LARGE DEVIATIONS FOR DYNAMICAL SYSTEMS WITH STRETCHED EXPONENTIAL DECAY OF CORRELATIONS

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ABSTRACT. We obtain large deviations estimates for systems with stretched exponential decay of correlations, which improve the ones obtained in [AFLV11]. As a consequence we obtain better large deviations estimates for Viana maps and get large deviations estimates for a class of intermittent maps with stretched exponential loss of memory.

1. INTRODUCTION

In the last two decades the study of statistical properties of non-uniformly dynamical systems has been capturing much interest and attention. The memory loss of the systems given in terms of decay of correlations and its connections with limiting laws, such as central limit theorems, invariance principles, extreme value distributions, and other properties such as large deviation principles have been investigated thoroughly. In order to prove such properties, several different techniques have been used, such as: inducing, coupling, spectral analysis of transfer operators or renewal equations. One of the tools that has revealed to be very powerful is the construction of Gibbs-Markov structures called Young towers whose inducing times allow to describe the rates of mixing of the system ([You98, You99]) and ultimately obtain estimates for large deviations ([MN08]) and several types of limiting laws.

In [AFLV11], the authors studied the relations between the rates of decay of correlations, large deviation estimates and the tail of the inducing times of Young towers. In particular, they proved a sort of reciprocal of Young's results, namely, they have shown that if the system has a certain rate of decay of correlations, then it admits a Young tower whose inducing times' tail decays at a similar rate. This was done both for local diffeomorphisms and maps with critical/singular sets. The construction of the Young tower and the estimates on the induced time tail follow from large deviations estimates for the expansion function and for the distance to the critical/singular set. Hence, at the core of that paper is a result ([AFLV11, Theorem D]) that establishes a connection between the rates for large deviations estimates of observable functions and the rates of decay of correlations of those observables

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against essentially bounded functions. Two types of decay rates were considered: polynomial and stretched exponential (which under certain certain more restrictive assumptions allowed also to deal with exponential rates, as well). To be more precise, this result asserted that if a systems has decay of correlations for observables on a certain Banach space against all essentially bounded functions at a polynomial rate or at a stretched exponential rate, then the large deviations estimates for those particular observables decay, respectively, at a polynomial or stretched exponential rate. Moreover, the dependence of the exponents and constants was clearly stated. We remark that the polynomial case had already been proved in [Mel09]. However, as a corollary from the stretched exponential counterpart, in [AFLV11], the authors obtained, for the first time, stretched exponential estimates for the large deviations of Hölder continuous observables evaluated along the orbits of Viana maps.

The main goal of this note is to improve the large deviations estimates for the stretched exponential case obtained in [AFLV11, Theorem D]. Namely, we obtain a smaller loss in the exponent of the stretched exponential rate when one goes from decay correlations to large deviations estimates. We apply this result to Viana maps in order to obtain the best rates of large deviations estimates for these maps available in the literature. We also apply it to some some non-uniformly expanding maps introduced in [CDKM18], which result from a modification of the intermittent maps studied in [LSV99] carried out in order to produce stretched exponential decay of correlations.

The proof of the result is based on a technique introduced by Gordin that allows to write the sum of the random variables generated by the dynamics as a sum of reverse time martingale differences plus a coboundary. Then the problem is reduced to control the large deviations of the sum of martingale differences. In the polynomial case, in [Mel09], this is done using Rio's inequality. In the stretched exponential case, in [AFLV11], the Azuma-Hoeffding inequality was used, instead. Here, we use again Rio's inequality and a power series expansion of the exponential moments of the sum of martingale differences in order to improve the exponent in the large deviation estimates obtained in [AFLV11, Theorem D] .

This short paper is organised as follows. In Section 2 we state the main result and give some applications. Section 3 is dedicated to the proofs.

2. Statement of results and applications

Let $T : X \to X$ be a dynamical system with an ergodic invariant probability measure μ. Let $\varphi \in L^1(\mu)$ be an integrable observable with $\int_X \varphi d\mu = 0$, and let

$$\varphi_n = \sum_{k=0}^{n-1} \varphi \circ T^k.$$

We state our main result:

Theorem 1. *Let $\varphi \in L^\infty(\mu)$. Suppose that there exist $C_\varphi > 0$, $\tau > 0$ and $\theta \in (0,1]$ such that*

$$\left| \int_X \varphi.\psi \circ T^n d\mu \right| \leq C_\varphi \|\psi\|_{L^\infty_\mu} e^{-\tau n^\theta}, \ \forall \psi \in L^\infty(\mu), \ n \geq 0. \tag{1}$$

Then there exists $c > 0$ such that for all $n \geq 1$ and $\epsilon > 0$,

$$\mu(|\varphi_n| > n\epsilon) \leq 2' e^{-\tau' \epsilon^{2\theta'} n^{\theta'}},$$

with $\theta' = \frac{\theta}{\theta+1}$, $\tau' = c\widetilde{C}_\varphi^{-2\theta'}$ and $\widetilde{C}_\varphi = \max\{\|\varphi\|_{L^\infty_\mu}, C_\varphi\}$.

Note that the exponent $\theta' = \frac{\theta}{\theta+1}$ that we obtain for the large deviations estimate is larger than the one obtained in [AFLV11, Theorem D], where the exponent was equal to $\frac{\theta}{\theta+2}$.

Of course, this result gives rise to the natural question regarding to which extent the exponent for the stretched exponential large deviations estimate that we managed to improve here is optimal. We believe that using the Gordin's technique, unless some better estimates for the large deviations of the sum of martingale differences are obtained, there is not much room for improvement.

2.1. **Large deviations for Viana maps.** In [Via97], Viana introduced an important class of nonuniform expanding dynamical systems with critical sets in dimension greater than one. This class of maps can be described as follows. Let $a_0 \in (1,2)$ be such that the critical point $x = 0$ is pre-periodic for the quadratic map $Q(x) = a_0 - x^2$. Let $S^1 = \mathbb{R}/\mathbb{Z}$ and $b : S^1 \to \mathbb{R}$ be a Morse function, for instance, $b(s) = \sin(2\pi s)$. For fixed small $\alpha > 0$, consider the map

$$\begin{array}{rccc} \hat{f}: & S^1 \times \mathbb{R} & \longrightarrow & S^1 \times \mathbb{R} \\ & (s,x) & \longmapsto & \big(\hat{g}(s), \hat{q}(s,x)\big) \end{array}$$

where $\hat{q}(s,x) = a(s) - x^2$ with $a(s) = a_0 + \alpha b(s)$, and $\hat{g}$ is the uniformly expanding map of the circle defined by $\hat{g}(s) = ds \pmod{\mathbb{Z}}$ for some integer $d \geq 2$. It is easy to check that for $\alpha > 0$ small enough there is an interval $I \subset (-2,2)$ for which $\hat{f}(S^1 \times I)$ is contained in the interior of $S^1 \times I$. Hence, $S^1 \times I$ is a forward invariant region for any map f sufficiently close to $\hat{f}$ in the C^0 topology. Any such map restricted to $S^1 \times I$ is called a *Viana map*. It was shown in [Alv00] that Viana maps have a unique ergodic expanding absolutely continuous invariant probability measure (acip) μ. In [Gou06], it was proved that every Viana map exhibits stretched exponential decay of correlations, with $\theta = 1/2$, for Hölder continuous functions against $L^\infty(\mu)$ functions. Consequently, the following corollary is a direct application of Theorem 1.

Corollary 2. *Let f be a Viana map and let μ be its unique expanding acip. Then, for every Hölder continuous observable φ and every $\epsilon > 0$, there exist $\tau = \tau(\varphi, \epsilon) > 0$ and $C = C(\varphi, \epsilon) > 0$ such that $\mu\left(\left|\frac{1}{n}\varphi_n - \int \varphi \, d\mu\right| > \epsilon\right) \leq Ce^{-\tau n^{1/3}}$.*

For comparison purposes, we remark that the large deviations estimate obtained in [AFLV11, Theorem G] was of the order of $e^{-\tau' n^{1/5}}$.

2.2. **Large deviations for intermittent maps with stretched exponential decay of correlations.** We consider the family of interval maps $f_\gamma : [0,1] \to [0,1]$, with $\gamma \geq (0,1]$, introduced in [CDKM18, Appendix A], which result from adapting the intermittent maps studied in [LSV99] so that the contact between the graph of f_β and the identity at the fixed point creates a stretched exponential fast accumulation of the pre-orbit of $1/2$ at 0, which eventually is responsible for stretched exponential decay of correlations. Namely, consider that

$$f_\gamma(x) = \begin{cases} x\left(1+\frac{(\log 2)^{\gamma^{-1}-1}}{|\log x|^{\gamma^{-1}-1}}\right), & x \leq 1/2 \\ 2x-1, & x > 1/2 \end{cases}. \tag{2}$$

From [CDKM18, Theorem A.1], it follows that f_γ has an absolutely continuous invariant measure μ and exhibits stretched exponential decay of correlations, with $\theta = \gamma$, for Hölder continuous functions against $L^\infty(\mu)$ functions. Hence, as consequence of Theorem 1, we obtain:

Corollary 3. *Let f_γ be as in (2) and let μ be its acip. Then, for every Hölder continuous observable φ and every $\epsilon > 0$, there exist $\tau = \tau(\varphi, \epsilon) > 0$ and $C = C(\varphi, \epsilon) > 0$ such that $\mu\left(\left|\frac{1}{n}\varphi_n - \int \varphi \, d\mu\right| > \epsilon\right) \leq Ce^{-\tau n^{\gamma/(\gamma+1)}}$.*

3. Proof

To prove Theorem 1, we will estimate all the moments of order q of φ_n.

Lemma 4. *There exists $K > 0$ such that for all $q > 0$ and $n \geq 1$, and all $\varphi \in L^\infty(\mu)$ satisfying (1), we have*

$$\left(\int_X |\varphi_n|^q d\mu\right)^{\frac{1}{q}} \leq K\widetilde{C}_\varphi q^{\frac{1}{2}\left(1+\frac{1}{\theta}\right)} n^{\frac{1}{2}}.$$

Proof. In this proof, we will use K to designate a generic constant whose value can change from one occurence to the other. The value of K depends only on the dynamical system (X, T, μ), and in particular is independent from n, q and the observable φ.

We will follow closely the proof of [Mel09, Lemma 2.1], adapted to our assumption of stretched exponential decay, keeping a precise track of the dependence in q of all the estimates.

Let $\mathcal{L} : L^1(\mu) \to L^1(\mu)$ be the transfer operator of T, *i.e.* the unique operator satisfying

$$\int_X \varphi.\psi \circ T d\mu = \int_X \mathcal{L}\varphi.\psi d\mu, \ \ \forall \varphi \in L^1(\mu), \, \forall \psi \in L^\infty(\mu).$$

Since $L^\infty(\mu)$ is the dual of $L^1(\mu)$, (1) implies that $\|\mathcal{L}^n\varphi\|_{L^1_\mu} \leq C_\varphi e^{-\tau n^\theta}$. Hence, for all $q \geq 1$,

$$\int_X |\mathcal{L}^n\varphi|^q d\mu \leq \|\mathcal{L}^n\varphi\|_{L^\infty_\mu}^{q-1} \int_X |\mathcal{L}^n\varphi| d\mu \leq C_\varphi \|\varphi\|_{L^\infty_\mu}^{q-1} e^{-\tau n^\theta}.$$

We then have

$$\|\mathcal{L}^n\varphi\|_{L^q_\mu} \le C_\varphi^{\frac{1}{q}}\|\varphi\|_{L^\infty_\mu}^{1-\frac{1}{q}} e^{-\frac{\tau}{q}n^\theta} \le \widetilde{C}_\varphi e^{-\frac{\tau}{q}n^\theta}.$$

Defining $\chi = \sum_{n=1}^\infty \mathcal{L}^n\varphi$, we get, for all $q \ge 1$,

$$\|\chi\|_{L^q_\mu} \le \sum_{n=1}^\infty \|\mathcal{L}^n\varphi\|_{L^q_\mu} \le \widetilde{C}_\varphi \sum_{n=1}^\infty e^{-\frac{\tau}{q}n^\theta}.$$

Since we have

$$\sum_{n=1}^\infty e^{-\frac{\tau}{q}n^\theta} \le \sum_{n=1}^\infty \int_{n-1}^n e^{-\frac{\tau}{q}t^\theta}dt = \int_0^\infty e^{-\frac{\tau}{q}t^\theta}dt = \frac{1}{\theta}\left(\frac{q}{\tau}\right)^{\frac{1}{\theta}} \int_0^\infty e^{-s}s^{\frac{1}{\theta}-1}ds = q^{\frac{1}{\theta}}\frac{1}{\theta}\frac{1}{\tau^{\frac{1}{\theta}}}\Gamma\left(\frac{1}{\theta}\right),$$

where we have performed the change of variables $s = \frac{\tau}{q}t^\theta$, we obtain $\|\chi\|_{L^q_\mu} \le K\widetilde{C}_\varphi q^{\frac{1}{\theta}}$.

Defining $\phi = \varphi + \chi - \chi\circ T$, we also have $\|\phi\|_{L^q_\mu} \le \|\varphi\|_{L^q_\mu} + 2\|\chi\|_{L^q_\mu} \le \|\varphi\|_{L^\infty_\mu} + 2K\widetilde{C}_\varphi q^{\frac{1}{\theta}} \le K\widetilde{C}_\varphi q^{\frac{1}{\theta}}$.

It is immediate to verify that $\mathcal{L}\phi = 0$, and so $\{\phi\circ T^k;\ k = 0,1,2,\ldots\}$ is a sequence of reverse martingale differences. Passing to the natural extension, we can assume without loss of generality that $\{\phi\circ T^k;\ k = 0,1,2,\ldots\}$ is a sequence of martingale differences with respect to a filtration $\{\mathcal{F}_k;\ k = 0,1,2,\ldots\}$.

By Rio's inequality [Rio00, Theorem 2.5], we have for all $q \ge 1$:

$$\|\varphi_n\|_{L^{2q}_\mu}^{2q} \le \left(4q\sum_{i=1}^n b_{i,n}\right)^q$$

where

$$b_{i,n} = \max_{i\le j\le n}\left\|\varphi\circ T^i \sum_{k=i}^j \mathbb{E}(\varphi\circ T^k|\mathcal{F}_i)\right\|_{L^q_\mu} \le \|\varphi\|_{L^\infty_\mu} \max_{i\le j\le n}\left\|\sum_{k=i}^j \mathbb{E}(\varphi\circ T^k|\mathcal{F}_i)\right\|_{L^q_\mu}$$

From the definition of ϕ and the martingale property, it follows

$$\sum_{k=i}^j \mathbb{E}(\varphi\circ T^k|\mathcal{F}_i) = \phi\circ T^i - \mathbb{E}(\chi\circ T^i|\mathcal{F}_i) + \mathbb{E}(\chi\circ T^{j+1}|\mathcal{F}_i),$$

so that

$$\left\|\sum_{k=i}^j \mathbb{E}(\varphi\circ T^k|\mathcal{F}_i)\right\|_{L^q_\mu} \le \|\phi\|_{L^q_\mu} + 2\|\chi\|_{L^q_\mu} \le K\widetilde{C}_\varphi q^{\frac{1}{\theta}}.$$

Hence $b_{i,n} \le \|\varphi\|_{L^\infty_\mu}K\widetilde{C}_\varphi q^{\frac{1}{\theta}} \le K\widetilde{C}_\varphi^2 q^{\frac{1}{\theta}}$ and we thus obtain for all $q \ge 1$

$$\|\varphi_n\|_{L^{2q}_\mu}^{2q} \le \left(4K\widetilde{C}_\varphi^2 q^{1+\frac{1}{\theta}}n\right)^q,$$

which yields the required estimate for all $q \geq 2$. The general case of $q > 0$ is deduced by changing the value of the constant K, since the map $q \mapsto \left(\int_X |\varphi_n|^q d\mu\right)^{\frac{1}{q}}$ is non decreasing for $q \in (0, \infty)$. $\square$

Lemma 5. *There exists $c > 0$ such that for all $\varphi \in L^\infty(\mu)$ satisfying* (1)*:*

$$\sup_{n \geq 1} \int_X e^{\tau' n^{-\theta'} |\varphi_n|^{2\theta'}} d\mu \leq 2,$$

with $\tau' = c\widetilde{C}_\varphi^{-2\theta'}$.

Proof. By expanding the exponential in power series and using Fubini's theorem and Lemma 4, we have for all $\tau' > 0$ and $n \geq 1$:

$$\int_X e^{\tau' n^{-\theta'} |\varphi_n|^{2\theta'}} d\mu = \sum_{k=0}^{\infty} \frac{(\tau')^k n^{-k\theta'}}{k!} \int_X |\varphi_n|^{2k\theta'} d\mu \leq \sum_{k=0}^{\infty} \left(2\theta' \tau' K^{2\theta'} \widetilde{C}_\varphi^{2\theta'}\right)^k \frac{k^k}{k!}.$$

Since $k! \geq \left(\frac{k}{3}\right)^k$ for all $k \geq 1$, we have

$$\int_X e^{\tau' n^{-\theta'} |\varphi_n|^{2\theta'}} d\mu \leq \sum_{k=0}^{\infty} \left(6\theta' K^{2\theta'} \widetilde{C}_\varphi^{2\theta'} \tau'\right)^k = 2,$$

if we set $\tau' = c\widetilde{C}_\varphi^{-2\theta'}$ for $c = (12\theta' K^{2\theta'})^{-1}$. $\square$

Proof of Theorem 1. For all $n \geq 1$ and $\epsilon > 0$, by Markov's inequality and Lemma 5,

$$\begin{aligned} \mu(|\varphi_n| > n\epsilon) &= \mu(e^{\tau' n^{-\theta'} |\varphi_n|^{2\theta'}} > e^{\tau' n^{\theta'} \epsilon^{2\theta'}}) \\ &\leq e^{-\tau' n^{\theta'} \epsilon^{2\theta'}} \int_X e^{\tau' n^{-\theta'} |\varphi_n|^{2\theta'}} d\mu \\ &\leq 2e^{-\tau' n^{\theta'} \epsilon^{2\theta'}}. \end{aligned}$$

$\square$

Remark 6. By changing the value of c, we can replace the constant 2 by any constant of the form $1 + \delta$, $\delta > 0$.

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