le 2^{n+1}.$$

The maps T_1 and T_2 have decay of correlations of BV observables against $L^1(\text{Leb})$, with rate functions $\rho_n^1 = (1/3)^{k_1 n}$ and $\rho_n^2 = (1/3)^{k_2 n}$.
It is necessary to show that conditions (2) thru (4) of Theorem 4.2.2 hold.
Making the necessary substitutions, there exists constants $C, C', C'' > 0$ such that,

$$\lim_{n\to\infty} \left\|\mathbf{1}_{U_n^{T_1}}\right\|_{BV} \text{Leb}(U_n^{T_1}) \sum_{j=q_n}^{\infty} \rho_j^1 \le C \lim_{n\to\infty} \frac{4^n}{3^{n(1+k_1/k)}} = 0$$
$$\lim_{n\to\infty} \left\|\mathbf{1}_{U_n^{T_2}}\right\|_{BV} \text{Leb}(U_n^{T_2}) \sum_{j=q_n}^{\infty} \rho_j^2 \le C' \lim_{n\to\infty} \frac{4^n}{3^{n(1+k_2/k)}} = 0$$

and

$$\lim_{n\to\infty} \left\|\mathbf{1}_{U_n^{T_1}}\right\|_{BV} \left\|\mathbf{1}_{U_n^{T_2}}\right\|_{BV} \sum_{j=q_n}^{\infty} \rho_j^1 \rho_j^2 \le C'' \lim_{n\to\infty} \frac{4^n}{3^{n(k_1/k+k_2/k)}} = 0.$$

Therefore, considering the restrictions imposed on k_1 and k_2, conditions (2) thru (4) hold.
The next step is to prove that condition (1) of Theorem 4.2.2 holds. For that purpose, it is necessary to achieve a value for $\|\mathbf{1}_{\mathscr{A}_{q_n,n}}\|_\alpha$.

Let $C(A)$ denote the number of connected components of a set A and P denote the maximum perimeter of the connected components of $\mathscr{A}_{q_n,n}$.
Then, for a given $0 < \alpha \leq 1$ and $\varepsilon_0 > 0$,

$$|\mathscr{A}_{q_n,n}|_\alpha \leq \sup_{0<\varepsilon\leq\varepsilon_0} \varepsilon^{-\alpha}\left(\varepsilon C(\mathscr{A}_{q_n,n})P\right).$$

Hence, we achieve that

$$|\mathscr{A}_{q_n,n}|_\alpha \leq PC(\mathscr{A}_{q_n,n})$$

and

$$\|\mathbf{1}_{\mathscr{A}_{q_n,n}}\|_\alpha \leq \mathrm{Leb}^2(\mathscr{A}_{q_n,n}) + PC(\mathscr{A}_{q_n,n}).$$

We now estimate the values of $\mathrm{Leb}^2(\mathscr{A}_{q_n,n})$, $C(\mathscr{A}_{q_n,n})$ and P.
Set $\Sigma = \mathfrak{C}$ then, following the proof of Theorem 4.2.2, we can write,

$$\mathscr{A}_{q_n,n} = \mathfrak{C}_n \setminus (\mathscr{C}_{n+k_1} \times \mathscr{C}_{n+k_2})$$

and we obtain

$$\mathrm{Leb}^2(\mathscr{A}_{q_n,n}) = (2/3)^{2n}(1-(2/3)^{k_1+k_2}).$$

Each connected component of the set $\mathfrak{C}_n$ is a square with side length equal to $1/3^n$ and each connected component of $\mathscr{C}_{n+k_1} \times \mathscr{C}_{n+k_2}$ is a rectangle, where the sides measure $1/3^{n+k_1}$ and $1/3^{n+k_2}$. This implies that, each candidate to a connected component of $\mathscr{A}_{q_n,n}$ is a square, with side length $1/3^n$, with rectangular holes where the sides of each hole measure $1/3^{n+k_1}$ and $1/3^{n+k_2}$. Figure 4.5 aims to represent this reasoning when $k_1 = 1$ and $k_2 = 2$.
It is necessary to show that, this candidates to connected components form indeed one single connected component of $\mathscr{A}_{q_n,n}$. To make such verification, it is enough to note that, due to scaling properties of the ternary Cantor set, the pattern of the rectangular holes in each square is similar to the scheme of the connected components of the set $\mathscr{C}_{k_1} \times \mathscr{C}_{k_2}$. Since $\mathscr{C}_{k_1}$ and $\mathscr{C}_{k_2}$ always have a gap between each interval that belongs to the set, this suffices to show that

$$C(\mathscr{A}_{q_n,n}) = C(\mathfrak{C}_n) = 4^n.$$

Moreover, the regularity of the connected components of $\mathscr{A}_{q_n,n}$ allows to estimate its maximum perimeter. Each connected component of $\mathscr{A}_{q_n,n}$ is a square with $2^{k_1+k_2}$ rectangular holes. Since each hole is a rectangle contained in $[0,1]\times[0,1]$ the maximum perimeter of each hole is 4. Hence, the maximum perimeter of each connected component is $4(2^{k_1+k_2}+1)$.
Making the necessary substitutions, we obtain that

$$\|\mathbf{1}_{\mathscr{A}_{q_n,n}}\|_\alpha \leq \mathrm{Leb}^2(\mathscr{A}_{q_n,n}) + 4(2^{k_1+k_2}+1)C(\mathscr{A}_{q_n,n}). \tag{4.2.7}$$

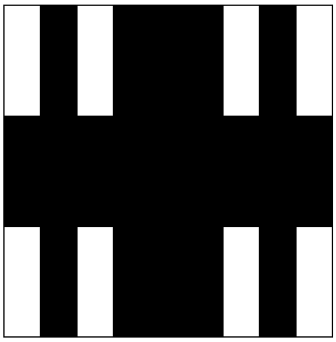

Fig. 4.5 Representation of each connected component of the set $\mathscr{A}_{q_n,n}$ when $k_1 = 1$ and $k_2 = 2$. The white rectangular holes in the picture correspond to the connected components of the set $\mathscr{C}_{n+1} \times \mathscr{C}_{n+2}$ that we delete from each connected component of $\mathfrak{C}_n$. The remaining part of each connected component of $\mathfrak{C}_n$ forms a connected component of the set $\mathscr{A}_{q_n,n}$.

The map T has decay of correlations for quasi-Hölder observables against $L^1(\text{Leb}^2)$ with rate function $\rho_n = 1/3^n$.
Let $t_n = n^2$, then $t_n = o(w_n)$ and there exists a constant $C''' > 0$ such that

$$\begin{aligned}\lim_{n\to\infty} \|\mathbf{1}_{\mathscr{A}_{q_n,n}}\|_\alpha w_n \rho_{t_n} &\leq \lim_{n\to\infty} \Big(\text{Leb}^2(\mathscr{A}_{q_n,n}) + 4(2^{k_1+k_2}+1)C(\mathscr{A}_{q_n,n})\Big) w_n \rho_{t_n} \\ &\leq \lim_{n\to\infty} (C'''(2/3)^{2n} + (2^{k_1+k_2}+1)4^{n+1})(\tau(2/3)^{-2n})\frac{1}{3^{n^2}} \\ &= 0\end{aligned}$$

and condition (1) of 4.2.2 holds.
To finish the proof, we use O'Brien's formula to establish that

$$\begin{aligned}\theta &= \lim_{n\to\infty} \frac{\text{Leb}^2(\mathfrak{C}_n \setminus (\mathscr{C}_{n+k_1} \times \mathscr{C}_{n+k_2}))}{\text{Leb}^2(\mathfrak{C}_n)} \\ &= \lim_{n\to\infty} \frac{(2/3)^{2n}(1-(2/3)^{k_1+k_2})}{(2/3)^{2n}} \\ &= 1-(2/3)^{k_1+k_2}.\end{aligned}$$

□

Chapter 5

Absence of Clustering with Fractal Maximal Sets

In the last chapter, we established the existence of an extreme value law for the stochastic process X_n constructed using an observable maximized in the ternary Cantor set $\mathscr{C}$. The dynamics used was the map $mx \mod 1$, where $m = 3^k$ for some integer k. This choice of m allowed us to guarantee the preservation, by the dynamics, of the maximal set $\mathscr{C}$ leading to the appearance of an extremal index lower than 1.
Following the same setting introduced in Chapter 4, we now generalize this result by proving the following theorem.

Theorem 5.0.1. *Let* $(X_n)_{n\in\mathbb{N}}$ *be the stochastic process constructed as in* (2.2.1) *for a dynamical system* T *defined in* (4.0.2)*, with* $\mathbb{N} \ni m \neq 3^k$ *for all* $k \in \mathbb{N}$*. Consider a sequence of thresholds* $(u_n)_{n\in\mathbb{N}}$ *such that* $u_n = n$ *and a sequence of times* $(w_n)_{n\in\mathbb{N}}$ *such that* $w_n = \lfloor \tau (3/2)^n \rfloor$*.*
Then, condition (3.0.5) *holds and, moreover*

$$\lim_{n\to\infty} \mu(M_{w_n} \leq n) = \mathrm{e}^{-\tau}.$$

We saw that, when $m = 3^k$ the maximal set was acting like a periodic point for the dynamics, *i.e* $T^j(\mathscr{M}) = \mathscr{M}$ for all $j \in \mathbb{N}$. This led to a big recurrence of the maximal set to itself resulting in the existence of clusters of exceedances.
When $m \neq 3^k$ for all $k \in \mathbb{N}$, although $T^j(\mathscr{M}) \neq \mathscr{M}$ for all $j \in \mathbb{N}$, one can easily check that, most of the times, we have that $T^{-j}(\mathscr{M}) \cap \mathscr{M} \neq \emptyset$ and this was enough to create clustering when $\mathscr{M}$ was a finite or countable set (see [3, 4]). However, here, the maximal set has a much more complex structure and one needs to evaluate how relevant the intersections $T^{-j}(\mathscr{M}) \cap \mathscr{M}$ are when compared with $\mathscr{M}$ itself, which translates to how compatible the dynamics of T is with the fractal structure of $\mathscr{M}$. We will see that, since $\mathscr{C}$ has a thickness not less than 1 (*i.e.*, the extractions in the construction of the Cantor set are relatively not too large), then the relevance of the intersections (or the compatibility between T and $\mathscr{M}$) can be measured by the box dimension of the intersections $T^{-j}(\mathscr{M}) \cap \mathscr{M}$, when compared

with the box dimension of $\mathscr{M}$ itself. We will show that the box dimension of $T^{-j}(\mathscr{M})\cap\mathscr{M}$ is strictly smaller than that of $\mathscr{M}$ (Proposition 5.2.1), which means that the possible clustering created by the fact that $T^{-j}(\mathscr{M})\cap\mathscr{M}\neq\emptyset$ is negligible and, in the limit, the EI is still 1.
The computation of the EI is much more subtle than the previous cases. We need results from fractal geometry in order to compute the dimension of the intersections $T^{-j}(\mathscr{M})\cap\mathscr{M}$ and then we need to study its impact on O'Brien's formula (3.1), for which we will perform a finer analysis using the notion of thickness of dynamically defined Cantor sets introduced by Newhouse in [36].
The proof of Theorem 5.0.1 will be done by ensuring the existence of an EI equal to 1 and the validity of conditions $Д_{q_n}$ and $Д'_{q_n}$. This will be done throughout Section 5.2.
The computation of the EI will be performed using the box dimension of the sets $T^{-j}(\mathscr{M})\cap\mathscr{M}$. For the purpose of calculating the box dimension of such sets, we start by introducing the concept of *Digraph Iterated Function Systems*.

5.1 Digraph IFS and Intersection of Sets

The notion of Digraph IFS generalizes the most common set up of Iterated Function Systems and it was introduced in [32]. A Digraph IFS consists of a digraph G where the set of vertices is denoted by V and the set of edges is denoted by E. To each of the vertices, we associate a metric space X_v. Furthermore to each of the edges between two vertices u and v, denoted by $e\in E_{uv}$, we associate a similarity $f_e : X_v \to X_u$ with ratio r_e.
For every path α in the graph G, we form the function f_α by composing the functions f_e along the path in reverse order. The ratio r_α of f_α is just the multiplication of the ratios of the composed function.
If every r_α is less than one, then, there exists a set W which is a union of compact sets W_v, one for every vertex, such that for every $u\in V$,

$$W_u = \bigcup_{v\in V}\bigcup_{e\in E_{uv}} f_e(W_v). \tag{5.1.1}$$

This invariant set W is called the attractor of the Digraph IFS. The existence of this set W is guaranteed if the similarities f_e have ratios smaller than one (see [12]).
It is possible to represent a Digraph IFS in matrix notation. For that purpose, we construct a Digraph IFS matrix M^* with entries indexed by $(u,v)\in V\times V$. The value of each entry will be the set of edges that link one vertex to another. To a Digraph IFS, G, we also associate a Digraph IFS substitution matrix M which is no more than the adjacency matrix of the digraph G.

If E is an attractor of a standard IFS, Mark McClure found in [33] a way to represent the set $E\cap g(E)$, where g is bijection, as an attractor of a Digraph IFS.
For that purpose he proved the following theorem.

Theorem 5.1.1. *Let E be an attractor of an IFS, $\{f_i\}_{i=1}^m$, such that all functions f_i are bijective contractions. Assume that exists a finite set of bijections S such that, for all $g\in S$ satisfying $E\cap g(E)\neq\emptyset$*

and for all $i, j = 1, \ldots, m$ satisfying $E \cap f_i^{-1} g f_j(E) \neq \emptyset$ then $f_i^{-1} g f_j \in S$. Under this condition the list of sets $\{E \cap g(E) : g \in S\}$ forms the attractor of a Digraph IFS.

To construct the Digraph IFS whose attractor is $E \cap g(E)$ it is necessary to use an iterative process to find the set of functions S.
We start with a set $S_0 = \{g\}$, then we define the set

$$S_{k+1} = S_k \cup \left\{ f_i^{-1} h f_j : h \in S_k \text{ and } i, j = 1, \ldots, m \right\}.$$

To fulfill the hypotheses of Theorem 5.1.1, in each step, we select only those functions h such that, $E \cap h(E) \neq \emptyset$. We continue the procedure until no new function is produced.
The functions in S will work as the vertices of the Digraph IFS while the edges will be labeled by the functions f_i. So each row and line of the matrix M^* can be labeled by a function that belongs to the set S. Each entry of this matrix can then be represented by a par of functions $(g, h) \in S \times S$. Each of the entries (g, h) will be a finite set of functions

$$\{f_1, f_2, \ldots, f_k\},$$

whose cardinal is the number of directed edges from g to h, where a function f_i belongs to the set if and only if

$$h = f_i^{-1} g f_j,$$

for some j.
The substitution matrix M of the Digraph IFS, will be the matrix M^* but with each entry replaced by the cardinal of the corresponding set.

Definition 5.1.2 (Open Set Condition)**.** A Digraph IFS satisfies the open set condition if and only if there exists open sets $\Omega_v \in X_v$ such that, for every $u, v \in V$ and $e \in E_{uv}$,

$$f_e(\Omega_v) \subseteq \Omega_u$$

and for all $u, v, v' \in V$, $e \in E_{uv}$ and $e' \in E_{uv}$ with $e' \neq e$,

$$f_e(\Omega_v) \cap f_{e'}(\Omega_{v'}) = \emptyset.$$

From [32], one has that the Hausdorff dimension of the attractor W of a Digraph IFS satisfying the open set condition, which by [10] and under certain conditions that are verified in our setting, is equal to its box dimension, can be written in terms of the spectral radius of M and the common ratios of the similarities of the Digraph IFS. We refer the reader to Appendix A for more a detailed discussion on the fractal dimension of attractors of Digraph IFS and spectral radius of matrices.

5.2 Dimension Estimates and Absence of Clustering

This section is dedicated to the proof of Theorem 5.0.1. The first step will be to estimate the dimension of the sets $T^{-q}(\mathscr{C}) \cap \mathscr{C}$ using the procedure described in Section 5.1. Then, using the notion of thickness, we will link dimension estimates to EI estimates. The proof will become complete by verifying conditions $Д_{q_n}$ and $Д'_{q_n}$.

5.2.1 Dimension Estimates

We will apply the procedure described in Section 5.1 to estimate the box dimension of the set $T^{-q}(\mathscr{C}) \cap \mathscr{C}$. Our main goal is to show the following proposition.

Proposition 5.2.1. *Let* $T = mx \mod 1$*, where* $\mathbb{N} \ni m \neq 3^k$ *for any* $k \in \mathbb{N}$*. Then, for all* $q \in \mathbb{N}$*, we have:*

$$\dim_H(T^{-q}(\mathscr{C}) \cap \mathscr{C}) = \dim_B(T^{-q}(\mathscr{C}) \cap \mathscr{C}) \leq \frac{1}{2}.$$

We start by noting that, for any q integer,

$$T^q(x) = m^q x \mod 1.$$

Therefore, the set $T^{-q}(\mathscr{C})$ is a union of sets formed by taking the preimage of $\mathscr{C}$ by each one of the branches of T^q. When restricted to each of the branches T^q is a bijection. Therefore, we will use the algorithm described in the previous section to represent the intersection of the preimage of $\mathscr{C}$, by each one of the branches of T^q, with $\mathscr{C}$ as a Digraph IFS attractor.
Due to the form of T, the functions g of interest to us to start the algorithm described in Theorem 5.1.1 are of the form

$$g = \frac{1}{m^q}x + b_g, \tag{5.2.1}$$

where b_g is of the form $k/(m^q)$ with k an integer less than m^q.
The algorithm leads to the construction of m^q sets of functions, which we denote by S_q^k for $k \in \{0, \dots, m^q - 1\}$, depending on the constant term of the function g that initiates the algorithm. Each of the sets S_q^k yields a substitution matrix M_q^k associated with the respective Digraph IFS.

Define the functions $f_i = x/3 + b_i$, where b_i is either 0 or $2/3$. This set of functions forms an IFS whose attractor is the ternary Cantor set $\mathscr{C}$.
For a given q and k, the functions $h = f_i^{-1} g f_j$ that belong to the set S_q^k, are of the form,

$$f_i^{-1} g f_j = \frac{1}{m^q}x + 3\left(\frac{1}{m^q}b_j + b_g - b_i\right),$$

where g already belongs to the set in question. Hence, for h to belong to S_q^k, it is necessary that its constant term satisfies

$$3\left(\frac{1}{m^q}b_j+b_g-b_i\right)\in\left\{\frac{-1}{m^q},0,\frac{1}{m^q},\frac{2}{m^q},\dots,1\right\}. \tag{5.2.2}$$

This implies that, all the functions in S_q^k are of the form,

$$\frac{1}{m^q}x+\frac{s}{m^q},$$

where $s\in\{-1,0,\dots,m^q\}$.
For the computation of the box dimension of a Digraph IFS attractor, it is necessary to caracterize the respective substitution matrix. Hence, we proceed now to the characterization of the matrices M_q^k. For better understanding, we divided this process into the following smaller results.

Lemma 5.2.2. *Let $q\in\mathbb{N}_0$ and $k\in\{0,\dots,m^q-1\}$, then, every entry of the matrix M_q^k is either* 0 *or* 1.

Proof. Fix q, k and let g be a function in S_q^k. As seen in (5.2.2), any other function h that belongs to the set S_q^k must be equal to

$$h=\frac{1}{m^q}x+b_h,$$

where b_h is of the form $s/(m^q)$ with $s\in\{-1,\dots,m^q\}$.
To prove the Lemma, we will need to address two different cases, each with two different possibilities:

- If $h=f_1^{-1}gf_2$ then $h\neq f_2^{-1}gf_1$,

- If $h=f_1^{-1}gf_2$ then $h\neq f_2^{-1}gf_2$,

- If $h=f_1^{-1}gf_1$ then $h\neq f_2^{-1}gf_1$,

- If $h=f_1^{-1}gf_1$ then $h\neq f_2^{-1}gf_2$.

For the first case, assume that $h=f_1^{-1}gf_2$ and $h=f_2^{-1}gf_1$. Then, we would obtain

$$\frac{1}{m^q}x+3\left(\frac{1}{m^q}b_1+b_g-b_2\right)=\frac{1}{m^q}x+3\left(\frac{1}{m^q}b_2+b_g-b_1\right).$$

Since $b_1=0$ and $b_2=2/3$, we are led to $-1=\frac{1}{m^q}$, which is an a contradiction.
Consider that $h=f_1^{-1}gf_2$ and $h=f_2^{-1}gf_2$. Then,

$$\frac{1}{m^q}x+3\left(\frac{1}{m^q}b_1+b_g-b_2\right)=\frac{1}{m^q}x+3\left(\frac{1}{m^q}b_2+b_g-b_2\right),$$

and $2/m^q = 0$, which is a contradiction.
Consider that $h = f_1^{-1} g f_1$ and $h = f_2^{-1} g f_1$. Then,

$$\frac{1}{m^q}x + 3\left(\frac{1}{m^q}b_1 + b_g - b_1\right) = \frac{1}{m^q}x + 3\left(\frac{1}{m^q}b_2 + b_g - b_1\right),$$

and $2/m^q = 0$ which is, again, a contradiction.
For the last case, assume that $h = f_1^{-1} g f_1$ and $h = f_2^{-1} g f_2$ and observe that

$$\frac{1}{m^q}x + 3\left(\frac{1}{m^q}b_1 + b_g - b_1\right) = \frac{1}{m^q}x + 3\left(\frac{1}{m^q}b_2 + b_g - b_2\right)$$

implies $1 = 1/m^q$, which is a contradiction and the Lemma is proved. □

Lemma 5.2.3. *Let $q \in \mathbb{N}_0$ and $k \in \{0, \dots, m^q - 1\}$, then the sum of the elements of each row of the matrix M_q^k is at most* 2.

Proof. Consider a function g in S_q^k. As seen before, any other function $h \in S_q^k$ must be of the form

$$h = \frac{1}{m^q}x + b_h,$$

where $b_h = s/(m^q)$ with $s \in \{-1, \dots, m^q\}$.
Accordingly to the possible values of b_i and b_j there are four different possibilities for the line of M_q^k indexed by g to have entries equal to 1. We will prove that these cases form two disjoint groups of two elements, which will prove the claim of the Lemma.

Assume that $b_i = 0$ and $b_j = 0$ and that $h = f_i^{-1} g f_j$ belongs to S_q^k. By (5.2.2), the constant term of h satisfies

$$3b_g \in \left\{\frac{-1}{m^q}, 0, \frac{1}{m^q}, \frac{2}{m^q}, \dots, 1\right\}. \tag{5.2.3}$$

By contradiction, assume that $h^* = f_i^{-1} g f_j$ belongs to S_q^k with $b_i = 2/3$ and $b_j = 0$. Again, by (5.2.2), the constant term of h^* satisfies

$$3b_g - 2 \in \left\{\frac{-1}{m^q}, 0, \frac{1}{m^q}, \frac{2}{m^q}, \dots, 1\right\}. \tag{5.2.4}$$

Since 2 is larger than the length of the interval $\left[\frac{-1}{m^q}, 1\right]$, for any m considered, then the two conditions, (5.2.3) and (5.2.4), cannot be simultaneously fulfilled for any b_g and the two cases are therefore necessarily disjoint.
Now, assume that $b_i = 0$ and $b_j = 2/3$ and that $h = f_i^{-1} g f_j$ belongs to S_q^k.
By (5.2.2), we obtain that the constant term of h satisfies

$$\frac{2}{m^q} + 3b_g \in \left\{\frac{-1}{m^q}, 0, \frac{1}{m^q}, \frac{2}{m^q}, \dots, 1\right\}. \tag{5.2.5}$$

Assume further that $h^* = f_i^{-1} g f_j$ belongs to S_q^k with $b_i = 2/3$ and $b_j = 2/3$. Again, by (5.2.2), the constant term of h^* satisfies

$$\frac{2}{m^q} + 3b_g - 2 \in \left\{ \frac{-1}{m^q}, 0, \frac{1}{m^q}, \frac{2}{m^q}, \ldots, 1 \right\}. \tag{5.2.6}$$

Since 2 is larger than the length of the interval $\left[\frac{-1}{m^q}, 1\right]$, for any possible m, then the two conditions, (5.2.5) and (5.2.6), cannot be simultaneously fulfilled for any b_g and the two cases are, again, disjoint and the Lemma is proved. □

Lemmas 5.2.2 and 5.2.3 allow us to characterize the substitution matrices M_q^k. Each matrix M_q^k is a $(0,1)$-matrix, whose spectral radius is less or equal to 2. We will show that, in fact, the spectral radius is strictly less than 2. In order to do that, we will consider a matrix N^q. This matrix will correspond to the substitution matrix of the Digraph IFS, D, whose nodes are all possible functions of the form

$$\frac{1}{m^q} x + \frac{s}{m^q},$$

where $s \in \{-1, \ldots, m^q\}$.
The Digraph IFS, D, has an edge from a node g to a node h if

$$h = f_i^{-1} g f_j, \tag{5.2.7}$$

which implies that the entry (g,h) of the matrix N^q will be different from zero.
Note that, due to relation (5.2.7), then, relation (5.2.2) holds for the constant term of h and Lemmas 5.2.2 and 5.2.3 apply to the matrix N^q without any change in the respective proof. So, N^q is a $(0,1)$-matrix whose row entries sum at most 2.
Furthermore, under these assumptions, the matrices M_q^k are principal submatrices of N^q, which means that if we are able to bound the spectral radius of N^q away from 2, uniformly on q, then, by A.3.1, the same will apply to M_q^k.

In what follows, we will use the notation $a \equiv b \mod p$, to express the fact that a and b are congruent modulo p.

Lemma 5.2.4. *Assume that m in the definition of T, in* (4.0.2)*, is such that m is not divisible by* 3*,* i.e.*,* $m \not\equiv 0 \mod 3$*. Then, the matrix N^q has a spectral radius less or equal to $\sqrt{3}$,* i.e.*,*

$$\rho(N^q) \leq \sqrt{3}.$$

Proof. Let g be a function of the form

$$\frac{1}{m^q} x + \frac{s}{m^q},$$

for $s \in \{-1, \ldots, m^q\}$.
If the entry (g,h) of N^q is different from zero, then $h = f_i^{-1} g f_j$ and relation (5.2.2) holds. This means

that the constant term of h satisfies

$$3\left(\frac{1}{m^q}b_j + b_g - b_i\right) \in \left\{\frac{-1}{m^q}, 0, \frac{1}{m^q}, \frac{2}{m^q}, \dots, 1\right\}.$$

Depending on the value of b_i and b_j, we have four different cases.

If $b_j = 0$ and $b_i = 2/3$, then, the constant term of h satisfies

$$\frac{3s - 2m^q}{m^q} \in \left\{\frac{-1}{m^q}, 0, \frac{1}{m^q}, \frac{2}{m^q}, \dots, 1\right\}.$$

If $b_j = 0$ and $b_i = 0$, then, the constant term of h satisfies

$$\frac{3s}{m^q} \in \left\{\frac{-1}{m^q}, 0, \frac{1}{m^q}, \frac{2}{m^q}, \dots, 1\right\}.$$

If $b_j = 2/3$ and $b_i = 0$, then, the constant term of h satisfies

$$\frac{3s + 2}{m^q} \in \left\{\frac{-1}{m^q}, 0, \frac{1}{m^q}, \frac{2}{m^q}, \dots, 1\right\}.$$

If $b_j = 2/3$ and $b_i = 2/3$, then, the constant term of h satisfies

$$\frac{3s - 2m^q + 2}{m^q} \in \left\{\frac{-1}{m^q}, 0, \frac{1}{m^q}, \frac{2}{m^q}, \dots, 1\right\}.$$

Up to this point, every entry of the matrix is indexed by functions of the form $\frac{1}{m^q}x + \frac{s}{m^q}$, with $s \in \{-1, \dots, m^q\}$. Hence, we can associate to the entry of the matrix (g, h) the index (s, s^*), where s and s^* are the numerators of the constant terms of g and h, respectively. An entry (s, s^*) of N^q is nonzero if and only if s is such that one of the above cases is verified.

If the first case occurs, then $3s - 2m^q \in \{-1, \dots, m^q\}$. Hence, if $N^q_{ss^*} \neq 0$, we have that $s^* = 3s - 2m^q$. If the second case occurs, then $3s \in \{-1, \dots, m^q\}$ and if $N^q_{ss^*} \neq 0$ then $s^* = 3s$. For the third case, we need $3s + 2$ to belong to the set $\{-1, \dots, m^q\}$ and if $N^q_{ss^*} \neq 0$ then $s* = 3s + 2$. If the last case is verified, then $3s - 2m^q + 2$ belongs to $\{-1, \dots, m^q\}$ and if $N^q_{ss^*} \neq 0$ then $s^* = s - 2m^q + 2$.

Changing the indices for the more usual set $\{1, \dots, m^q + 2\}$, we obtain that N^q can be characterized by

$$\begin{cases} N^q_{i,3i-2} = 1 & \text{if } i, 3i-2 \in \{1, \dots, m^q + 2\} \\ N^q_{i,3i-4} = 1 & \text{if } i, 3i-4 \in \{1, \dots, m^q + 2\} \\ N^q_{i,3i-2m^q-4} = 1 & \text{if } i, 3i - 2m^q - 4 \in \{1, \dots, m^q + 2\} \\ N^q_{i,3i-2m^q-2} = 1 & \text{if } i, 3i - 2m^q - 2 \in \{1, \dots, m^q + 2\} \\ N^q_{i,j} = 0 & \text{otherwise.} \end{cases} \tag{5.2.8}$$

For a more visual representation of N^q, we may write:

$$N^q = \begin{pmatrix} 1 & 0 & 0 & \dots & \dots & \dots & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & \dots & \dots & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & \dots & \dots & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & \dots & \dots & \dots & 0 & 0 & 1 \end{pmatrix}. \tag{5.2.9}$$

The shape of the matrix N^q will depend on how many sequences of $(1,0,1)$ will fit in m^q+1 columns. Since m is not divisible by 3, by Fermat's Little Theorem, we have $m^2 \equiv 1 \mod 3$.
Let $q = 2k$, for some $k \in \mathbb{N}$, then

$$m^{2k} + 1 \equiv 2 \mod 3.$$

If $q = 2k+1$ for some $k \in \mathbb{N}$, then, we have $m^{2k+1} + 1 \equiv m+1 \mod 3$. Hence, in this case

$$m^{2k+1} + 1 \equiv 0 \mod 3 \text{ or } m^{2k+1} + 1 \equiv 2 \mod 3,$$

depending on whether $m \equiv 2 \mod 3$ or $m \equiv 1 \mod 3$, respectively.
This means that we have two different cases to address, either $m^q + 1 \equiv 0 \mod 3$ or $m^q + 1 \equiv 2 \mod 3$.

Let q be such that $m^q + 1 \equiv 2 \mod 3$. Denote by $x = (x_1, x_2, \dots, x_{m^q+2})$ a vector in $\mathbb{R}^{m^q+2}$.
Note that there is no i such that $N^q_{i,3i-2} = 1$ and $N^q_{i,3i-2m^q-2} = 1$, in conjunction, or $N^q_{i,3i-2} = 1$ and $N^q_{i,3i-2m^q-4} = 1$, together. Similarly, there is also no i such that $N^q_{i,3i-4} = 1$ and $N^q_{i,3i-2m^q-4} = 1$, together, or $N^q_{i,3i-4} = 1$ and $N^q_{i,3i-2m^q-2} = 1$.

Hence, we can write

$$N^q x = \begin{pmatrix} x_1 \\ x_2 + x_4 \\ \dots \\ x_{2+3\alpha} + x_{4+3\alpha} \\ x_{m^q+1} \\ 0 \\ \dots \\ 0 \\ x_2 \\ x_{m^q-1-3\beta} + x_{m^q+1-3\beta} \\ \dots \\ x_{m^q-1} + x_{m^q+1} \\ x_{m^q+2} \end{pmatrix},$$

where α, β are integers that satisfy $0 < \alpha, \beta < (m^q + 1)/3$.
Consider the sets

$$\mathscr{A} = \{i \in \mathbb{N} : i = 2 + 3\alpha \text{ and } 0 < \alpha < (m^q + 1)/3\}$$

and

$$\mathscr{B} = \{i \in \mathbb{N} : i = m^q - 1 - 3\beta \text{ and } 0 < \beta < (m^q + 1)/3\}.$$

Then, $\|N^q x\|^2$ can be written as

$$x_1^2 + x_2^2 + \sum_{i \in \mathscr{A}} (x_i + x_{i+2})^2 + \sum_{j \in \mathscr{B}} (x_j + x_{j+2})^2 + x_{m^q+1} + x_{m^q+2}. \tag{5.2.10}$$

For simplicity, let

$$A := \sum_{i \in \mathscr{A}} (x_i + x_{i+2})^2$$

and

$$B := \sum_{i \in \mathscr{B}} (x_j + x_{j+2})^2.$$

Each coordinate of the vector x appears, at most, once in A and once in B. Let i^* be such that, x_{i^*} appears both in A and B and assume that $i^* \in \mathscr{A}$. Then $i^* = j + 2$ for some j in $\mathscr{B}$. To prove this, we proceed by contradiction.
Assume that $i^* = j$ for some $j \in \mathscr{B}$. Then,

$$2 + 3\alpha = m^q - 1 - 3\beta,$$

for some integers α, β. But, this implies that 3 divides m which is a contradiction. With a similar argument, we prove that, if $i^* = i + 2$ for some $i \in \mathscr{A}$, then $i^* = j$ for some $j \in \mathscr{B}$.

Now, let i^* be such that x_{i^*} appears in A and in B and assume that $i^* \in \mathscr{A}$. Then $i^* = j + 2$ for some $j \in \mathscr{B}$. We will show that x_{i^*+2} does not appear more than once in (5.2.10).

If $x_{i^*+2} = x_1$ or $x_{i^*+2} = x_3$, then $i^*+2 = 1$ or $i^*+2 = 3$ and i^* cannot belong to $\mathscr{A}$. On other hand, if $x_{i^*+2} = x_{m^q+2}$ then

$$i^*+2 = m^q+2$$

and for some $\beta \in \mathbb{N}$,

$$m^q+1-3\beta = m^q+2$$

which implies that $3\beta = -1$. This is a contradiction.
A similar argument can be made to show that $x_{i^*+2} \neq x_{m^q}$.
To finish, assume that exists a $j^* \in \mathscr{B}$ such that $i^*+2 = j^*$ or $i^*+2 = j^*+2$. If $i^*+2 = j^*$, then there exist integers β and β^* such that

$$m^q+1-3\beta = m^q-1-3\beta^*.$$

Hence, $2 = 3(\beta-\beta^*)$ which is also a contradiction. If $i^*+2 = j^*+2$, then $j^* \in \mathscr{A}$. This is impossible as proved earlier.
Similarly, if x_{i^*} appears in A and B and $i^* = i+2$ for some $i \in \mathscr{A}$, then x_i cannot appear more than once in (5.2.10).

Using Proposition A.3.2, we can write that, for any $\varepsilon > 0$,

$$(x_i + x_{i+2})^2 \leq (1+\varepsilon)x_i^2 + (1+1/\varepsilon)x_{i+2}^2.$$

As proved above and due to the matrix pattern, for x_l to appear in the sum A and B then $l = i$ for some $i \in \mathscr{A}$ and $l = j+2$ for some in $j \in \mathscr{B}$. Hence, for all $\varepsilon > 0$,

$$\|N^q x\|^2 \leq x_1^2 + x_2^2 + \sum_{i\in\mathscr{A}} (1+\varepsilon)x_i^2 + (1+1/\varepsilon)x_{i+2}^2 + \sum_{i\in\mathscr{B}} (1+1/\varepsilon)x_j^2 + (1+\varepsilon)x_{j+2}^2 + x_{m^q+1}^2 + x_{m^q+2}^2 \quad (5.2.11)$$

and we can establish that there are coefficients c_l such that

$$\|N^q x\|^2 \leq \sum_{l=1}^{m^q+2} c_l x_l^2. \quad (5.2.12)$$

So, choosing $\varepsilon = 0.5$ and if x_l appears in both sums A and B, then $c_l = 2(1+\varepsilon) = 3$. Furthermore, if x_{l^*} is another coordinate such that the term $(x_l + x_{l^*})^2$ appears only in A or in B, then $c_{l^*} = (1+1/\varepsilon) \leq 3$, since x_{l^*} does not appear anywhere else in (5.2.11). On other hand, if x_1 appears either in A or B then c_1 is equal to $1+1+\varepsilon \leq 3$ and the same conclusion holds for c_2, c_{m^q+1} or c_{m^q+2}.
Hence, for every $1 \leq l \leq m^q+2$, we have $c_l \leq 3$ and, therefore

$$\|N^q x\|^2 \leq 3 \sum_{l=1}^{m^q+2} x_l^2 \leq 3\|x\|^2.$$

Consequently, by (A.3.1),

$$\rho(N^q) \leq \sqrt{3}.$$

If $m^q + 1 \equiv 0 \mod 3$, in a very similar way, we obtain

$$\|N^q x\|^2 = x_1^2 + \sum_{i\in\mathscr{A}} (x_i + x_{i+2})^2 + \sum_{j\in\mathscr{B}} (x_j + x_{j+2})^2 + x_{m^q+2}.$$

Then using the inequality

$$\|N^q x\|^2 \leq x_1^2 + \sum_{i\in\mathscr{A}} (1 + 1/\varepsilon)x_i^2 + (1 + \varepsilon)x_{i+2}^2 + \sum_{j\in\mathscr{B}} (1 + \varepsilon)x_j + (1 + 1/\varepsilon)x_{j+2}^2 + x_{m^q+2},$$

with $\varepsilon = 0.5$, the proof follows for all q. □

We point out that each of the Digraph IFS associated to the intersection $T^{-q}(\mathscr{C}) \cap \mathscr{C}$ satisfies the open set condition. Each digraph is composed of only two different similarities, $x/3$ and $x + 2/3$. Hence, choosing $\Omega_v = (0,1)$ for every $v \in V$, we can check that the conditions in Definition 5.1.2 are easily satisfied. Using this remark and the discussion made in Section A.2 of the Appendix A regarding the fractal dimension of attractors of Digraph IFS, we can proceed to the proof of Proposition 5.2.1.

Proof of Proposition 5.2.1. Recalling that the matrices M_q^k of each Digraph IFS associated to the intersection $T^{-q}(\mathscr{C}) \cap \mathscr{C}$ are principal submatrices of N^q, then A.3.1 implies that

$$\rho(M_q^k) \leq \rho(N^q).$$

Hence, if m is not divisible by 3, Lemma 5.2.4 gives us $\rho(M_q^k) \leq \sqrt{3}$. Consequently, noting that each Digraph IFS associated with a $(0,1)$-matrix M_q^k satisfies the open set condition and is composed of only two different similarities, $x/3$ and $x + 2/3$, both with ratio $1/3$, we can apply Theorem A.2.1 and A.2.2 to estimate the Hausdorff dimension of $T^{-q}(\mathscr{C}) \cap \mathscr{C}$, namely,

$$\dim_H(T^{-q}(\mathscr{C}) \cap \mathscr{C}) \leq \frac{\log\sqrt{3}}{-\log 1/3} = \frac{1}{2},$$

for all $q \in \mathbb{N}$.

Moreover, as checked in Section A.2 of the Appendix A, conditions of Theorem A.2.4 and Theorem A.2.3 are satisfied in our setting and therefore $\dim_H(T^{-q}(\mathscr{C}) \cap \mathscr{C}) = \dim_B(T^{-q}(\mathscr{C}) \cap \mathscr{C})$, which allows us to obtain:

$$\dim_B(T^{-q}(\mathscr{C}) \cap \mathscr{C}) \leq \frac{\log\sqrt{3}}{-\log 1/3} = \frac{1}{2} < \frac{\log 2}{\log 3} = \dim_B(\mathscr{C}). \tag{5.2.13}$$

So far, m is not divisible by 3. Using the self-similarity of the Cantor set $\mathscr{C}$ it is possible to extend our findings to the cases where $m = 3^k c$, for some integers $c, k > 1$ such that c is not divisible by 3. Figure 5.1 intends to illustrate our reasoning for the case where $k = 1$, $c = 2$ and $q = 1$.

Consider the map $\tilde{T}(x) = cx \mod 1$. We claim that

$$\dim_B(T^{-q}(\mathscr{C}) \cap \mathscr{C}) = \dim_B(\tilde{T}^{-q}(\mathscr{C}) \cap \mathscr{C}).$$

Let $g_\gamma \colon \mathbb{R} \to \mathbb{R}$ be given by $g(x) = \gamma x$ for all $x \in \mathbb{R}$. The set $T^{-q}(\mathscr{C}) \cap \mathscr{C}$ is obtained by intersecting $3^{kq}c^q$ copies of the set $g_{3^{-kq}c^{-q}}(\mathscr{C})$ distributed side by side along the interval $[0,1]$, with the set $\mathscr{C}$.

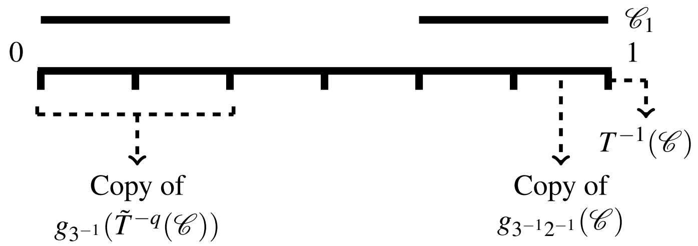

Fig. 5.1 For $c = 2$ and $k = 1$ this figure illustrates the relation between $\tilde{T}^{-1}(\mathscr{C}) \cap \mathscr{C}$ and $T^{-1}(\mathscr{C}) \cap \mathscr{C}$.

Note that because of the self-similarity of $\mathscr{C}$, the intersection of each of the 2^{kq} connected components of $\mathscr{C}_{kq}$ with $\mathscr{C}$ is a copy of $g_{3^{-kq}}(\mathscr{C})$. Moreover, each of the 2^{kq} connected components of $\mathscr{C}_{kq}$ meets exactly c^q of the copies of the set $g_{3^{-kq}c^{-q}}(\mathscr{C})$ that constitute $T^{-q}(\mathscr{C})$. Therefore, the intersection of the set $T^{-q}(\mathscr{C}) \cap \mathscr{C}$ with each of the 2^{kq} connected components of $\mathscr{C}_{kq}$ is a copy of $g_{3^{-kq}}(\tilde{T}^{-q}(\mathscr{C}) \cap \mathscr{C})$ and the claim follows. $\square$

Remark 5.2.5. We note that when $m = 3^k$ for some $k \in \mathbb{N}$, one can check that the matrices N^q have a spectral radius equal to 2, which means that the box dimension of $T^{-q}(\mathscr{C}) \cap \mathscr{C}$ is equal to the box dimension of $\mathscr{C}$. For example, if $m = 3$ and $q = 1$ then

$$N^q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

which can be easily checked to have a spectral radius equal to 2. This is consistent with what we proved in Theorem 4.0.1.

5.2.2 From Dimension Estimates to EI estimates

In this section, we show how to make use of the information regarding the irrelevance of the intersection of $\mathscr{C}$ with its iterates, in order to compute the EI of the limiting law stated in Theorem 5.0.1. Essentially, we have to translate the difference between the box dimension of $\mathscr{C}$ and $T^{-q}(\mathscr{C}) \cap \mathscr{C}$ to the difference between the Lebesgue measure of the respective convex hull approximations of decreasing size. We will use some ideas used by Newhouse in [36], such as *thickness*, to study invariants of Cantor sets to prove the abundance of wild hyperbolic sets.

Again, let $\mathscr{C}$ denote the ternary Cantor set and $\mathscr{C}_n$ its n-th approximation consisting of 2^n disjoint intervals of length 3^{-n} and let $\mathfrak{C}_n$ denote the collection of intervals whose disjoint union forms $\mathscr{C}_n$. Note that $\mathscr{C}_n$ is a set while $\mathfrak{C}_n$ is a collection of sets. Consider that $T^{-q}(\mathfrak{C}_n)$ is the collection of all the connected components of $T^{-q}(\mathscr{C}_n)$. We consider the set $\mathscr{C} \cap T^{-q}(\mathscr{C})$. Note that $\mathscr{C}_n \cap T^{-q}(\mathscr{C}_n) \downarrow \mathscr{C} \cap T^{-q}(\mathscr{C})$.
Let $\overline{A}$ denote the closure of A, $\mathring{A}$ its interior and A^c its complement. Define

$$N_{3^{-n}} = \#\{I \in \mathfrak{C}_n : \mathring{I} \cap (\mathscr{C} \cap T^{-q}(\mathscr{C})) \neq \emptyset\}, \tag{5.2.14}$$

$$N^*_{3^{-n}} = \#\{I \in \mathfrak{C}_n : \mathring{I} \cap (\mathscr{C}_n \cap T^{-q}(\mathscr{C}_n)) \neq \emptyset\}. \tag{5.2.15}$$

Since $\mathscr{C} \cap T^{-q}(\mathscr{C}) \subset \mathscr{C}_n \cap T^{-q}(\mathscr{C}_n)$, then $N_{3^{-n}} \leq N^*_{3^{-n}}$. However, one can prove that something stronger.

Proposition 5.2.6. *If n is sufficiently large so that $3^{-n} \leq m^{-q}$, we have that $N_{3^{-n}} = N^*_{3^{-n}}$.*

In order to prove the proposition, we need the following result which follows from the thickness property of the ternary Cantor set $\mathscr{C}$.

Definition 5.2.7. Let Λ be a Cantor set (not necessarily the ternary Cantor set $\mathscr{C}$ but homeomorphic to $\mathscr{C}$). To define thickness, we consider the gaps of Λ: a gap of Λ is a connected component of $\mathbb{R} \setminus \Lambda$; a bounded gap is a bounded connected component of $\mathbb{R} \setminus \Lambda$.
Let U be any bounded gap and u be a boundary point of U such that $u \in \Lambda$. Let B be a bridge of Λ at u, *i.e.* the maximal interval in $\mathbb{R}$ such that

- u is a boundary point of B;
- B contains no point of a gap U' whose length $|U'|$ is at least the length of U.

The thickness of Λ at u is defined as $\rho(\Lambda, u) = |B|/|U|$. The thickness of Λ, denoted by $\rho(\Lambda)$, is the infimum over these $\rho(\Lambda, u)$ for all boundary points u of bounded gaps.

In the construction of the ternary Cantor set $\mathscr{C}$ the gaps created at the $n-th$ step of its construction have the exact same size as the connected components of $\mathscr{C}_n$, therefore, the thickness of $\mathscr{C}$ is equal to 1.
The next lemma resembles the Gap Lemma in [36, 38], which was stated for two Cantor sets Λ_1 and Λ_2 such that $\rho(\Lambda_1) \cdot \rho(\Lambda_2) > 1$. The conclusion was that either their intersection is nonempty or one of them is contained inside a gap of the other. Since, here, we need to consider two Cantor sets, both with thickness equal to 1, we prove the following result which allows, in particular, to generalise the statement of the Gap Lemma.
Note that, if the maximal set $\mathscr{M}$ was such that $\rho(\mathscr{M}) > 1$ then, we could skip this step.

Lemma 5.2.8. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be two affine transformations such that $f([0,1]) \cap g([0,1]) \neq \emptyset$, and let $\Lambda_1 = f(\mathscr{C})$ and $\Lambda_2 = g(\mathscr{C})$. Then, either $\Lambda_1 \cap \Lambda_2 \neq \emptyset$ or one of them is contained inside a gap of the other (*i.e.*, $f([0,1])$ is contained inside a gap of Λ_2 or $g([0,1])$ is contained inside a gap of Λ_1).*

Proof. Let us assume that neither Λ_1 nor Λ_2 are contained inside a gap of the other and assume that $\Lambda_1 \cap \Lambda_2 = \emptyset$, and derive a contradiction. Let us denote by G_1 a gap of Λ_1 and G_2 a gap of Λ_2. We say that (G_1, G_2) form a *gap pair* if G_2 contains exactly one boundary point of G_1, which also contains exactly one boundary point of G_2. Recall that the boundary points of G_1 belong to Λ_1, as the boundary points of G_2 must belong to Λ_2. Observe that such a gap pair must always exist because $f([0,1]) \cap g([0,1]) \neq \emptyset$ and otherwise the points of Λ_2 could never be removed from $f([0,1])$ in order to have that $\Lambda_1 \cap \Lambda_2 = \emptyset$ (having in mind that neither $f([0,1])$ nor $g([0,1])$ are contained inside a gap o Λ_2 and Λ_1, respectively). Given such a pair we build a sequence of gap pairs $(G_1^{(i)}, G_2^{(i)})_{i\in\mathbb{N}}$ such that either $|G_1^{(i+1)}| < |G_1^{(i)}|$ or $|G_2^{(i+1)}| < |G_2^{(i)}|$. Let $\left(G_1^{(i)}, G_2^{(i)}\right)$ be given. Let $m, p \in \mathbb{N}$ be the smallest integers such that $G_1^{(i)}, G_2^{(i)}$ appear as gaps of $f(\mathscr{C}_m), g(\mathscr{C}_p)$, respectively. Let C_1^ℓ, C_1^r and C_2^ℓ, C_2^r be the intervals of $f(\mathcal{C}_m), g(\mathcal{C}_p)$, respectively, that appear to the left and right of the gaps $G_1^{(i)}$ and $G_2^{(i)}$ and share the respective border points. Note that by construction we have that $|C_1^\ell| = |C_1^r| = |G_1^{(i)}|$ and $|G_2^{(i)}| = |C_2^\ell| = |C_2^r|$. Therefore, we must have that the right endpoint of $G_2^{(i)}$ belongs to C_1^r or the left endpoint of $G_1^{(i)}$ belongs to C_2^ℓ, or both. Let us assume *w.l.o.g.* that the first case happens and denote by T the right endpoint of $G_2^{(i)}$. Clearly, $T \in \Lambda_2$ and, since we are assuming that $\Lambda_1 \cap \Lambda_2 = \emptyset$, we have $T \notin \Lambda_1$. Hence, T must fall into some gap of Λ_1 inside C_1^r, which we will denote by $G_1^{(i+1)}$. Since $|C_1^r| = |G_1^{(i)}|$, it follows that $|G_1^{(i+1)}| < |G_1^{(i)}|$. In this case, we set $G_2^{(i+1)} = G_2^{(i)}$ and define $\left(G_1^{(i+1)}, G_2^{(i+1)}\right)$ as the new gap pair. It follows that $\lim_{i\to\infty} |G_1^{(i)}| = 0$ or $\lim_{i\to\infty} |G_2^{(i)}| = 0$, or both. Assume the first and let $y_i \in G_1^{(i)}$ and y be an accumulation point of $(y_i)_{i\in\mathbb{N}}$. Then y is also an accumulation point of the right endpoints of $G_1^{(i)}$, which belong to Λ_1, and of the left endpoints of $G_2^{(i)}$, which belong to Λ_2 and, by definition of gap pair, are all inside $G_1^{(i)}$. But since Λ_1 and Λ_2 are compact sets, then $y \in \Lambda_1 \cap \Lambda_2$, which is a contradiction. □

Proof of Proposition 5.2.6. Observe that $T^{-q}(\mathscr{C}_n)$ corresponds to m^q copies of $\mathscr{C}_n$ contracted by the factor m^{-q} and placed side by side on $[0,1]$. Let $I \in \mathcal{C}_n$ be an interval such that $\mathring{I} \cap (\mathscr{C}_n \cap T^{-q}(\mathscr{C}_n)) \neq \emptyset$. Let $J \in T^{-q}(\mathcal{C}_n)$ be an interval of $T^{-q}(\mathscr{C}_n)$ such that $J \cap \mathring{I} \neq \emptyset$. Note that $\Lambda_1 := I \cap \mathscr{C}$ and $\Lambda_2 := J \cap T^{-q}(\mathscr{C})$ are copies of the original Cantor set, due to its self-similarity property. In fact, if we let f, g to be affine transformations such that $f([0,1]) = I$ and $g([0,1]) = J$, then $I \cap \mathscr{C} = f(\mathscr{C})$ and $J \cap T^{-q}(\mathscr{C}) = g(\mathscr{C})$. Noting that $|J| \leq |I|$, then if J is not contained in any gap of Λ_1, by Lemma 5.2.8, it follows that $\Lambda_1 \cap \Lambda_2 \neq \emptyset$ and therefore $I \cap (\mathscr{C} \cap T^{-q}(\mathscr{C})) \neq \emptyset$. If J is contained in some gap of Λ_1, we consider J_1 to be the interval of $T^{-q}(\mathcal{C}_{n-1})$ that contains J. If J_1 is not contained entirely in the same gap in which J is contained, then, since by the structure of the Cantor sets we must still have that $|J_1| \leq |I|$, then by the argument above we are lead to the same conclusion that $I \cap (\mathscr{C} \cap T^{-q}(\mathscr{C})) \neq \emptyset$. If J_1 is still contained in the same gap of Λ_1, we define J_2 as the interval of $T^{-q}(\mathcal{C}_{n-2})$ that contains J and so on, until, eventually, we find some $k \leq n$ so that J_k is not contained entirely in the same gap in which J is contained and $|J_k| \leq |I|$. This is guaranteed because the size of the gap of $\Lambda_1 \subset I$ is smaller than $3^{-n} \leq |J_n|$. □

Using the results above, we can proceed with the computation of the Extremal Index θ.

Let $\tilde{N}_{3^{-n}}$ denote the minimum number of balls of radius 3^{-n} to cover the set $\mathscr{C}\cap T^{-q}(\mathscr{C})$. By definition of box dimension stated in Appendix A, we have that

$$\lim_{n\to\infty}\frac{\log\tilde{N}_{3^{-n}}}{\log 3^n}\le\frac{\log\sqrt{3}}{\log 3}=\frac{1}{2}.$$

Hence, there exists an $\varepsilon>0$ such that

$$\gamma:=\frac{1}{2}+\varepsilon<\frac{\log 2}{\log 3},$$

and, for n sufficiently large,

$$\tilde{N}_{3^{-n}}<e^{\gamma n\log 3}. \tag{5.2.16}$$

Observe that $3\tilde{N}_{3^{-n}}$ balls of radius 3^{-n} are enough to cover the set $\mathscr{C}_{n-1}\cap T^{-q}(\mathscr{C}_{n-1})$. Since $\mathscr{C}_n\cap T^{-q}(\mathscr{C}_n)\subseteq\mathscr{C}_{n-1}\cap T^{-q}(\mathscr{C}_{n-1})$, we have

$$\tilde{N}_{3^{-n}}\le N_{3^{-n}}\le 3\tilde{N}_{3^{-n}}.$$

Applying Proposition 5.2.6, we obtain that, for n sufficiently large (in particular, such that $3^{-n}<m^{-q}$),

$$\tilde{N}_{3^{-n}}\le N^*_{3^{-n}}\le 3\tilde{N}_{3^{-n}}.$$

This implies that

$$\mu(\mathscr{C}_n\cap T^{-q}(\mathscr{C}_n))=\frac{1}{3^n}N^*_{3^{-n}}\le\frac{3\tilde{N}_{3^{-n}}}{3^n}.$$

Hence, by 5.2.16,

$$\mu(\mathscr{C}_n\cap T^{-q}(\mathscr{C}_n))\le\frac{3e^{\gamma n\log 3}}{3^n}. \tag{5.2.17}$$

Recall that the sequence of thresholds $(u_n)_{n\in\mathbb{N}}$ is such that $u_n=n$, which means that $U_n=\mathscr{C}_n$ and then O'Brien's formula (3.1) gives:

$$\lim_{n\to\infty}1-\theta_n=\lim_{n\to\infty}\frac{\mu(\mathscr{C}_n\setminus\mathscr{A}_{q_n,n})}{\mu(\mathscr{C}_n)}. \tag{5.2.18}$$

The set $\mathscr{A}_{q_n,n}$ depends on a sequence $(q_n)_{n\in\mathbb{N}}$ that we are going to choose in the following way:

$$q_n:=\left\lceil n\frac{\log 3}{\log m}\right\rceil. \tag{5.2.19}$$

Note that $q_n=o(w_n)$ as required and, moreover, we have $3^{-n}\le m^{-q_n}$, for all $n\in\mathbb{N}$, which guarantees that we can apply Proposition 5.2.6 and estimate (5.2.17) holds, for all $q\le q_n$. Then, observing that $\mathscr{C}_n\setminus\mathscr{A}_{q_n,n}\subseteq\bigcup_{i=1}^{q_n}(\mathscr{C}_n\cap T^{-i}(\mathscr{C}_n))$, we get

$$\mu\left(\mathscr{C}_n\setminus\mathscr{A}_{q_n,n}\right)=\mu\left(\bigcup_{q=1}^{q_n}(\mathscr{C}_n\cap T^{-q}(\mathscr{C}_n))\right)\le\sum_{q=1}^{q_n}\mu(\mathscr{C}_n\cap T^{-q}(\mathscr{C}_n))\le 3q_n\frac{e^{\gamma n\log 3}}{3^n}.$$

Picking up on (5.2.18) and since $\gamma \log 3 < \log 2$, we finally obtain

$$\lim_{n\to\infty} 1-\theta_n = \lim_{n\to\infty} \frac{\mu(\mathscr{C}_n \setminus \mathscr{A}_{q_n,n})}{\mu(\mathscr{C}_n)} \leq \lim_{n\to\infty} 3q_n \frac{\frac{e^{\gamma n \log 3}}{3^n}}{(\frac{2}{3})^n} \leq 3 \lim_{n\to\infty} q_n e^{n(\gamma \log 3 - \log 2)} = 0.$$

Therefore, $\theta = 1$.

5.2.3 The Existence of Limiting Laws

To finish the proof of Theorem 5.0.1, it is only necessary to guarantee that conditions Д_{q_n} and $\text{Д}'_{q_n}$ are satisfied. We recall that the system $([0,1],T,\mu)$ has exponential decay of correlations of BV observables against L^1 observables, *i.e.*, , for all $\phi \in BV$ and $\psi \in L^1(\mu)$, there exist $C>0$ and $r=\frac{1}{m}$ such that

$$\mathrm{Cor}_\mu(\phi,\psi,n) \leq Cr^n.$$

The BV norm of $\mathbf{1}_{\mathscr{A}_{q_n,n}}$ is directly related with the number of connected components of $\mathscr{A}_{q_n,n}$, which we need to control. In order to do that, we start by estimating, for each $q=1,\ldots,q_n$, the number of intervals of $T^{-q}(\mathscr{C}_n^c)$ that intersect a single connected component of $\mathscr{C}_n$, which we will denote by I.

Recall that our choice for q_n made in (5.2.19) guarantees that $|I| = 3^{-n} \leq m^{-q}$, for all $q \leq q_n$. This implies that an interval I from $\mathscr{C}_n$ can intersect at most 2 of the m^q copies of $\mathscr{C}_n$ that were contracted to fit on equally sized intervals of length m^{-q} which form the set $T^{-q}(\mathscr{C}_n)$. We also note that $\mathscr{C}_n$ is built in a symmetrical way by choosing 2^n intervals of equal length, 3^{-n}, which alternate with 2^n-1 holes of different sizes. This means that the number of holes of $\mathscr{C}_n$ is just about its number of connected components. In order to estimate the maximum number of connected components of $T^{-q}(\mathscr{C}_n)$ (with length $m^{-q}3^{-n}$) that intersect the interval I, we define $\kappa \in \mathbb{N}$ such that

$$3^{\kappa-1}m^{-q} \leq 1 \leq 3^{\kappa}m^{-q},$$

i.e., we take $\kappa = \left\lceil q\frac{\log m}{\log 3}\right\rceil$.
As represented by Figure 5.2, the structure of the Cantor set $\mathscr{C}$ dictates that the maximum number of components of size $m^{-q}3^{-n}$ of one of the m^q copies of $\mathscr{C}_n$ that compose $T^{-q}(\mathscr{C}_n)$ and fit into the interval I is at most 2^κ.

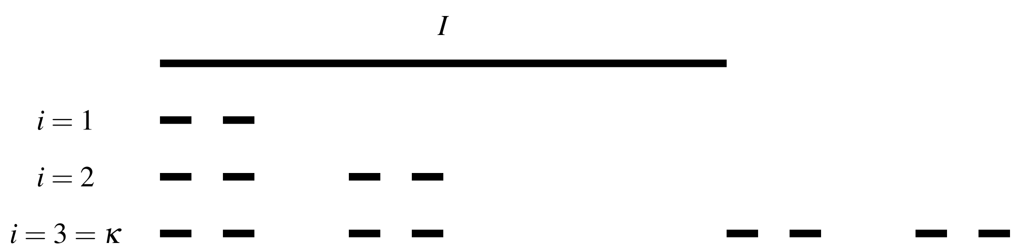

Fig. 5.2 The impact of the structure of $\mathscr{C}$ on the maximum number of connected components of $T^{-q}(\mathscr{C}_n)$ that fit into each interval I.

As seen above, the number of holes of one of the m^q copies of $\mathscr{C}_n$ (or connected components of $T^{-q}(\mathscr{C}_n^c)$) that fit into the interval I is bounded above also by 2^κ. Since there are at most 2 of the m^q copies of $\mathscr{C}_n$ that form $T^{-q}(\mathscr{C}_n)$ which intersect I, then the maximum number of connected components of $T^{-q}(\mathscr{C}_n^c)$ that intersect I is $2^{\kappa+1} = 2^{\left\lceil q\frac{\log m}{\log 3}\right\rceil+1}$.

Observing that the intersection of a collection of i subintervals of I with another collection of j subintervals of I produces at most $i+j$ connected components, then $\mathscr{C}_n$ is formed by 2^n intervals like I. Having in mind the choice of q_n in (5.2.19), then the number of connected components of $\mathscr{A}_{q_n,n} = \mathscr{C}_n \cap T^{-1}(\mathscr{C}_n^c) \cap \ldots \cap T^{-q_n}(\mathscr{C}_n^c)$ is bounded above by

$$2^n \sum_{q=1}^{q_n} 2^{\left\lceil q\frac{\log m}{\log 3}\right\rceil+1} = 2^{n+2} \sum_{q=1}^{q_n} 2^{\left\lfloor q\frac{\log m}{\log 3}\right\rfloor} \leq 2^{n+2} \sum_{q=1}^{q_n} m^{q\frac{\log 2}{\log 3}} \leq 2^{n+3} m^{q_n\frac{\log 2}{\log 3}}$$
$$\leq 2^{n+3} m^{(n\frac{\log 3}{\log m}+1)\frac{\log 2}{\log 3}} \leq 8m4^n$$

and, consequently

$$\|\mathbf{1}_{\mathscr{A}_{q_n,n}}\|_{BV} \leq 16m4^n + 1 \leq 32m4^n.$$

Choosing, for example, $t_n = n^2$ then $t_n = o(w_n)$ and

$$\lim_{n\to\infty} \left\|\mathbf{1}_{\mathscr{A}_{q_n,n}}\right\|_{BV} w_n r^{t_n} \leq \lim_{n\to\infty} \left\lfloor \tau\left(\frac{3}{2}\right)^n \right\rfloor 32m4^n m^{-n^2} = 0,$$

which implies that condition $Д_{q_n}(u_n, w_n)$ holds by Theorem 3.2.3.

Observe that the choice of q_n implies that for $q \geq q_n > n\frac{\log 3}{\log m}$ we have $m^{-q} < 3^{-n}$. Recall that $T^{-q}(\mathscr{C}_n)$ corresponds to m^q copies of $\mathscr{C}_n$ contracted by the factor m^{-q} and placed side by side on $[0,1]$ and, since $\mu(T^{-q}(\mathscr{C}_n)) = \mu(\mathscr{C}_n) = (2/3)^n$, then each such copy has a measure equal to $m^{-q}\mu(\mathscr{C}_n) = m^{-q}(2/3)^n$. We point out that each of the 2^n connected components of $\mathscr{C}_n$ intersects at most $\left\lfloor \frac{3^{-n}}{m^{-q}} \right\rfloor + 2$ intervals of size m^{-q}. Hence,

$$\mu(\mathscr{C}_n \cap T^{-q}(\mathscr{C}_n)) \leq m^{-q}\left(\frac{2}{3}\right)^n \left(\left\lfloor \frac{3^{-n}}{m^{-q}} \right\rfloor + 2\right) 2^n \leq m^{-q}\left(\frac{2}{3}\right)^n \left(\frac{3^{-n}}{m^{-q}} + 2\right) 2^n$$
$$\leq \left(\frac{2}{3}\right)^{2n} + 2\left(\frac{2}{3}\right)^n m^{-q} 2^n \leq 3\left(\frac{2}{3}\right)^{2n}. \tag{5.2.20}$$

We choose $k_n = n$. Note that $k_n \xrightarrow[n\to\infty]{} \infty$ and $k_n t_n = n^3 = o(w_n)$. Now, observing that $\mathscr{A}_{q_n,n} \subset \mathscr{C}_n$, then (5.2.20) implies that:

$$w_n \sum_{j=q_n+1}^{\lfloor w_n/k_n \rfloor - 1} \mu\left(\mathscr{A}_{q_n,n} \cap T^{-j}\left(\mathscr{A}_{q_n,n}\right)\right) \leq w_n \sum_{j=q_n+1}^{\lfloor w_n/k_n \rfloor - 1} \mu\left(\mathscr{C}_n \cap T^{-j}\left(\mathscr{C}_n\right)\right)$$
$$\leq w_n \sum_{j=q_n+1}^{\lfloor w_n/k_n \rfloor - 1} 3\left(\frac{2}{3}\right)^{2n} \leq 3\frac{w_n^2}{k_n}\left(\frac{2}{3}\right)^{2n} \leq \frac{3}{k_n}\tau^2\left(\frac{3}{2}\right)^{2n}\left(\frac{2}{3}\right)^{2n} = \frac{3\tau^2}{k_n} \xrightarrow[n\to\infty]{} 0$$

and, therefore $Д'_{q_n}(u_n, w_n)$ also holds. Since we have already proved that $\theta = 1$, by Theorem 3.1.1, we conclude the claim of Theorem 5.0.1, *i.e.*,

$$\lim_{n\to\infty} \mu(M_{w_n} \leq n) = e^{-\tau}, \tag{5.2.21}$$

when $m \in \mathbb{N}$ is not a power of 3.

5.3 Absence of Clustering and Two Dimensional Uniformly Expanding Maps

The starting point of this section is the stochastic process X_n, constructed using the two dimensional setting presented in Section 4.2 of Chapter 4.
We again consider the map T given by,

$$\begin{aligned} T\colon\ [0,1]^2 &\longrightarrow [0,1]^2 \\ (x,y) &\mapsto (m_1 \cdot x \mod 1, m_2 \cdot y \mod 1), \end{aligned} \tag{5.3.1}$$

and the observable

$$\psi(x,y) = \begin{cases} n, & \text{if } \min(\varphi(x), \varphi(y)) = n \\ \infty, & \text{otherwise,} \end{cases} \tag{5.3.2}$$

whose maximal set is $\mathfrak{C} = \mathscr{C} \times \mathscr{C}$.
If $m_1 = 3^{k_1}$ and $m_2 = 3^{k_2}$, we are able to demonstrate in Section 4.2 that there exists full compatibility between $\mathfrak{C}$ and $T^{-j}(\mathfrak{C})$. This enabled us to demonstrate the existence of a limiting law for the stochastic process X_n, constructed using T and ψ, with an Extremal Index strictly smaller than 1.
We will now show that if $m_1 = m_2$ cannot be written as 3^k for any integer k, then the compatibility between T and $\mathfrak{C}$ is broken and $T^{-j}(\mathfrak{C}) \cap (\mathfrak{C})$ will have a smaller box dimension than $\mathfrak{C}$. The result is an insignificant clustering effect which, in the limit, will lead to a Extremal Index equal to 1.

Theorem 5.3.1. *Consider* $(X_n)_{n\in\mathbb{N}}$ *to be the stochastic process constructed as in* (2.2.1) *for the dynamical system* T *defined in* (5.3.1) *and the observable* ψ *defined in* (5.3.2). *Assume that* $m_1 = m_2$ *cannot be written as* 3^k*, for any* $k \in \mathbb{N}$. *Set* $u_n = n$ *and let* $(w_n)_{n\in\mathbb{N}}$ *be a sequence of times, such that* $w_n = \left\lfloor \tau (3/2)^{2n} \right\rfloor$.
Then, condition (3.0.5) *holds and*

$$\lim_{n\to\infty} Leb^2(M_{w_n} \leq n) = e^{-\tau}.$$

We also will be considering the case where the map T is such that $m_1 = 3^k$ for some $k \in \mathbb{N}$, and m_2 cannot be written in the form 3^j for all $j \in \mathbb{N}$. This case represents a middle ground between Theorems 4.2.1 and 5.3.1. If for one side we should expect clustering to appear, due to compatibility between

T_1 and the ternary Cantor set $\mathscr{C}$, the reality is that the incompatibility between T_2 and $\mathscr{C}$ is enough to guarantee the absence of clustering in the stochastic process X_n.

Theorem 5.3.2. *Consider $(X_n)_{n\in\mathbb{N}}$ to be the stochastic process constructed as in* (2.2.1) *for the dynamical system T defined in* (5.3.1) *and the observable ψ defined in* (5.3.2)*. Assume that $m_1 = 3^k$ and m_2 cannot be written as 3^j, for all $j \in \mathbb{N}$. Moreover, assume that k and m_2 satisfy the inequality*

$$\left\lceil n\frac{\log 3}{\log m_2}\right\rceil \leq \left\lfloor \frac{n}{k}\right\rfloor. \tag{5.3.3}$$

Set $u_n = n$ and let $(w_n)_{n\in\mathbb{N}}$ be a sequence of times, such that $w_n = \left\lfloor \tau\,(3/2)^{2n}\right\rfloor$.
Then, condition (3.0.5) *holds and*

$$\lim_{n\to\infty} Leb^2(M_{w_n} \leq n) = \mathrm{e}^{-\tau}.$$

To prove the results stated in Theorem 5.3.1 and Theorem 5.3.2, we will rely on the formula for the lower bound of the Extremal Index θ, achieved in Theorem 3.3.5. This formula, when used in conjunction with Theorem 5.0.1, is sufficient to guarantee that θ is equal to 1. The proof becomes complete by showing that conditions $Д_{q_n}(u_n, w_n)$ and $Д'_{q_n}(u_n, w_n)$ hold.

Proof of Theorem 5.3.1. Let T be the dynamical system, presented in (5.3.1), where $m_1 = m_2$ cannot be written as 3^k, for any $k \in \mathbb{N}$. Consider the stochastic process $X_n = \psi \circ T$.
We point out that, the invariant measures associated with T and with $T_1 = m_1 x \mod 1$ and $T_2 = m_1 y \mod 1$ are Leb^2 and Leb, respectively.
Let $u_n = n$ be the sequence of thresholds and set $w_n = \left\lfloor \tau\,(3/2)^{2n}\right\rfloor$. Due to the construction of the observable ψ, we have that

$$U_n^{T_1} = U_n^{T_2} = \mathscr{C}_n \qquad \text{and} \qquad U_n = \mathfrak{C}_n.$$

Checking that,

$$w_n \text{Leb}^2(U_n) = \left\lfloor \tau\,(3/2)^{2n}\right\rfloor \left(\frac{2}{3}\right)^{2n} \xrightarrow[n\to\infty]{} \tau,$$

we obtain that condition (3.0.5) is verified.
Now, we apply Theorem 3.3.5 to prove that $\theta = 1$. Set $q_n = q_n^* = \left\lceil n\frac{\log 3}{\log m_1}\right\rceil$. Then, $q_n = o(w_n)$ and Theorem 5.0.1 guarantees that

$$\theta_1 = 1 \qquad \text{and} \qquad \theta_2 = 1.$$

Applying formula (3.3.11) of Theorem 3.3.5 for the Extremal Index, we achieve that

$$1 \geq \theta \geq \theta_1 + \theta_2 - \theta_1\theta_2 = 1.$$

The map T has decay of correlations for quasi-Hölder observables against $L^1(\text{Leb}^2)$, with rate function $\rho_n = 1/m_1^n$. Moreover, the maps T_1 and T_2 have decay of correlations for BV observables against $L^1(\text{Leb})$, with rate functions $\rho_n^1 = \rho_n^2 = \rho_n$. Hence, to prove the validity of conditions $Д_{q_n}(u_n, w_n)$ and $Д'_{q_n}(u_n, w_n)$ it is only necessary to check hypothesis (1) thru (4) of Theorem 3.3.2.
Since $T_1 = T_2$ then, using Remark 3.3.4, hypothesis (2) thru (4) of Theorem 3.3.2 are reduced to

$$\lim_{n\to\infty} \left\| \mathbf{1}_{U_n^{T_1}} \right\|_{BV} \text{Leb}(U_n^{T_1}) \sum_{j=q_n}^{\infty} \rho_j^1 = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \left\| \mathbf{1}_{U_n^{T_1}} \right\|_{BV}^2 \sum_{j=q_n}^{\infty} \rho_j^1 \rho_j^2 = 0.$$

Since $\|\mathbf{1}_{\mathscr{C}_n}\|_{BV} \leq 2^{n+1}$, there exists a constant $C > 0$ such that

$$\begin{aligned} \lim_{n\to\infty} \|\mathbf{1}_{\mathscr{C}_n}\|_{BV} \leq 2^{n+1} \text{Leb}(\mathscr{C}_n) \sum_{j=q_n}^{\infty} \rho_j^1 &\leq C \lim_{n\to\infty} \frac{4^n}{3^n} \frac{1}{m_1^{q_n}} \\ &\leq C m_1 \lim_{n\to\infty} \frac{4^n}{3^{2n}} \\ &= 0. \end{aligned}$$

Similarly, there exists a constant $C' > 0$ such that

$$\begin{aligned} \lim_{n\to\infty} \|\mathbf{1}_{\mathscr{C}_n}\|_{BV} \sum_{j=q_n}^{\infty} \rho_j^1 \rho_j^2 &\leq C \lim_{n\to\infty} \frac{4^n}{m_1^{2q_n}} \\ &\leq C m_1^2 \lim_{n\to\infty} \frac{4^n}{3^{2n}} \\ &= 0. \end{aligned}$$

To prove hypothesis (1), we will show that exists a sequence $(t_n)_{n\in\mathbb{N}}$, such that $t_n = o(w_n)$ and

$$\lim_{n\to\infty} w_n \left(\|\mathbf{1}_{\mathfrak{C}_n}\|_\alpha \rho(t_n) + 2\text{Leb}^2(\mathfrak{C}_n \setminus \mathscr{A}_{q_n,n}) \right) = 0.$$

Following the same reasoning as in the proof of Theorem 4.2.1, we have that

$$\|\mathbf{1}_{\mathfrak{C}_n}\|_\alpha \leq \text{Leb}^2(\mathfrak{C}_n) + PC(\mathfrak{C}_n),$$

where P denotes the maximum perimeter of the connected components of $\mathfrak{C}_n$ and $C(\mathfrak{C}_n)$ represents the maximum number of connected components of $\mathfrak{C}_n$.
Since, $\text{Leb}^2(\mathfrak{C}_n) = (2/3)^{2n}$, $C(\mathfrak{C}_n) = 4^n$ and $P = 4/3^n$, we can write that,

$$\|\mathbf{1}_{\mathfrak{C}_n}\|_\alpha \leq \left(\frac{2}{3}\right)^{2n} + \frac{4^{n+1}}{3^n}. \tag{5.3.4}$$

It is necessary to estimate $\text{Leb}^2(\mathfrak{C}_n \setminus \mathscr{A}_{q_n,n})$. For that purpose, we point out that

$$\text{Leb}^2(\mathfrak{C}_n \cap T^{-q}(\mathfrak{C}_n)) = \text{Leb}(\mathscr{C}_n \cap T_1^{-q}(\mathscr{C}_n))\text{Leb}(\mathscr{C}_n \cap T_2^{-q}(\mathscr{C}_n)) = \left(\text{Leb}(\mathscr{C}_n \cap T_1^{-q}(\mathscr{C}_n))\right)^2$$

and consequently,

$$\text{Leb}^2\left(\bigcup_{q=1}^{q_n}\mathfrak{C}_n\cap T^{-q}(\mathfrak{C}_n)\right)\leq\sum_{q=1}^{q_n}\left(\text{Leb}(\mathscr{C}_n\cap T_1^{-q}(\mathscr{C}_n))\right)^2.$$

Using the estimative (5.2.17) of Chapter 5, we get that

$$\text{Leb}^2\left(\bigcup_{q=1}^{q_n}\mathfrak{C}_n\cap T^{-q}(\mathfrak{C}_n)\right)\leq q_n\frac{9e^{2\gamma n\log 3}}{3^{2n}}$$

and since $\mathfrak{C}_n\setminus\mathscr{A}_{q_n,n}\subseteq\bigcup_{q=1}^{q_n}\mathfrak{C}_n\cap T^{-q}(\mathfrak{C}_n)$, we obtain that

$$\text{Leb}^2(\mathfrak{C}_n\setminus\mathscr{A}_{q_n,n})\leq q_n\frac{9e^{2\gamma n\log 3}}{3^{2n}},\tag{5.3.5}$$

where γ satisfies the inequality $\gamma\log 3<\log 2$.
Let $t_n=n^2$, then $t_n=o(w_n)$. Using (5.3.4) and (5.3.5), we get that, for all $m_1>1$,

$$\begin{aligned}
&\lim_{n\to\infty}w_n\left(\|\mathbf{1}_{\mathfrak{C}_n}\|_\alpha\rho(t_n)+2\text{Leb}^2(\mathfrak{C}_n\setminus\mathscr{A}_{q_n,n})\right)\\
\leq&\lim_{n\to\infty}\tau\left(\frac{3}{2}\right)^{2n}\left(\left(\frac{2}{3}\right)^{2n}+\frac{4^{n+1}}{3^n}\right)\frac{1}{m_1^{n^2}}+\tau\left(\frac{3}{2}\right)^{2n}q_n\frac{9e^{2\gamma n\log 3}}{3^{2n}}\\
\leq&\lim_{n\to\infty}\left(\tau+\tau\frac{4^{n+1}9^n}{3^n4^n}\right)\frac{1}{m_1^{n^2}}+9\tau q_ne^{2n(\gamma\log 3-\log 2)}\\
=&0.
\end{aligned}$$

This implies that hypothesis (1) of Theorem 3.3.2 holds and Theorem 5.3.1 follows. □

Theorem 5.3.1 is an example where there is no clustering associated with either X_n^1 or X_n^2. For that reason, the existence of clustering in the process X_n was not expected.
A more interesting situation is the setting presented in Theorem 5.3.2. In this case, the map T is composed of two unidimensional dynamics with one of them preserving the structure of the Cantor set, $\mathscr{C}$. We already saw that it is this compatibility, between the dynamics and the maximal set of the observable function, that leads to clustering. However, due to the structure product in $\mathfrak{C}$, the presence of a second dynamics that does not preserve the structure of $\mathscr{C}$ is enough to guarantee that $T^{-j}(\mathfrak{C}_n)\cap\mathfrak{C}$ as low relevance in the set $\mathfrak{C}$. In fact, formula 3.3.11 is the mathematical translation of this thought. This low relevance is then translated into a Extremal Index of 1.
One could use the strategy followed in Theorem 5.0.1 and prove that the box dimension of $T^{-j}(\mathfrak{C}_n)\cap\mathfrak{C}$ is indeed smaller than the box dimension $\mathfrak{C}$ and from here establish the value of θ. But, the nature of $\mathfrak{C}$, allows a more direct calculation of θ using the formula in (3.3.11). With this remark, we proceed to the proof of Theorem 5.3.2.

Proof of Theorem 5.3.2. This proof follows the same structure as the proof of Theorem 5.3.1. Consider the dynamical system T given by (5.3.1) and the stochastic process $X_n = \psi \circ T$. Let $m_1 = 3^k$ mod 1 for some $k \in \mathbb{N}$ and assume that m_2 cannot be written as 3^j for any integer j. Moreover, assume that k and m_2 are such that

$$\left\lceil n\frac{\log 3}{\log m_2} \right\rceil \leq \left\lfloor \frac{n}{k} \right\rfloor. \tag{5.3.6}$$

The sequence of thresholds is $u_n = n$ and we set $w_n = \left\lfloor \tau (3/2)^{2n} \right\rfloor$. Again, we have that

$$U_n^{T_1} = U_n^{T_2} = \mathscr{C}_n \qquad \text{and} \qquad U_n = \mathfrak{C}_n,$$

which implies that condition 3.0.5 is satisfied.
Again, we use Theorem 3.3.5 to show that $\theta = 1$. Make $q_n = q_n^* = \left\lceil n\frac{\log 3}{\log m_2} \right\rceil$. Under the hypothesis in (5.3.6) and due to Theorem 4.1.2, we have that

$$\mathscr{A}_{q_n,n}^{T_1} = \mathscr{C}_n \setminus \mathscr{C}_{n+k} \qquad \text{and} \qquad \theta_1 = 1 - (2/3)^k.$$

Moreover, Theorem 5.0.1 guarantees that $\theta_2 = 1$.
Applying formula (3.3.11) of Theorem 3.3.5 for θ, we achieve that

$$1 \geq \theta \geq \theta_1 + \theta_2 - \theta_1\theta_2 = 1.$$

Put $r = \min\{3^k, m_2\}$, then T has decay of correlations for quasi-Hölder observables against $L^1(\text{Leb}^2)$ with rate function $\rho_n = 1/r^n$. The maps $T_1 = 3^k x \mod 1$ and $T_2 = m_2 y \mod 1$ have decay of correlations for BV observables against $L^1(\text{Leb})$ with rate functions $\rho_n^1 = (1/3^k)^n$ and $\rho_n^2 = (1/m_2)^n$.
We will check conditions (1) thru (4) of Theorem 3.3.2 to prove $Д_{q_n}(u_n, w_n)$ and $Д'_{q_n}(u_n, w_n)$.

To prove hypothesis (1), we will show that exists a sequence $(t_n)_{n\in\mathbb{N}}$, such that $t_n = o(w_n)$ and

$$\lim_{n\to\infty} w_n \left(\|\mathbf{1}_{\mathfrak{C}_n}\|_\alpha \rho(t_n) + 2\text{Leb}^2(\mathfrak{C}_n \setminus \mathscr{A}_{q_n,n}) \right) = 0.$$

Similarly to the proof of Theorem 5.3.1, we can write that

$$\|\mathbf{1}_{\mathfrak{C}_n}\|_\alpha \leq \text{Leb}^2(\mathfrak{C}_n) + PC(\mathfrak{C}_n),$$

where P denotes the maximum perimeter of the connected components of $\mathfrak{C}_n$ and $C(\mathfrak{C}_n)$ represents the maximum number of connected components of $\mathfrak{C}_n$.
Again, as in the proof of Theorem 5.3.1, $\text{Leb}^2(\mathfrak{C}_n) = (2/3)^{2n}$, $C(\mathfrak{C}_n) = 4^n$ and $P = 4/3^n$.
Therefore,

$$\|\mathbf{1}_{\mathfrak{C}_n}\|_\alpha \leq \left(\frac{2}{3}\right)^{2n} + \frac{4^{n+1}}{3^n}. \tag{5.3.7}$$

We need an estimated value for $\mathrm{Leb}^2(\mathfrak{C}_n \setminus \mathscr{A}_{q_n,n})$. We can put that,

$$\mathrm{Leb}^2(\mathfrak{C}_n \cap T^{-q}(\mathfrak{C}_n)) = \mathrm{Leb}(\mathscr{C}_n \cap T_1^{-q}(\mathscr{C}_n))\mathrm{Leb}(\mathscr{C}_n \cap T_2^{-q}(\mathscr{C}_n)) \le \mathrm{Leb}(\mathscr{C}_n \cap T_1^{-q}(\mathscr{C}_n))\mathrm{Leb}(\mathscr{C}_n)$$

and consequently,

$$\mathrm{Leb}^2\left(\bigcup_{q=1}^{q_n} \mathfrak{C}_n \cap T^{-q}(\mathfrak{C}_n)\right) \le \sum_{q=1}^{q_n} \mathrm{Leb}(\mathscr{C}_n \cap T_1^{-q}(\mathscr{C}_n))\mathrm{Leb}(\mathscr{C}_n).$$

Using the estimative (5.2.17) of Chapter 5, we get that

$$\mathrm{Leb}^2\left(\bigcup_{q=1}^{q_n} \mathfrak{C}_n \cap T^{-q}(\mathfrak{C}_n)\right) \le q_n \left(\frac{3e^{\gamma n \log 3}}{3^n}\right)\left(\frac{2}{3}\right)^n$$

and since $\mathfrak{C}_n \setminus \mathscr{A}_{q_n,n} \subseteq \bigcup_{q=1}^{q_n} \mathfrak{C}_n \cap T^{-q}(\mathfrak{C}_n)$, we obtain that

$$\mathrm{Leb}^2(\mathfrak{C}_n \setminus \mathscr{A}_{q_n,n}) \le q_n \left(\frac{3e^{\gamma n \log 3}}{3^n}\right)\left(\frac{2}{3}\right)^n, \tag{5.3.8}$$

where γ satisfies the inequality $\gamma \log 3 < \log 2$.
Set $t_n = n^2$, then $t_n = o(w_n)$ and using (5.3.7) and (5.3.8), we obtain that,

$$\begin{aligned}
&\lim_{n\to\infty} w_n \left(\|\mathbf{1}_{\mathfrak{C}_n}\|_\alpha \rho(t_n) + 2\mathrm{Leb}^2(\mathfrak{C}_n \setminus \mathscr{A}_{q_n,n})\right)\\
\le &\lim_{n\to\infty} \tau \left(\frac{3}{2}\right)^{2n}\left(\left(\frac{2}{3}\right)^{2n} + \frac{4^{n+1}}{3^n}\right)\frac{1}{r^{n^2}} + \tau\left(\frac{3}{2}\right)^{2n} q_n \left(\frac{3e^{\gamma n \log 3}}{3^n}\right)\left(\frac{2}{3}\right)^n\\
\le &\lim_{n\to\infty} \left(\tau + \tau\frac{4^{n+1}9^n}{3^n 4^n}\right)\frac{1}{r^{n^2}} + 3\tau q_n e^{n(\gamma\log 3 - \log 2)}\\
= &0.
\end{aligned}$$

Before proving hypothesis (2) thru (4), we point out that

$$\left\lceil n\frac{\log 3}{\log m_2}\right\rceil \le \left\lfloor \frac{n}{k} \right\rfloor \Rightarrow k \ge \frac{1}{\log_{m_2}(3)}.$$

With this assumption on k there exists a constant $C' > 0$ such that

$$\begin{aligned}
\lim_{n\to\infty} \|\mathbf{1}_{\mathscr{C}_n}\|_{BV} \mathrm{Leb}(\mathscr{C}_n) \sum_{j=q_n}^{\infty} \rho_j^1 &\le C' \lim_{n\to\infty} 2^n (2/3)^n \left(\frac{1}{3^k}\right)^{n\log_{m_2}(3)}\\
&\le C' \lim_{n\to\infty} \frac{4^n}{3^{n(1+k\log_{m_2}(3))}}\\
&= 0.
\end{aligned}$$

Similarly, there exists a constant $C'' > 0$ such that

$$\begin{aligned}
\lim_{n\to\infty} \|\mathbf{1}_{\mathscr{C}_n}\|_{BV} \operatorname{Leb}(\mathscr{C}_n) \sum_{j=q_n}^{\infty} \rho_j^2 &\leq C'' \lim_{n\to\infty} 2^n (2/3)^n \left(\frac{1}{m_2}\right)^{n\log_{m_2}(3)} \\
&\leq C'' \lim_{n\to\infty} \frac{4^n}{9^n} \\
&= 0.
\end{aligned}$$

To finish, there exists another constant $C''' > 0$ such that

$$\begin{aligned}
\lim_{n\to\infty} \|\mathbf{1}_{\mathscr{C}_n}\|_{BV}^2 \sum_{j=q_n}^{\infty} \rho_j^1 \rho_j^2 &\leq C''' \lim_{n\to\infty} 4^n \frac{1}{m_2}^{n\log_{m_2}(3)} \left(\frac{1}{3^k}\right)^{n\log_{m_2}(3)} \\
&\leq C'' \lim_{n\to\infty} \frac{4^n}{3^{n(1+k\log_{m_2}(3))}} \\
&= 0.
\end{aligned}$$

This computation shows that conditions (2) thru (4) of Theorem 3.3.2 hold and Theorem 5.3.2 follows.

$\square$

Chapter 6

The Extremal Index as a Geometrical Indicator of Compatibility

In the chapters above, we have seen that the compatibility between the dynamics and the fractal structure of the maximal set plays a big role in the determination of the Extremal Index. In this chapter, we intend to illustrate the viability of the EI as an indicator between the compatibility of the fractal structure of a set and a certain dynamics. We perform several numerical simulations using different dynamical systems and fractal sets. We began by testing numerically some of the theoretical results stated in Chapter 4 and Chapter 5. Then, we kept the same maximal set and tested several different uniformly expanding and non-uniformly expanding dynamical systems and even irrational rotations. Finally, we considered a different maximal set, which consisted on a dynamically defined Cantor set obtained from a quadratic map, and tested it against both linear dynamics (which should be incompatible) and systems compatible with the one that generated the Cantor set.

We remark that in some cases (such as with irrational rotations), the systems are outside the scope of application of the theory considered earlier. In other cases, with some adjustments to the arguments, one could actually check that conditions $Д_{q_n}$ and $Д'_{q_n}$ hold.

We will use the EI estimator introduced by Hsing in [24]. Namely, we will consider:

$$\hat{\theta}_n(u,q) = \frac{\sum_{i=0}^{n-1} \mathbf{1}_{T^{-i}(\mathscr{A}_q(u))}}{\sum_{i=0}^{n-1} \mathbf{1}_{T^{-i}(U(u))}}, \tag{6.0.1}$$

where the sets $U(u)$ and $\mathscr{A}_q(u)$ are defined in (3.1.2).

The parameters u and q are tuning parameters which determine the quality of the estimate. In principle, one should consider high values of u so that the tail behaviour is captured by the quantities in $\hat{\theta}_n(u,q)$. But, if u is too high there may not be enough information to estimate accurately the EI. Since when $Д'_{q*}(u_n,w_n)$ holds for some fixed $q* \in \mathbb{N}$ then $Д'_q(u_n,w_n)$ holds for all $q > q*$, then the parameter q should not affect as much the quality of the estimator. We will test several values of u and a few for q and then we analyse the data to identify regions of stability of the estimator.

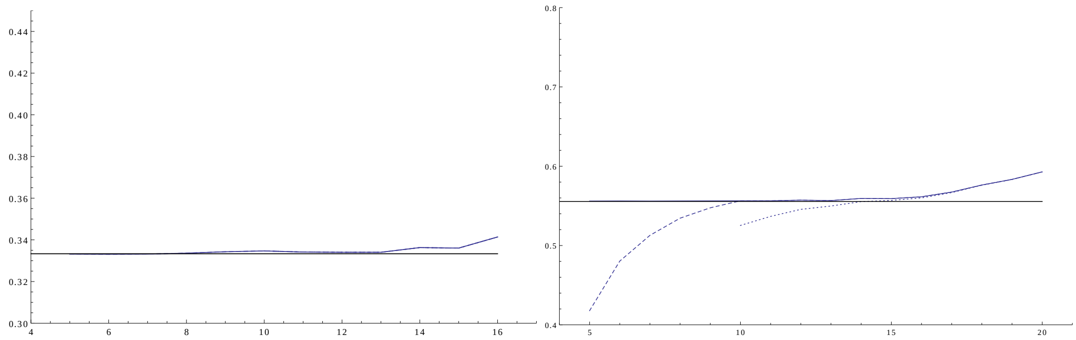

Fig. 6.1 On the y-axis, mean values of $\hat{\theta}_n(u,q)$ for each u of the x-axis, with $n = 50.000$ and $\ell = 500$. The full line corresponds to $q = 1$, the dashed line to $q = 5$ and the dotted line to $q = 10$. The black horizontal line represents the exact value of the EI given by Theorem 4.0.1. On the left, we have $T(x) = 3x \mod 1$ and, on the right, $T(x) = 9x \mod 1$.

6.1 The Ternary Cantor Set and Linear Dynamics

We numerically illustrate the existence of an EI equal to 1 when m is not a power of 3, as stated in Theorem 5.0.1, and the validity of the formula for the EI stated in Theorem 4.0.1, when $m = 3^k$ for some $k \in \mathbb{N}$, in which case the Cantor set is invariant by the dynamics.

The numerical simulations performed consisted in randomly generating ℓ uniformly distributed points on $[0,1]$ (recall that Lebesgue measure is invariant for the linear maps considered in Theorems 5.0.1 and 4.0.1) and, for each one, compute the first n iterates of the respective orbit and evaluate the observable function φ, defined in (4.0.1), along each orbit. Then, for each the ℓ time series obtained as described above, we compute $\hat{\theta}_n(u,q)$ for several values of u and q, which are adequately chosen for the range of u values. We observe an excellent agreement between the theoretical value of θ and the observed estimates of $\hat{\theta}_n(u,q)$, in the regions of stability which correspond to the values of u in $[5,15]$, in the case $m = 3$, and $[10,15]$, in the case $m = 9$.

In the case $m = 5$, there is also an excellent agreement between the theoretical value $\theta = 1$ and the observed estimates of $\hat{\theta}_n(u,q)$, in the regions of stability which correspond to higher values of u, namely, for $u \in [15,28]$. We note that the agreement improves considerably when we increase the number of iterations n, which allows to have more information on the tails. The simulations results show an excellent performance of the EI in order to distinguish between the compatibility and incompatibility of the dynamics with the structure of the Cantor set.

In the previous cases, either $T(\mathscr{C}) = \mathscr{C}$ or $T(\mathscr{C}) \cap \mathscr{C}$ is negligible. We consider a case where we have a relevant intersection $T(\mathscr{C}) \cap \mathscr{C}$, although $T(\mathscr{C}) \neq \mathscr{C}$. The idea is to consider a map that maps the left side component of $\mathscr{C}$ onto $\mathscr{C}$, while the right side component is sent to a set with a negligible intersection with $\mathscr{C}$.

Let $T : [0,1] \to [0,1]$ be the linear map whose first branch coincides with the first branch of $3x \mod 1$ and the others send each of the 5 equally lengthed subintervals of $[2/3,1]$ onto $[0,1]$. See Figure 6.3. Although this map was not considered in the previous sections, it is easy to adjust the arguments to show that an EVL applies with an EI, which is the mean between $1/3$ (the contribution from the left

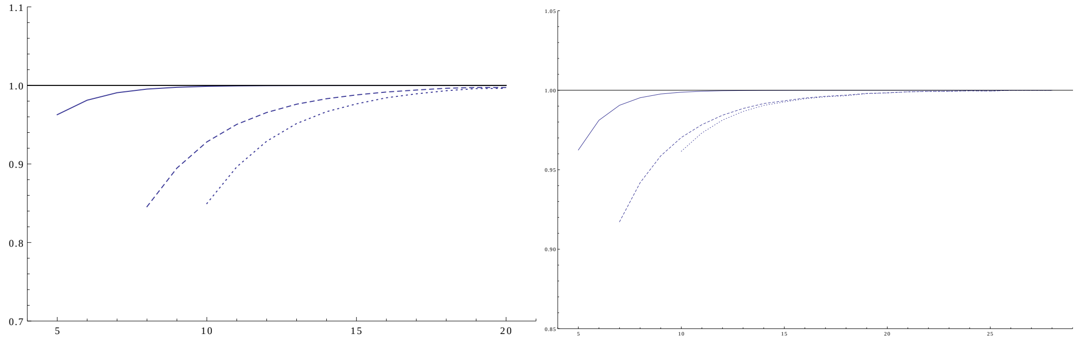

Fig. 6.2 On the y-axis, mean values of $\hat{\theta}_n(u,q)$ for each $5 \leq u \leq 20$ of the x-axis, with $n = 50.000$. The full line corresponds to $q = 1$, the dashed line to $q = 5$ and the dotted line to $q = 10$. The black horizontal line represents the exact value of the EI given by Theorem 5.0.1. The dynamics is $T(x) = 5x$ mod 1. On the left, $n = 50.000$ and $\ell = 500$. On the right, $n = 500.000$ and $\ell = 100$.

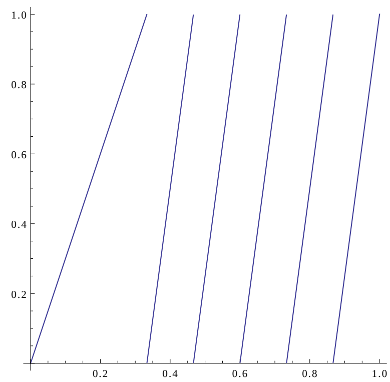

Fig. 6.3 Mixed linear map

side) and 1 (the contribution from the right side). Namely, using the estimates in Section 5.2.2, one can show that

$$\lim_{n\to\infty} \frac{\mu(\mathscr{A}_{q_n,n} \cap [2/3,1])}{\mu(\mathscr{C}_n \cap [2/3,1])} = 1$$

and, as in Section 4.1, one can show that

$$\mathscr{A}_{q_n,n} \cap [0,1/3] = (\mathscr{C}_n \setminus \mathscr{C}_{n+1}) \cap [0,1/3],$$

which imply:

$$\theta = \lim_{n\to\infty} \frac{\mu(\mathscr{A}_{q_n,n})}{\mu(U_n)} = \lim_{n\to\infty} \frac{\mu((\mathscr{C}_n \setminus \mathscr{C}_{n+1}) \cap [0,1/3]) + \mu(\mathscr{C}_n \cap [2/3,1])}{\mu(\mathscr{C}_n)} = \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{2}\cdot 1 = \frac{2}{3}. \quad (6.1.1)$$

As it can be seen in Figure 6.4, the numerical estimates for the EI point to the theoretical value $\theta = 2/3$ and the performance of the EI estimator improves when n is increased, as expected.

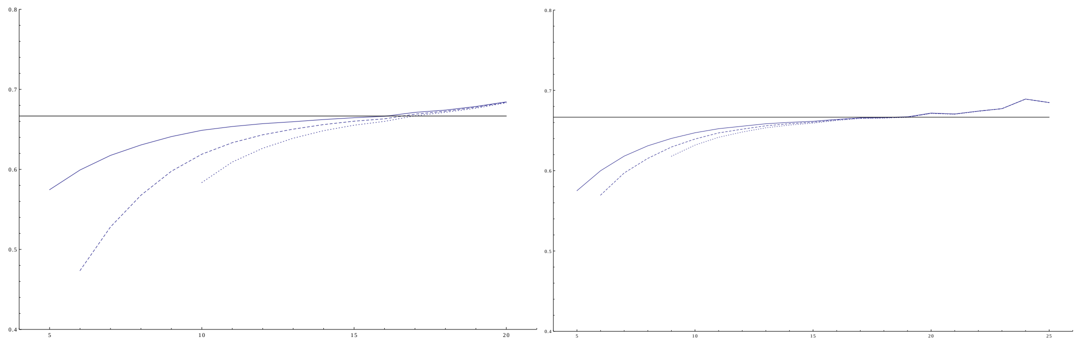

Fig. 6.4 On the y-axis, mean values of $\hat{\theta}_n(u,q)$ for each u of the x-axis. The full line corresponds to $q=1$, the dashed line to $q=5$ and the dotted line to $q=10$. The black horizontal line represents the exact value of the EI given in (6.1.1). The dynamics is described in Figure 6.3. On the left, $n=50.000$ and $\ell=500$. On the right, $n=500.000$ and $\ell=100$.

6.2 The Ternary Cantor Set, Nonlinear Dynamics and Irrational Rotations

We considered two different nonlinear dynamics and an ergodic rotation. The first is a uniformly expanding map resemblant to the doubling map but in which the branches are convex curves.
Namely, we let

$$\begin{aligned} T\colon [0,1] &\longrightarrow [0,1] \\ x &\longmapsto \begin{cases} \frac{4}{3}x(x+1) & 0\le x<\frac{1}{2} \\ \frac{4}{3}\left(x-\frac{1}{2}\right)\left(x+\frac{1}{2}\right) & \frac{1}{2}\le x\le 1 \end{cases} \end{aligned} \tag{6.2.1}$$

This map does not seem to have any compatibility with the ternary Cantor set and in fact the simulation results illustrate an EI estimate equal to 1, which is observed for high values of $u\approx 20$. (See top left panel in Figure 6.5). We note that, for this particular map, some adjustments to the arguments presented in 5.2.3 would allow to check conditions $Д(u_n,w_n)$ and $Д'(u_n,w_n)$. However, the box dimension estimates used in Section 5.2.1 cannot be easily adapted and therefore we cannot state that the EI is indeed 1, despite the numerical evidence.
Then, we also considered the Gauss map, which is a non-uniformly expanding map, but still with very good mixing properties,

$$\begin{aligned} T\colon \quad [0,1] &\longrightarrow [0,1] \\ x &\longmapsto \tfrac{1}{x}-\left\lfloor \tfrac{1}{x}\right\rfloor . \end{aligned} \tag{6.2.2}$$

We remark that for this map is not possible to adapt easily the arguments used in 5.2.3 in order to check conditions $Д(u_n,w_n)$ and $Д'(u_n,w_n)$, since it has countably many branches, which makes the estimates for the number of connected components of $\mathscr{A}_{q_n,n}$, obtained earlier, useless. Nonetheless, the numerical simulations also reveal that, on the region of stability of the estimator (for high values of u), one gets an EI equal to 1, which indicates that the dynamics is incompatible with the structure of the ternary Cantor set $\mathscr{C}$ (See top right panel in Figure 6.5).

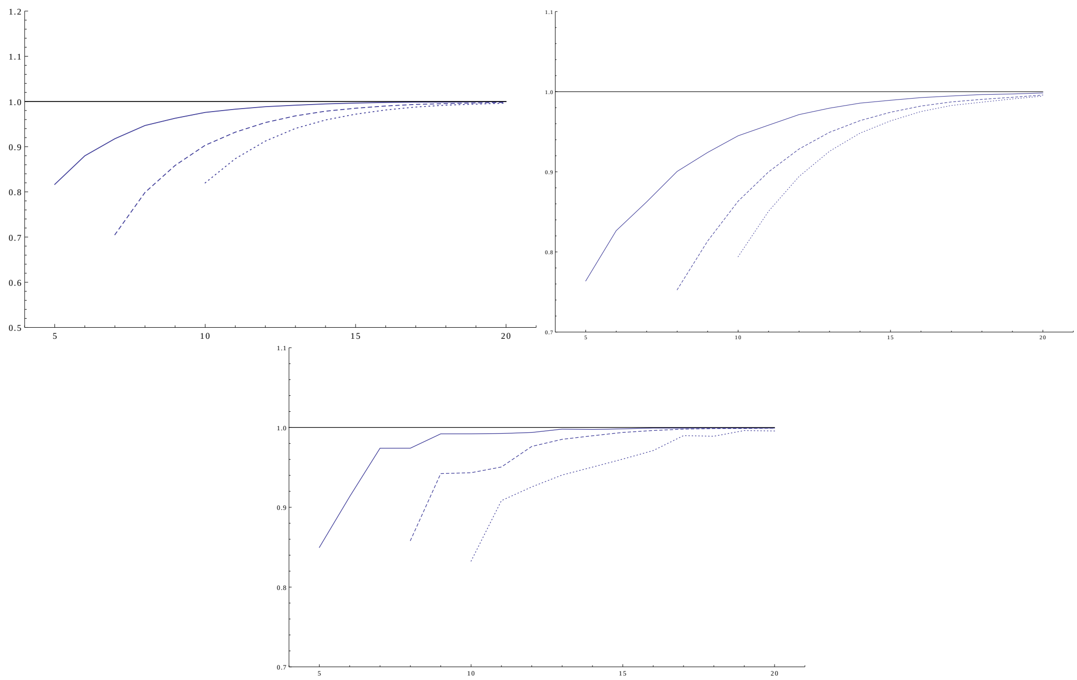

Fig. 6.5 On the y-axis, mean values of $\hat{\theta}_n(u,q)$ for each $5 \leq u \leq 20$ of the x-axis, with $n = 50.000$ and $\ell = 500$. The full line corresponds to $q = 1$, the dashed line to $q = 5$ and the dotted line to $q = 10$. The black horizontal line represents the expected value for the EI. On the top left T is given by (6.2.1), on the top right T is given by (6.2.2) and on the bottom $T(x) = x + \pi/3 \mod 1$.

Finally, we also considered an irrational rotation $T : [0,1] \to [0,1]$ given by $T(x) = x + \frac{\pi}{3} \mod 1$, as in [31], and, in coherence with the numerical simulations performed there, we also obtain a numerical evidence that the EI is 1. We remark that irrational rotations are completely outside the scope of application of the theoretical results obtained here, which depend heavily on the exponential decay of correlations of the systems considered.

6.3 A Different Cantor Set

In this section, we consider for maximal set a dynamically defined Cantor set as described in Section 4.1.1. In this case, Λ is generated by the quadratic dynamical system $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = 6x(1-x)$, *i.e.*,

$$\Lambda = \{x \in [0,1] \colon\ g^n(x) \in [0,1] \text{ for all } n \in \mathbb{N}\}.$$

In this case, we define the observable map

$$\begin{aligned} \varphi \colon [0,1] &\longrightarrow [0,1] \\ x &\longmapsto \begin{cases} n, & \text{if } n = \inf\{j \in \mathbb{N} \colon\ g^j(x) \notin [0,1]\} \\ \infty, & x \in \Lambda \end{cases} . \end{aligned} \tag{6.3.1}$$

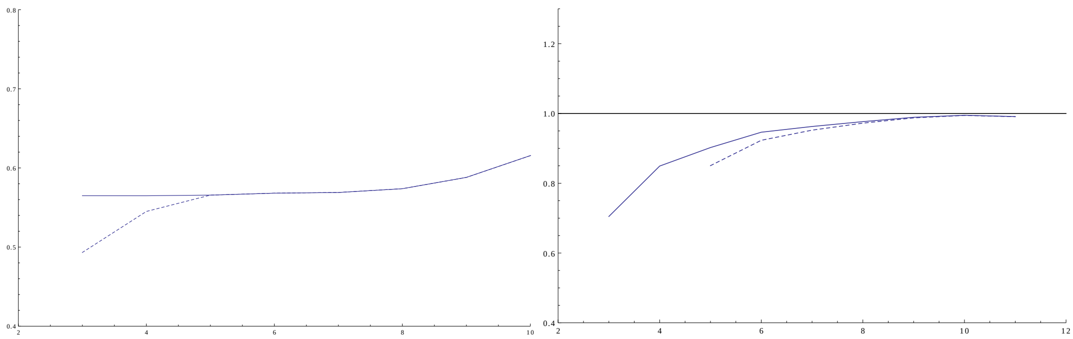

Fig. 6.6 On the y-axis, mean values of $\hat{\theta}_n(u,q)$ for each $5 \leq u \leq 20$ of the x-axis, with $n = 50.000$ and $\ell = 500$. The full line corresponds to $q = 1$, the dashed line to $q = 5$ and the dotted line to $q = 10$. On the left T is given by (6.3.2) and on the right T is given by $T(x) = 5x \mod 1$.

We studied numerically the behaviour of two systems.
The first one is defined by

$$\begin{aligned} T\colon\ [0,1] &\longrightarrow [0,1] \\ x &\longmapsto \begin{cases} g(x) & 0 \leq x < \frac{1}{6}\left(3-\sqrt{3}\right) \\ \frac{x+\frac{1}{6}\left(\sqrt{3}-3\right)}{\frac{1}{6}\left(\sqrt{3}-3\right)+\frac{1}{6}\left(3+\sqrt{3}\right)} & \frac{1}{6}\left(3-\sqrt{3}\right) \leq x < \frac{1}{6}\left(3+\sqrt{3}\right) \\ g(x) & \frac{1}{6}\left(3+\sqrt{3}\right) \leq x < 1 \end{cases}, \end{aligned} \tag{6.3.2}$$

which is compatible with the structure of the Cantor set Λ since its left and right branches coincide with the map g that generated Λ, just as F was compatible with G in Section 4.1.2. The second is the linear system $T : [0,1] \to [0,1]$, where $T(x) = 5x \mod 1$, which, *a priori*, has no reason to be compatible with the geometric structure of Λ. Both these systems are full branched Markov maps, which means that have decay of correlations against L^1 observables.
We note that, if we adapt the the arguments presented in Sections 4.1 and 5.2.3, one could check that conditions $Д(u_n, w_n)$ and $Д'(u_n, w_n)$ hold for these systems and the observable φ defined in (6.3.1). Hence, these examples fit the theory and we expect the existence of an EVL, but the analytical computation of the EI is much more complicated and cannot be carried as for the ternary Cantor set, in Sections 4.1 and 5.2.2.
As in the usual ternary Cantor set and the linear dynamics, the EI easily detects the compatibility between the dynamics and the fractal structure of Λ. In the first case, where T is given by (6.3.2), the numerical simulations reveal an EI approximately equal to 0.61, which is consistent with the expected connection between g and T, while in the second case, where $T(x) = 5x \mod 1$, we obtain an EI equal to 1 (see Figure 6.6).

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Appendix A

Definitions and Preliminary Results

In this appendix, we will present some definitions and results that are necessary to the theoretical coherence of the statements proved through this doctoral thesis. We divide the appendix in three different sections. In the first section, we state the definition of box dimension and Hausdorff dimension. We also state some necessary properties of these two concepts of fractal dimension. In the second section, we present results that allow to characterize the fractal dimension of an attractor of a Digraph Iterated Function System. For last, we state results that allow to better comprehend the spectral radius of a matrix.
Most of the time, the theorems are written without the respective proof. Nevertheless, a reference for such proof will always be provided.

A.1 Fractal Dimension

The concept of fractal dimension is widely used to help characterize a set that presents some fractal structure. There exists more than one definition of fractal dimension, each one having its advantages and disadvantages. In here, we state two definitions of fractal dimension: box dimension and Hausdorff dimension.
For a more detailed analysis of fractal geometry, the reader is referred to [14].

Definition A.1.1 (Box Dimension)**.** Let F de a subset of $\mathbb{R}^d$, then, the box dimension of F is defined as

$$\dim_B(F) = \lim_{\varepsilon \to 0} \frac{\log N_\varepsilon(F)}{-\log \varepsilon}, \tag{A.1.1}$$

where $N_\varepsilon(F)$ denotes the smallest number of balls of radius ε that cover F.

Definition A.1.2 (Hausdorff Dimension)**.** Let F be a subset of $\mathbb{R}^d$ and $\{F_i\}_{i\in\mathbb{N}}$ be a countable collection of sets, with diameter at most δ, that cover F. For $\alpha \geq 0$, we define the α - dimensional Hausdorff measure of F as

$$H^{\alpha}(U) = \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} |F_i|^{\alpha} : \text{where } \{F_i\} \text{ is a } \delta - \text{cover of } F \right\}.$$

The Hausdorff dimension of F is defined as

$$\dim_H(F) = \inf\{\alpha : H^{\alpha}(F) = 0\} = \sup\{\alpha : H^{\alpha}(F) = \infty\}.$$

The computation of the dimension of a set, using the definition of box dimension or the definition of Hausdorff dimension, leads often to different results. For example, if we consider the set of the rationals in $[0,1]$, its Hausdorff dimension will be 0 since it is a countable set. However, if we calculate the box dimension of the same set, we will obtain the value of 1.
Nevertheless, it exists a huge number of sets where the dimensions coincide. For fractal sets displaying some self-similar structure, as the attractor of an IFS satisfying the open set condition, we can prove that the box dimension and the Hausdorff dimension have the same value.

To finish this small tour in fractal dimensions, we state a property of both definitions that will be very useful in Chapter 5.
The box dimension and the Hausdorff dimension are finitely stable. If we consider a finite collection of subsets of $\mathbb{R}^d$, denoted by $\{E_1, \ldots, E_n\}$, then

$$\dim_B \left(\bigcup_{i=1}^{n} E_i \right) = \max_i \dim_B(E_i)$$

and

$$\dim_H \left(\bigcup_{i=1}^{n} E_i \right) = \max_i \dim_H(E_i).$$

A.2 Fractal Dimension and Digraph Iterated Function Systems

As introduced in Chapter 5, a Digraph IFS is constituted by a digraph G, where the set of vertices is denoted by V and the set of edges is denoted by E. To each of the vertices is associated a metric space X_v and to each edge that links the vertice u to the vertice v, denoted by $e \in E_{uv}$, we associate a similarity $f_e : X_v \to X_u$ with ratio r_e.
If r_e is smaller than 1 for every edge of G, then there exists a unique attractor W which is the union of compact sets W_v, one for every vertex, such that for every $u \in V$,

$$W_u = \bigcup_{v \in V} \bigcup_{e \in E_{uv}} f_e(W_v). \tag{A.2.1}$$

To each Digraph IFS, G, we associate a substitution matrix M that corresponds to the adjacency matrix of the digraph. For simplicity, we are going to assume in this discussion, that M is a $(0,1)$-matrix and that every similarity has a common ratio r smaller than 1. Note that, this assumption on M implies that E_{uv} consists of only, at most, one element for each possible (u,v).
In [32], the authors found a way to compute the Hausdorff dimension of the attractor W of a Digraph IFS satisfying the open set condition stated in Definition 5.1.2. For that purpose, we define the construction matrix A_G associated with a Digraph IFS, G. When the Digraph IFS satisfies the assumptions stated above, this matrix A_G is given by,

$$A_G = rM. \tag{A.2.2}$$

For a given $\delta > 0$, let $A_{G,\delta}$ denote the matrix $r^{\delta}M$ and let $\Phi(\delta)$ denote its spectral radius.

Theorem A.2.1 ([32], Theorem 3)**.** *For each strongly connected Digraph IFS, G, the Hausdorff dimension of the attractor W is the number* ε *such that* $\Phi(\varepsilon) = 1$*. Moreover, the* ε*- dimensional Hausdorff measure of W is finite.*

The theorem above implies that the Hausdorff dimension of an attractor W, of a strongly connected Digraph IFS G, satisfying the assumptions stated above, is given by

$$\dim_H(W) = \frac{-\log(\lambda)}{\log r}, \tag{A.2.3}$$

where λ denotes the spectral radius of M.
When the Digraph IFS is not strongly connected Mauldin and Williams, again in [32], discovered that $\dim_H(W)$ is the maximum of the Hausdorff dimensions of the attractors of the Digraph IFS created by considering the strongly connected components of G.
To be more formal, let $SC(G)$ denote the set of the strongly connected components of G and consider ε_H to be the Hausdorff dimension, as given by Theorem A.2.1, of the attractor of a Digraph IFS H, contained in $SC(G)$.

Theorem A.2.2 ([32], Theorem 4)**.** *For a Digraph IFS, G, the Hausdorff dimension of the attractor W is given by the number,*

$$\varepsilon = max\{\varepsilon_H : H \in SC(G)\}.$$

These two results allow us to characterize the Hausdorff Dimension of the attractor of a Digraph IFS. The next step is to characterize its box dimension.
In the article [10], Manav Das and Sze-Man Ngai proved a theorem that characterizes the box dimension of the attractor W of a Digraph IFS satisfying the so called *graph finite type condition*. Here, we will not state in detail the definition of the graph finite type condition due to its complexity. It suffices to say that this condition was devised to allow the computation of the fractal dimension of an attractor even if the correspondent Digraph IFS has overlaps that make impossible for the open set condition to hold. We refer the reader to [10] for more information on this topic and for the proof of the following theorem.

Theorem A.2.3 ([10], Theorem 1.1)**.** *Let G be a Digraph IFS of contractive similarities on* $\mathbb{R}^n$*, satisfying the graph finite type condition. Then, the correspondent attractor W, satisfies:*

- $dim_H(W) = dim_B(W)$,
- $H^{\alpha}(W) > 0$, *where* $\alpha := dim_H(W)$,
- *If G is strongly connected, then* $H^{\alpha}(W)$ *is finite.*

Recall that, an algebraic integer $\beta > 1$ is called a Pisot number if all of its algebraic conjugates are in modulus less than one. Manav Das and Sze-Man Ngai, again in [10], proved that a Digraph IFS, where the similarities have a specific form, satisfies the graph finite type condition.

Theorem A.2.4 ([10], Theorem 2.7)**.** *Let G be a Digraph IFS of contracting similarities in* $\mathbb{R}^n$ *such that, for every* $e \in E$*, each similarity,* f_e *can be written as*

$$\beta^{-s_e} R_e x + b_e,$$

where $\beta > 1$ *is a Pisot number,* s_e *is a positive integer,* R_e *is an orthogonal transformation and* $b_e \in \mathbb{R}^n$*. Assume that,* $\{R_e\}$ *generates a finite group H and*

$$H\{b_e : e \in E\} \subseteq k_1\mathbb{Z}[\beta] \times \ldots \times k_n\mathbb{Z}[\beta]$$

for some $k_1, \ldots, k_d$ *in* $\mathbb{R}$*.*
Then, G satisfies the graph finite type condition.

As a title of example, we can show that a Digraph IFS G, constituted by the similarities $x/3$ and $x/3 + 2/3$ satisfies the conditions of Theorem A.2.4.
Since, accordingly with [5], all integers bigger than 1 are Pisot numbers, it is only necessary to consider $\beta_e = 3$, $s_e = 1$ and R_e equal to the identity transformation on $\mathbb{R}$ for both similarities. Clearly, R_e generates a finite group and choosing $k_1 = 2/3$, we obtain $H\{0, 2/3\} \subseteq 2/3\mathbb{Z}[3]$ which implies that all of the assumptions of the Theorem are satisfied.
The Theorems A.2.1, A.2.2, A.2.3 and A.2.4, when used together, allows us to compute the box dimension and Hausdorff dimension of an attractor W of a Digraph IFS, G, that meets the following assumptions:

- G satisfies the open set condition,
- all of the similarities have a common contractive ratio r,
- the set E_{uv} is formed by one single element,
- the similarities can be written as in Theorem A.2.4.

For such Digraph IFS, the box dimension and the Hausdorff dimension coincide and are equal to

$$\dim_B(W) = \dim_H(W) = \frac{-\log(\lambda)}{\log r},$$

where λ denotes the spectral radius of the substitution matrix M of G.
The Digraph IFS systems considered in Chapter 5, which are constituted by the similarities $x/3$ and $x/3+2/3$ considered in the example above, satisfy all of the conditions stated here. It is precisely that fact that allows us to compute the box dimension of the correspondent attractors.

A.3 Spectral Radius of a Matrix

In this section, we introduce the concept of spectral radius of a matrix.
Start by considering a complex matrix $A \in \mathscr{M}^n(\mathbb{C})$. The spectral radius of A is defined as

$$\rho(A) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}.$$

To aid in the computation of such value, we recall that the Euclidean norm of a complex matrix $A \in \mathscr{M}^n(\mathbb{C})$ is defined as

$$\|A\|_2 = \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2},$$

where $\|x\|_2$ is the usual Euclidean vector norm.
This norm is multiplicative, in some literature also called consistent or sub-multiplicative, in the sense that satisfies

$$\|AB\|_2 \leq \|A\|_2 \|B\|_2,$$

for arbitrary matrices A and B.
Following a result from [11], we have that the spectral radius of any complex matrix satisfies

$$\rho(A) \leq \|A\|_2. \tag{A.3.1}$$

Recall that, a submatrix of a matrix A is a matrix created by deletion of some rows and columns that have the same index in the original matrix. It is possible to characterize the spectral radius of a submatrix of a matrix A, where each entry of A is either positive or zero. The proof of this statement can be found in [1].

Proposition A.3.1. *Assume that A is a nonnegative matrix, that is, every entry is either positive or* 0*, then, if B is a principal submatrix of A the spectral radius of B satisfies,*

$$\rho(B) \leq \rho(A).$$

To finish this appendix, we prove a result that, in spite of not being related to this subject at first glance, it will be useful to bound the spectral radius of the matrices in Chapter 5.

Proposition A.3.2. *Let a, b by any positive real numbers and consider $\varepsilon > 0$. Then,*

$$2ab \leq \frac{a^2}{\varepsilon} + \varepsilon b^2.$$

Proof. Young's inequality states that, for any positive reals c and d,

$$cd \leq \frac{c^2}{2} + \frac{d^2}{2}.$$

Let a, b be any positive reals and consider $\varepsilon > 0$.
Put $c = a/\sqrt{\varepsilon}$ and $d = b\sqrt{\varepsilon}$. Applying Young's inequality, we obtain that,

$$ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}.$$

Hence,

$$2ab \leq \frac{a^2}{\varepsilon} + \varepsilon b^2.$$

□